Conserved quantities for asymptotically AdS spacetimes in quadratic curvature gravity in terms of a rank-4 tensor

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We investigate the conserved quantities associated to Killing isometries for asymptotically AdS spacetimes within the framework of quadratic-curvature gravity. By constructing a rank-4 tensor possessing the same index symmetries as the ones of the Riemann tensor, we propose a 2-form potential resembling the Noether one for quadratic-curvature gravity. Such a potential is compared with the results via other methods existing in the literature to establish the equivalence. Then this potential is adopted to define conserved quantities of asymptotically AdS spacetimes. As applications, we explicitly compute the mass of static spherically symmetric spacetimes, as well as the mass and the angular momentum for rotating spacetimes, such as the four(higher)-dimensional Kerr-AdS black holes and black strings embedded in quadratic-curvature gravities. Particularly, we emphasize the conserved charges of Einstein-Gauss-Bonnet, Weyl, and critical gravities, together with the ones for the asymptotically AdS solutions satisfying vacuum Einstein field equations.

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I. INTRODUCTION

The theories of quadratic(-curvature) gravity (it is sometimes referred to as higher derivative gravity or R^2-gravity), such as Einstein-Gauss-Bonnet gravity [1,2], Weyl gravity and critical gravity [3–5], can be regarded as the higher-order derivative generalizations of the well-known Einstein gravity theory. They are generally described by the Einstein-Hilbert Lagrangian with or without cosmological constant plus at least one of the square of the Riemann tensor, the square of the Ricci tensor, and the quadratic Ricci curvature scalar, or by the Lagrangian merely made up of no less than one of the three aforementioned quadratic curvature terms. From a mathematical perspective, such a Lagrangian can be viewed as a functional for the tensors of the Riemann and the metric. Since it was discovered in [6] that the inclusion of quadratic curvature terms renders the gravity theory perturbatively renormalizable in the quantization process, quadratic-curvature gravity has been treated as a viable candidate for a theory of quantum gravity, and it has attracted significant attention [7–12].

Conventionally, to understand fully the physical and geometric properties for a given gravity theory, a prominent task is to search for solutions of this theory. For quadratic-curvature gravities, in contrast with Einstein gravity, the involvement of the quadratic curvature terms renders it more difficult to handle the field equations. However, if some symmetries are allowed to enter into the metrics so that they are static and diagonal in form, it becomes much more practicable to analytically solve the field equations. As a consequence, a lot of static solutions with various asymptotic structures have been found in the past few decades. Among them, here we mention the ones presented by the works [13–45]. On the other hand, although it is of great difficulty to construct exact rotating solutions in quadratic-curvature gravities without systematical methods generating solutions. One feasible way to achieve this is to embed the rotating solutions obeying the vacuum Einstein field equations into such theories. For example, the four-dimensional stationary and axially symmetric Kerr-AdS black hole solution [46] is likewise the one for Einstein-Gauss-Bonnet, Weyl and critical gravities. Besides, it will be demonstrated below that some quadratic-curvature gravities are able to embrace the higher-dimensional generalizations of the four-dimensional Kerr-AdS solution [47,48]. With those solutions in hand, as usual, a necessary procedure is to give their conserved charges for the sake of understanding thermodynamic properties, which are of considerable interest at the present stage. As a matter of fact, there exist a number of approaches for conserved charges in the literature, such as the covariant phase space method [49–51] and its development [52],

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the Abbott-Deser-Tekin (ADT) formalism [53–57], the Ashtekar-Magnon-Das (AMD) method [58,59] and the field-theoretic approach [60–62]. In the last few years, in the spirit of these methods or other ones, many efforts have been given from diverse perspectives to explore the conserved quantities of various spacetimes in quadratic-curvature gravities, particularly the asymptotically AdS ones [63–81].

Within this paper, in spite of the fact that a lot of methods have been devoted to the definition for the conserved quantities of quadratic-curvature gravities, we attempt to propose a simple and convenient formulation of conserved quantities for asymptotically AdS spacetimes within the framework of these theories and then make use of the formulation to explicitly illustrate how the quadratic curvature terms correct the mass and the angular momentum of black holes in the context of Einstein gravity. For this purpose, we shall follow the work [65] to construct a rank-4 tensor that exhibits the same index symmetries with those of the Riemann tensor. The linear combination of such a tensor with another rank-4 one that is defined by the derivative of the Lagrangian with respect to the Riemann tensor further gives rise to a 2-form potential associated to an arbitrary Killing vector field, which takes a similar structure as the Noether one. Of particular interest will be the applications of this potential in Einstein-Gauss-Bonnet, Weyl and critical gravities. By virtue of the comparison of the potential with the results via other methods, it will be demonstrated that its integral on a codimension-2 surface can bring about an appropriate formula for conserved charges of asymptotically AdS spacetimes according to Stokes' theorem. Furthermore, by utilizing this formula, we shall calculate the mass for static and spherically symmetric spacetimes, as well as the mass and the angular momentum of four(higher)-dimensional Kerr-AdS black holes corrected by terms in quadratic curvatures.

The rest of the present paper is organized as follows. In Sec. II, starting with the general form for the Lagrangian of quadratic-curvature gravities, we shall introduce a rank-4 tensor to construct the potentials associated to conserved quantities of these theories. Such potentials will be compared with the ones via other methods in the literature, and their applications in some typical quadratic-curvature gravities will be strengthened. In Sec. III, we will apply the formula for conserved charges to compute the mass of static spherically symmetric spacetimes with the asymptotically AdS structure, including the ones in general relativity and Einstein-Gauss-Bonnet gravity in arbitrary dimensions, as well as the ones in four-dimensional Weyl and critical gravities. In Sec. IV, we shall compute the mass and angular momenta of four(higher)-dimensional rotating Kerr-AdS black holes and black strings embedded into the theories of quadratic-curvature gravity. The last section is devoted to our conclusions.

II. THE GENERAL FORMALISM

In this section, we shall investigate potentials defined in terms of a rank-4 tensor with the same index symmetries as those of the Riemann tensor within the framework of the theories of quadratic-curvature gravity. It will be demonstrated that such potentials are equivalent to the ones via other methods, such as the (off-shell) ADT formalism, the covariant phase space approach, the generalized Komar integral and the field-theoretic method. As a consequence, it is allowed to apply them to define conserved quantities of asymptotically AdS spacetiems in these gravity theories. In particular, we are going to analyze the applications of the potentials in general relativity, Einstein-Gauss-Bonnet gravity, Weyl gravity and critical gravity.

In the present work, the integer D stands for the dimensions of spacetimes. We adopt the notations in [82] to define the Riemann curvature tensor $R_{\mu\nu\rho\sigma}$ through $(\nabla_\mu\nabla_\nu - \nabla_\nu\nabla_\mu)V_\rho = R_{\mu\nu\rho\sigma}V^\sigma$ (here V_ρ denotes an arbitrary vector field), while $R_{\mu\nu} = g^{\rho\sigma}R_{\rho\mu\sigma\nu}$ and $R = g^{\rho\sigma}R_{\rho\sigma}$ represent the Ricci tensor and its scalar curvature respectively. For generality, we take into consideration of the usual Einstein-Hilbert Lagrangian in the presence of a negative cosmological constant Λ plus the linear combination for all the quadratic curvature terms R^2, $R^{\alpha\beta}R_{\alpha\beta}$ and $R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma}$, written as the following form

$$\sqrt{-g}L = \sqrt{-g}(R - 2\Lambda + c_1R^2 + c_2R^{\alpha\beta}R_{\alpha\beta} + c_3R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma}). \tag{2.1}$$

Here (c_1, c_2, c_3) represent coupling constants. With different choices of these constants, Eq. (2.1) admits the Lagrangians for some typical quadratic-curvature gravities. For example, it includes Weyl gravity as a special case, which is regarded as a compelling alternative to Einstein gravity. Specifically, the Weyl tensor in D-dimensional spacetime is given by [82]

$$C^{\mu\nu}_{\rho\sigma} = \frac{R}{(D-1)(D-2)}\delta^{\mu\nu}_{\rho\sigma} - \frac{4}{D-2}R^{[\mu}_{[\rho}\delta^{\nu]}_{\sigma]} + R^{\mu\nu}_{\rho\sigma}. \tag{2.2}$$

Here and in what follows, we follow the convention whereby the square brackets enclosing indices denote antisymmetrization of them, such that $R^{[\mu}_{[\rho}\delta^{\nu]}_{\sigma]} = (R^{[\mu}_{\rho}\delta^{\nu]}_{\sigma} - R^{[\mu}_{\sigma}\delta^{\nu]}_{\rho})/2$, and the generalized Kronecker delta symbol $\delta^{\mu_1\cdots\mu_m}_{\nu_1\cdots\nu_m} = m!\delta^{[\mu_1}_{[\nu_1}\cdots\delta^{\mu_m]}_{\nu_m]}$. An arbitrary rank-4 tensor $X^{\mu\nu}_{\rho\sigma} = g^{\mu\alpha}g^{\nu\beta}X_{\alpha\beta\rho\sigma}$. It is easy to check that the Weyl tensor $C^{\mu\nu}_{\rho\sigma}$ is traceless, namely, $C^{\mu\rho}_{\nu\rho} = C^{\rho\mu}_{\rho\nu} = 0$. The contraction between two Weyl tensors is read off as

$$C^{\mu\nu}_{\rho\sigma}C^{\rho\sigma}_{\mu\nu} = \frac{1}{4}\delta^{\gamma\lambda\mu\nu}_{\alpha\beta\rho\sigma}C^{\alpha\beta}_{\gamma\lambda}C^{\rho\sigma}_{\mu\nu} = C^{\mu\nu}_{\rho\sigma}R^{\rho\sigma}_{\mu\nu}$$
$$= \frac{2}{(D-1)(D-2)}R^2 - \frac{4}{D-2}R^\rho_\sigma R^\sigma_\rho + R^{\mu\nu}_{\rho\sigma}R^{\rho\sigma}_{\mu\nu}. \tag{2.3}$$

For the purpose to obtaining the first equality in Eq. (2.3), we have made use of the expansion for the generalized Kronecker delta symbol $\delta^{\gamma\lambda\mu\nu}_{\alpha\beta\rho\sigma}$, that is,

$$\delta^{\gamma\lambda\mu\nu}_{\alpha\beta\rho\sigma} = \delta^{\gamma\lambda}_{\alpha\beta}\delta^{\mu\nu}_{\rho\sigma} - \delta^{\gamma\mu}_{\alpha\beta}\delta^{\lambda\nu}_{\rho\sigma} - \delta^{\gamma\nu}_{\alpha\beta}\delta^{\mu\lambda}_{\rho\sigma} - \delta^{\mu\lambda}_{\alpha\beta}\delta^{\gamma\nu}_{\rho\sigma} - \delta^{\nu\lambda}_{\alpha\beta}\delta^{\mu\gamma}_{\rho\sigma} + \delta^{\mu\nu}_{\alpha\beta}\delta^{\gamma\lambda}_{\rho\sigma}. \tag{2.4}$$

In the absence of the $R-2\Lambda$ term, together with all the three coupling constants satisfying $c_2 = -2(D-1)c_1$ and $c_3 = (D-1)(D-2)c_1/2$, the Lagrangian (2.1) becomes the one for D-dimensional Weyl gravity, namely,

$$\sqrt{-g}L_W = \frac{c_1(D-1)(D-2)}{2}\sqrt{-g}C^{\mu\nu\rho\sigma}C_{\mu\nu\rho\sigma}, \tag{2.5}$$

where Eq. (2.3) has been used. Besides, when $c_2 = -4c_1$ and $c_3 = c_1$, Eq. (2.1) transforms into the Lagrangian for Einstein-Gauss-Bonnet gravity [1,2], taking the form

$$\sqrt{-g}L_{\mathrm{EGB}} = \sqrt{-g}(R - 2\Lambda + c_1 L_{GB}), \tag{2.6}$$

in which the Gauss-Bonnet invariant L_{GB} is read off as

$$L_{GB} = R^2 - 4R^{\alpha\beta}R_{\alpha\beta} + R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma} = \frac{1}{4}\delta^{\gamma\lambda\mu\nu}_{\alpha\beta\rho\sigma}R^{\alpha\beta}_{\gamma\lambda}R^{\rho\sigma}_{\mu\nu}. \tag{2.7}$$

In order to arrive at the last equality in Eq. (2.7), we have utilized Eq. (2.4). As a consequence of the Gauss-Bonnet-Chern theorem (see Refs. [83–85] for this theorem in the context of pseudo-Riemann manifolds and its implications in gravity theories), the integration of the Gauss-Bonnet invariant on $D=4$ compact manifold gives rise to a constant with a value relying on the four-dimensional Euler characteristic of the manifold. When $D=4$, the Gauss-Bonnet term is often referred to as a topological invariant, the addition of which to the Lagrangian makes no contribution to the modification of the bulk dynamics. However, this term is of great importance in the renormalization of the Einstein gravity theory. Besides, it can be adopted to define the conserved charges in asymptotically (locally) AdS spaces within the context of the four-dimensional Einstein gravity or quadratic gravity [73,86]. What is more, due to the vanishing of the generalized Kronecker delta symbol $\delta^{\gamma\lambda\mu\nu}_{\alpha\beta\rho\sigma}$ in three dimensions, Eq. (2.7) leads to the identity $R^2 - 4R^{\alpha\beta}R_{\alpha\beta} + R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma} = 0$ or $C^{\mu\nu}_{\rho\sigma}C^{\rho\sigma}_{\mu\nu} = 0$ in three dimensions, which can be reproduced by means of using the equation $C^{\mu\nu}_{\rho\sigma}(D=3) = 0$ derived from the expansion of the identity $\delta^{\gamma\lambda\mu\nu}_{\alpha\beta\rho\sigma}R^{\alpha\beta}_{\gamma\lambda} = 0$ $(D=3)$ in terms of Eq. (2.4). The substitution of the three-dimensional vacuum Einstein field equations $R_{\mu\nu} = 2\Lambda g_{\mu\nu}$ into $C^{\mu\nu}_{\rho\sigma}(D=3) = 0$ yields $R^{\mu\nu}_{\rho\sigma} = -\Lambda\delta^{\mu\nu}_{\rho\sigma}$. This implies that all the vacuum solutions in three-dimensional Einstein gravity with the cosmological constant are locally equivalent to the ones with maximal symmetries, whose Riemann curvature tensor obeys the relation $R^{\mu\nu}_{\rho\sigma} \propto \delta^{\mu\nu}_{\rho\sigma}$.

To connect more straightforwardly the Lagrangian (2.1) with some typical quadratic gravities, when the dimension $D \geq 4$, with Eqs. (2.3) and (2.7), the Lagrangian (2.1) can be reexpressed as the form consisting of the usual Einstein-Hilbert part, the square of the Weyl tensor, the Gauss-Bonnet invariant and the quadratic Ricci curvature scalar term, namely,

$$L = R - 2\Lambda + \frac{(D-2)(c_2+4c_3)}{4(D-3)}C^{\mu\nu}_{\rho\sigma}C^{\rho\sigma}_{\mu\nu} - \frac{(D-2)c_2+4c_3}{4(D-3)}L_{GB} + \left(c_1 + \frac{Dc_2+4c_3}{4D-4}\right)R^2. \tag{2.8}$$

One can take advantage of Eq. (2.8) in the description for some specifical quadratic-curvature gravities. For example, within the case where $D=4$, $c_2 = -3c_1$, and $c_3 = 0$, Eq. (2.8) gives rise to the Lagrangian $\mathcal{L}^{(4D)}_{CG}$ for four-dimensional critical gravity [3], having the form

$$\mathcal{L}^{(4D)}_{CG} = \sqrt{-g}\left[R - 2\Lambda + \frac{3}{2}c_1(L_{GB} - C^{\mu\nu}_{\rho\sigma}C^{\rho\sigma}_{\mu\nu})\right]. \tag{2.9}$$

Here the constant parameter c_1 is imposed to take the specifical value $c_1 = -1/(2\Lambda)$. It should be pointed out that the expression (2.9) differs from the original one for the Lagrangian given by [3]. Such an expression renders it convenient to reveal the relationships among critical gravity, Weyl gravity, and Einstein-Gauss-Bonnet gravity, as well as to achieve the higher-dimensional generalization according to Eq. (B7) in Appendix B. Apart from the theory of critical gravity characterized by the Lagrangian (2.9), neglecting the $R-2\Lambda$ part and letting $c_1 = 3\alpha$, $c_2 = -12\alpha$ and $c_3 = 6\alpha$ in Eq. (2.8), where α represents an arbitrary constant parameter, one acquires another type of four-dimensional critical gravity proposed in terms of the four-dimensional scale invariant gravity in [5], whose Lagrangian is the linear combination of the one for Weyl gravity with a quadratic Ricci curvature scalar R^2 term, namely,

$$\begin{aligned}\tilde{\mathcal{L}}^{(4D)}_{CG} &= \alpha\sqrt{-g}(R^2 + 6C^{\mu\nu}_{\rho\sigma}C^{\rho\sigma}_{\mu\nu}) \\ &= -3\alpha\sqrt{-g}(R^2 - 4R^{\rho\sigma}R_{\rho\sigma}) + 6\alpha\sqrt{-g}L_{GB}.\end{aligned} \tag{2.10}$$

It will be demonstrated in Appendix A that the Lagrangian (2.10) allows for the existence of asymptotically AdS solutions, and its higher-dimensional generalization will be given by Eq. (B10) in Appendix B. In addition, when $c_1 = -D^2/[8\Lambda(D-2)^2]$, $c_2 = -4(D-1)c_1/D$, and $c_3 = 0$, the Lagrangian (2.1) or (2.8) becomes the one in Eq. (B7), which can be thought of as the higher-dimensional generalization of the Lagrangian (2.9) for four-dimensional critical gravity [4].

Next, we take into account the variation of the Lagrangian (2.1) with respect to the metric tensor $g_{\mu\nu}$. We write down

$$\delta(\sqrt{-g}L) = \sqrt{-g}E_{\mu\nu}\delta g^{\mu\nu} + \sqrt{-g}\nabla_{\mu}\Theta^{\mu}. \qquad (2.11)$$

In the above equation, the expression for field equations $E_{\mu\nu}$ is presented by [87]

$$\begin{aligned} E_{\mu\nu} &= \left(\frac{\partial L}{\partial g^{\mu\nu}}\right)_{R_{\cdots\cdot}} - \frac{1}{2}Lg_{\mu\nu} - R_{(\mu}{}^{\lambda\rho\sigma}P_{\nu)\lambda\rho\sigma} - 2\nabla^{\rho}\nabla^{\sigma}P_{\rho(\mu\nu)\sigma} \\ &= R_{(\mu}{}^{\lambda\rho\sigma}P_{\nu)\lambda\rho\sigma} - 2\nabla^{\rho}\nabla^{\sigma}P_{\rho(\mu\nu)\sigma} - \frac{1}{2}Lg_{\mu\nu} \\ &= R_{\mu}{}^{\lambda\rho\sigma}P_{\nu\lambda\rho\sigma} - 2\nabla^{\rho}\nabla^{\sigma}P_{\rho\mu\nu\sigma} - \frac{1}{2}Lg_{\mu\nu}. \end{aligned} \qquad (2.12)$$

Here $R_{\cdots\cdot}$ stands for the covariant rank-4 Riemann curvature tensor. The expression $(\partial L/\partial g^{\mu\nu})_{R_{\cdots\cdot}} = 2R_{(\mu}{}^{\lambda\rho\sigma}P_{\nu)\lambda\rho\sigma}$ [87–89] and Eq. (A8) have been used to gain the second and the last equalities, respectively. The surface term Θ^{μ} in Eq. (2.11) is written as

$$\Theta^{\mu} = 2P^{\mu\nu\rho\sigma}\nabla_{\sigma}\delta g_{\rho\nu} - 2\delta g_{\nu\rho}\nabla_{\sigma}P^{\mu\nu\rho\sigma}. \qquad (2.13)$$

In Eqs. (2.12) and (2.13), the tensor $P^{\mu\nu\rho\sigma}$ is defined in terms of the derivative with respect to the Riemann tensor, that is, $P^{\mu\nu}_{\rho\sigma} = (\partial L/\partial R^{\rho\sigma}_{\mu\nu})_{g_{\alpha\beta},g^{\gamma\lambda}}$, which is referred to as entropy tensor in [87], attributed to the fact that its integral on the Killing horizon can give rise to the entropy of some higher-order derivative modified gravity theories [50]. By virtue of Eqs. (A1) and (A2) in Appendix A, the tensor $P^{\mu\nu}_{\rho\sigma}$ takes the following form

$$\begin{aligned} P^{\mu\nu}_{\rho\sigma} &= \left(\frac{\partial L}{\partial R^{\rho\sigma}_{\mu\nu}}\right)_{g_{\alpha\beta},g^{\gamma\lambda}} \\ &= \frac{1}{2}\delta^{\mu\nu}_{\rho\sigma} + c_1 R\delta^{\mu\nu}_{\rho\sigma} + 2c_2 R^{[\mu}_{[\rho}\delta^{\nu]}_{\sigma]} + 2c_3 R^{\mu\nu}_{\rho\sigma}. \end{aligned} \qquad (2.14)$$

Particularly, within the context of Weyl and Einstein-Gauss-Bonnet gravities, the tensor $P^{\mu\nu}_{\rho\sigma}$ is represented by $P^{\mu\nu}_{W\rho\sigma}$ and $P^{\mu\nu}_{EGB\rho\sigma}$, respectively, which are expressed as

$$\begin{aligned} P^{\mu\nu}_{W\rho\sigma} &= \frac{\partial L_W}{\partial R^{\rho\sigma}_{\mu\nu}} = c_1(D-1)(D-2)C^{\mu\nu}_{\rho\sigma}, \\ P^{\mu\nu}_{EGB\rho\sigma} &= \frac{\partial L_{\rm EGB}}{\partial R^{\rho\sigma}_{\mu\nu}} = \frac{1}{2}\delta^{\mu\nu}_{\rho\sigma} + \frac{c_1}{2}\delta^{\gamma\lambda\mu\nu}_{\alpha\beta\rho\sigma}R^{\alpha\beta}_{\gamma\lambda}. \end{aligned} \qquad (2.15)$$

Substituting Eqs. (2.14) and (A14) into Eq. (2.12), we further write down the expression for the field equations $E_{\mu\nu}$, being of the form

$$\begin{aligned} E_{\mu\nu} = {}& R_{\mu\nu} + 2c_1 RR_{\mu\nu} + 2(c_2+2c_3)R_{\mu\rho\nu\sigma}R^{\rho\sigma} \\ & - 4c_3 R_{\mu\lambda}R^{\lambda}_{\nu} + 2c_3 R_{\mu}{}^{\lambda\rho\sigma}R_{\nu\lambda\rho\sigma} - \frac{1}{2}Lg_{\mu\nu} \\ & + \frac{1}{2}(4c_1+c_2)g_{\mu\nu}\Box R + (c_2+4c_3)\Box R_{\mu\nu} \\ & - (2c_1+c_2+2c_3)\nabla_{\mu}\nabla_{\nu}R. \end{aligned} \qquad (2.16)$$

Here $E_{\mu\nu}$ can be also found in the works [11,80,90,91] and Eq. (A22) demonstrates that $E_{\mu\nu}$ is divergence-free. It should be pointed out that $E_{\mu\nu}$ might shed light on the understanding for the field equations of higher-order gravities as it has been demonstrated in [11] that the linearized expressions for the equations of motion in any higher-order gravity characterized by the Lagrangian made up of the Riemann tensor can be always mapped to those in the quadratic-curvature gravity theory.

With the expression $E_{\mu\nu}$ for the field equations in hand, let us explore its AdS space solutions before proceeding any further. To do so, it is assumed that the Lagrangian (2.1) admits D-dimensional maximally symmetric AdS spacetimes with the line element

$$d\bar{s}^2 = -(1-\hat{\Lambda}r^2)dt^2 + \frac{dr^2}{1-\hat{\Lambda}r^2} + r^2 d\Omega^2_{D-2}. \qquad (2.17)$$

Here and in what follows, quantities with an overline refer to the AdS background metric (2.17). In the above expression, t and r represent the time and radial coordinate respectively, $d\Omega^2_{D-2}$ denotes the line element for the $(D-2)$-dimensional unit sphere, and the constant parameter $\hat{\Lambda}$ could be thought of as an effective cosmological constant with the general form $\hat{\Lambda} = \hat{\Lambda}(\Lambda, c_1, c_2, c_3)$. According to Eq. (2.17), the Riemann curvature tensor $\bar{R}^{\mu\nu}_{\rho\sigma}$, the Ricci curvature tensor $\bar{R}^{\mu}_{\rho}$, and scalar $\bar{R}$ are respectively given by

$$\bar{R}^{\mu\nu}_{\rho\sigma} = \hat{\Lambda}\delta^{\mu\nu}_{\rho\sigma}, \qquad \bar{R}^{\mu}_{\rho} = (D-1)\hat{\Lambda}\delta^{\mu}_{\rho}, \qquad \bar{R} = D(D-1)\hat{\Lambda}. \qquad (2.18)$$

The substitution of Eq. (2.18) into Eq. (2.16) results in the expression $\bar{E}^{\mu}_{\nu}$ for the field equations of the AdS spacetimes, being of the form

$$\begin{aligned} \bar{E}^{\mu}_{\nu} &= \bar{R}^{\mu\lambda}_{\rho\sigma}\bar{P}^{\rho\sigma}_{\nu\lambda} - \frac{1}{4}\delta^{\mu}_{\nu}\bar{R}^{\rho\sigma}_{\alpha\beta}\bar{P}^{\alpha\beta}_{\rho\sigma} - \frac{1}{4}\bar{R}\delta^{\mu}_{\nu} + \Lambda\delta^{\mu}_{\nu} \\ &= -\frac{1}{4}[(D-1)(2Dk-8k+D)\hat{\Lambda} - 4\Lambda]\delta^{\mu}_{\nu}. \end{aligned} \qquad (2.19)$$

To simplify the calculations, in the first equality of Eq. (2.19), we have made use of the identity given by Eq. (A18). The constant k in Eq. (2.19) is defined through the value of the tensor $P^{\mu\nu}_{\rho\sigma}$ on the D-dimensional AdS spacetimes, that is,

$$\bar{P}^{\mu\nu}_{\rho\sigma} = P^{\mu\nu}_{\rho\sigma}(g_{\alpha\beta} \to \bar{g}_{\alpha\beta}) = k\delta^{\mu\nu}_{\rho\sigma}, \tag{2.20}$$

and it is read off as

$$k = \frac{1}{2} + [(D-1)(Dc_1 + c_2) + 2c_3]\hat{\Lambda}. \tag{2.21}$$

For instance, $k = 1/2$ for Einstein gravity, $k = 0$ for both Weyl gravity in any dimension and four-dimensional critical gravity described by the Lagrangian (2.9), and $k_{\rm EGB} = k(c_2 = -4c_1, c_3 = c_1)$ for Einstein-Gauss-Bonnet gravity. For the purpose to guaranteeing that the Lagrangian (2.1) allows for the AdS space solution (2.17), it is required that $\bar{E}^{\mu}_{\nu} = 0$. In this regard, Eq. (2.19) shows that the constant parameter $\hat{\Lambda}$ has to be constrained by the following condition [11,31,70,78,80]

$$\frac{D-4}{D-2}[(D-1)(Dc_1+c_2)+2c_3]\hat{\Lambda}^2+\hat{\Lambda}-\hat{\Lambda}_{gr}=0, \tag{2.22}$$

in which the constant $\hat{\Lambda}_{gr}$ is defined by

$$\hat{\Lambda}_{gr} = \frac{2\Lambda}{(D-1)(D-2)}. \tag{2.23}$$

Furthermore, by the aid of Eq. (2.21), the constraint (2.22) is reexpressed as

$$(D-4)k = (D-2)\frac{\hat{\Lambda}_{gr}}{\hat{\Lambda}} - \frac{D}{2}. \tag{2.24}$$

According to Eq. (2.22) or Eq. (2.24), $\hat{\Lambda} = \hat{\Lambda}_{gr}$ for Einstein gravity. In the case of Weyl gravity, the constraint $(D-1)(Dc_1 + c_2) = -2c_3$ leads to $\hat{\Lambda} = \hat{\Lambda}_{gr}$. As a result, the D-dimensional AdS spacetimes in general relativity are also the solutions for Weyl gravity. In the case for Einstein-Gauss-Bonnet gravity, the constraint (2.22) is rewritten as $\tilde{c}_1\hat{\Lambda}^2 + \hat{\Lambda} - \hat{\Lambda}_{gr} = 0$. Here and in what follows, $\tilde{c}_1 = (D-3)(D-4)c_1$. Its solutions are $\hat{\Lambda} = [-1 \pm (1 + 4\tilde{c}_1\hat{\Lambda}_{gr})^{1/2}]/(2\tilde{c}_1)$. What is more, in $D = 4$ dimensions, $\hat{\Lambda} = \hat{\Lambda}_{gr} = \Lambda/3$ holds for all four-dimensional quadratic-curvature gravities. This means that the four-dimensional AdS space is an exact solution of all these gravities.

In the remainder of the present section, we are going to follow the work [65] to investigate the definition of the conserved quantities associated to Killing isometries for asymptotically AdS spacetimes in the framework of quadratic-curvature gravity theories depicted generally by the Lagrangian (2.1). For this purpose, a crucial procedure is to construct a rank-4 tensor $P^{\mu\nu}_{(\mathrm{ref})\rho\sigma}$ inheriting the index symmetries of the Riemann curvature tensor, which is of the form

$$P^{\mu\nu}_{(\mathrm{ref})\rho\sigma} = \frac{1}{4(D-3)\hat{\Lambda}}\delta^{\gamma\lambda\mu\nu}_{\alpha\beta\rho\sigma}R^{\alpha\beta}_{\gamma\lambda} - \frac{D-4}{2}\delta^{\mu\nu}_{\rho\sigma}. \tag{2.25}$$

It can be proven that $P^{[\mu\nu\rho]\sigma}_{(\mathrm{ref})} = 0$ and $P^{\mu\nu\rho\sigma}_{(\mathrm{ref})}$ is conserved identically, namely, $\nabla_\mu P^{\mu\nu\rho\sigma}_{(\mathrm{ref})} = 0$. Especially, for the case of the AdS spacetime (2.17), one observes that $\bar{P}^{\mu\nu}_{(\mathrm{ref})\rho\sigma} = P^{\mu\nu}_{(\mathrm{ref})\rho\sigma}|_{g=\bar{g}} = \delta^{\mu\nu}_{\rho\sigma}$. Moreover, through the linear combination of both the tensors $P^{\mu\nu}_{\rho\sigma}$ and $P^{\mu\nu}_{(\mathrm{ref})\rho\sigma}$, another rank-4 tensor $\mathcal{P}^{\mu\nu}_{\rho\sigma}$ is defined as

$$\mathcal{P}^{\mu\nu}_{\rho\sigma} = P^{\mu\nu}_{\rho\sigma} - kP^{\mu\nu}_{(\mathrm{ref})\rho\sigma}. \tag{2.26}$$

Apparently, the tensor $\mathcal{P}^{\mu\nu}_{\rho\sigma}$ inherits all the index symmetries of the Riemann curvature tensor and it disappears on the AdS spacetime, namely, $\mathcal{P}^{\mu\nu}_{\rho\sigma}|_{g=\bar{g}} = 0$. It is further assumed that the asymptotically AdS spacetimes admits the symmetry generated by a Killing vector field ξ^μ, which can be associated to a conserved current J^μ in accordance with Noether theorem. According to Poincare lemma, J^μ corresponding to the Killing vector ξ^μ is read off as $J^\mu = \nabla_\nu K^{\mu\nu}$, where the 2-form potential $K^{\mu\nu}$ is proposed as [65]

$$K^{\mu\nu} = \mathcal{P}^{\mu\nu}_{\rho\sigma}\nabla^\rho\xi^\sigma - 2\xi^\sigma\nabla^\rho\mathcal{P}^{\mu\nu}_{\rho\sigma}, \tag{2.27}$$

in terms of the rank-4 tensor $\mathcal{P}^{\mu\nu}_{\rho\sigma}$. Here we point out that $K^{\mu\nu}$ is our suggested potential that is appropriate for the definition of conserved charges of asymptotically AdS spacetimes within the framwork of quadratic-curvature gravities. By the aid of Eq. (A14), the equations of motion $E^\mu_\nu = 0$ and the identity $\nabla_\mu\nabla_\nu\xi_\rho = R_{\rho\nu\mu\sigma}\xi^\sigma$ for the Killing vector ξ^μ, the substitution of Eq. (2.27) into the expression for the conserved current J^μ yields

$$\begin{aligned} J^\mu &= \xi^\nu(P^{\mu\lambda}_{\rho\sigma}R^{\rho\sigma}_{\nu\lambda} - 2\nabla^\rho\nabla^\sigma P^{\mu}{}_{\rho\sigma\nu}) + kP^{\mu\nu}_{(\mathrm{ref})\rho\sigma}R^{\rho\sigma}_{\nu\kappa}\xi^\kappa \\ &= \frac{1}{2}L\xi^\mu + \frac{k}{4(D-3)\hat{\Lambda}}\delta^{\gamma\lambda\mu\nu}_{\alpha\beta\rho\sigma}R^{\alpha\beta}_{\gamma\lambda}R^{\rho\sigma}_{\nu\kappa}\xi^\kappa + (D-4)kR^\mu_\nu\xi^\nu. \end{aligned} \tag{2.28}$$

Here the second equality is achieved under the on-shell condition for the metric tensor. With the help of $\nabla_\mu(L\xi^\mu) = \xi^\mu\nabla_\mu L = 0$, $2\nabla_\mu(R^\mu_\nu\xi^\nu) = 2R_{\mu\nu}\nabla^\mu\xi^\nu + \xi^\mu\nabla_\mu R = 0$ and $\sqrt{-g}\nabla_\mu(\delta^{\gamma\lambda\mu\nu}_{\alpha\beta\rho\sigma}R^{\alpha\beta}_{\gamma\lambda}R^{\rho\sigma}_{\nu\kappa}\xi^\kappa) = \partial_\mu\partial_\nu(\sqrt{-g}\delta^{\gamma\lambda\mu\nu}_{\alpha\beta\rho\sigma}R^{\alpha\beta}_{\gamma\lambda}\nabla^\rho\xi^\sigma) = 0$, one can verify that J^μ is conserved, namely, $\nabla_\mu J^\mu = 0$. At the same time, one observes that J^μ vanishes on the AdS spacetimes, namely, $J^\mu|_{g=\bar{g}} = \bar{J}^\mu = 0$, or equivalently, the exterior derivative for the Hodge dual of the two-form potential $K^{\mu\nu}$ fulfills $(\mathrm{d}{\star}K)|_{g=\bar{g}} = 0$. On the other hand, as usual, by means of the variation for the Lagrangian (2.1) with respect to the metric tensor, together with the Lie derivative with regard to the diffeomorphism symmetry generated by the Killing vector field ξ^μ, one obtains the 2-form Noether potential $K_R^{\mu\nu}$, given by

$$K_R^{\mu\nu} = P^{\mu\nu}_{\rho\sigma}\nabla^\rho\xi^\sigma - 2\xi^\sigma\nabla^\rho P^{\mu\nu}_{\rho\sigma}. \tag{2.29}$$

Obviously, $K^{\mu\nu}$ resembles the Noether potential $K_R^{\mu\nu}$ due to the fact that the replacement of the tensor $P^{\mu\nu\rho\sigma}$ with $\mathcal{P}^{\mu\nu\rho\sigma}$ in Eq. (2.29) makes $K_R^{\mu\nu}$ coincide with $K^{\mu\nu}$ and both the tensors $P^{\mu\nu\rho\sigma}$ and $\mathcal{P}^{\mu\nu\rho\sigma}$ have the same index symmetries. In terms of $K_R^{\mu\nu}$, the potential $K^{\mu\nu}$ can be further expressed as an alternative form

$$K^{\mu\nu} = K_R^{\mu\nu} - kP^{\mu\nu}_{(\mathrm{ref})\rho\sigma}\nabla^\rho\xi^\sigma. \tag{2.30}$$

In this respect, the potential $K^{\mu\nu}$ can be decomposed into two components. The first one is the usual Noether potential $K_R^{\mu\nu}$, while the second one $kP^{\mu\nu}_{(\mathrm{ref})\rho\sigma}\nabla^\rho\xi^\sigma$ is responsible for curing divergences appearing in the calculations of $K_R^{\mu\nu}$. The existence of the factor $\hat{\Lambda}^{-1}$ in the rank-4 tensor $P^{\mu\nu}_{(\mathrm{ref})\rho\sigma}$ may render the second component divergent when $\hat{\Lambda} \to 0$. However, since the other quantities $K_R^{\mu\nu}$, $R^{\mu\nu}_{\rho\sigma}$, and $\nabla^\rho\xi^\sigma$ involved in the potential $K^{\mu\nu}$ generally depend on the parameter $\hat{\Lambda}$ for asymptotically AdS spacetimes, their total contribution can guarantee the convergence of $K^{\mu\nu}$ under the limit $\hat{\Lambda} \to 0$. This situation happens also to the potentials via the topological regulation method [72,73], as well as to the potential for four-dimensional Einstein gravity proposed in [86], since such potentials explicitly contain the inverse of the cosmological constant. Therefore, the above suggests that it is feasible to take the limit $\hat{\Lambda} \to 0$ on the potential $K^{\mu\nu}$ so as to extend $K^{\mu\nu}$ to compute the conserved charges of asymptotically flat counterparts for asymptotically AdS spacetimes.

Remarkably, the structure of $K^{\mu\nu}$ is similar as the Komar-type potentials for the theories of higher-order derivative gravity proposed in [74,75]. In fact, according to these works, those Komar-type potentials can be generally expressed as the form $K_{gK}^{\mu\nu} = K_R^{\mu\nu} - B^{\mu\nu}$ with the anti-symmetric tensor $B^{\mu\nu}$ defined through $\nabla_\nu B^{\mu\nu} = 1/2L\xi^\mu$. Here the 2-form $B^{\mu\nu}$ is determined up to a divergence-free 2-form, and its local existence is always guaranteed attributed to the fact that the divergence of $L\xi^\mu$ vanishes identically for the diffeomorphism invariant Lagrangian L and the Killing vector field ξ^μ. The divergence $\nabla_\nu K_R^{\mu\nu} = 1/2L\xi^\mu$ under the on-shell condition, cancelling out the divergence of the 2-form $B^{\mu\nu}$ in the conserved current $J_{gK}^\mu = \nabla_\nu K_{gK}^{\mu\nu}$. As a consequence, one obtains $J_{gK}^\mu = 0$. In this regard, both the currents J^μ and J_{gK}^μ coincides with each other on the AdS spacetimes, rendering it of possibility for the integrals of $K^{\mu\nu}$ and $K_{gK}^{\mu\nu}$ on the codimension-2 surfaces at infinity to yield the same asymptotic charges. Due to the above, the $kP^{\mu\nu}_{(\mathrm{ref})\rho\sigma}\nabla^\rho\xi^\sigma$ ingredient in $K^{\mu\nu}$ can be interpreted as a prospective substitution for the 2-form $B^{\mu\nu}$ at infinity, and the potential $K^{\mu\nu}$ could be regarded as a Komar-like potential for the theories of quadratic-curvature gravity.

Before any further process, here we give three significant examples on the applications of the potential $K^{\mu\nu}$. As a direct application to the AdS spacetime (2.17), $K^{\mu\nu}$ turns into the one $\bar{K}^{\mu\nu} = K^{\mu\nu}|_{g=\bar{g}} = 0$, implying that $K^{\mu\nu}$ vanishes identically on the AdS spacetime. In addition, when the potential (2.27) is applied to the theory of Weyl gravity described by the Lagrangian (2.5), one substitutes the tensor $P^{\mu\nu}_{W\rho\sigma}$ in Eq. (2.15) into the Noether potential $K_R^{\mu\nu}$ to acquire the potential $K_{\mathrm{Weyl}}^{\mu\nu}$, given by

$$K_{\mathrm{Weyl}}^{\mu\nu} = c_1(D-1)(D-2)(C^{\mu\nu}_{\rho\sigma}\nabla^\rho\xi^\sigma - 2\xi^\sigma\nabla^\rho C^{\mu\nu}_{\rho\sigma}), \tag{2.31}$$

which is just the Noether potential for Weyl gravity, arising from that $k = 0$. It can be tested that the perturbation of the potential $K_{\mathrm{Weyl}}^{\mu\nu}$ on four-dimensional AdS spaces is equivalent to the one given by Eq. (23) in [79], which was acquired via the (off-shell) ADT formalism [53–57]. Moreover, for the D-dimensional Einstein-Gauss-Bonnet gravity with the Lagrangian (2.6), its potential $K_{\mathrm{EGB}}^{\mu\nu}$, derived from Eq. (2.27) without the on-shell condition for the metric, takes the following form

$$K_{\mathrm{EGB}}^{\mu\nu} = \frac{1+2\tilde{c}_1\hat{\Lambda}}{2}\left[(D-2)\nabla^\mu\xi^\nu - \frac{1}{4(D-3)\hat{\Lambda}}\delta^{\gamma\lambda\mu\nu}_{\alpha\beta\rho\sigma}R^{\alpha\beta}_{\gamma\lambda}\nabla^\rho\xi^\sigma\right], \tag{2.32}$$

which is proportional to the potential for Einstein gravity given in [63–65] by the factor $(1 + 2\tilde{c}_1\hat{\Lambda})$ regardless of the on-shell condition. As a matter of fact, since it will be demonstrated below that the linear perturbation of $K^{\mu\nu}$ with respect to the decomposition to the metric $g_{\mu\nu} = \delta g_{\mu\nu} + \bar{g}_{\mu\nu}$, where $\bar{g}_{\mu\nu}$ is the metric of the AdS space (2.17), is consistent with the potential defined by the ADT formalism, the same perturbation for $K_{\mathrm{EGB}}^{\mu\nu}$, being a special case of $K^{\mu\nu}$, is equivalent to the superpotential (3.11) on the AdS space proposed in [60]. The latter was obtained via the field-theoretic approach [60–62] and was verified to coincide with the ADT potential in [60]. Apart from this, the linear perturbation of $K_{\mathrm{EGB}}^{\mu\nu}$ on AdS spaces is equivalent to the ADT potential given by Eq. (41) in [79]. However, $K_{\mathrm{EGB}}^{\mu\nu}$ here is much simpler than the superpotentials in [60,79]. In particular, under a critical condition of Eq. (2.32), where $1 + 2\tilde{c}_1\hat{\Lambda} = 0$ or both the constant parameters c_1 and Λ are related to each other through

$$c_1 = -\frac{(D-1)(D-2)}{8(D-3)(D-4)\Lambda}, \tag{2.33}$$

the potentials for $D(D\geq 5)$-dimensional Einstein-Gauss-Bonnet gravities vanish identically, leading to zero conserved charges.

For convenience to calculations, with the help of the expression (2.4) for the expansion of the generalized Kronecker-delta symbol $\delta^{\gamma\lambda\mu\nu}_{\alpha\beta\rho\sigma}$, as well as Eq. (A13) for divergence of the rank-4 tensor $P^{\mu\nu}_{\rho\sigma}$, the potential $K^{\mu\nu}$ given by Eq. (2.27) is rewritten as

$$\begin{aligned}K^{\mu\nu} &= [1+(D-4)k+(2c_1-\hat{k})R]\nabla^{[\mu}\xi^{\nu]} \\ &\quad -2(c_2+2\hat{k})R^{[\mu}_{\rho}\nabla^{\nu]}\xi^{\rho}+(2c_3-\hat{k})R^{\mu\nu}_{\rho\sigma}\nabla^{\rho}\xi^{\sigma} \\ &\quad +(4c_1+c_2)\xi^{[\mu}\nabla^{\nu]}R-2(c_2+4c_3)\xi^{\rho}\nabla^{[\mu}R^{\nu]}_{\rho}. \end{aligned} \tag{2.34}$$

Here the constant parameter $\hat{k}$ is presented by

$$\hat{k}=\frac{k}{(D-3)\hat{\Lambda}}. \tag{2.35}$$

It is worth to mentioning that the potential $K^{\mu\nu}$ in Eq. (2.34) can go further under the on-shell condition that the metric tensor is the solution of the field equations $E_{\mu\nu}=0$. In terms of the following identities associated with the Killing vector ξ^{ν}

$$\begin{aligned}\Box\nabla^{\mu}\xi^{\nu} &= -R^{\mu\nu}_{\rho\sigma}\nabla^{\rho}\xi^{\sigma}-\xi^{\rho}\nabla^{[\mu}R^{\nu]}_{\rho}, \\ \nabla^{\mu}\Box\xi^{\nu} &= -\xi^{\rho}\nabla^{\mu}R^{\nu}_{\rho}-R^{\nu}_{\rho}\nabla^{\mu}\xi^{\rho},\end{aligned} \tag{2.36}$$

the potential $K^{\mu\nu}$ in Eq. (2.34) can be reformulated in the language of differential forms into an alternative form

$$\begin{aligned}K &= \frac{1}{2}[1+(D-4)k+(6c_1+c_2-\hat{k})R]\mathrm{d}\xi \\ &\quad -(c_2+2\hat{k})\mathrm{d}\Box\xi+2(c_2+2c_3+\hat{k})\Box\mathrm{d}\xi \\ &\quad +\left(2c_2+5c_3+\frac{3}{2}\hat{k}\right)R^{\rho\sigma}_{\mu\nu}\nabla_{\rho}\xi_{\sigma}\mathrm{d}x^{\mu}\wedge\mathrm{d}x^{\nu} \\ &\quad -\frac{1}{2}(4c_1+c_2)\mathrm{d}(R\xi).\end{aligned} \tag{2.37}$$

Equation (2.37) demonstrates that the potential $K^{\mu\nu}$ could be reproduced through the action of the differential operators on the Killing vector field. This can be seen more clearly due to the one-form current $J=-\star\mathrm{d}\star K$ [92]. In comparison, the differential form (2.37) for the potential $K^{\mu\nu}$ can be regarded as a special case of the general potential proposed in [93].

We move on to present the concrete expressions for the potential $K^{\mu\nu}$ given by Eq. (2.34) in some special cases of the Lagrangian (2.1). When it becomes the Einstein-Hilbert one $L_{\mathrm{EH}}=R-(D-1)(D-2)\hat{\Lambda}$ with $\hat{\Lambda}=\hat{\Lambda}_{gr}$ under $c_1=c_2=c_3=0$, by the aid of the equations of motion $R_{\mu\nu}=(D-1)\hat{\Lambda}g_{\mu\nu}$, the potential $K^{\mu\nu}$ is simplified as[1]

$$K^{\mu\nu}_{gr}=\frac{1}{D-3}\left(\nabla^{[\mu}\xi^{\nu]}-\frac{1}{2\hat{\Lambda}}R^{\mu\nu}_{\rho\sigma}\nabla^{\rho}\xi^{\sigma}\right), \tag{2.38}$$

which is consistent with the usual Komar potential modified by an additional second-order derivative term proportional to the Riemann curvature tensor, given by Eq. (2.15) in [92]. When $c_3=0$, the potential $K^{\mu\nu}$ in $D=4$ dimensions coincides with the one given by Eq. (16) in [71] or Eq. (26) in [72], and $K^{\mu\nu}$ in $D=2(n+2)$ dimensions is equivalent to the potential given by Eq. (23) in [71] or Eq. (14) in [73], acquired via the topological regularization method. When the Lagrangian (2.1) admits the solutions obeying $R_{\mu\nu}=(D-1)\hat{\Lambda}g_{\mu\nu}$, for such solutions, the potential $K^{\mu\nu}$ takes the form

$$K^{\mu\nu}_{\mathrm{Ric}}=2[k-2(D-3)c_3\hat{\Lambda}]K^{\mu\nu}_{gr}. \tag{2.39}$$

In particular, when $D=4$, one obtains $K^{\mu\nu}_{\mathrm{Ric}}|_{D=4}=2[1+6(4c_1+c_2)\hat{\Lambda}]K^{\mu\nu}_{gr}|_{D=4}$, implying that the potential is irrelevant to the coupling constant c_3 or the scalar term $R^{\mu\nu}_{\rho\sigma}R^{\rho\sigma}_{\mu\nu}$. Obviously, the potential $K^{\mu\nu}_{\mathrm{Ric}}$ is proportional to $K^{\mu\nu}_{gr}$, supporting the finding that the ADT potentials of quadratic curvature gravities are proportional to the one for Einstein gravity in [64,78]. At the linearized level, the potential $K^{\mu\nu}_{\mathrm{Ric}}$ completely agrees with the one given by Eq. (6) in [67], and the conserved charge defined in terms of $K^{\mu\nu}_{\mathrm{Ric}}$ is consistent with the one given by Eq. (2.3) in [66]. What is more, in the case of the four-dimensional critical gravity depicted by the

[1]The potential $K^{\mu\nu}_{gr}$ can be interpreted as the generalization of the usual Komar potential $\nabla^{[\mu}\xi^{\nu]}$ in asymptotically AdS spacetimes. In contrast with the latter, the former incorporates an additional second-order derivative term $(2\hat{\Lambda})^{-1}R^{\mu\nu}_{\rho\sigma}\nabla^{\rho}\xi^{\sigma}$. Such a term was also introduced in the potential given by Eq. (8) in [86], which was proposed to define the conserved charges of four-dimensional Einstein gravity with cosmological constant. The $(2\hat{\Lambda})^{-1}R^{\mu\nu}_{\rho\sigma}\nabla^{\rho}\xi^{\sigma}$ term plays the main role in eliminating the divergent terms appearing within $\nabla^{[\mu}\xi^{\nu]}$ to render a finite result. Apart from this, it corrects the normalization factors in the explicit calculations for the conserved quantities of spacetimes. In particular, $K^{\mu\nu}_{gr}$ vanishes on the AdS spaces in Eq. (2.17). Additionally, for ultrastatic spacetimes with the metric ansatz $ds^2=-dt^2+g_{ij}dx^idx^j$, where the Riemannian metric g_{ij} on the $(D-1)$-dimensional space is independent of the time coordinate t [94], due to the fact that $R^{t}_{\rho\mu\nu}=0$ and the conserved quantities are defined via the integral of $K^{\mu\nu}_{gr}$ over the surface at $t=\mathrm{Const}$, the $(2\hat{\Lambda})^{-1}R^{\mu\nu}_{\rho\sigma}\nabla^{\rho}\xi^{\sigma}$ term actually makes no contribution to the conserved quantities. Thus, $K^{\mu\nu}_{gr}$ is equivalent to the Komar potential, rendering it feasible to use $K^{\mu\nu}_{gr}$ to yield the Komar charges for asymptotically flat ultrastatic spacetimes. Since the timelike Killing vector $\xi^{\mu}=-\delta^{\mu}_{t}$ of such spacetimes obeys $\nabla^{\mu}\xi^{\nu}=0$, further yielding $K^{\mu\nu}_{gr}=0$, the energy of ultrastatic spacetimes vanishes.

Lagrangian (2.9), the constraints $c_2 = -3c_1$ and $c_3 = 0$ enable us to express $K^{\mu\nu}$ as a simpler form

$$\begin{aligned} K^{\mu\nu}_{\rm 4DCG} &= \nabla^{[\mu}\xi^{\nu]} - \frac{1}{2\Lambda}(2R\nabla^{[\mu}\xi^{\nu]} + 6R^{[\mu}_{\rho}\nabla^{\nu]}\xi^{\rho} \\ &\quad + \xi^{[\mu}\nabla^{\nu]}R + 6\xi^{\rho}\nabla^{[\mu}R^{\nu]}_{\rho}) \\ &= -3\nabla^{[\mu}\xi^{\nu]} - \frac{3}{\Lambda}(R^{[\mu}_{\rho}\nabla^{\nu]}\xi^{\rho} + \xi^{\rho}\nabla^{[\mu}R^{\nu]}_{\rho}), \end{aligned} \tag{2.40}$$

or equivalently formulated in the language of exterior algebra as follows

$$\begin{aligned} K_{\rm 4DCG} &= -\frac{3}{2}\mathrm{d}\xi - \frac{3}{2\Lambda}\mathrm{d}\Box\xi + \frac{3}{\Lambda}\Box\mathrm{d}\xi \\ &\quad + \frac{3}{\Lambda}R^{\rho\sigma}_{\mu\nu}\nabla_{\rho}\xi_{\sigma}\mathrm{d}x^{\mu}\wedge\mathrm{d}x^{\nu} \\ &= \frac{1}{2}\mathrm{d}\xi - \frac{3}{2\Lambda}\mathrm{d}\Box\xi + \frac{3}{\Lambda}\Box\mathrm{d}\xi - 4K_{gr}. \end{aligned} \tag{2.41}$$

In Eq. (2.40), we have used the result $R = 4\Lambda$ derived from Eq. (A21) to arrive at the second equality. It can be verified that the perturbation of the potential $K^{\mu\nu}_{\rm 4DCG}$ on the AdS spacetime (2.17) vanishes.

In the above, some special cases of the potential $K^{\mu\nu}$ have been compared with the ones via other methods (the results will be summarized in Appendix D). Apart from this, for generality, it is of great necessity to build the relation between $K^{\mu\nu}$ and the well-known Iyer-Wald potential defined by the covariant phase space method [49–51] in the context of the Lagrangian (2.1). By virtue of following the covariant phase space approach, the Iyer-Wald potential associated with the Killing vector ξ^{μ} is read off as [11]

$$Q^{\mu\nu}_{\rm IW} = \delta K^{\mu\nu}_R + \frac{1}{2}K^{\mu\nu}_R g^{\rho\sigma}\delta g_{\rho\sigma} - \xi^{[\mu}\Theta^{\nu]}, \tag{2.42}$$

which coincides with the off-shell generalized ADT potential [57]. In Eq. (2.42), the surface term Θ^{μ} and the Noether potential $K^{\mu\nu}_R$ are given by Eqs. (2.13) and (2.29) respectively. As a special case of (2.42), where the background spacetime is the fixed D-dimensional AdS space (2.17) and the linear perturbation of the metric tensor is defined through $\delta g_{\mu\nu} = g_{\mu\nu} - \bar{g}_{\mu\nu}$, the potential $Q^{\mu\nu}_{\rm IW}$ becomes

$$\begin{aligned} \bar{Q}^{\mu\nu}_{\rm IW} &= \delta K^{\mu\nu}_R + \frac{1}{2}g^{\alpha\beta}\delta g_{\alpha\beta}\bar{P}^{\mu\nu}_{\rho\sigma}\overline{\nabla}^{\rho}\bar{\xi}^{\sigma} - 2\bar{\xi}^{[\mu}\bar{P}^{\nu]\lambda}_{\rho\sigma}\bar{g}^{\rho\alpha}\overline{\nabla}^{\sigma}\delta g_{\alpha\lambda} \\ &= \delta K^{\mu\nu}_R + k\bar{g}^{\alpha\beta}\delta g_{\alpha\beta}\overline{\nabla}^{[\mu}\bar{\xi}^{\nu]} - 2k\bar{\xi}^{[\mu}\delta^{\nu]\lambda}_{\rho\sigma}\bar{g}^{\rho\alpha}\overline{\nabla}^{\sigma}\delta g_{\alpha\lambda}, \end{aligned} \tag{2.43}$$

which is consistent with the ADT potential for the Lagrangian (2.1) [55,56,78]. On the other hand, with the help of Eq. (2.43), the linear perturbation of the potential $K^{\mu\nu}$ on the AdS space (2.17) gives rise to the following result

$$\delta K^{\mu\nu} = \bar{Q}^{\mu\nu}_{\rm IW} - \hat{k}\overline{\nabla}_{\gamma}\bar{U}^{\gamma\mu\nu}. \tag{2.44}$$

In Eq. (2.44), the 3-form $\bar{U}^{\gamma\mu\nu}$ is presented by

$$\bar{U}^{\gamma\mu\nu} = \frac{1}{2}\delta^{\lambda\gamma\mu\nu}_{\alpha\beta\rho\sigma}\bar{g}^{\beta\eta}(\overline{\nabla}^{\alpha}\delta g_{\eta\lambda})\overline{\nabla}^{\rho}\bar{\xi}^{\sigma}. \tag{2.45}$$

As a consequence of Eq. (2.44), the perturbation of the potential $K^{\mu\nu}$ on the AdS spacetime background is equivalent to the Iyer-Wald potential $\bar{Q}^{\mu\nu}_{\rm IW}$ on the same background, as well as to the ADT potential. This is attributed to the fact that the integral of $\mathrm{d}\star\bar{U}$, which stands for the exterior derivative for the Hodge dual of the 3-form $\bar{U}$, vanishes according to Stokes' theorem. In this regard, the $kP^{\mu\nu}_{({\rm ref})\rho\sigma}\nabla^{\rho}\xi^{\sigma}$ component in $K^{\mu\nu}$ plays the role of compensating the contribution from the $\xi^{[\mu}\Theta^{\nu]}$ part in Iyer-Wald potential $Q^{\mu\nu}_{\rm IW}$. So the potential $K^{\mu\nu}$ could be thought of as the "integration form" of $Q^{\mu\nu}_{\rm IW}$ on the fixed AdS background. Moreover, the perturbation of the potential $K^{\mu\nu}$ on the AdS spacetime background is equivalent to the ones presented in [64,66–68]. The potential $K^{\mu\nu}$ in four dimensions or in odd dimensions is perhaps equivalent to the one proposed in [69] in the absence of torsion, or the one obtained via counterterm method in [70] respectively.

At the end, we are going to make use of the potential $K^{\mu\nu}$ given by Eq. (2.27) or (2.34) to define the conserved charges of quadratic-curvature gravities described by the Lagrangian (2.1). To achieve this, as usual, it is assumed that there exists a $(D-1)$-dimensional hypersurface Σ with the boundary $\partial\Sigma$. In terms of a $(D-2)$-form $\star K$, which is the Hodge dual of the 2-form potential $K^{\mu\nu}$, according to Stokes' theorem, a formula for the conserved charges of such gravity theories can be put forward as the integral of the potential $\star K$ over the $(D-2)$-dimensional surface $\partial\Sigma$, that is,

$$Q = \frac{1}{8\pi}\int_{\partial\Sigma}\star K. \tag{2.46}$$

Here let us make some remarks on the formula (2.46) for conserved quantities. First, according to the equivalence relation displayed by Eq. (2.44), the potential $K^{\mu\nu}$ can be used to replace the ADT and the Iyer-Wald ones when the latter two are applied to compute the conserved quantities of asymptotically AdS spacetimes in general relativity and quadratic gravities. However, in contrast with the conserved quantities by means of the latter two, the formula (2.46) has the merit of avoiding the computation for the perturbation of the potential, greatly simplifying the calculations. Second, $K^{\mu\nu}$ resembles the Noether potential $K^{\mu\nu}_R$ as well as the potential $K^{\mu\nu}_{gK}$. Accordingly, the formula (2.46) takes a similar structure as the Komar-type integral and the Wald entropy formula, providing a simple and convenient formulation for the conserved quantities of asymptotically AdS spacetimes in quadratic gravities. However, unlike the original Komar integral, which suffers the disadvantage to present an expression for the mass differing from that for the angular momentum by an anomalous factor in form, the formula (2.46) gives a unified form of the conserved charges. Third, Eq. (2.30) shows that the potential $K^{\mu\nu}$ can be decomposed into two parts. One is the Noether potential $K_R^{\mu\nu}$, while the other is the antisymmetric rank-2 tensor $-kP^{\mu\nu}_{(\mathrm{ref})\rho\sigma}\nabla^\rho\xi^\sigma$. However, when the formula (2.46) is adopted to compute the mass of asymptotically AdS spacetimes, it cannot be split into two parts according to the decomposition of $K^{\mu\nu}$, attributed to the fact that each separate part of the formula suffers from divergence at spatial infinity although their combination is convergent.

III. MASS FOR STATIC AND SPHERICALLY SYMMETRIC SPACETIMES

In this section, as an application for the formula of conserved charges, we shall compute the mass for general static and spherically symmetric spacetimes endowed with an asymptotically AdS structure. Such spacetimes are solutions obeying the field equation $E_{\mu\nu}=0$, and they cover the ones in general relativity, four-dimensional Weyl and critical gravities, and Einstein-Gauss-Bonnet gravity in some special cases.

We begin with the general metric ansatz for D-dimensional asymptotically AdS spacetimes with spherical symmetry. In the coordinate system $\{t,r,y^i\}$ $(i=1,2,\cdots,D-2)$, where y^is denote the coordinates parametrizing a $(D-2)$-dimensional unit sphere, the line element of spacetime can be always expressed as the following general form

$$ds^2=-f(r)dt^2+\frac{dr^2}{f(r)}+B^2(r)d\Omega^2_{D-2},\\ d\Omega^2_{D-2}=h_{ij}(y)dy^idy^j. \tag{3.1}$$

In the above expression, h_{ij} represents the metric tensor for the codimension-2 unit sphere. However, here we point out that all the results related to the curvature tensors below hold for an arbitrary codimension-2 spacial compact manifold. In order to maintain the asymptotically AdS structure (2.17), the functions $f(r)$ and $B(r)$ in Eq. (3.1) are required to behave at the asymptotic infinity as $f(\infty)\to 1-\hat{\Lambda}r^2$ and $B(\infty)\to r$, respectively. Moreover, under the coordinate transformation $\varrho=\varrho(r)$ determined by $\varrho=B(r)$, the line element (3.1) is recast into another common form:

$$ds^2=-F(\varrho)dt^2+\frac{d\varrho^2}{H(\varrho)}+\varrho^2d\Omega^2_{D-2}. \tag{3.2}$$

Here the functions $F(\varrho)$ and $H(\varrho)$ are given respectively by

$$F(\varrho)=f[r(\varrho)],\qquad H(\varrho)=f[r(\varrho)]\left(\frac{dB}{dr}\right)^2. \tag{3.3}$$

In terms of the line element (3.1) or (3.2), there have been a lot of works devoting to seeking exact static and spherically symmetric solutions within the context of higher-derivative gravity theories. For example, see the works [13–32] and related references therein.

As an attempt to acquire the concrete expressions for the mass of the general static spherically symmetric black holes in terms of the formula (2.46) for the conserved charges, it is of great necessity to compute the relevant curvature tensors for the spacetimes described by Eq. (3.1). Implementing computations to the (t,r,ρ,σ) components of the Riemann curvature tensor $R^{\mu\nu}_{\rho\sigma}$ gives rise to

$$R^{tr}_{\rho\sigma}=-f''\delta^t_{[\rho}\delta^r_{\sigma]}. \tag{3.4}$$

Here and in what follows, the quantity with the prime "′" denotes its derivative with respect to the radial coordinate r, such as $f'=df/dr$ and $f''=d^2f/dr^2$. Furthermore, after some complicated calculations, we obtain the related components of the Ricci tensor, which are read off as $R_{ti}=R_{ri}=0$, together with the ones

$$R^t_t=-\frac{1}{2B}[Bf''+(D-2)B'f'],\\ R^r_r=-\frac{1}{2B}[Bf''+(D-2)(2fB''+B'f')],\\ R^j_i=\frac{1}{B^2}[R^j_{hi}-\delta^j_i(BfB''+BB'f'+(D-3)fB'^2)]. \tag{3.5}$$

In the above equation, R^j_{hi} represents the Ricci curvature tensor for the $(D-2)$-dimensional line element $d\Omega^2_{D-2}$, whose Ricci curvature scalar is defined through $R_h=h^{ij}R_{hij}$. On the basis of Eq. (3.5), the Ricci curvature scalar R, defined as $R=R^\mu_\mu=R^t_t+R^r_r+R^i_i$, is presented by

$$R=\frac{1}{B^2}[R_h-B^2f''-(D-2)(2BfB''+2BB'f'\\ +(D-3)fB'^2)]. \tag{3.6}$$

Particularly, in the case where h_{ij} is the metric tensor for the $(D-2)$-dimensional unit sphere, the Ricci curvature scalar R_h in Eq. (3.6) takes the value $R_h=(D-2)(D-3)$.

With Eqs. (3.4)–(3.6) in hand, wo process to compute the 2-form potentials in terms of $K^{\mu\nu}$ given by Eq. (2.34). As usual, the Killing vector corresponding to the mass for the static and spherically symmetric spacetimes characterized by the line element (3.1) is chosen as $\xi^\mu=(-1,0,\cdots,0)$.

With such a Killing vector, we calculate the tr component of the potential $K^{\mu\nu}$, yielding

$$K^{tr} = \frac{c_2+2\hat{k}}{2}(R^t_t + R^r_r)f' - \frac{2c_3-\hat{k}}{2}f'f'' - \frac{4c_1+c_2}{2}f\partial_r R + \frac{f'}{2}[1+(D-4)k+(2c_1-\hat{k})R] - \frac{c_2+4c_3}{2B}[2Bf\partial_r R^t_t + (D-2)ff'B'']. \tag{3.7}$$

Subsequently, we take into account several special aspects of K^{tr}. First, when the Lagrangian (2.1) turns into the one for Einstein gravity, in which $c_1 = c_2 = c_3 = 0$ and $k = 1/2$, the component K^{tr} coincides with the one given by a generalized Komar formulation [92]. It is simplified as

$$K^{tr}_{gr} = \frac{D-2}{4}\left(f' - \frac{f'}{\hat{\Lambda}B^2} + \frac{ff'B'^2}{\hat{\Lambda}B^2}\right). \tag{3.8}$$

Second, in the framework of Einstein-Gauss-Bonnet gravity, where $c_2 = -4c_1$ and $c_3 = c_1$, the tr component K^{tr} is expressed as

$$K^{tr}_{\mathrm{EGB}} = \frac{f'}{2}[1+(D-4)k_{\mathrm{EGB}}] - \frac{k_{\mathrm{EGB}} - 2c_1(D-3)\hat{\Lambda}}{2(D-2)^{-1}\hat{\Lambda}B^2}(1-fB'^2)f' = [1+2c_1(D-3)(D-4)\hat{\Lambda}]K^{tr}_{gr}. \tag{3.9}$$

In the above equation, the constant k_{EGB} is determined by plugging $c_2 = -4c_1$ and $c_3 = c_1$ into Eq. (2.21), being of the form $k_{\mathrm{EGB}} = 1/2 + (D-2)(D-3)c_1\hat{\Lambda}$. Apparently, the second equality in Eq. (3.9) verifies the observation in Eq. (2.32) that the potential for Einstein-Gauss-Bonnet gravity is proportional to the one for Einstein gravity. Third, in the case for the four-dimensional Weyl gravity characterized by the Lagrangian (2.5), the component K^{tr} becomes

$$K^{tr}_{\mathrm{4DWG}} = \frac{c_1}{B^3}K^{tr}_{qc}. \tag{3.10}$$

In Eq. (3.10), the quantity K^{tr}_{qc} represents the contribution from all the quadratic curvature terms in the Lagrangian (2.5), being of the form

$$K^{tr}_{qc} = 2Bf' - 4fB' - 6Bff'B'^2 + 4f^2B'B'B' + (2ff''' - f'f'')B^3 + 2(fB'f'' - 2f^2B''' - 3ff'B'' + B'f'^2)B^2. \tag{3.11}$$

Fourth, for the four-dimensional critical gravity with the Lagrangian (2.9) ($c_1 = -(2\Lambda)^{-1}$), which is equivalent to $\sqrt{-g}[L_{\mathrm{EH}} - 1/2L_W(D=4)]$, with the help of Eq. (3.11), K^{tr} given by Eq. (3.7) is presented by

$$K^{tr}_{\mathrm{4DCG}} = \left(K^{tr}_{gr} - \frac{1}{2}K^{tr}_{\mathrm{4DWG}}\right)_{D=4} = K^{tr}_{gr}(D=4) - \frac{c_1}{2B^3}K^{tr}_{qc}. \tag{3.12}$$

Particularly, when the metric tensor is the solution of the vacuum Einstein field equations $R_{\mu\nu} = \Lambda g_{\mu\nu}$, calculations show that the potential K^{tr}_{4DCG} disappears. As a matter of fact, this can be seen straightforwardly from the general expression for the potential $K^{\mu\nu}_{\mathrm{4DCG}}$ given by Eq. (2.40).

In the remainder of this section, according to K^{tr}, together with its concrete expressions in various specifical quadratic-curvature gravities, we shall give the mass formulations for the static, spherically symmetric and asymptotically AdS spacetimes within such gravity theories. With the metric ansatz (3.1), by means of plugging Eq. (3.7) in Eq. (2.46), the general formulation for the mass M of these spacetimes is defined through

$$M = \frac{1}{8\pi}\int_{r=\infty}\sqrt{h}B^{D-2}K^{tr}d^{D-2}y, \tag{3.13}$$

where $h = \det(h_{ij})$ is the determinant of the codimension-2 metric h_{ij}. As a special case of Eq. (3.13), for Einstein gravity, Eq. (3.8) sends the mass M into the form

$$M_{gr} = \frac{V_{D-2}}{8\pi}\lim_{r\to\infty}\frac{K^{tr}_{gr}}{B^{2-D}} = \frac{V_{D-2}}{32\pi(D-3)\hat{\Lambda}}\lim_{r\to\infty}\frac{(2\hat{\Lambda}+f'')f'}{B^{2-D}}. \tag{3.14}$$

In Eq. (3.14), $V_{D-2} = \int\sqrt{h}d^{D-2}y = 2\pi^{(D-1)/2}/\Gamma((D-1)/2)$ is the volume of the $(D-2)$-dimensional unit sphere. In order to get the second equality, the equation $f'' + 2\hat{\Lambda} = (D-2)(D-3)(\hat{\Lambda}B^2 + fB'B' - 1)/B^2$ derived from the vacuum Einstein field equations $R^\nu_\mu = (D-1)\hat{\Lambda}\delta^\nu_\mu$ has been used. As an application of Eq. (3.14), it can be utilized to evaluate the mass of the D-dimensional Schwarzschild-AdS black holes, in the line element of which $f(r)$ and $B(r)$ are given by

$$f_{\mathrm{SAdS}} = 1 - \frac{2m}{r^{D-3}} - \hat{\Lambda}r^2, \qquad B_{\mathrm{SAdS}} = r, \tag{3.15}$$

respectively. Substituting Eq. (3.15) into Eq. (3.14) produces the following mass

$$M_{\mathrm{SAdS}} = \frac{m(D-2)V_{D-2}}{8\pi}, \tag{3.16}$$

coinciding with the standard result in the literature. Besides, Eq. (3.14) can be also adopted to compute the mass for four-dimensional spherical MadMax AdS black holes constructed quite recently in [95].

In the situation for D-dimensional Einstein-Gauss-Bonnet gravity, the potential K^{tr} in Eq. (3.13) is substituted by the one $K^{tr}_{\rm EGB}$, leading to the mass

$$M_{\rm EGB} = \frac{[1+2c_1(D-3)(D-4)\hat{\Lambda}]V_{D-2}}{8\pi}\lim_{r\to\infty}\frac{K^{tr}_{gr}}{B^{2-D}} = [1+2c_1(D-3)(D-4)\hat{\Lambda}]M_{gr}. \tag{3.17}$$

Here M_{gr} is defined in terms of the first equality in Eq. (3.14) and the mass $M_{\rm EGB}$ agrees with the one via the field-theoretic approach [60–62]. Obviously, when $c_1 = 0$ or $D = 4$, $M_{\rm EGB} = M_{gr}$, implying that the inclusion of the Gauss-Bonnet term in four dimensions does not affect the mass. For instance, let us apply Eq. (3.17) to compute the mass of the static spherically symmetric asymptotically AdS black holes in D-dimensional ($D > 4$) Einstein-Gauss-Bonnet gravity [15–19]. For such black holes, both the functions $B(r)$ and $f(r)$ in their line elements are read off as $B_{\rm EGB} = r$ and

$$f_{\rm EGB} = 1 + \frac{r^2}{2\tilde{c}_1} - \frac{r^2}{2\tilde{c}_1}\sqrt{1 + 4\tilde{c}_1\hat{\Lambda}_{gr} + \frac{4\tilde{c}_1 m}{r^{D-1}}}, \tag{3.18}$$

respectively, where $\tilde{c}_1 = (D-3)(D-4)c_1$ as before. Eq. (3.17) gives rise to the mass

$$\mathcal{M}_{\rm EGB} = \frac{m(D-2)V_{D-2}}{16\pi}. \tag{3.19}$$

In the case of the four-dimensional Weyl gravity, the substitution of Eq. (3.10) into Eq. (3.13) gives rise to the mass

$$M_{\rm 4DWG} = \frac{c_1 V_2}{8\pi}\lim_{r\to\infty}\frac{K^{tr}_{qc}}{B}. \tag{3.20}$$

As a consequence of Eq. (3.12), the mass $M_{\rm 4DCG}$ for the four-dimensional critical gravity described by the Lagrangian (2.9) can be expressed as the linear combination of M_{gr} and $M_{\rm 4DWG}$, that is,

$$M_{\rm 4DCG} = \left(M_{gr} - \frac{1}{2}M_{\rm 4DWG}\right)_{D=4, c_1=-1/(2\Lambda)}. \tag{3.21}$$

Eventually, we take into consideration of some other applications for the mass formulations (3.17), (3.20), and (3.21). First, the formulation (3.17) is applicable for the definition of the mass for the Bardeen-type static spherically symmetric black holes found in [37–39], as well as the black holes in five-dimensional Chern-Simons gravity [42]. Second, as what is shown in Appendix A, Weyl gravity admits the four-dimensional Schwarzschild-AdS black hole as its exact solution. Hence, it is allowed to adopt Eq. (3.20) to define its mass, which coincides with the one via the (off-shell) ADT and covariant phase space methods [79]. This holds true for the black holes given in [13,14]. Third, the four-dimensional Schwarzschild-AdS black hole solution can be embedded into the four-dimensional critical gravity. For such a solution, the potential $K^{tr}_{\rm 4DCG} = 0$ results in its vanishing mass, verifying the conclusion in [3], as well as the one via the Ashtekar-Magnon-Das method [76,77]. Apart from the applications mentioned above, it deserves a further investigation to verify wether the formula (2.46) for conserved charges can be applicable for the so-called Buchdahl-inspired metrics presented in [34–36], as well as the (charged) Lifshitz-type black holes with quadratic-curvature corrections [40,90,96,97].

IV. MASS AND ANGULAR MOMENTUM FOR ROTATING SPACETIMES

In this section, for the completeness of this study, we shall calculate the mass and angular momentum of the four-dimensional rotating Kerr-AdS black hole in the framework of the four-dimensional Weyl, critical, and Einstein-Gauss-Bonnet gravities, respectively. What is more, the mass and angular momentum for the higher-dimensional generalizations of the Kerr-AdS black hole will be computed in the context of the quadratic-curvature gravities with R^2 and $R^{\alpha\beta}R_{\alpha\beta}$ terms. In parallel with the analysis for Kerr-AdS black holes, we shall take into account black strings in asymptotically AdS spacetimes.

We adopt the line element for the four-dimensional Kerr-AdS black holes in a non-rotating frame at infinity, taking the following form in Boyer-Lindquist coordinates (t, r, θ, ϕ) in $(-,+,+,+)$ notation [46]

$$ds^2 = -\frac{\Delta_r}{\Sigma}\left[dt - a\sin^2\theta\left(\frac{d\phi}{\Xi} - a\ell^2\frac{dt}{\Xi}\right)\right]^2 + \frac{\Sigma}{\Delta_r}dr^2 + \frac{\Sigma}{\Delta_\theta}d\theta^2 + \frac{\Delta_\theta\sin^2\theta}{\Sigma}\left[a dt - (r^2+a^2)\left(\frac{d\phi}{\Xi} - a\ell^2\frac{dt}{\Xi}\right)\right]^2. \tag{4.1}$$

In Eq. (4.1), the functions Δ_θ, Σ, and Δ_r are given by

$$\Delta_\theta = 1 - a^2\ell^2\cos^2\theta,$$
$$\Sigma = r^2 + a^2\cos^2\theta,$$
$$\Delta_r = (r^2+a^2)(1+\ell^2 r^2) - 2mr, \tag{4.2}$$

respectively. The constant ℓ, whose inverse ℓ^{-1} stands for the radius of curvature for the maximally symmetric AdS spaces, is read off as $\ell^2 = -\hat{\Lambda}_{gr}(D=4) = -\Lambda/3$. The constant parameter $\Xi = 1 - \ell^2 a^2$, and (m, a) are integration constants related to the mass and angular momentum, respectively. Due to the fact that the four-dimensional Kerr-AdS black hole is the solution of the vacuum Einstein field equation $R_\mu^\nu = -3\ell^2\delta_\mu^\nu$, the field equations given by

Eq. (2.16) support that such a black hole is also the exact solution of the quadratic gravity characterized by the Lagrangian

$$\sqrt{-g}L_{\rm Ric}^{(4D)} = \sqrt{-g}(R + 6\ell^2 + c_1 R^2 + c_2 R^{\alpha\beta}R_{\alpha\beta}), \quad (4.3)$$

where both the coupling constants c_1 and c_2 are allowed to be arbitrary in form.

Substituting $\hat{\Lambda} = -\ell^2$ into the 2-form potential $K^{\mu\nu}_{\rm Ric}$ given by Eq. (2.39), we acquire the potential $K^{\mu\nu}_{\rm KAdS}$ to define the conserved charges of the four-dimensional Kerr-AdS black hole (4.1) corrected by quadratic curvature terms, being of the form

$$\begin{aligned} K^{\mu\nu}_{\rm KAdS} &= [1 - 6(4c_1 + c_2)\ell^2]K^{\mu\nu}_{gr}(D = 4, \hat{\Lambda} = -\ell^2) \\ &= [1 - 6(4c_1 + c_2)\ell^2]\left(\nabla^{[\mu}\xi^{\nu]} + \frac{1}{2\ell^2}R^{\mu\nu}_{\rho\sigma}\nabla^\rho\xi^\sigma\right), \end{aligned} \quad (4.4)$$

where the potential $K^{\mu\nu}_{gr}$ for general relativity is presented by Eq. (2.38). Equation (4.4) demonstrates that the conserved charges in the framework of quadratic gravity described by the Lagrangian (4.3) are proportional to the ones in Einstein gravity. As a consequence, by virtue of the mass $M_{\rm KAdS} = m/\Xi^2$ and the angular momentum $J_{\rm KAdS} = ma/\Xi^2$ for the four-dimensional Kerr-AdS black hole in general relativity [92,98–102], we directly present its mass $M^{(4\rm DQG)}_{KAdS}$ and angular momentum $J^{(4\rm DQG)}_{\rm KAdS}$ in the quadratic gravity frame as

$$M^{(4\rm DQG)}_{\rm KAdS} = \frac{[1 - 6(4c_1 + c_2)\ell^2]m}{\Xi^2}, \quad J^{(4\rm DQG)}_{\rm KAdS} = \frac{[1 - 6(4c_1 + c_2)\ell^2]ma}{\Xi^2}. \quad (4.5)$$

Particularly, as is shown in Appendix A, the four-dimensional Kerr-AdS black hole is also the solution of four-dimensional Weyl gravity ($c_2 = -6c_1$) described by the Lagrangian (2.5) that does not incorporate the Einstein-Hilbert one $\sqrt{-g}(R - 2\Lambda)$. Thus, neglecting all the contributions from the Einstein-Hilbert Lagrangian, that is, throwing away m/Ξ^2 in $M^{(4\rm DQG)}_{\rm KAdS}$ and ma/Ξ^2 in $J^{(4\rm DQG)}_{\rm KAdS}$, one obtains the mass $M^{(4\rm DWG)}_{\rm KAdS}$ and the angular momentum $J^{(4\rm DWG)}_{\rm KAdS}$ of the Kerr-AdS black hole in the framework of Weyl gravity, read off as

$$M^{(4\rm DWG)}_{\rm KAdS} = \frac{12c_1\ell^2 m}{\Xi^2}, \qquad J^{(4\rm DWG)}_{\rm KAdS} = \frac{12c_1\ell^2 ma}{\Xi^2}, \quad (4.6)$$

respectively. In fact, the mass and angular momentum given by the above equation are in agreement with those for the four-dimensional charged rotating black hole in Weyl gravity [79,103], which turns into the Kerr-AdS black hole in the absence of the charge parameter. This is attributed to the fact that the U(1) gauge field falls off fast at infinity so that it makes no contribution to the conserved charges.

In the case for the theory of critical gravity with the Lagrangian (2.9), where $c_1 = 1/(6\ell^2)$ and $c_2 = -3c_1$, the factor $1 - 6(4c_1 + c_2)\ell^2 = 0$. Thus, both the mass and angular momentum for the four-dimensional Kerr-AdS spacetimes vanish. This holds true for four-dimensional critical gravity described by the Lagrangian (2.10). To illustrate this, we substitute $c_1 = 3\alpha$, $c_2 = -12\alpha$ and $c_3 = 6\alpha$ into Eq. (2.34) to obtain the potential

$$\tilde{K}^{\mu\nu}_{\rm 4DCG} = -6\alpha(R\nabla^{[\mu}\xi^{\nu]} + 4R^{[\mu}_\rho\nabla^{\nu]}\xi^\rho + 4\xi^\rho\nabla^{[\mu}R^{\nu]}_\rho). \quad (4.7)$$

Apparently, for the solutions satisfying $R_{\mu\nu} = 3\hat{\Lambda}g_{\mu\nu}$, the potential $\tilde{K}^{\mu\nu}_{\rm 4DCG} = 0$, resulting in identically vanishing conserved charges, which holds true for the conserved charges defined in terms of the AMD method [76,77]. This supports the observation on both the energy and the angular momentum of the four-dimensional Kerr-AdS black holes in [5]. What is more, in the case for four-dimensional Einstein-Gauss-Bonnet gravity, the quadratic-curvature Gauss-Bonnet term makes no contribution to the mass and the angular momentum of the Kerr-AdS black hole, arising from that the coupling constants c_1 and c_2 are constrained by $4c_1 + c_2 = 0$ in such a gravity theory. This is also supported by the covariant phase space and the (off-shell) ADT methods. However, it was demonstrated that the Gauss-Bonnet term is able to result in a correction to the Bekenstein-Hawking entropy in [104–107].

Without loss of generality, we turn our attention toward the case for the higher-dimensional Kerr-AdS black holes embedded into the theory of quadratic-curvature gravity. As is demonstrated in Appendix B, the solutions satisfying $R_{\mu\nu} = (D-1)\hat{\Lambda}g_{\mu\nu}$ ($D > 4$) are also the ones corresponding to the following Lagrangian

$$\sqrt{-g}L_2 = \sqrt{-g}\left(R - 2\Lambda + c_1R^2 - Dc_1R^{\alpha\beta}R_{\alpha\beta} + \frac{(D-2)(\hat{\Lambda}_{gr} - \hat{\Lambda})}{(D-1)(D-4)\hat{\Lambda}^2}R^{\alpha\beta}R_{\alpha\beta}\right). \quad (4.8)$$

Here all three constant parameters Λ, c_1 and $\hat{\Lambda}$ are allowed to be very general, which guarantee the constraint (2.22) to all the constant parameters to hold identically. For such solutions, the 2-form potential $K^{\mu\nu}_{\rm Ric}$ in Eq. (2.39), adopted to define their conserved charges, turns into

$$K^{\mu\nu}_{\rm Ric} \to \frac{2(D-2)\hat{\Lambda}_{gr} - D\hat{\Lambda}}{(D-4)\hat{\Lambda}}K^{\mu\nu}_{gr}. \quad (4.9)$$

In light of the above equation, one sees that the potential associated with the Lagrangian (4.8) is proportional to the one for Einstein gravity and it is irrelevant to the coupling constant c_1. This is attributed to the fact that the potential corresponding to the $c_1\sqrt{-g}(R^2 - DR^{\alpha\beta}R_{\alpha\beta})$ part in the Lagrangian (4.8) vanishes identically according to Eq. (B11).

As is evident, the D-dimensional Kerr-AdS black hole [47,48], which obeys the field equation $R_{\mu\nu} = -(D-1)\ell^2 g_{\mu\nu}$ derived from the Einstein-Hilbert Lagrangian $\mathcal{L}_{\rm EH} = \sqrt{-g}[R + (D-1)(D-2)\ell^2]$, is an exact solution for the Lagrangian (4.8) with $\hat{\Lambda} = -\ell^2$. Such a solution has $n = (D-\varepsilon-1)/2$ ($\varepsilon = 1$ for D even and $\varepsilon = 0$ for D odd) independent rotations characterized by n parameters a_i $(1 \le i \le n)$ in n orthogonal 2-planes with 2π-periodic azimuthal angles ϕ_i. In the coordinate system $\{t, r, \mu_1, \ldots, \mu_{n+\varepsilon-1}, \phi_1, \ldots, \phi_n\}$, the line element for the D-dimensional Kerr-AdS black hole is read off as [47,48]

$$\begin{aligned} ds^2 = {} & -HWdt^2 + \frac{r^2UV}{H(V-2m)}dr^2 + \sum_{i=1}^{n+\varepsilon}(r^2+a_i^2)\frac{d\mu_i^2}{\Xi_i} \\ & - \frac{\ell^2}{HW}\left(\sum_{i=1}^{n+\varepsilon}\frac{r^2+a_i^2}{\Xi_i}\mu_i d\mu_i\right)^2 \\ & + \frac{2mH}{r^2UV}\left(Wdt - \sum_{i=1}^{n} a_i\mu_i^2\frac{d\phi_i}{\Xi_i}\right)^2 \\ & + \sum_{i=1}^{n}\mu_i^2(r^2+a_i^2)\frac{d\phi_i^2}{\Xi_i}, \end{aligned} \tag{4.10}$$

where the four functions (H, U, V, W) are presented respectively by

$$\begin{aligned} & H = 1 + \ell^2 r^2, \qquad U = \sum_{i=1}^{n+\varepsilon}\frac{\mu_i^2}{r^2+a_i^2}, \\ & V = r^{\varepsilon-2}H\prod_{i=1}^{n}(r^2+a_i^2), \qquad W = \sum_{i=1}^{n+\varepsilon}\frac{\mu_i^2}{\Xi_i}. \end{aligned} \tag{4.11}$$

In Eqs. (4.10) and (4.11), m denotes an integral constant related to the mass, and the constant parameters Ξ_i $(1 \le i \le n)$ are associated with the rotation parameters a_i $(1 \le i \le n)$ through $\Xi_i = 1 - a_i^2\ell^2$, while $\Xi_{n+1} = 1$, arising from that $a_{n+1} = 0$ for even D. The μ_i variables are constrained by $\sum_{i=1}^{n+\varepsilon}\mu_i^2 = 1$. By making use of the potential (4.9) to compute the mass M_{qc} and the angular momenta $J_{qc}^{(i)}$ $(i = 1, \cdots, n)$ for the higher-dimensional Kerr AdS black holes (4.10) corrected by quadratic curvature terms, we acquire the results that are proportional to the ones [92,98–102] for their counterparts in the framework of general relativity by the factor $-[2(D-2)\hat{\Lambda}_{gr} + D\ell^2]/[(D-4)\ell^2]$, that is,

$$\begin{aligned} M_{qc} &= -\frac{[2(D-2)\hat{\Lambda}_{gr} + D\ell^2]V_{D-2}}{4\pi(D-4)\ell^2}\frac{m}{\prod_{j=1}^{n}\Xi_j} \\ & \quad \times \left(\sum_{i=1}^{n}\frac{1}{\Xi_i} - \frac{1-\varepsilon}{2}\right), \\ J_{qc}^{(i)} &= -\frac{[2(D-2)\hat{\Lambda}_{gr} + D\ell^2]V_{D-2}}{4\pi(D-4)\ell^2}\frac{ma_i}{\Xi_i\prod_{j=1}^{n}\Xi_j}. \end{aligned} \tag{4.12}$$

As it is mentioned in Appendix C, the above conserved charges satisfy both the differential and integral forms for the first law of thermodynamics of black holes. Particularly, when $\Lambda = -D(D-1)\ell^2/4$ or $\hat{\Lambda} = -\ell^2$, yielding $\hat{\Lambda}_{gr} = -D\ell^2/[2(D-2)]$, the D-dimensional Kerr-AdS black hole (4.10) is also the exact solution for D-dimensional critical gravity described by the Lagrangian (B7). In such a case, Eq. (4.12) indicates that all the mass and the angular momenta of this black hole vanish identically. In the absence of the cosmological constant, it is worth mentioning that the higher-derivative corrections to the conserved quantities of four-dimensional Kerr black holes and static spherically symmetric black holes in arbitrary dimensions were taken into account by virtue of the Euclidean action in [108] and [109] respectively.

Moreover, we follow the similar procedure to take into consideration of the case for black strings in asymptotically AdS spacetimes. It can be verified that the four-dimensional rotating black strings in Einstein gravity given by the work [110] and their higher-dimensional generalization found in [111] satisfy respectively the field equations derived from the Lagrangian (4.3) with $\ell = 1/l$ and the Lagrangian (4.8) with $\hat{\Lambda} = -1/l^2$, where the constant l denotes the radius of AdS spaces in [110,111]. By analogy with the above case for the Kerr-AdS black holes, we adopt the 2-form potential $K_{\rm Ric}^{\mu\nu}$ in Eq. (2.39) to define the mass and angular momentum for these black strings. The results for the four-dimensional black strings and their higher-dimensional generalizations in the framework of the quadratic-curvature gravity are proportional to the ones for their counterparts in the context of general relativity (see the works [110–112] for the mass and angular momentum of such black strings) by the factors $1 - 6(4c_1 + c_2)/l^2$ and $-[2(D-2)l^2\hat{\Lambda}_{gr} + D]/(D-4)$, respectively.

V. SUMMARY

In this paper, we explored the conserved charges of asymptotically AdS spacetimes in the context of quadratic-curvature gravities described by the generic Lagrangian (2.1), which consists of the Ricci scalar R, the cosmological constant Λ, as well as the quadratic curvature terms R^2, $R^{\mu\nu}R_{\mu\nu}$, and $R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma}$. In order to achieve this, we first analyzed the structure of the Lagrangian (2.1) and derived the expression for the equations of motion via the variation of this Lagrangian. Next, through the linear combination of the rank-4 tensors $P^{\mu\nu}_{\rho\sigma}$ and $P^{\mu\nu}_{(\mathrm{ref})\rho\sigma}$, we defined the rank-4 tensor $\mathcal{P}^{\mu\nu}_{\rho\sigma}$ given by Eq. (2.26), which inherits the index symmetries of the Riemann curvature tensor and disappears on the AdS spacetimes. By the aid of $\mathcal{P}^{\mu\nu}_{\rho\sigma}$, a 2-form potential $K^{\mu\nu}$ associated to the Killing vector field was proposed in Eq. (2.27), which resembles the Noether potential $K^{\mu\nu}_{R}$ in Eq. (2.29). To demonstrate that $K^{\mu\nu}$ is suitable for the definition of conserved charges of asymptotically AdS spacetimes, we compared it with the results via other methods, such as the covariant phase space approach, the (off-shell) ADT formalism, the generalized Komar integral and the field-theoretic method. Furthermore, in terms of the potential $K^{\mu\nu}$, the formula (2.46) for conserved charges was presented. Finally, as applications, we derived the mass formula (3.13) for static and spherically symmetric spacetimes in four dimensions and above. Besides, the formula (2.46) was applied to compute the mass and the angular momentum of the four(higher)-dimensional Kerr-AdS black holes and black strings embedded into quadratic-curvature gravities, which are proportional to the ones in Einstein gravity. All the results reveal that the potential $K^{\mu\nu}$ can successfully give rise to a simple formula for the conserved charges of asymptotically AdS spacetimes in the theories of quadratic-curvature gravity.

Particularly, we stressed on the definition of conserved quantities for Weyl gravity, Einstein-Gauss-Bonnet gravity and critical gravity. For these theories, the potentials adopted to define conserved charges were given respectively by Eqs. (2.31), (2.32), and (2.40). What is more, for the asymptotically AdS spacetimes satisfying the vacuum Einstein field equations, the potential $K^{\mu\nu}_{\mathrm{Ric}}$ given by Eq. (2.39) takes a simple form, which is proportional to the one for Einstein gravity.

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APPENDIX A: THE PROPERTIES FOR THE TENSOR $P^{\mu\nu\rho\sigma}$ AND THE EQUATIONS OF MOTION

We devote this appendix to the properties of the rank-4 tensor $P^{\mu\nu\rho\sigma}$ and the divergence for the expression of the equations of motion, together with the field equations associated with four-dimensional Weyl and critical gravities and Einstein-Gauss-Bonnet gravity in arbitrary dimensions. Some of the following results for $P^{\mu\nu\rho\sigma}$ overlap the ones in [87].

In order to acquire the rank-4 tensor $P^{\mu\nu\rho\sigma}$ given by Eq. (2.14), we evaluate the derivative of the Riemann tensor $R_{\alpha\beta\gamma\lambda}$ with respect to $R^{\rho\sigma}_{\mu\nu}$, yielding

$$\frac{\partial R_{\alpha\beta\gamma\lambda}}{\partial R^{\rho\sigma}_{\mu\nu}} = \frac{1}{8}(g_{\alpha\kappa}g_{\beta\eta}\delta^{\kappa\eta}_{\rho\sigma}\delta^{\mu\nu}_{\gamma\lambda} + g_{\gamma\kappa}g_{\lambda\eta}\delta^{\kappa\eta}_{\rho\sigma}\delta^{\mu\nu}_{\alpha\beta}), \qquad (A1)$$

whose contraction with the metric tensor further gives rise to

$$\frac{\partial R_{\gamma\lambda}}{\partial R^{\rho\sigma}_{\mu\nu}} = g^{\alpha\beta}\frac{\partial R_{\alpha\gamma\beta\lambda}}{\partial R^{\rho\sigma}_{\mu\nu}} = -\frac{1}{4}(g_{\gamma[\rho}\delta^{\mu\nu}_{\sigma]\lambda} + g_{\lambda[\rho}\delta^{\mu\nu}_{\sigma]\gamma}),$$
$$\frac{\partial R}{\partial R^{\rho\sigma}_{\mu\nu}} = g^{\gamma\lambda}\frac{\partial R_{\gamma\lambda}}{\partial R^{\rho\sigma}_{\mu\nu}} = \frac{1}{2}\delta^{\mu\nu}_{\rho\sigma}. \qquad (A2)$$

According to the definition for the rank-4 tensor $P^{\mu\nu\rho\sigma}$, it is required to at least inherit the following algebraic symmetries for the Riemann tensor $R^{\mu\nu\rho\sigma}$, that is,

$$P^{\mu\nu\rho\sigma} = P^{\rho\sigma\mu\nu} = -P^{\nu\mu\rho\sigma} = -P^{\mu\nu\sigma\rho}. \qquad (A3)$$

Particularly, it can be proved that the tensor $P^{\mu\nu\rho\sigma}$ also fulfills the algebraic Bianchi-type identities $P^{\mu[\nu\rho\sigma]} = 0 = P^{[\mu\nu\rho]\sigma}$ in the case of quadratic-curvature gravities. Besides, by merely making use of the properties of the Riemann tensor and the index symmetries of $P_{\mu\nu\rho\sigma}$ presented by Eq. (A3), we have

$$2\nabla^{[\rho}\nabla^{\sigma]}P_{\mu\nu\rho\sigma} = -2R_{[\mu}{}^{\lambda\rho\sigma}P_{\nu]\lambda\rho\sigma} + R_{\rho}{}^{\lambda\rho\sigma}P_{\mu\nu\lambda\sigma} + R_{\sigma}{}^{\lambda\rho\sigma}P_{\mu\nu\rho\lambda},$$
$$= -2R_{[\mu}{}^{\lambda\rho\sigma}P_{\nu]\lambda\rho\sigma} + 2R^{\rho\sigma}P_{\mu\nu\rho\sigma}$$
$$= -2R_{[\mu}{}^{\lambda\rho\sigma}P_{\nu]\lambda\rho\sigma}, \qquad (A4)$$

together with the following identity

$$2\nabla^{[\rho}\nabla^{\sigma]}P_{\rho\mu\nu\sigma} = R_{[\mu}{}^{\lambda\rho\sigma}P_{\nu]\sigma\rho\lambda} + R_{[\mu}{}^{\lambda\rho\sigma}P_{\nu]\rho\lambda\sigma}$$
$$= R_{[\mu}{}^{\lambda\rho\sigma}P_{\nu]\lambda\rho\sigma}. \qquad (A5)$$

In order to gain the last equality in Eq. (A5), we have used the Bianchi identity $R_{\mu[\lambda\rho\sigma]} = 0$. As a consequence of Eqs. (A4) and (A5), unlike in [87], without the requirement that $R_{[\mu}{}^{\lambda\rho\sigma}P_{\nu]\lambda\rho\sigma} = 0$ and $P^{\rho[\mu\nu\sigma]} = 0$, their combination brings about another identity

$$\nabla^{\rho}\nabla^{\sigma}P_{\mu[\nu\rho\sigma]} = 0, \qquad (A6)$$

or $\nabla^{\rho}\nabla^{\sigma}P_{\mu\nu\rho\sigma} = -2\nabla^{\rho}\nabla^{\sigma}P_{\rho[\mu\nu]\sigma}$. This can be also obtained by using $P^{[\mu\nu\rho\sigma]} = P^{\mu[\nu\rho\sigma]}$ and $\partial_{\rho}\partial_{\sigma}(\sqrt{-g}P^{[\mu\nu\rho\sigma]}) = 0$. Moreover, by the aid of the equality (A5), as well as the decompositions to the rank-2 tensors $R_{\mu}{}^{\lambda\rho\sigma}P_{\nu\lambda\rho\sigma}$ and $\nabla^{\rho}\nabla^{\sigma}P_{\rho\mu\nu\sigma}$, that is,

$$R_{\mu}{}^{\lambda\rho\sigma}P_{\nu\lambda\rho\sigma} = R_{(\mu}{}^{\lambda\rho\sigma}P_{\nu)\lambda\rho\sigma} + R_{[\mu}{}^{\lambda\rho\sigma}P_{\nu]\lambda\rho\sigma},$$
$$\nabla^{\rho}\nabla^{\sigma}P_{\rho\mu\nu\sigma} = \nabla^{\rho}\nabla^{\sigma}P_{\rho(\mu\nu)\sigma} + \nabla^{\rho}\nabla^{\sigma}P_{\rho[\mu\nu]\sigma}, \qquad (A7)$$

we obtain the following identity

$$R_\mu{}^{\lambda\rho\sigma}P_{\nu\lambda\rho\sigma} - 2\nabla^\rho\nabla^\sigma P_{\rho\mu\nu\sigma} = R_{(\mu}{}^{\lambda\rho\sigma}P_{\nu)\lambda\rho\sigma} - 2\nabla^\rho\nabla^\sigma P_{\rho(\mu\nu)\sigma} \tag{A8}$$

under the only requirement that $P_{\mu\nu\rho\sigma}$ exhibits the index symmetries in Eq. (A3). As a result of Eq. (A8), the round brackets in the expression (2.12) for the field equations can be omitted.

In the case where $P_{\mu\nu\rho\sigma}$ is not required to satisfy $P_{\mu[\nu\rho\sigma]} = 0$, we can introduce an auxiliary rank-4 tensor $\tilde{P}_{\mu\nu\rho\sigma}$ to decompose $P_{\mu\nu\rho\sigma}$ as

$$P_{\mu\nu\rho\sigma} = \tilde{P}_{\mu\nu\rho\sigma} + P_{\mu[\nu\rho\sigma]}. \tag{A9}$$

Obviously, $\tilde{P}_{\mu\nu\rho\sigma}$ possesses the index symmetries given by Eq. (A3) and satisfies the Bianchi-type identity $\tilde{P}_{\mu[\nu\rho\sigma]} = 0$. By means of $R_\mu{}^{\lambda\rho\sigma}P_{\nu[\lambda\rho\sigma]} = R_\mu{}^{[\lambda\rho\sigma]}P_{\nu[\lambda\rho\sigma]} = 0$ and Eq. (A6), we have

$$R_\mu{}^{\lambda\rho\sigma}P_{\nu\lambda\rho\sigma} - 2\nabla^\rho\nabla^\sigma P_{\rho\mu\nu\sigma} = R_\mu{}^{\lambda\rho\sigma}\tilde{P}_{\nu\lambda\rho\sigma} - 2\nabla^\rho\nabla^\sigma \tilde{P}_{\rho\mu\nu\sigma}. \tag{A10}$$

As a matter of fact, the expression (2.12) for the field equations is applicable to the more general Lagrangian depending on the metric and the Riemann tensor. In terms of Eq. (A10), we conclude that the equations of motion associated to such types of Lagrangian are irrelevant to $P_{\mu[\nu\rho\sigma]}$. This holds true for the surface term Θ^μ arising from that $P^{\mu[\nu\rho\sigma]}\nabla_{[\sigma}\delta g_{\rho\nu]} = 0$ and $\delta g_{[\nu\rho}\nabla_{\sigma]}P^{\mu[\nu\rho\sigma]} = 0$. Thus, the variation of the Lagrangian that is the functional of the metric and the Riemann tensor is alternatively given by

$$\delta(\sqrt{-g}L) = \sqrt{-g}\tilde{E}_{\mu\nu}\delta g^{\mu\nu} + \sqrt{-g}\nabla_\mu\tilde{\Theta}^\mu, \tag{A11}$$

where $\tilde{E}_{\mu\nu} = E_{\mu\nu}(P\to\tilde{P})$ and $\tilde{\Theta}^\mu = \Theta^\mu(P\to\tilde{P})$. According to Eq. (A11), one can follow the conventional Noether procedure to obtain the Noether current and potential unrelated to $P_{\mu[\nu\rho\sigma]}$, as well as the Iyer-Wald potential. Hereto we stress that all the above results hold true for an arbitrary rank-4 tensor $P_{\mu\nu\rho\sigma}$ only armed with the index symmetries given by Eq. (A3). If we further make use of the result $R_\mu{}^{\lambda\rho\sigma}P_{\nu\lambda\rho\sigma} = R_\nu{}^{\lambda\rho\sigma}P_{\mu\lambda\rho\sigma}$, which was explicitly proved under the conditions that the Lagrangian preserves diffeomorphism invariance and the P-tensor exhibits the algebraic symmetries presented by Eq. (A3) in [87], Eqs. (A4) and (A5) turn into

$$\nabla^\rho\nabla^\sigma P_{\mu\nu\rho\sigma} = 0, \qquad \nabla^\rho\nabla^\sigma P_{\rho[\mu\nu]\sigma} = 0. \tag{A12}$$

The results in Eq. (A12) will be verified below in the framework of quadratic-curvature gravities. Specifically, the divergence of $P^{\mu\nu\rho\sigma}$ takes the form

$$\nabla^\rho P^{\mu\nu}_{\rho\sigma} = \frac{1}{4}(4c_1 + c_2)\delta^{\mu\nu}_{\rho\sigma}\nabla^\rho R + (c_2 + 4c_3)\nabla^{[\mu}R^{\nu]}_\sigma. \tag{A13}$$

Calculations on the divergence of $\nabla^\rho P^{\mu\nu}_{\rho\sigma}$ give rise to

$$\begin{aligned}\nabla_\rho\nabla^\sigma P^{\mu\rho}_{\sigma\nu} = &-\frac{1}{4}(4c_1 + c_2)\delta^\mu_\nu\Box R - \frac{1}{2}(c_2 + 4c_3)\Box R^\mu_\nu \\ &+\frac{1}{2}(c_2 + 4c_3)(R^\mu_\lambda R^\lambda_\nu - R^\mu{}_{\rho\nu\sigma}R^{\rho\sigma}) \\ &+\frac{1}{2}(2c_1 + c_2 + 2c_3)\nabla^\mu\nabla_\nu R,\end{aligned} \tag{A14}$$

as well as $\nabla^\rho\nabla^\sigma P^{\mu\nu}_{\rho\sigma} = \nabla^{[\rho}\nabla^{\sigma]}P^{\mu\nu}_{\rho\sigma} = 0$. The contraction between the indices μ and ν in $\nabla_\rho\nabla^\sigma P^{\mu\rho}_{\sigma\nu}$ yields the scalar

$$g^{\mu\nu}\nabla^\rho\nabla^\sigma P_{\mu\rho\sigma\nu} = -\frac{1}{4}[4(D-1)c_1 + Dc_2 + 4c_3]\Box R. \tag{A15}$$

In terms of Eq. (A4), the equality $\nabla^\rho\nabla^\sigma P_{\mu\nu\rho\sigma} = 0$ renders us to arrive at $R_{[\mu}{}^{\lambda\rho\sigma}P_{\nu]\lambda\rho\sigma} = 0$, or

$$R_\mu{}^{\lambda\rho\sigma}P_{\nu\lambda\rho\sigma} = R_\nu{}^{\lambda\rho\sigma}P_{\mu\lambda\rho\sigma}. \tag{A16}$$

Due to Eq. (A5) or (A6), Eq. (A16) further results in

$$\nabla^\rho\nabla^\sigma P_{\rho\mu\nu\sigma} = \nabla^\rho\nabla^\sigma P_{\rho\nu\mu\sigma}, \tag{A17}$$

or $\nabla_\rho\nabla_\sigma P^{\rho[\mu\nu]\sigma} = \nabla_{[\rho}\nabla_{\sigma]}P^{\rho\mu\nu\sigma} = 0$. Therefore, we have the conclusion that $\nabla^\rho\nabla^\sigma P_{\mu\nu\rho\sigma} = 0$ is the necessary and sufficient condition for the result that each of the rank-2 tensors $\nabla^\rho\nabla^\sigma P_{\rho\mu\nu\sigma}$ and $R_\mu{}^{\lambda\rho\sigma}P_{\nu\lambda\rho\sigma}$ is symmetric with respect to both the indices μ and ν.

With the help of the tensor $P^{\mu\nu\rho\sigma}$, the Lagrangian density L can be reexpressed as

$$2L = R - 4\Lambda + R^{\rho\sigma}_{\mu\nu}P^{\mu\nu}_{\rho\sigma}. \tag{A18}$$

When the metric satisfies the field equations $E_{\mu\nu} = 0$, by the aid of Eq. (A15), L is required to fulfill the following on-shell condition

$$(D-4)L = [4(D-1)c_1 + Dc_2 + 4c_3]\Box R - 2R + 8\Lambda, \tag{A19}$$

or equivalently,

$$\begin{aligned}(D-4)R^{\rho\sigma}_{\mu\nu}P^{\mu\nu}_{\rho\sigma} = &[8(D-1)c_1 + 2Dc_2 + 8c_3]\Box R \\ &- DR + 4D\Lambda.\end{aligned} \tag{A20}$$

In particular, when the dimension of spacetimes $D = 4$, the constraint (A19) or (A20) for all the quadratic-curvature gravities is simplified as

$$2[3c_1 + c_2 + c_3]\Box R = R - 4\Lambda. \tag{A21}$$

For four-dimensional Weyl gravity coupled with the Einstein-Hilbert Lagrangian and critical gravity described by the Lagrangian (2.9), one obtains $R = 4\Lambda$.

In terms of the above properties of the tensor $P^{\mu\nu\rho\sigma}$, the divergence of the expression for the equations of motion $E_{\mu\nu}$ is read off as

$$\begin{aligned}\nabla_\nu E^\nu_\mu &= \frac{1}{2}P^{\lambda\nu}_{\rho\sigma}\nabla_\lambda R^{\rho\sigma}_{\mu\nu} - \frac{1}{4}R^{\rho\sigma}_{\alpha\beta}\nabla_\mu P^{\alpha\beta}_{\rho\sigma} - \frac{1}{4}\nabla_\mu R \\ &= \frac{1}{4}[(2c_1R+1)(2\nabla_\nu R^\nu_\mu - \nabla_\mu R) \\ &\quad + 2c_2R^\nu_\rho(\delta^{\gamma\lambda}_{\mu\nu}\nabla_\lambda R^\rho_\gamma - \nabla_\sigma R^{\rho\sigma}_{\mu\nu}) \\ &\quad + 2c_3R^{\gamma\lambda}_{\rho\sigma}(2\nabla_\gamma R^{\rho\sigma}_{\mu\lambda} - \nabla_\mu R^{\rho\sigma}_{\gamma\lambda})] \\ &\equiv 0. \end{aligned} \tag{A22}$$

In order to obtain the last identity in the above equation, the Bianchi identity $\nabla_{[\mu}R^{\rho\sigma}_{\gamma\lambda]} = 0$ has been used. Equation (A22) shows that the expression for the equations of motion is conserved.

In the remainder of this appendix, to demonstrate some typical cases of the general expression (2.16) for the field equations, we take into account the equations of motion for the four-dimensional Weyl and critical gravities, as well as the one for the Einstein-Gauss-Bonnet gravity in arbitrary dimensions. It can be proven that the Gauss-Bonnet term (2.7) fulfills identically

$$\begin{aligned} g_{\mu\nu}L_{GB} &= 4RR_{\mu\nu} - 8R_{\mu\rho\nu\sigma}R^{\rho\sigma} - 8R_{\mu\lambda}R^\lambda_\nu + 4R_\mu{}^{\lambda\rho\sigma}R_{\nu\lambda\rho\sigma} \\ &\quad + \frac{1}{4}g_{\nu\zeta}\delta^{\zeta\gamma\lambda\kappa\eta}_{\mu\alpha\beta\rho\sigma}R^{\alpha\beta}_{\gamma\lambda}R^{\rho\sigma}_{\kappa\eta}. \end{aligned} \tag{A23}$$

Attributed to the fact that $\delta^{\zeta\gamma\lambda\kappa\eta}_{\mu\alpha\beta\rho\sigma} = 0$ when $D \le 4$, Eq. (A23) can be used to simplify the expression $E_{\mu\nu}$ for field equations in four dimensions. Doing so yields $E^{(4D)}_{\mu\nu} = E_{\mu\nu}|_{c_3=0}(c_1, c_2 \to \hat{c}_1, \hat{c}_2)$, where $\hat{c}_1 = c_1 - c_3$ and $\hat{c}_2 = c_2 + 4c_3$, namely,

$$\begin{aligned} E^{(4D)}_{\mu\nu} &= E^{(gr)}_{\mu\nu} + 2\hat{c}_1RR_{\mu\nu} + 2\hat{c}_2R_{\mu\rho\nu\sigma}R^{\rho\sigma} \\ &\quad - \frac{1}{2}g_{\mu\nu}(\hat{c}_1R^2 + \hat{c}_2R_{\alpha\beta}R^{\alpha\beta}) \\ &\quad + \frac{1}{2}(4\hat{c}_1 + \hat{c}_2)g_{\mu\nu}\Box R + \hat{c}_2\Box R_{\mu\nu} \\ &\quad - (2\hat{c}_1 + \hat{c}_2)\nabla_\mu\nabla_\nu R. \end{aligned} \tag{A24}$$

Here $E^{(gr)}_{\mu\nu}$ denotes the expression for field equations corresponding to the Einstein-Hilbert Lagrangian $\sqrt{-g}L_{\rm EH} = \sqrt{-g}(R - 2\Lambda)$, and it reads

$$E^{(gr)}_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu}. \tag{A25}$$

From Eq. (A24), one observes that the solution satisfying $E^{(gr)}_{\mu\nu} = 0$ or $R_{\mu\nu} = \Lambda g_{\mu\nu}$ in four dimensions must guarantee that $E^{(4D)}_{\mu\nu} = 0$. That is to say, the vacuum solution to four-dimensional general relativity must be the exact one for four-dimensional quadratic gravity [43]. As a result of Eq. (A25), the expression for the field equations of four-dimensional Weyl gravity, corresponding to $E^{(4D)}_{\mu\nu}$ with $c_2 = -6c_1$ and $c_3 = 3c_1$, or with $\hat{c}_1 = -2c_1$ and $\hat{c}_2 = 6c_1$, is given by

$$\begin{aligned} E^{(\rm 4DW)}_{\mu\nu} &= c_1(R^2 - 3R_{\rho\sigma}R^{\rho\sigma} - \Box R)g_{\mu\nu} + 6c_1\Box R_{\mu\nu} \\ &\quad + 2c_1(6R_{\mu\rho\nu\sigma}R^{\rho\sigma} - 2RR_{\mu\nu} - \nabla_\mu\nabla_\nu R) \\ &= -6c_1(2\nabla^\rho\nabla^\sigma + R^{\rho\sigma})C_{\mu\rho\sigma\nu}, \end{aligned} \tag{A26}$$

where Eq. (A23) has been adopted to expand the four-dimensional Bach tensor $(\nabla^\rho\nabla^\sigma + R^{\rho\sigma}/2)C_{\mu\rho\nu\sigma}$. According to Eq. (A26), one finds that $g^{\mu\nu}E^{(\rm 4DW)}_{\mu\nu} = 0$ and the condition $R_{\mu\nu} = \lambda g_{\mu\nu}$, where λ is an arbitrary constant, must give rise to $E^{(\rm 4DW)}_{\mu\nu} = 0$. This indicates that the four-dimensional vacuum general relativistic solution is the one for the four-dimensional Weyl gravity as well.

When $c_1 = \beta$, $c_2 = \alpha$ and $c_3 = 0$, Eq. (2.16) becomes the expression for the field equation of the four-dimensional critical gravity given in [3]. Specifically, according to the Lagrangian (2.9), the expression $E^{(\rm 4DCG)}_{\mu\nu}$ for the field equation of the four-dimensional critical gravity can be related to the one $E^{(\rm 4DW)}_{\mu\nu}$ for the four-dimensional Weyl gravity through

$$E^{(\rm 4DCG)}_{\mu\nu} = E^{(gr)}_{\mu\nu} + E^{(\rm 4DW)}_{\mu\nu}\left(c_1 = \frac{1}{4\Lambda}\right). \tag{A27}$$

It should be pointed out that the higher-dimensional generalization of Eq. (A27) can be found in the work [4] (see also Eq. (B7) in Appendix B). What is more, with the help of Eq. (A23), neglecting the $E^\nu_{(gr)\mu}$ part and letting $c_1 = 3\alpha$, $c_2 = -12\alpha$ and $c_3 = 6\alpha$ in Eq. (2.16), we get the expression for the field equation of the critical gravity described by the Lagrangian (2.10), being of the form

$$\begin{aligned} \tilde{E}^{(\rm 4DCG)}_{\mu\nu} &= 6\alpha(4R_{\mu\rho\nu\sigma}R^{\rho\sigma} - RR_{\mu\nu} - \nabla_\mu\nabla_\nu R) \\ &\quad + \frac{3}{2}\alpha(R^2 - 4R_{\rho\sigma}R^{\rho\sigma})g_{\mu\nu} + 12\alpha\Box R_{\mu\nu}. \end{aligned} \tag{A28}$$

Due to the fact that $g^{\mu\nu}\tilde{E}^{(\rm 4DCG)}_{\mu\nu} = 6\alpha\Box R$, one necessary condition for $\tilde{E}^{(\rm 4DCG)}_{\mu\nu} = 0$ is $\Box R = 0$. With Eqs. (A26) and (A27), solutions obeying $E^\nu_{(gr)\mu}(D=4) = 0$ must be the ones of the critical gravity relative to the Lagrangian (2.9). This can also be seen from the Lagrangian (2.9) attributed to the fact that the Gauss-Bonnet term in four dimensions is a topological surface term, which makes no contribution to the equations of motion. According to Eq (A28), the critical gravity with the Lagrangian (2.10) allows for the solutions satisfying $R_{\mu\nu} = \lambda g_{\mu\nu}$. As a result, in light of all the above in this appendix, one observes that the AdS black hole solution with cylindrical symmetry found in [113] can be embedded in the four-dimensional Weyl and critical gravities as well.

In addition, in the case for Einstein-Gauss-Bonnet gravity ($c_2 = -4c_1,\ c_3 = c_1$), Eq. (2.16) coincides with the field equation in [79], which can be further simplified as the following form by virtue of Eq. (A23)

$$E^{\nu}_{(GB)\mu} = E^{\nu}_{(gr)\mu} - \frac{c_1}{8}\delta^{\nu\gamma\lambda\kappa\eta}_{\mu\alpha\beta\rho\sigma} R^{\alpha\beta}_{\gamma\lambda} R^{\rho\sigma}_{\kappa\eta}. \tag{A29}$$

In terms of Eq. (A29), it is easy to see that the Gauss-Bonnet term is nondynamical in four dimensions due to the fact that $\delta^{\nu\gamma\lambda\kappa\eta}_{\mu\alpha\beta\rho\sigma} \equiv 0$ in four dimensions.

APPENDIX B: EMBEDDING HIGHER-DIMENSIONAL SOLUTIONS TO VACUUM EINSTEIN FIELD EQUATIONS INTO QUADRATIC GRAVITY

In order to test the definition for conserved quantities in quadratic gravity described by the Lagrangian $\sqrt{-g}L$, enough exact solutions of this theory in various dimensions are desired. Unfortunately, since the field equations $E_{\mu\nu} = 0$ are nonlinear fourth-order partial differential equations (PDE), solving them to obtain solutions is extremely difficult. However, within the context of some special cases of quadratic gravity or it coupled with matter fields, such as Einstein-Gauss-Bonnet gravity, Weyl gravity and critical gravity, some exact static spherically symmetric solutions were found in the literature. Particularly, it has been demonstrated in Appendix A that all the vacuum solutions to four-dimensional Einstein gravity automatically become the ones for four-dimensional quadratic gravity. Some non-Einstein vacuum solutions to four-dimensional quadratic gravity were obtained in [28–30,43]. Apart from the aforementioned solutions, to our knowledge, within the framework of the full theory of quadratic gravity, all exact solutions in any dimension found so far are Kundt spacetimes with constant Ricci scalar constructed in [44,45], which can be put into Kerr-Schild form. Due to the complexity of the field equations, exact rotating solutions in $D > 4$ dimensions are still absent up till now.

In the present appendix, let us focus on the solutions to the field equations $E_{\mu\nu} = 0$ in Eq. (2.16) under the sole constraint condition that they obey the vacuum Einstein field equations $R_{\mu\nu} = \lambda g_{\mu\nu}$ in arbitrary dimensions, where λ represents a constant. Such a constraint to them gets rid of all the fourth-order derivative terms $\Box R$, $\Box R_{\mu\nu}$ and $\nabla_\mu \nabla_\nu R$ in the field equations. Hence, the original fourth-order field equations reduce to the much simpler second-order ones depending only on the metric and curvature terms. Then the substitution of $R_{\mu\nu} = \lambda g_{\mu\nu}$ into the resulting field equations gives rise to

$$E^{\nu}_{\mu}|_{R_{\rho\sigma}=\lambda g_{\rho\sigma}} = -\frac{1}{2}[(D-4)(Dc_1 + c_2)\lambda^2 + (D-2)\lambda - 2\Lambda]\delta^{\nu}_{\mu} + \frac{c_3}{2}(4R^{\rho\sigma}_{\mu\lambda}R^{\nu\lambda}_{\rho\sigma} - R^{\rho\sigma}_{\alpha\beta}R^{\alpha\beta}_{\rho\sigma}\delta^{\nu}_{\mu}) = 0. \tag{B1}$$

The four free parameters (Λ, c_1, c_2, c_3) in field equations allow for their appropriate values to guarantee that Eq. (B1) holds without any further requirement for the solutions except for the existing constraint $R_{\mu\nu} = \lambda g_{\mu\nu}$. [this will be illustrated below by Eqs. (B4), (B9), (B10) and (B12)]. That is to say, like in the four-dimensional case, it is also possible to embed the vacuum solutions in higher-dimensional Einstein gravity into quadratic gravities in higher dimensions. Under the guidance of the above, we will demonstrate in detail that the solutions of the vacuum Einstein equations $R_{\mu\nu} = (D-1)\hat{\Lambda} g_{\mu\nu}$ (here the dimension of spacetime $D > 4$ and the non-vanishing constant $\hat{\Lambda}$ is allowed to be arbitrary) belong to the quadratic gravities with the Lagrangian (4.8) as well.

Substituting equations $R_{\mu\nu} = (D-1)\hat{\Lambda} g_{\mu\nu}$ and $R = D(D-1)\hat{\Lambda}$ into Eq. (4.8), we have

$$L_2 = D(D-1)[(D-1)(Dc_1 + \tilde{c}_2)\hat{\Lambda}^2 + \hat{\Lambda}] - 2\Lambda, \tag{B2}$$

where the constant $\tilde{c}_2$ is read off as

$$\tilde{c}_2 = \frac{(D-2)(\hat{\Lambda}_{gr} - \hat{\Lambda})}{(D-1)(D-4)\hat{\Lambda}^2} - Dc_1. \tag{B3}$$

Under the condition that $R_{\mu\nu} = (D-1)\hat{\Lambda} g_{\mu\nu}$, the equations of motion for the Lagrangian (4.8), being a special case of Eq. (2.16) or (B1), take the form

$$\begin{aligned} E^{\nu}_{(2)\mu} &= R^{\nu}_{\mu} + 2c_1 R R^{\nu}_{\mu} + 2\tilde{c}_2 R^{\nu\sigma}_{\mu\rho} R^{\rho}_{\sigma} - \frac{1}{2}L_2 \delta^{\nu}_{\mu} \\ &= -\frac{D-4}{D-2}(D-1)(Dc_1 + \tilde{c}_2)\hat{\Lambda}^2 \delta^{\nu}_{\mu} - (\hat{\Lambda} - \hat{\Lambda}_{gr})\delta^{\nu}_{\mu}. \end{aligned} \tag{B4}$$

Substituting Eqs. (B2) and (B3) into Eq. (B4), we obtain further $E^{\nu}_{(2)\mu} = 0$. This implies that the general relativistic solutions fulfilling $R_{\mu\nu} = (D-1)\hat{\Lambda} g_{\mu\nu}$ must be the ones of the field equations derived from the Lagrangian (4.8), which is the usual Einstein-Hilbert Lagrangian supplemented with the quadratic curvature terms R^2 and $R^{\alpha\beta}R_{\alpha\beta}$. For instance, the D-dimensional Schwarzschild-AdS black holes in general relativity are solutions of the quadratic gravity described by the Lagrangian (4.8) [114]. In addition, when $\hat{\Lambda} = 0$, the quadratic-curvature Lagrangian associated with the solutions satisfying $R_{\mu\nu} = 0$ in any dimension can be expressed as

$$\sqrt{-g}L_3 = \sqrt{-g}(R + c_1 R^2 + c_2 R^{\alpha\beta} R_{\alpha\beta}). \tag{B5}$$

Apparently, since the D-dimensional Kerr-AdS black hole given by Eq. (4.10) fulfills the field equations $R_{\mu\nu} = -(D-1)\ell^2 g_{\mu\nu}$ derived from the Einstein-Hilbert Lagrangian $\mathcal{L}_{\rm EH} = \sqrt{-g}[R + (D-1)(D-2)\ell^2]$ [47,48], the expression (B4) for field equations supports that such a black hole solution is also an exact one for the Lagrangian (4.8) with $\hat{\Lambda} = -\ell^2$. Apart from this, it was shown in [115] that a solution in three-dimensional new massive gravity with constant scalar curvature can be embedded into five-dimensional quadratic gravity through dimension lifting.

Subsequently, we move on to demonstrate that the Lagrangian (4.8) includes the one describing D-dimensional critical gravity proposed in [4]. To do this, we let both the arbitrary constants c_1 and $\hat{\Lambda}$ in the Lagrangian (4.8) take the following values

$$c_1 = -\frac{D^2}{4(D-1)(D-2)^3 \hat{\Lambda}_{gr}} = -\frac{D^2}{8(D-2)^2 \Lambda},$$
$$\hat{\Lambda} = \frac{2(D-2)}{D}\hat{\Lambda}_{gr} = \frac{4\Lambda}{D(D-1)}, \tag{B6}$$

respectively, which obey the constraint (2.22) to the constant parameters. Then we arrive at the Lagrangian (4.8) for the higher-dimensional critical gravity [4], that is,

$$\mathcal{L}_{CG} = \sqrt{-g}\left[R - 2\Lambda - \frac{D^2}{8(D-2)^2\Lambda}\left(R^2 - \frac{4D-4}{D} R_{\rho\sigma}R^{\rho\sigma}\right)\right]$$
$$= \sqrt{-g}\left[R - 2\Lambda - \frac{D(D-1)}{8(D-2)(D-3)\Lambda}(L_{GB} - C^{\mu\nu}_{\rho\sigma} C^{\rho\sigma}_{\mu\nu})\right]. \tag{B7}$$

The last expression in Eq. (B7) can be acquired directly by setting $c_1 = -D^2/[8(D-2)^2\Lambda]$, $c_2 = -4(D-1)c_1/D$ and $c_3 = 0$ in the Lagrangian (2.8). It reveals that critical gravity can be understood as the linear combination of Einstein-Gauss-Bonnet gravity and Weyl gravity. According to Eq. (2.16), the expression of the equations of motion for the D-dimensional critical gravity is read off as

$$E^{(DDCG)}_{\mu\nu} = E^{(gr)}_{\mu\nu} - \frac{D}{16\Lambda(D-2)^2}$$
$$\times [4DRR_{\mu\nu} - 16(D-1)R_{\mu\rho\nu\sigma}R^{\rho\sigma}$$
$$+ (4(D-1)R_{\rho\sigma}R^{\rho\sigma} - DR^2 + 4\Box R)g_{\mu\nu}$$
$$- 8(D-1)\Box R_{\mu\nu} + 4(D-2)\nabla_\mu \nabla_\nu R]. \tag{B8}$$

When $\Lambda = -D(D-1)\ell^2/4$ or $\hat{\Lambda} = -\ell^2$, it can be verified that the D-dimensional Kerr-AdS black hole solution given by Eq. (4.10) indeed obeys the field equations $E^{(DDCG)}_{\mu\nu} = 0$.

At last, let us make some remarks on the Lagrangian $\sqrt{-g}L_2$. Such a Lagrangian has three free parameters denoted by $(\Lambda, c_1, \hat{\Lambda})$ or $(\Lambda, c_1, \tilde{c}_2)$. It covers the Lagrangians for Einstein gravity $(c_1 = 0, \hat{\Lambda} = \hat{\Lambda}_{gr})$ and higher-dimensional critical gravity [c_1 and $\hat{\Lambda}$ are given by Eq. (B6)], which only possess one parameter. In fact, within the framework of quadratic gravities, there exist other single-parameter Lagranians that admit the solutions satisfying $R_{\mu\nu} = (D-1)\hat{\Lambda}g_{\mu\nu}$, such as

$$\sqrt{-g}L_2|_{\Lambda,\tilde{c}_2=0} = \sqrt{-g}\left(R - \frac{D-2}{D(D-1)(D-4)\hat{\Lambda}}R^2\right),$$
$$\sqrt{-g}L_2|_{\Lambda,c_1=0} = \sqrt{-g}\left(R - \frac{D-2}{(D-1)(D-4)\hat{\Lambda}}R^{\alpha\beta}R_{\alpha\beta}\right), \tag{B9}$$

together with the following Lagrangian

$$\tilde{\mathcal{L}}^{(DD)}_{CG} = \sqrt{-g}(L_2 - R + 2\Lambda)|_{\hat{\Lambda}=\hat{\Lambda}_{gr}}$$
$$= c_1\sqrt{-g}(R^2 - DR^{\alpha\beta}R_{\alpha\beta}), \tag{B10}$$

which coincides with the Lagrangian (2.10) for four-dimensional critical gravity proposed in [5] when $D = 4$ and $c_1 = -3\alpha$, arising from that the Gauss-Bonnet term in four dimensions is nondynamical. Due to the fact that the potential for the Lagrangian (B10), being of the form

$$\tilde{K}^{\mu\nu}_{\rm DDCG} = 2c_1(R\nabla^{[\mu}\xi^{\nu]} + DR^{[\mu}_\rho \nabla^{\nu]}\xi^\rho + D\xi^\rho\nabla^{[\mu}R^{\nu]}_\rho)$$
$$- c_1(D-4)\xi^{[\mu}\nabla^{\nu]}R, \tag{B11}$$

vanishes identically for the solutions obeying $R_{\mu\nu} = (D-1)\hat{\Lambda}g_{\mu\nu}$, $\tilde{\mathcal{L}}^{(DD)}_{CG}$ may be interpreted as the higher-dimensional generalization for the Lagrangian (2.10). Besides, there also exist Lagrangians with double free parameters, for instance,

$$\sqrt{-g}L_2|_{\tilde{c}_2=0} = \sqrt{-g}\left(R - 2\Lambda + \frac{2\Lambda - (D-1)(D-2)\hat{\Lambda}}{D(D-4)(D-1)^2\hat{\Lambda}^2}R^2\right),$$
$$\sqrt{-g}L_2|_{c_1=0} = \sqrt{-g}\left(R - 2\Lambda + \frac{2\Lambda - (D-1)(D-2)\hat{\Lambda}}{(D-4)(D-1)^2\hat{\Lambda}^2}R^{\alpha\beta}R_{\alpha\beta}\right), \tag{B12}$$

as well as the Lagrangian without any coupling parameter, namely, $\sqrt{-g}(R^2 - DR^{\alpha\beta}R_{\alpha\beta})$. What is more, maybe one expects to acquire a more general Lagrangian than $\sqrt{-g}L_2$ that admits the solutions fulfilling $R_{\mu\nu} = (D-1)\hat{\Lambda}g_{\mu\nu}$ by incorporating the square of the Riemann tensor $R^{\rho\sigma}_{\alpha\beta}R^{\alpha\beta}_{\rho\sigma}$ within the Lagrangian. In doing so, one could take into consideration of the Lagrangian $\sqrt{-g}L$. For such a Lagrangian, apart from the necessary condition $R_{\mu\nu} = (D-1)\hat{\Lambda}g_{\mu\nu}$, the field equations given by Eq. (B1) yields another constraint for the metric tensor $g_{\mu\nu}$, that is,

$$\frac{c_3}{D-1}(4R^{\rho\sigma}_{\mu\lambda}R^{\nu\lambda}_{\rho\sigma} - \delta^\nu_\mu R^{\rho\sigma}_{\alpha\beta}R^{\alpha\beta}_{\rho\sigma}) = (D-1)(D-4)(Dc_1+c_2)\times\hat{\Lambda}^2\delta^\nu_\mu + (D-2)(\hat{\Lambda}-\hat{\Lambda}_{gr})\delta^\nu_\mu. \tag{B13}$$

Substituting $c_2 = \tilde{c}_2$ into the above equation leads to $c_3 = 0$ or $4R^{\rho\sigma}_{\mu\lambda}R^{\nu\lambda}_{\rho\sigma} = \delta^\nu_\mu R^{\rho\sigma}_{\alpha\beta}R^{\alpha\beta}_{\rho\sigma}$ for the Riemann tensor. Since the latter cannot be always guaranteed to hold under the condition $R_{\mu\nu} = (D-1)\hat{\Lambda}g_{\mu\nu}$, a simple and feasible setting is $c_3 = 0$ to render the solutions of $R_{\mu\nu} = (D-1)\hat{\Lambda}g_{\mu\nu}$ as the ones of $E_{\mu\nu} = 0$. This implies that the Lagrangian $\sqrt{-g}L_2$ with three parameters can be regarded as the most general one for quadratic gravities that embrace solutions only required to obey $R_{\mu\nu} = (D-1)\hat{\Lambda}g_{\mu\nu}$. For example, although Einstein-Gauss-Bonnet gravity ($c_3 \neq 0$) is a natural generalization of general relativity, the higher-dimensional Kerr-AdS black hole solutions in the context of the latter cannot be embedded into the former because of the failure of those solutions to satisfy Eq. (B13).

APPENDIX C: THE FIRST LAW OF THERMODYNAMICS FOR KERR-AdS BLACK HOLES WITHIN THE FRAMEWORK OF QUADRATIC-CURVATURE GRAVITIES

In the present appendix, we investigate the first law of Kerr-AdS black holes in the theory of Einstein gravity corrected by the curvature terms R^2 and $R^{\alpha\beta}R_{\alpha\beta}$.

Within the framework of the Einstein gravity theory, the mass M and all the angular momenta $J^{(i)}$ s for D-dimensional stationary and axially symmetric Kerr-AdS black holes in Eq. (4.10) are given by [92,98–102]

$$M = \frac{V_{D-2}}{4\pi}\frac{m}{\prod_{j=1}^n \Xi_j}\left(\sum_{i=1}^n \frac{1}{\Xi_i} - \frac{1-\epsilon}{2}\right),$$
$$J^{(i)} = \frac{V_{D-2}}{4\pi}\frac{ma_i}{\Xi_i\prod_{j=1}^n \Xi_j}. \tag{C1}$$

The angular velocities Ω_i corresponding to the angular momentum $J^{(i)}$, the entropy S and the surface gravity κ are presented by [47,98]

$$\Omega_i = \frac{a_i H(r_H)}{r_H^2 + a_i^2}, \qquad S = \frac{V_{D-2}}{4} r_H^{\varepsilon-1}\prod_i^n \frac{r_H^2 + a_i^2}{\Xi_i},$$
$$\kappa = r_H H(r_H)\sum_i^n \frac{1}{r_H^2 + a_i^2} - \frac{1}{r_H}\left(\frac{H(r_H)}{2}\right)^\varepsilon, \tag{C2}$$

respectively. In Eq. (C2), the outer horizon radius r_H is the largest root of $V - 2m = 0$. The Hawking temperature is then read off as $T = \kappa/(2\pi)$. It has been demonstrated in [98,99] that all the quantities in Eqs. (C1) and (C2) satisfy the differential form for the first law of thermodynamics, that is,

$$dM = TdS + \sum_i^n \Omega_i dJ^{(i)}, \tag{C3}$$

together with the Smarr formula

$$\frac{D-3}{D-2}M = TS + \sum_i^n \Omega_i J^{(i)}. \tag{C4}$$

On the other hand, in the context of the Lagrangian (4.8) with $\hat{\Lambda} = -\ell^2$, the mass M_{qc} and the angular momenta $J^{(i)}_{qc}$s for the Kerr-AdS black holes corrected by quadratic-curvature terms are presented by Eq. (4.12). By virtue of the following relation

$$\frac{\partial L_2}{\partial R^{\rho\sigma}_{\mu\nu}} = -\frac{2(D-2)\hat{\Lambda}_{gr} + D\ell^2}{(D-4)\ell^2}\frac{\partial R}{\partial R^{\rho\sigma}_{\mu\nu}}, \tag{C5}$$

we obtain the entropy S_{qc} via the Wald's entropy formula for black holes [50,105,116], being of the form

$$S_{qc} = -\frac{2(D-2)\hat{\Lambda}_{gr} + D\ell^2}{(D-4)\ell^2}S. \tag{C6}$$

The angular velocities, the entropy and the surface gravity are still presented by Eq. (C2). Consequently, by letting Eqs. (C3) and (C4) multiplied by the factor $-[2(D-2)\hat{\Lambda}_{gr} + D\ell^2]/[(D-4)\ell^2]$, respectively, we obtain the first law of thermodynamics for the Kerr-AdS black holes in the gravity theory with quadratic-curvature terms, namely,

$$dM_{qc} = TdS_{qc} + \sum_i^n \Omega_i dJ^{(i)}_{qc}, \tag{C7}$$

as well as the following Smarr formula

$$\frac{D-3}{D-2}M_{qc} = TS_{qc} + \sum_{i}^{n}\Omega_i J_{qc}^{(i)}. \qquad \text{(C8)}$$

APPENDIX D: COMPARISON BETWEEN VARIOUS POTENTIALS

In Sec. II, we put forward the potential $K^{\mu\nu}$ in Eq. (2.27) or (2.34) for the computation of the conserved charges of asymptotically AdS spacetimes in the context of quadratic gravities described by the Lagrangian $\sqrt{-g}L$ in Eq. (2.1). To illustrate its significance, we also gave some special cases of $K^{\mu\nu}$, including $K^{\mu\nu}_{\text{Weyl}}$, $K^{\mu\nu}_{\text{EGB}}$, $K^{\mu\nu}_{gr}$ and $K^{\mu\nu}_{\text{Ric}}$, corresponding to the Lagrangians $\sqrt{-g}L_W$ in Eq. (2.5), $\sqrt{-g}L_{\text{EGB}}$ in Eq. (2.6), $\sqrt{-g}L_{\text{EH}} = \sqrt{-g}[R-(D-1)(D-2)\hat{\Lambda}]$ and $\sqrt{-g}L$ constrained by the solutions satisfying $R_{\mu\nu} = (D-1)\hat{\Lambda}g_{\mu\nu}$, respectively. We compared those potentials with some existing ones defined through various methods developed in the literature, such as the (off-shell generalized) ADT method, the field-theoretic approach and the topological regularization method. Some equivalences to the potentials proposed in this work were presented. For convenience to see all the results, we summarize them in Table I. For example, in the third row, the linear perturbation of the two-form $K^{\mu\nu}_{\text{Weyl}}$ on the AdS spaces, $\delta K^{\mu\nu}_{\text{Weyl}}$, is consistent with the ADT potential given by Eq. (23) in [79]. In the seventh row, the potential $K^{\mu\nu}$ (even D) is equivalent to the one given by Eq. (23) in [71] and Eq. (14) in [73], obtained by means of the topological regularization method.

TABLE I. Results from comparison between various potentials.

Lagrangian	Potential	Equivalences of the potential
$\sqrt{-g}L$	$\delta K^{\mu\nu}$	Eq. (2.43) via the ADT method
$\sqrt{-g}L_W$	$\delta K^{\mu\nu}_{\text{Weyl}}$	Eq. (23) in [79]
$\sqrt{-g}L_{\text{EGB}}$	$\delta K^{\mu\nu}_{\text{EGB}}$	Eq. (3.11) in [60], Eq. (41) in [79]
$\sqrt{-g}L_{\text{EH}}$	$K^{\mu\nu}_{gr}$	Eq. (2.15) in [92]
$\sqrt{-g}L$	$(\delta)K^{\mu\nu}_{\text{Ric}}$	Eq. (2.3) in [66], Eq. (6) in [67]
$\sqrt{-g}L$	$K^{\mu\nu}$ (even D)	Eq. (23) in [71], Eq. (14) in [73]

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