Moment of inertia for axisymmetric neutron stars in the standard model extension

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We develop a consistent approach to calculate the moment of inertia (MOI) for axisymmetric neutron stars (NSs) in the Lorentz-violating Standard-Model Extension (SME) framework. To our knowledge, this is the first relativistic MOI calculation for axisymmetric NSs in a Lorentz-violating gravity theory other than deformed, rotating NSs in general relativity. Under Lorentz violation, there is a specific direction in the spacetime, and NSs get stretched or compressed along that direction. When a NS is spinning stationarily along this direction, a conserved angular momentum and the concept of MOI are well defined. In the SME framework, we calculate the partial differential equation governing the rotation and solve it numerically with the finite element method to get the MOI for axisymmetric NSs caused by Lorentz violation. Besides, we study an approximate case where the correction to the MOI is regarded solely from the deformation of the NS and compare it with its counterpart in the Newtonian gravity. Our formalism and the numerical method can be extended to other theories of gravity for static axisymmetric NSs.

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I. INTRODUCTION

At the classical level, gravitational phenomena are well described by general relativity (GR), which has withstood various experimental tests over the past century with flying colors [1,2]. At the quantum level, the Standard Model (SM) of particle physics provides an accurate description of interactions between microscopic particles. Together, GR and SM form the foundation for our contemporary understanding of the nature. However, there has been a longstanding quest to a final theory, so-called quantum gravity, that can consistently describe all phenomena. Quantum gravity is expected to exhibit unique behaviors different from GR at the Planck energy scale, but testing theories at the Planck scale is challenging if possible [3,4]. Therefore, physicists have turned their attention to searching for relic effects of quantum gravity at low energy scales, and Lorentz violation is one possible relic effect [3–8]. A field-theoretic approach, the Standard-Model Extension (SME), collects all possible operators of Lorentz violation in a Lagrangian [8–11],

$$\mathcal{L}_{\mathrm{SME}} = \mathcal{L}_{\mathrm{GR}} + \mathcal{L}_{\mathrm{SM}} + \mathcal{L}_{\mathrm{LV}} + \mathcal{L}_{\mathrm{k}}, \tag{1}$$

where $\mathcal{L}_{\mathrm{GR}}$ represents the Einstein-Hilbert term for GR, $\mathcal{L}_{\mathrm{SM}}$ is the Lagrangian of the SM, $\mathcal{L}_{\mathrm{LV}}$ is the Lorentz-violating term, and $\mathcal{L}_{\mathrm{k}}$ describes the dynamics of the Lorentz-violating fields. For the term $\mathcal{L}_{\mathrm{LV}}$, in this study we consider the minimal gravitational Lorentz violation with operators of mass dimension four [8],

$$\mathcal{L}_{\mathrm{LV}}^{(4)} = \frac{1}{16\pi}\left(-uR + s^{\mu\nu}R^{\mathrm{T}}_{\mu\nu} + t^{\alpha\beta\gamma\delta}C_{\alpha\beta\gamma\delta}\right), \tag{2}$$

where R is the Ricci scalar, $R^{\mathrm{T}}_{\mu\nu}$ is the trace-free Ricci tensor, $C_{\alpha\beta\gamma\delta}$ is the Weyl conformal tensor, and u, $s^{\mu\nu}$, $t^{\alpha\beta\gamma\delta}$ are the Lorentz-violating fields. In the SME framework, we can describe the Lorentz-violating fields by introducing their vacuum expectation values, $\bar{u}$, $\bar{s}^{\mu\nu}$, and $\bar{t}^{\alpha\beta\gamma\delta}$, which are then called the Lorentz-violation coefficients [8]. Extensive experiments have been conducted to constrain the Lorentz-violation coefficients [12–21].

In this work we will consider neutron stars (NSs) in the SME framework. NSs are ideal laboratories for testing fundamental theories and principles, including the Lorentz symmetry [22–29]. Pulsars, which are rotating NSs, provide us a superb opportunity to test theories of gravity [22,23,27] including the Lorentz symmetry in circumstances of strong gravitational field [13,14,18,19]. In some cases, the uncertain equation of state (EOS) for dense nuclear matter of NSs could introduce degeneracy with gravity tests [30–32]. Nevertheless, measurements of NS properties, such as mass, radius, moment of inertia (MOI), and tidal Love number offer us an avenue to study the

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EOS [33–40]. Through high-precision pulsar timing observations [27,37,39,41], gravitational-wave detections of binary NS mergers [42–44], and multiwavelength observations of x-ray pulsars [45,46], we can obtain high-precision measurements of the structure of NSs. These constraints on the EOS of NSs help us gain insights into the physics of dense nuclear matter, as well as gravity tests.

With Lorentz violation, NSs undergo nonspherical deformations. Studying the structure of NSs under Lorentz violation can help us test the SME framework and offer the potential for identifying additional observable effects [47–49]. Xu *et al.* [47] employed a method similar to the post-Tolman-Oppenheimer-Volkoff (post-TOV) approach [50] to deal with the effects from Lorentz violation, and obtained the leading-order corrections to the structure of NSs caused by Lorentz violation. In this paper, we attempt to extend the study of the structure of NSs under Lorentz violation. In particular, we focus on the MOI of NSs.

MOI is one of the crucial structural parameters of NSs, as it characterizes the rotational properties of NSs. MOI is closely connected to the central issues in NS physics. Firstly, observations and studies of MOI can help us constrain the EOS of NSs [27,51,52]. The MOI of NSs varies with different EOSs, and it can also be measured directly from high-precision observations of binary pulsars [27,41]. Owing to the high precision of pulsar timing, there is the potential to detect orbital effects related to the MOI, e.g. through the periastron advance caused by the spin-orbit coupling. Currently, with a 16-year data span, an upper limit of the MOI for PSR J0737-3039A in the Double Pulsar system has been obtained [27]. With the advent of the next generation radio telescopes, such as the Square Kilometre Array, there is hope for direct measurements of the MOI of NSs [41,53], offering us a means to study the EOS of NSs. Secondly, glitch phenomena in pulsar timing observations are also believed to be related to the MOI of NSs. Glitches are one type of timing irregularities in pulsar timing observations, which manifest as sudden changes in rotation frequencies of pulsars, and are often followed by a relaxation [54–56]. There are a lot of theoretical models that aim to explain glitches, such as models involving superfluid or crustquake [57–60]. Investigating the origins of glitches contributes to our comprehension of the physics within NSs. Considering that the angular momentum of a NS is conserved or almost conserved, any changes in the MOI will result in variations in the angular velocity, leading to noticeable observational effects for pulsars. If we intend to explain glitches through deformations of NSs, we need to calculate the MOI and infer the variation in the MOI from the change of angular velocity. Some studies have also attempted to explain the unexplained issues in glitches, such as the deficiency of MOI contributed by the NS crust [61], e.g. with a modified gravity [62]. Additionally, considering the precision of pulsar spin measurements, other MOI-related physical processes affecting NS rotation may also be measurable. In this context, calculating the MOI corrections induced by these physical processes is essential.

Research on the MOI of NSs in a relativistic setting can be traced back to the 1960s when Hartle and Thorne [63,64] calculated the structure of slowly rotating NSs in GR and computed the MOI for spherically symmetric NSs. Their results showed a significant difference between the calculations in GR and those in the Newtonian gravity. Another important theoretical work related to the MOI of NSs is the discovery of the so-called I-Love-Q relation, which is one of the most famous universal relations for NSs [65,66]. Numerical calculations revealed that the relations between any two of the dimensionless MOI, the dimensionless tidal Love number, and the dimensionless quadrupole moment are insensitive to the EOS of NSs. The I-Love-Q relation provides us a way to test gravity theories independently of the EOS [32]. In addition, calculations have also been performed on the MOI for NSs in alternative gravity theories [67–69], but they are limited to the assumption of the spherical background configuration.

It is worth noting that previous calculations of the MOI of NSs have been based on the assumption of spherical symmetry. To our knowledge, no calculations in the relativistic setting have been performed yet regarding the correction to the MOI caused by nonspherical deformations other than rotation itself. Indeed, considering the MOI of nonspherical NSs is meaningful. Firstly, there exist various physical processes that can induce nonspherical deformations in NSs, such as crustal deformations, magnetic field effects and so on [70,71]. Exploring the corrections to MOI caused by nonspherical deformations can provide valuable insights into the structure and dynamics of NSs, and contribute to our understanding of complex behaviors of NSs. Secondly, from the perspective of gravity theories, there are some modified gravity theories breaking the spherical symmetry, such as the bumblebee theory [8] and the Einstein-Æther theory [72]. In these gravity theories, there may exist axisymmetric solutions for NSs which are more stable than the spherical ones. In that case, studying the structure of nonspherical NSs helps us understand these theories better.

In this context, we present a consistent calculation of the MOI for axisymmetric NSs in the SME framework. The organization of the paper is as follows. In Sec. II, we introduce the calculation of MOI for spherical NSs in GR to lay the groundwork. In Sec. III, we first review the deformed NSs in the SME found in Ref. [47] in Sec. III A. Then in Sec. III B, we obtain the partial differential equation (PDE) that describes the rotational metric in the SME, retaining the correction terms up to the first order in the Lorentz-violation coefficients. In Sec. III C, we solve the PDE numerically with the finite element method to get the MOI for NSs. Finally, we summarize in Sec. IV. In this paper, we adopt the units where $G = c = 1$.

II. MOI OF SPHERICAL NSs IN GR

In GR, the definition of MOI is based on the definitions of angular velocity and angular momentum. To obtain the MOI of a NS, we need to calculate the gravitational field equation to get the metric of the rotating spacetime [63,64]. We begin with the metric of a stationary, axially symmetric system,

$$ds^2 = -H^2dt^2 + Q^2dr^2 + r^2K^2[d\theta^2 + \sin^2\theta(d\varphi - Ldt)^2], \tag{3}$$

where H, Q, K, and L are functions of r and θ. The corresponding four-velocity of the fluid reads as

$$u^\mu = (u^t, 0, 0, \Omega u^t). \tag{4}$$

We adopt the assumption of slow rotation, where the effects on pressure, energy density, and gravitational field caused by the rotation can be treated as perturbations. In this case, we can expand L in orders of Ω,

$$L(r,\theta) = \omega(r,\theta) + \mathcal{O}(\Omega^3). \tag{5}$$

Then, we can solve ω from the following field equation:

$$G^t_\varphi = 8\pi T^t_\varphi. \tag{6}$$

It is worth noting that the leading-order correction to L is of order Ω but the leading-order corrections of H, Q, and K are of order Ω^2. If we want to calculate the leading-order effect, we can consider all diagonal components of the metric in Eq. (6) as the background solution of a spherical NS, whose metric is commonly written as

$$ds^2 = -e^{\nu(r)}dt^2 + e^{\lambda(r)}dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2). \tag{7}$$

Finally, we can express the field equation (6) in the form of [63,64]

$$\frac{1}{r^4}\frac{\partial}{\partial r}\left(r^4 j\frac{\partial\bar{\omega}}{\partial r}\right) + \frac{4}{r}\frac{\mathrm{d}j}{\mathrm{d}r}\bar{\omega} + \frac{e^{(\lambda-\nu)/2}}{r^2}\frac{1}{\sin^3\theta}\frac{\partial}{\partial\theta}\left(\sin^3\theta\frac{\partial\bar{\omega}}{\partial\theta}\right) = 0, \tag{8}$$

where

$$\bar{\omega}(r,\theta) \equiv \Omega - \omega(r,\theta), \tag{9}$$

and $j(r) \equiv e^{(\lambda+\nu)/2}$. The regular condition and boundary condition are given by

$$\bar{\omega}|_{r=0} = \text{constant}, \tag{10}$$

$$r^2(\Omega - \bar{\omega})|_{r\to\infty} = 0. \tag{11}$$

The separation of variables method is the most straightforward approach to solve the PDE. Fortunately, Eq. (8) can be separated using vector spherical harmonics,

$$\bar{\omega}(r,\theta) = \sum_{l=1}^{\infty}\bar{\omega}_l(r)\left(-\frac{1}{\sin\theta}\frac{\mathrm{d}P_l(\cos\theta)}{\mathrm{d}\theta}\right), \tag{12}$$

where $P_l(\cos\theta)$ is the Legendre function, and $\bar{\omega}_l(r)$ satisfies

$$\frac{1}{r^4}\frac{\mathrm{d}}{\mathrm{d}r}\left[r^4 j(r)\frac{\mathrm{d}\bar{\omega}_l}{\mathrm{d}r}\right] + \left[\frac{4}{r}\frac{\mathrm{d}j}{\mathrm{d}r} - e^{(\lambda-\nu)/2}\frac{l(l+1)-2}{r^2}\right]\bar{\omega}_l = 0. \tag{13}$$

With the boundary condition and the regular condition, it is proved that $\bar{\omega}_l(r)$ vanishes except for $l = 1$ and thus $\bar{\omega}$ is independent of θ [63,64]. Equation (8) reduces to an ordinary differential equation (ODE),

$$\frac{1}{r^4}\frac{d}{dr}\left(r^4 j\frac{d\bar{\omega}}{dr}\right) + \frac{4}{r}\frac{dj}{dr}\bar{\omega} = 0. \tag{14}$$

The solution outside the star has the form

$$\bar{\omega}(r) = \Omega - \frac{2J}{r^3}, \tag{15}$$

where J is just the angular momentum of the NS. Finally, the definition of MOI reads as $I \equiv J/\Omega$.

We can get a more compact equation for the MOI with some further definitions. First, we define an angular momentum function of the variable r,

$$\mathcal{J}(r) \equiv \frac{1}{6}r^4\left(\frac{d\bar{\omega}(r)}{dr}\right), \tag{16}$$

and a corresponding MOI function

$$\mathcal{I}(r) \equiv \frac{\mathcal{J}(r)}{\Omega}. \tag{17}$$

When r is larger than the radius of the NS, R, the angular momentum function $\mathcal{J}$ and the MOI function $\mathcal{I}$ are equal to J and I, respectively.

With the above definitions and Eq. (14), we can obtain the ODE of $\mathcal{I}(r)$ [29],

$$\frac{\mathrm{d}\mathcal{I}}{\mathrm{d}r} = \frac{8}{3}\pi r^4\rho\left(1+\frac{p}{\rho}\right)\left(1-\frac{5}{2}\frac{\mathcal{I}}{r^3}+\frac{\mathcal{I}^2}{r^6}\right)\left(1-\frac{2m}{r}\right)^{-1}, \tag{18}$$

where ρ and p are the energy density and the pressure of NSs respectively, and m is the mass function defined in the TOV equation. The form of Eq. (18) is similar to the TOV equation for p where the right-hand side is the Newtonian term times three dimensionless factors.

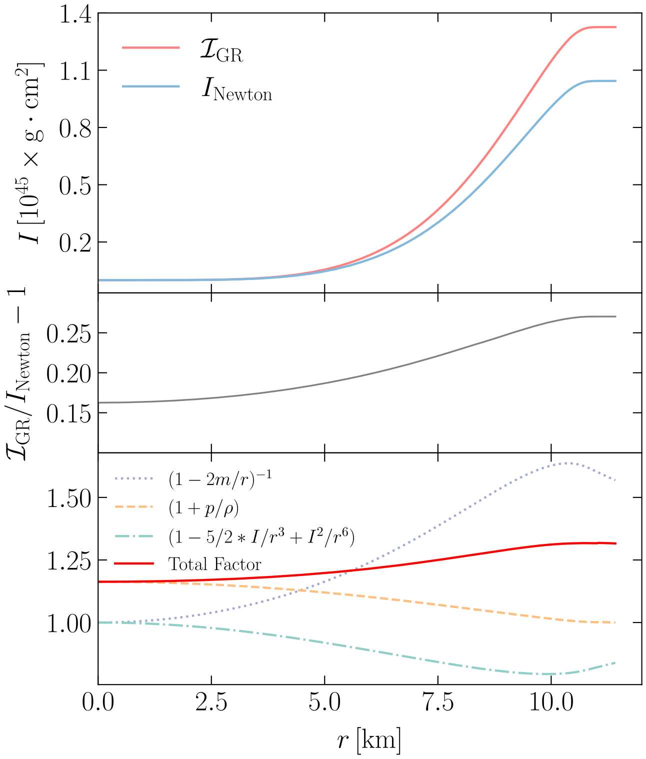

FIG. 1. The MOI function of a $1.4M_\odot$ NS with EOS AP4. The top panel represents the MOI function in GR, $\mathcal{I}_{\rm GR}$, and the MOI in Newtonian gravity, $I_{\rm Newton}$. The middle panel represents the relative difference between $\mathcal{I}_{\rm GR}$ and $I_{\rm Newton}$. The bottom panel represents the magnitude of the dimensionless correction factors in Eq. (18). The "Total Factor" is the product of three correction factors.

We consider a NS of $1.4M_\odot$ with EOS AP4 as an example, and calculate the MOI function $\mathcal{I}$ with respect to r. The result is shown in Fig 1. Besides, we calculate the MOI in the framework of Newtonian gravity according to

$$\frac{\mathrm{d}I_{\rm Newton}}{\mathrm{d}r} = \frac{8}{3}\pi r^4 \rho. \tag{19}$$

The difference in the MOI between GR and the Newtonian gravity reaches ~20%, which is consistent with the value of the dimensionless factors. Furthermore, the calculation of MOI in GR is no longer linear, because on the right-hand side of Eq. (18), there exists a factor related to $\mathcal{I}$ breaking the linearity. That is to say, if the MOI of a sphere is I_1 and the MOI of another concentric sphere is I_2, then the total MOI is not simply $I_1 + I_2$ as in the case of Newtonian gravity.

III. MOI OF AXISYMMETRIC NSs IN THE SME

In this section, we derive the modified PDE in the SME to calculate the MOI for axisymmetric NSs. In the axisymmetric case, the PDE becomes more complicated in its dependence on θ, making it difficult to solve the PDE through separation of variables. We analyze the asymptotic behavior of the solution and solve it numerically with the finite element method.

A. NSs in the SME

In the gravitational sector of the minimal SME, the linearized field equations can be written as [8,73]

$$G_{\mu\nu} = 8\pi T_{\mu\nu} - \bar{s}^{\alpha\beta} G_{\mu\alpha\beta\nu}, \tag{20}$$

where

$$\begin{aligned} G_{\alpha\beta\gamma\delta} = {} & -R_{\alpha\beta\gamma\delta} + g_{\alpha\gamma}R_{\beta\delta} + g_{\beta\delta}R_{\alpha\gamma} - g_{\alpha\delta}R_{\beta\gamma} \\ & - g_{\beta\gamma}R_{\alpha\delta} - \frac{1}{2}(g_{\alpha\gamma}g_{\beta\delta} - g_{\alpha\delta}g_{\beta\gamma})R, \end{aligned} \tag{21}$$

and $\bar{s}^{\alpha\beta}$ are the Lorentz-violation coefficients. By applying it to the strong-field regime, we acknowledge that higher-order corrections at $\mathcal{O}(s^2 \cdot h)$ and $\mathcal{O}(s \cdot h^2)$ might exist, where s is the typical value of $\bar{s}^{\alpha\beta}$ components and h is the typical metric deviation from the flat spacetime. For NSs, $h \sim 0.1$ while limits on $\bar{s}^{\alpha\beta}$ are as low as 10^{-11} [13,21]. If we briefly assume $s \lesssim h \sim 0.1$, then the relative error caused by applying Eq. (20) to NSs is less than $\mathcal{O}(s \cdot h) \lesssim 1\%$. As a starting study for testing Lorentz symmetry using future precise measurements of NSs' MOI, we can tolerate the error for now. Our formalism to be presented can be directly applied to a more general version of Eq. (20) where higher-order terms are included.

Using Eq. (20), the Lorentz-violating term $\bar{s}^{\alpha\beta}G_{\mu\alpha\beta\nu}$ results in corrections to the metric of the NS. The modified metric can be represented as

$$g_{\mu\nu} = g^{\rm LI}_{\mu\nu} + \delta g^{\rm LV}_{\mu\nu}, \tag{22}$$

where

$$g^{\rm LI}_{\mu\nu} = \mathrm{diag}\{-e^{\nu}, e^{\lambda}, r^2, r^2 \sin\theta\} \tag{23}$$

is the TOV solution for the NS in GR, and

$$\delta g^{\rm LV}_{\mu\nu} = \mathrm{diag}\{-\delta\phi(r,\theta), 0, 0, 0\} \tag{24}$$

represents the correction caused by Lorentz violation. We have [8]

$$\delta\phi(r,\theta) = -\bar{s}^{jk} U^{jk}, \tag{25}$$

and

$$U^{jk} = \int \frac{(x^j - x'^j)(x^k - x'^k)}{|\vec{x} - \vec{x}'|^3} \rho(\vec{x}') \mathrm{d}^3 x', \tag{26}$$

where ρ is the energy density distribution in the NS. Note that the repeated indices in Eq. (25) are summed over.

Under the perturbation of Lorentz violation, the structure of a NS is affected [47]. The energy density and the pressure are given by

$$\rho = \rho^{(0)} + \rho^{(1)}, \tag{27}$$

$$p = p^{(0)} + p^{(1)}, \tag{28}$$

where $\rho^{(0)}(r)$ and $p^{(0)}(r)$ are respectively the energy density and pressure of undisturbed NSs, and $\rho^{(1)}(\vec{x})$ and $p^{(1)}(\vec{x})$ are corresponding corrections caused by Lorentz violation, which are [47]

$$\rho^{(1)}(\vec{x}) = -\alpha(\theta,\varphi) r\partial_r \rho^{(0)}(r), \tag{29}$$

$$p^{(1)}(\vec{x}) = -\alpha(\theta,\varphi) r\partial_r p^{(0)}(r). \tag{30}$$

In the above equations, $\alpha(\theta,\varphi)$ is

$$\alpha(\theta,\varphi) = \frac{1}{2}\sum_{m=-2}^{2} s^{(s)}_{2m} Y_{2m}(\theta,\varphi), \tag{31}$$

where $Y_{lm}(\theta,\varphi)$ are the spherical harmonics, and $s^{(s)}_{2m}$ are

$$\begin{aligned} s^{(s)}_{2,-2} &= \sqrt{\frac{2\pi}{15}}(\bar{s}^{xx} - \bar{s}^{yy} + 2i\bar{s}^{xy}), \\ s^{(s)}_{2,-1} &= 2\sqrt{\frac{2\pi}{15}}(\bar{s}^{xz} + i\bar{s}^{yz}), \\ s^{(s)}_{2,0} &= \frac{2}{3}\sqrt{\frac{\pi}{5}}(-\bar{s}^{xx} - \bar{s}^{yy} + 2\bar{s}^{zz}), \\ s^{(s)}_{2,1} &= 2\sqrt{\frac{2\pi}{15}}(-\bar{s}^{xz} + i\bar{s}^{yz}), \\ s^{(s)}_{2,2} &= \sqrt{\frac{2\pi}{15}}(\bar{s}^{xx} - \bar{s}^{yy} - 2i\bar{s}^{xy}). \end{aligned} \tag{32}$$

We study the specific case with axial symmetry, so we choose $\bar{s}^{\mu\nu}$ to vanish except for $\bar{s}^{zz}$ where the z axis is the NS's spin direction. In other words, we assume that the specific direction of the Lorentz-violating field is parallel to the spinning axis of the NS.

In a brief summary, modifications from Lorentz violation are categorized into two aspects. Firstly, Lorentz violation modifies the gravitational field equations. Secondly, it induces deformations in the NS structure.

B. The modified PDE for MOI

Similar to the calculation in GR, the perturbative rotational properties in a stationary axisymmetric background spacetime are governed by the $t\varphi$ component of Eq. (20). We can begin with the general form of an axisymmetric metric as given in Eq. (3). We adopt the slow-rotation assumption, so that at the linear order of the angular velocity Ω, the functions for the diagonal metric components take the background result while the function L takes the form of Eq. (5), and the related function $\bar{\omega}$ can be defined as in Eq. (9). Furthermore, we now would like to consider the leading corrections due to the Lorentz-violation coefficient $\bar{s}^{zz}$, so we can treat $\bar{\omega}$ as having an expansion in terms of $\bar{s}^{zz}$,

$$\bar{\omega} = \bar{\omega}^{(0)} + \bar{\omega}^{(1)} + \mathcal{O}(|\bar{s}^{zz}|^2), \tag{33}$$

where $\bar{\omega}^{(0)}$ corresponds to the GR result with vanishing $\bar{s}^{zz}$, and $\bar{\omega}^{(1)}$ gives the leading-order correction to the metric component $g_{t\varphi}$ due to Lorentz violation.

To calculate $\bar{\omega}^{(1)}$, we write out the $t\varphi$ component of Eq. (20) and arrange it in orders of $\bar{s}^{zz}$. The zeroth-order equation in $\bar{s}^{zz}$ reads as

$$\begin{aligned} &\frac{1}{r^4}\frac{\partial}{\partial r}\left(r^4 j \frac{\partial\bar{\omega}^{(0)}}{\partial r}\right) + \frac{4}{r}\frac{\mathrm{d}j}{\mathrm{d}r}\bar{\omega}^{(0)} \\ &\quad + \frac{e^{(\lambda-\nu)/2}}{r^2}\frac{1}{\sin^3\theta}\frac{\partial}{\partial\theta}\left(\sin^3\theta\frac{\partial\bar{\omega}^{(0)}}{\partial\theta}\right) = 0, \end{aligned} \tag{34}$$

which is identical to Eq. (8) as expected, and therefore $\bar{\omega}^{(0)}$ is just the GR solution. The first-order equation in $\bar{s}^{zz}$ is

$$\begin{aligned} &\frac{1}{r^4}\frac{\partial}{\partial r}\left(r^4 j \frac{\partial\bar{\omega}^{(1)}}{\partial r}\right) + \frac{4}{r}\frac{\mathrm{d}j}{\mathrm{d}r}\bar{\omega}^{(1)} \\ &\quad + \frac{e^{(\lambda-\nu)/2}}{r^2}\frac{1}{\sin^3\theta}\frac{\partial}{\partial\theta}\left(\sin^3\theta\frac{\partial\bar{\omega}^{(1)}}{\partial\theta}\right) \\ &= S_1(r,\theta) + S_2(r,\theta), \end{aligned} \tag{35}$$

where

$$S_1(r,\theta) = \frac{4}{r}\frac{\mathrm{d}j}{\mathrm{d}r}\bar{\omega}^{(0)}\delta\phi e^{-\nu} + 16\pi(\rho^{(1)} + p^{(1)})e^{(\lambda-\nu)/2}\bar{\omega}^{(0)}, \tag{36}$$

$$\begin{aligned} S_2(r,\theta) = {}&(\delta\phi e^{-\nu} + \bar{s}^{zz}\sin^2\theta)\frac{1}{r^4}\frac{\partial}{\partial r}\left(r^4 j\frac{\partial\bar{\omega}^{(0)}}{\partial r}\right) \\ &+ \frac{1}{2}j\left[\delta\phi_{,r}e^{-\nu} - \delta\phi e^{-\nu}\frac{\mathrm{d}\nu}{\mathrm{d}r} + \frac{\bar{s}^{zz}}{r}(8e^{\lambda}\cos^2\theta - 2\sin^2\theta)\right] \\ &\times\frac{\partial\bar{\omega}^{(0)}}{\partial r}. \end{aligned} \tag{37}$$

Note that $S_1(r,\theta)$ arises from modifications in $T_{t\varphi}$ due to Lorentz violation while $S_2(r,\theta)$ arises from $G_{t\varphi}$ and the Lorentz-violating term $\bar{s}^{zz}G_{tzz\varphi}$ in the field equation (20). With the boundary conditions in Eqs. (10) and (11) for $\bar{\omega} \approx \bar{\omega}^{(0)} + \bar{\omega}^{(1)}$, $\bar{\omega}^{(1)}$ can be solved from Eq. (35) once we have the GR solution $\bar{\omega}^{(0)}$.

Numerical method needs to be employed to solve Eq. (35) as well as $\bar{\varpi}^{(0)}$ inside the NS. Instead of solving $\bar{\varpi}^{(0)}$ and $\bar{\varpi}^{(1)}$ one by one, it is straightforward to solve the combined PDE,

$$\frac{1}{r^4}\frac{\partial}{\partial r}\left(r^4 j\frac{\partial\bar{\varpi}}{\partial r}\right)+\frac{4}{r}\frac{\mathrm{d}j}{\mathrm{d}r}\bar{\varpi}+\frac{e^{(\lambda-\nu)/2}}{r^2}\frac{1}{\sin^3\theta}\frac{\partial}{\partial\theta}\left(\sin^3\theta\frac{\partial\bar{\varpi}}{\partial\theta}\right) = S_1'(r,\theta)+S_2'(r,\theta), \tag{38}$$

where

$$S_1'(r,\theta)=\frac{4}{r}\frac{\mathrm{d}j}{\mathrm{d}r}\bar{\varpi}\delta\phi e^{-\nu}+16\pi(\rho^{(1)}+p^{(1)})e^{(\lambda-\nu)/2}\bar{\varpi}, \tag{39}$$

$$S_2'(r,\theta)=(\delta\phi e^{-\nu}+\bar{s}^{zz}\sin^2\theta)\frac{1}{r^4}\frac{\partial}{\partial r}\left(r^4 j\frac{\partial\bar{\varpi}}{\partial r}\right) +\frac{1}{2}j\left[\delta\phi_{,r}e^{-\nu}-\delta\phi e^{-\nu}\frac{\mathrm{d}\nu}{\mathrm{d}r}+\frac{\bar{s}^{zz}}{r}(8e^{\lambda}\cos^2\theta-2\sin^2\theta)\right] \times\frac{\partial\bar{\varpi}}{\partial r}. \tag{40}$$

Before we proceed to the numerical results, let us clarify the asymptotic behavior of $\bar{\varpi}$ so that the angular momentum and hence the MOI can be defined.

We split the solution of Eq. (38) into two parts, $\bar{\varpi}=\bar{\varpi}_1+\bar{\varpi}_2$, where $\bar{\varpi}_1$ is the solution ignoring the source S_2', and it contains the zeroth-order solution in $\bar{s}^{zz}$, which represents the contribution from the matter, and $\bar{\varpi}_2$ is the solution ignoring the source S_1', and it does not contain the zeroth-order solution in $\bar{s}^{zz}$, which represents the contribution from the modification of the gravity theory. We discuss the asymptotic behaviors of $\bar{\varpi}_1$ and $\bar{\varpi}_2$ separately.

Outside the star, S_1' is zero, and hence $\bar{\varpi}_1$ has the exterior solution [63]

$$\bar{\varpi}_1=\sum_{l=1}^{\infty}\bar{\varpi}_{1l}\left(-\frac{1}{\sin\theta}\frac{dP_l}{d\theta}\right), \tag{41}$$

where P_l are the Legendre polynomials and $\bar{\varpi}_{1l}$ consists of the r^{-l-2} terms and the r^{l-1} terms. Because of the asymptotic flatness boundary condition, the r^{l-1} terms with $l\neq 1$ must have vanishing coefficients. Now very different from GR, the PDE for $\bar{\varpi}_1$ does not admit separation of variables inside the NSs due to the source term S_1', so for each l in $\bar{\varpi}_{1l}$, the coefficient for r^{-l-2} is no longer forced to be proportional to the coefficient for r^{l-1}, meaning that the r^{-l-2} terms can exist in spite of the absence of the r^{l-1} terms.

As axisymmetric NSs exhibit reflection symmetry about the equatorial plane, we have $\bar{\varpi}(r,\theta)=\bar{\varpi}(r,\pi-\theta)$, which implies that odd-power terms of $\cos\theta$ should not appear in $\bar{\varpi}$ and hence $\bar{\varpi}_1$, and it excludes the even l terms in Eq. (41). In conclusion, the expansion of $\bar{\varpi}_1$ outside the star takes the form

$$\bar{\varpi}_1=\mathcal{A}_1+\frac{\mathcal{B}_1}{r^3}+\frac{\mathcal{C}_1}{r^5}\left(-\frac{1}{\sin\theta}\frac{dP_3}{d\theta}\right)+O\left(\frac{1}{r^7}\right), \tag{42}$$

where $\mathcal{A}_1$, $\mathcal{B}_1$, $\mathcal{C}_1$ are constants.

For $\bar{\varpi}_2$, its PDE admits separation of variables neither outside the star nor inside the star. Then we have to assume its asymptotic expansion in terms of $1/r$ to be the general form,

$$\bar{\varpi}_2=\mathcal{A}_2+\frac{\mathcal{B}_2(\theta)}{r^3}+\frac{\mathcal{C}_2(\theta)}{r^4}+\ldots \tag{43}$$

Note that there is no $1/r$ term nor $1/r^2$ term because of the boundary condition of asymptotic flatness. For the very same boundary condition, the coefficient $\mathcal{A}_2$ has to be independent of θ and satisfies $\Omega=\mathcal{A}_1+\mathcal{A}_2$, where $\mathcal{A}_1$ is the constant in Eq. (42).

Combining the asymptotic expansions in Eqs. (42) and (43), substituting into Eq. (38), and arranging the equation in orders of $1/r$, we find at the leading order an equation for the coefficient $\mathcal{B}_2(\theta)$,

$$\frac{1}{\sin^3\theta}\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\sin^3\theta\frac{\mathrm{d}\mathcal{B}_2}{\mathrm{d}\theta}\right)=3\bar{s}^{zz}\mathcal{B}_1(\sin^2\theta-4\cos^2\theta), \tag{44}$$

and equations for the coefficient $\mathcal{C}_2(\theta)$ are at higher orders. Note that we have used $\bar{\varpi}_2/\bar{\varpi}_1\sim O(\bar{s}^{zz})$ and substituted $\bar{\varpi}$ with $\bar{\varpi}_1$ in S_2'. For $\bar{\varpi}_2$ to be smooth, we impose

$$\left.\frac{\mathrm{d}\mathcal{B}_2(\theta)}{\mathrm{d}\theta}\right|_{\theta=0}=\left.\frac{\mathrm{d}\mathcal{B}_2(\theta)}{\mathrm{d}\theta}\right|_{\theta=\pi}=0, \tag{45}$$

and get

$$\mathcal{B}_2(\theta)=-\frac{3}{2}\bar{s}^{zz}\mathcal{B}_1\sin^2\theta+\text{constant}. \tag{46}$$

Therefore, gathering Eqs. (42) and (43) we find the asymptotic behavior of $\bar{\varpi}$ to be

$$\bar{\varpi}=\Omega\left[1-\frac{2}{r^3}(p\sin^2\theta+q)+\mathcal{O}\left(\frac{1}{r^4}\right)\right], \tag{47}$$

where Ω, p, and q are constants.

With the asymptotic expression for $\bar{\varpi}$ in Eq. (47), the angular momentum of the spacetime is found to be [74]

$$J=\Omega\left(\frac{4}{5}p+q\right). \tag{48}$$

Then the MOI of the star is

$$I=\frac{J}{\Omega}=\left(\frac{4}{5}p+q\right). \tag{49}$$

Now we are ready to calculate the constants p, q and therefore the MOI for NSs by numerically solving Eq. (38).

C. Numerical calculation

To numerically solve the PDE for $\bar{\omega}$, we first perform a change of variables which is inspired by Cook *et al.* [75],

$$(r,\theta) \to \left(x \equiv \frac{r\cos\theta}{r+R}, y \equiv \frac{r\sin\theta}{r+R}\right), \tag{50}$$

where R is the radius of an unperturbed NS. Equation (38) then changes to

$$\begin{aligned} & f_1(x,y)(x\bar{\omega}_{,x} + y\bar{\omega}_{,y}) + f_2(x,y)(x^2\bar{\omega}_{,xx} + 2xy\bar{\omega}_{,xy} + y^2\bar{\omega}_{,yy}) + f_3(x,y)\bar{\omega} + f_4(x,y)(x\bar{\omega}_{,y} - y\bar{\omega}_{,x}) \\ & \quad + f_5(x,y)(y^2\bar{\omega}_{,xx} - 2xy\bar{\omega}_{,xy} + x^2\bar{\omega}_{,yy} - x\bar{\omega}_{,x} - y\bar{\omega}_{,y}) = 0, \end{aligned} \tag{51}$$

where

$$\begin{aligned} f_1(x,y) = {} & \left[4j(1-\bar{r})^3 + \bar{r}(1-\bar{r})^2 R\frac{\mathrm{d}j}{\mathrm{d}r} - 2j\bar{r}(1-\bar{r})^3\right](\bar{r}^2 - \bar{r}^2\delta\phi e^{-\nu} - \bar{s}^{zz}y^2) \\ & - \left[\frac{1}{2}\delta\phi_{,r}e^{-\nu}jR\bar{r}^3(1-\bar{r})^2 - \frac{1}{2}\delta\phi e^{-\nu}jR\frac{\mathrm{d}\nu}{\mathrm{d}r}\bar{r}^3(1-\bar{r})^2 - \frac{1}{2}\bar{s}^{zz}j(1-\bar{r})^3(-8e^{\lambda}x^2 + 2y^2)\right], \end{aligned} \tag{52}$$

$$f_2(x,y) = j(1-\bar{r})^4(\bar{r}^2 - \bar{r}^2\delta\phi e^{-\nu} - \bar{s}^{zz}y^2), \tag{53}$$

$$f_3(x,y) = 4\bar{r}^3(1-\bar{r})R\frac{\mathrm{d}j}{\mathrm{d}r}(1-\delta\phi e^{-\nu}) - 16\pi(\rho^{(1)} + p^{(1)})e^{(\lambda-\nu)/2}\bar{r}^4R^2, \tag{54}$$

$$f_4(x,y) = e^{(\lambda-\nu)/2}\bar{r}^2(1-\bar{r})^2\frac{3x}{y}, \tag{55}$$

$$f_5(x,y) = e^{(\lambda-\nu)/2}\bar{r}^2(1-\bar{r})^2, \tag{56}$$

with $\bar{r} = r/(r+R) = \sqrt{x^2+y^2}$ being a dimensionless variable. Axisymmetric NSs exhibit reflection symmetry about the equatorial plane, so the parameter space of (x,y) can be reduced to a quarter sector, satisfying $x >= 0$ and $y >= 0$. The boundary conditions become

$$\left.\frac{\partial\bar{\omega}}{\partial x}\right|_{x=0} = \left.\frac{\partial\bar{\omega}}{\partial y}\right|_{y=0} = 0, \tag{57}$$

$$\bar{\omega}|_{\bar{r}\to 1} = \Omega. \tag{58}$$

The advantage of the (x,y) variables is in two aspects. First, $r \in (0,+\infty)$ corresponds to $\bar{r} \in (0,1)$. We can use a finite sector of unit radius on the xy plane to represent the infinite (r,θ) plane, which is convenient for us to input boundary conditions at infinity in numerics. Additionally, $r = R$ corresponds to $\bar{r} = 1/2$. As we use the finite element method with uniform grids in the xy plane, the interior of the NS will be solved more meticulously compared to using the (r,θ) variables. This is beneficial for increasing the accuracy. Second, we choose (x,y) instead of $(\bar{r},\theta)$ to avoid inputting the regular condition similar to Eq. (10) when $\bar{r}$ tends to zero.

Using the variables (x,y), we find that the finite element method suffices to solve the PDE for $\bar{\omega}$ in Eq. (51). After each numerical solution is obtained, we fit it at large r according to Eq. (47) to extract the constants Ω, p, and q. Afterward the MOI for the star is calculated using Eq. (49).

We use the EOS AP4 as an example and calculate MOIs for NSs with different masses. The results are shown in Fig. 2. For illustration purposes, we have taken $\bar{s}^{zz} = 10^{-2}$ in our numerical calculation. From Fig. 2, we can see that the ratio $\delta I/I_{\rm GR}$ follows a relatively good linear relation. It may help us to quickly estimate δI.

We have completed the calculation of MOIs for axisymmetric NSs due to Lorentz violation. We find that it is also interesting to calculate the MOI solely from $\bar{\omega}_1$. By doing this we ignore the source term S_2' in Eq. (38), so the Lorentz-violating effect comes into play only through the energy-momentum tensor of the NS matter. The result can be compared with the estimation made by Xu *et al.* [47], where the correction in the MOI caused by Lorentz violation is calculated in the Newtonian way by only considering the change in matter distribution, via

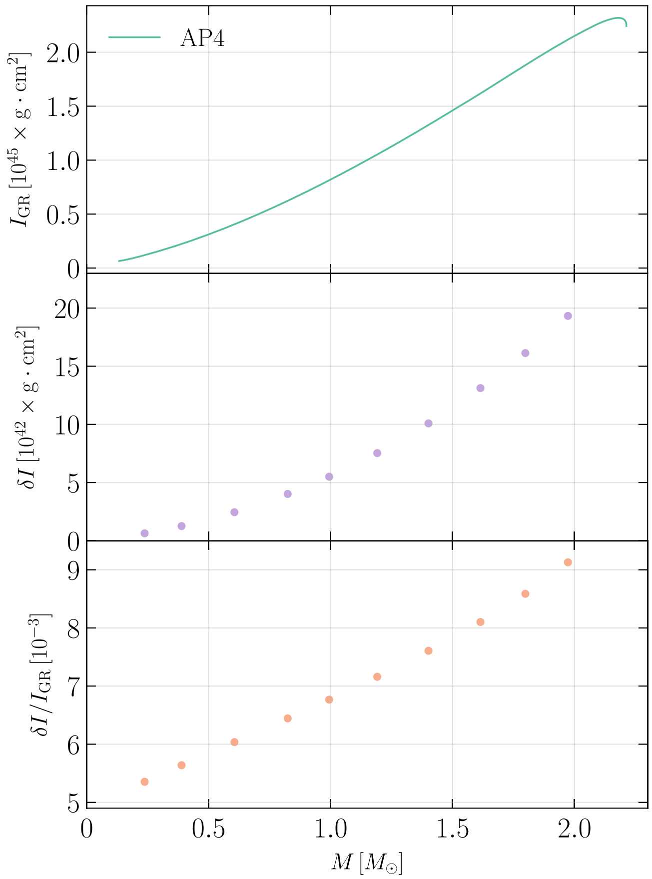

FIG. 2. Corrections to the MOIs of NSs caused by Lorentz violation with EOS AP4, as functions of the mass of the NS. The top panel shows the MOI of spherical NSs in GR, $I_{\rm GR}$. The middle panel shows the absolute correction to the MOI caused by Lorentz violation with $\bar{s}^{zz} = 10^{-2}$. The bottom panel shows the ratio $\delta I/I_{\rm GR}$.

$$\delta I_{\rm Newton} = \int \rho^{(1)}(r,\theta) r^2 \sin^2\theta \mathrm{d}^3x = -\frac{1}{3}\bar{s}^{zz} I_{\rm Newton}, \quad (59)$$

where $I_{\rm Newton}$ is the Newtonian MOI with the mass density $\rho^{(0)}$ in the absence of Lorentz violation.

The numerical approach to calculate $\bar{\omega}_1$ differs slightly from what we have done for calculating $\bar{\omega}$. We only need to remove the terms corresponding to S_2' in Eqs. (51)–(56). After obtaining numerical solutions and fitting them according to Eq. (42), the MOI is then calculated by $I_1 = -\mathcal{B}_1/2\mathcal{A}_1$. For NSs with different masses, results of I_1 are shown in Fig. 3 in terms of $\delta I_1 = I_1 - I_{\rm GR}$. Figure 3 also shows the change of a factor k defined in

$$\delta I_1 = -k\bar{s}^{zz} I_{\rm GR}. \quad (60)$$

As the mass of the NS decreases, we expect $I_{\rm GR} \to I_{\rm Newton}$ and $\delta I_1 \to \delta I_{\rm Newton}$ so that $k \to 1/3$. This is exactly what we see in Fig. 3.

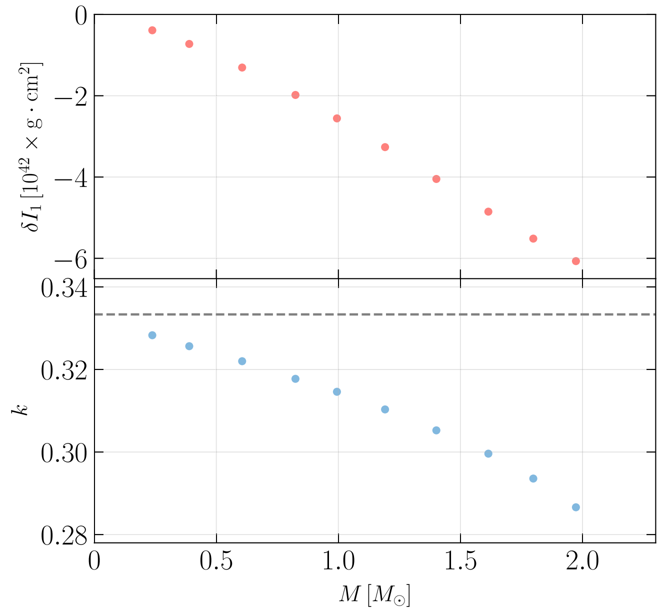

FIG. 3. Corrections to MOIs caused by deformations of NSs with EOS AP4, as functions of the NS mass. The top panel shows the correction to MOI caused by deformations of NSs with $\bar{s}^{zz} = 10^{-2}$. The bottom panel shows the value of k defined in Eq. (60). The gray dashed line represents $k = 1/3$, which is the Newtonian limit for k.

IV. SUMMARY

In this paper, we develop the method to calculate axisymmetric NSs' MOI in the presence of Lorentz violation in a relativistic setting. Solutions are worked out for the first time in the effective-field-theoretic framework of SME. We treat the effect of Lorentz violation as a perturbation and derive the modified PDE for the MOI from the gravitational field equations. Then, we discuss the asymptotic behavior of the solution analytically. After that, we perform a change of variables, solve the PDE with the finite element method and fit the numerical solutions with polynomials to get the MOI.

After obtaining the numerical results, we calculate corrections to the MOI of NSs caused by Lorentz violation. Besides that, we separately calculate corrections to the MOI caused by the deformation of the NS. We compare the ratio $\delta I_1/I_{\rm GR}$ with its counterpart in Newtonian gravity and show the difference. For a $1.4M_\odot$ NS with EOS AP4, the difference is at the level of $\sim 8\%$.

In the future, we can extend this method to study the structure of axisymmetric NSs in other modified gravity theories, e.g. the bumblebee theory [8,76] and the Einstein-Æther theory [72]. If stable axisymmetric NS solutions exist in these modified gravity theories, the calculation procedure outlined in this paper may serve as an important reference for calculating the MOI of axisymmetric NSs in these modified gravity theories. This assists us in studying the properties of NSs in the modified gravity theories. As a byproduct, we have also presented a consistent method for calculating the MOI corrections caused by axisymmetric deformations in NSs within the framework of GR, which may offer a more precise method for computing MOI corrections in some theoretical models as well. We are looking forward to calculating corrections to the MOI caused by general deformations of NSs in the framework of GR, to help us understand the physics of NSs better.

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