Schrödinger symmetry of Schwarzschild-(A)dS black hole mechanics

Jibril Ben Achour,[1,2,*] Etera R. Livine,[2,†] and Daniele Oriti[1]
[1]*Arnold Sommerfeld Center for Theoretical Physics, Munich, Germany*
[2]*Université de Lyon, ENS de Lyon, Laboratoire de Physique, CNRS UMR 5672, Lyon 69007, France*

(Received 18 September 2023; accepted 24 October 2023; published 14 November 2023)

We show that the dynamics of Schwarzschild-(anti–)de Sitter [(A)dS] black holes admits a symmetry under the 2D Schrödinger group, whatever the sign or value of the cosmological constant. This is achieved by reformulating the spherically symmetric reduction of general relativity as a 2D mechanical system with a nontrivial potential controlled by the cosmological constant, and explicitly identifying the conserved charges for black hole mechanics. We expect the Schrödinger symmetry to drive the dynamics of quantum Schwarzschild-(A)dS black holes. This suggests that Schrödinger-preserving nonlinear deformations (of the Gross-Piteavskii type) should capture universal quantum gravity corrections to the black hole geometry. Such a scenario could be realized in condensed matter analog models.

DOI: 10.1103/PhysRevD.108.104028

I. INTRODUCTION

Black holes are iconic predictions of general relativity which stand as a fantastic window to unravel the fundamental structure of space-time. Indeed, the laws of black hole mechanics and their thermodynamical interpretation have revealed that they are equipped with an entropy and a temperature [1,2]. It follows that black holes can be understood as many-body systems built from the collective behavior of (still unknown) microscopic degrees of freedom. Such a thermodynamical point of view on gravitational systems has been widely extended since then, to cosmological space-time, causal diamonds and light cones geometries. The key challenges in completing this picture are on the one hand, to identify the nature of these microscopic degrees of freedom, and on the other hand, to understand the emergence of classical geometries from such microscopic description. While there might be different ways to encode the microscopic degrees of freedom depending on the chosen model or theory, one expects that their dynamics, and thus the emergence of space-time in the continuum hydrodynamical approximation, to be governed by universal symmetries.

Dualities between gravitational and condensed matter systems, for which the mean-field approximation methods are well under control, provide a powerful avenue to shed light on these issues. Such mapping naturally emerged in the nonrelativistic regime of holographic gauge/gravity dualities such as the AdS/CFT correspondence. In view of the prominent role played by the Schrödinger equation and its nonlinear extensions in nonrelativistic physics, an important effort has been devoted to constructing cold atoms/gravity correspondence based on the Schrödinger group [3,4]. Concretely, nonrelativistic holography relates manifolds with Schrodinger isometries to nonrelativistic conformal field theory living on their boundary [5–7]. Condensed matter systems enjoying such nonrelativistic conformal symmetry are characterized by an anisotropic scaling invariance of the space-time coordinates of the form

$$t \to \lambda t, \qquad x^i \to \lambda^z x^i, \tag{1}$$

where $z = 2$ is the critical exponent. Such invariance appears in a variety of contexts, from strongly correlated fermions, vortices, monopoles, compressible fluid mechanics and in Bose-Einstein condensates. In particular, this conformal symmetry is realized for suitable nonlinear Schrödinger equations describing ultracold atom gases, such as the Gross-Piteavskii condensate and the Tonks-Girardeau gas [8,9]. While the construction of dualities between such condensed matter systems and gravity has mostly been investigated in the framework of nonrelativistic holography, it seems that dictionaries between nonlinear Schrödinger and gravity could be identified based directly on the shared symmetries of the two classes of systems.

The goal of this short paper is to develop this storyline for Schwarzschild-(anti–)de Sitter [(A)dS] black holes. Concretely, we consider the spherically symmetric stationary reduction of general relativity, that can be called more descriptively Schwarzschild-(A)dS black hole mechanics. And we show that this reduced gravitational model admits a symmetry under the 2D Schrödinger group, whatever the sign and value of the cosmological constant. This is achieved by explicitly identifying the conserved charges generating this symmetry. We expect

*j.benachour@lmu.de
†etera.livine@ens-lyon.fr

this symmetry to be conserved when quantizing the system. Indeed, standard quantization is meant to identify suitable representations (in the mathematical sense) of the symmetry group. Then breaking a classical symmetry at the quantum level usually reveals a deep physical phenomenon, with strong experimental signatures such as anomalies, phase transitions, emergent collective modes or new propagating degrees of freedom. Here the 2D Schrödinger group is the symmetry group of classical 2D mechanics, consisting of the Galilean relativity transformations plus conformal transformations. It is conserved by standard quantum mechanics. One can thus expect this symmetry to also be preserved when considering quantum gravity corrections to the black hole geometry. Deforming it, or breaking it, would signal a departure from the standard quantization scheme (e.g., such as noncommutative deformations [10]) and/or new physics for quantum black holes.

This sets a strong criterion to discriminate between regularized black hole metric proposals in quantum gravity phenomenology. For instance, assuming that quantum Schwarzschild-(A)dS black holes could be generally modeled as a nonlinear extension of 2D quantum mechanics with a self-interaction between black hole quanta, preserving the Schrödinger group symmetry fixes the self-interaction term to be in ψ^4, thus implying a universal UV behavior to quantum black holes and the existence of a dictionary between black hole quantum mechanics and the Gross-Pitaevskii equation. This scenario is especially interesting with respect to the possibility of imagining a new type of analog quantum black hole systems, e.g., with Bose-Einstein condensates, based on an exact mapping between dynamical conserved charges and no longer on mimicking the Schwarzschild space-time metric as for sonic black holes. This possibility could then be extended to a large class of cosmological dynamics following the symmetry and conserved charge analysis of [11–13].

We start by reviewing the Schrödinger symmetry of classical mechanics, which encodes its invariance under Galilean and conformal transformations, and showing that it is indeed preserved under quantization. The Casimirs of the Schrödinger group, initially vanishing at the classical level, acquire nonzero values in quantum mechanics and reflect the extra degrees of freedom represented by the wave function dressing the classical system. Although this material is not new, these aspects are often not emphasized. They are nevertheless crucial to the identification of the symmetry of the black hole dynamics. Then moving to Schwarzschild-(A)dS black holes, the spherically symmetric reduction of general relativity can be written as a mechanical system, with a nontrivial potential given by the cosmological constant term, and we show that this potential does not spoil the invariance under the Schrödinger symmetry. We underline an important difference with standard mechanics: the nonpositive signature of the kinetics, which actually comes from studying a relativistic system. In particular, we identify the black hole mass (thus its energy) as a boost generator within the symmetry algebra, thereby echoing discussions about the modular Hamiltonian for black hole thermodynamics. Finally, we argue that this should be a key symmetry for quantum black holes, and we discuss its relevance for quantum gravity.

II. SCHRÖDINGER SYMMETRY AND GALILEAN RELATIVITY

Let us start with reviewing the algebra of conserved charges for the classical mechanics of a free particle in d spatial dimensions, driven by the action

$$S[t, x^a] = \frac{m}{2}\int \mathrm{d}t\, \dot{x}^a\, \dot{x}_a, \tag{2}$$

where m is the particle's mass and the index a runs from 1 to d. The canonical analysis defines the conjugate momentum and Poisson bracket,

$$p_a = m\dot{x}_a, \qquad \{x^a, p_b\} = \delta^a_b, \tag{3}$$

and the Legendre transform gives the Hamiltonian

$$S[t, x^a] = \int \mathrm{d}t\,[p_a\dot{x}^a - H] \quad \text{with} \quad H = \frac{1}{2m} p_a p^a. \tag{4}$$

By the Noether theorem, symmetries are generated by conserved charges. In general, those conserved charges can depend explicitly on time and satisfy

$$\mathrm{d}_t\mathcal{O} = \partial_t\mathcal{O} + \{\mathcal{O}, H\} = 0. \tag{5}$$

The algebra of conserved charges for the free particle is well known. It leads to the Schrödinger algebra, which reflects the free particle's invariance under the Galilean transformations and conformal transformations. This construction is crucial, because this is the maximal symmetry preserved by the quantization. In more detail, a first set of conserved charges consists of the momentum p_a, the Galilean boost generator b_a and the angular momentum j_{ab},

$$b_a = \frac{1}{m}[mx_a - tp_a], \qquad j_{ab} = x_a p_b - x_b p_a, \tag{6}$$

which satisfy the Galilean algebra

$$\begin{aligned} &\{p_a, p_b\} = \{b_a, b_b\} = 0, \qquad \{b_a, p_b\} = \delta_{ab}, \\ &\{j_{ab}, p_c\} = \delta_{ac}p_b - \delta_{ab}p_c, \qquad \{j_{ab}, b_c\} = \delta_{ac}b_b - \delta_{ab}b_c, \\ &\{j_{ab}, j_{cd}\} = \delta_{ac}j_{bd} - \delta_{ad}j_{bc} - \delta_{ad}j_{bc} + \delta_{bd}j_{ac}. \end{aligned} \tag{7}$$

The momentum p_a generates the symmetry under space translations $x^a \mapsto x^a + w^a$, while the angular momentum j_{ab} generates the symmetry under $\mathrm{SO}(d)$ space rotations. The vector b_a depends explicitly on the time t; it is an evolving constant of motion, indicating the initial condition (at $t = 0$) for the particle position. It can be interpreted as an extra component of the angular momentum with respect to a pair of conjugate variables $(x^0, p_0) = (t, m)$. It generates the symmetry under translation by a fixed speed,

$$x^a \mapsto x^a + v^a t, \qquad p^a \mapsto p^a + m v^a. \tag{8}$$

Together, (p_a, b_a, j_{ab}) encode the Galilean relativity of the free classical particle. To these, we add three other conserved charges q_μ, defined as

$$\begin{aligned} q_+ &= mH, \\ 2q_0 &= D - 2Ht, \\ 2mq_- &= m x^a x_a - 2tD + 2t^2 H, \end{aligned} \tag{9}$$

where we have introduced the dilatation generator $D = x^a p_a$. These three observables form a $\mathfrak{sl}(2,\mathbb{R})$ Lie algebra,

$$\{q_0, q_\pm\} = \pm q_\pm, \qquad \{q_+, q_-\} = -2q_0, \tag{10}$$

and generate the conformal symmetry of the free particle: $q_+ \propto H$ generates time translations, q_0 is the initial condition for D and generates inverse rescalings of the position and momentum, and finally q_- gives the initial condition for the squared distance x^2 and generates special conformal transformations. This conformal symmetry is a universal feature of mechanical systems, leading for instance to the conformal structure of the Hydrogen atom spectrum (e.g., [14]).

The $\mathfrak{sl}(2,\mathbb{R})$ does not commute with the Galilean sector; the nonvanishing brackets are

$$\begin{aligned} \{q_0, p_a\} &= +\frac{1}{2} p_a, \qquad \{q_-, p_a\} = +b_a, \\ \{q_0, b_a\} &= -\frac{1}{2} b_a, \qquad \{q_+, b_a\} = -p_a. \end{aligned} \tag{11}$$

Putting all the conserved charges together, this algebra is known as the d-dimensional Schrödinger algebra $\mathfrak{sh}(d)$,

$$\mathfrak{sh}(d) = (\mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{so}(d)) \oplus_s (\mathbb{R}^d \oplus \mathbb{R}^d), \tag{12}$$

where $\oplus_s$ denotes a semidirect sum, where the $\mathfrak{sl}(2,\mathbb{R})$ sector generated by the q's and the $\mathfrak{so}(d)$ sector generated by the j's act nontrivially on the $\mathbb{R}^d \oplus \mathbb{R}^d$ sector consisting of the p's and b's. Once exponentiated, these charges give the Schrödinger symmetry group,

$$\mathrm{Sh}(d) = (\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SO}(d)) \ltimes (\mathbb{R}^d \times \mathbb{R}^d). \tag{13}$$

This is the key symmetry group of mechanics preserved by quantization.

An important remark is that, while there are 2D independent variables in the phase space, given by the pairs (x^a, p_a), we have identified $3 + d(d-1)/2 + 2d$ conserved charges. This means that these constants of motion are clearly redundant and that there exist relations between them. These relations are nevertheless not linear, and it is important to keep in mind that a nonlinear polynomial of Lie algebra generators does not automatically belong to that Lie algebra: the symmetry transformations generated by a conserved charge or a power of that charge are *a priori* not the same.

Let us focus here on the 2D case $d = 2$. A more systematic treatment for arbitrary dimension can be found in [15]. For $d = 2$, the angular momentum has a single component $j \equiv j_{12}$. A first relation expresses it in terms of the two pairs of constants of motion (b^a, p_a),

$$\mathcal{C}_2 \equiv b \wedge p - j = 0 \quad \text{with} \quad b \wedge p = (b_1 p_2 - b_2 p_1), \tag{14}$$

which reflects that the b_a's are simply the evolving constants of motion for the positions x^a. This actually is the quadratic Casimir of the Schrödinger algebra: it commutes with all the Schrödinger charges, and thus is invariant under translations, boosts, rotations and conformal transformations. Another set of conditions resulting from the expressions of the charges in terms of x's and p's gives the conformal charges in terms of the boost charges and momenta:

$$q_+ = \frac{p^2}{2}, \qquad q_0 = \frac{b^a p_a}{2}, \qquad q_- = \frac{b^2}{2}. \tag{15}$$

But these relations are not invariant under conformal transformations. Another important relation is the balance equation giving the $\mathfrak{sl}(2,\mathbb{R})$ Casimir in terms of the angular momentum,

$$q_+ q_- - q_0^2 = \frac{1}{4} j^2, \tag{16}$$

but it is not invariant under translations or boosts. It is nevertheless possible to repackage these relations in terms of the cubic Casimir of the Schrödinger algebra,

$$\begin{aligned} \mathcal{C}_3 \equiv q_0^2 - q_+ q_- + \frac{1}{4} j^2 + \frac{b^2}{2} q_+ \\ + \frac{p^2}{2} q_- - b^a p_a q_0 - \frac{b \wedge p}{2} j = 0, \end{aligned} \tag{17}$$

which is appropriately invariant under all Schrödinger symmetries.

Although the Schrödinger symmetry algebra is preserved by the quantization, and even characterizes the quantization procedure, these relations and vanishing

Casimir conditions, $\mathcal{C}_2 = \mathcal{C}_3 = 0$, are not valid at the quantum level anymore. Their nonzero values actually encode the dressing of the classical particle with quantum fluctuations and reveal the infinite tower of new degrees of freedom when upgrading the classical variables (x^a, p_a) to a wave function $\Psi(x^a)$.

The goal of the present paper is to show that the dynamics of (spherically symmetric) black holes in general relativity is also driven by the same Schrödinger symmetry charges, as pointed out in [11], to extend those previous results to include a nonvanishing cosmological constant, and to discuss its role in describing quantum black holes.

III. SYMMETRY OF QUANTUM MECHANICS

Before moving on to black holes, we discuss the fate of the Schrödinger symmetry in standard nonrelativistic quantum mechanics. We consider the free Schrödinger system in d-spatial dimension defined by the field theory Lagrangian:

$$S[\Psi, \bar{\Psi}] = \int \mathrm{d}t \mathrm{d}^d x \left[i\hbar \bar{\Psi} \partial_t \Psi - \frac{\hbar^2}{2m} \partial_a \Psi \partial^a \bar{\Psi} \right]. \tag{18}$$

The resulting field equation is the Schrödinger equation:

$$i\partial_t \Psi = -\frac{\hbar}{2m} \partial_a \partial^a \Psi, \tag{19}$$

which gives the equation of motion for the wave function Ψ in the x polarization. The canonical analysis of this action gives the pair of conjugate variables,

$$\{\Psi(x), \bar{\Psi}(y)\} = \frac{1}{i\hbar} \delta^{(d)}(x-y), \tag{20}$$

and the field theory Hamiltonian,

$$H = -\frac{\hbar^2}{2m} \int \mathrm{d}^d x \, \bar{\Psi} \partial_a \partial^a \Psi. \tag{21}$$

We introduce the probability integral $n = \int \mathrm{d}^d x \, \bar{\Psi}\Psi$, also understood as the number of particles, and the average position and momentum,

$$X^a = \int \mathrm{d}^d x \, \bar{\Psi} x^a \Psi, \qquad P_a = -i\hbar \int \mathrm{d}^d x \, \bar{\Psi} \partial_a \Psi, \tag{22}$$

as well as the quadratic moments of the wave function,

$$J_{ab} = -i\hbar \int \mathrm{d}^d x \, \bar{\Psi}(x_a \partial_b - x_b \partial_a)\Psi, \tag{23}$$

$$D = \frac{-i\hbar}{2} \int \mathrm{d}x^d \, \bar{\Psi}(x^a \partial_a + \partial_a x^a)\Psi, \tag{24}$$

$$\mathcal{X} = \int \mathrm{d}^d x \, \bar{\Psi} x^a x_a \Psi. \tag{25}$$

The angular momentum J_{ab}, the expectation value C of the dilatation generator $\vec{x}\cdot\vec{p}$ and the position uncertainty $\mathcal{X}$ characterize the shape of the wave packet.

The integrals, n, P_a and J_{ab}, have vanishing Poisson brackets with the Hamiltonian, and are thus constants of motion, $\{n, H\} = \{P_a, H\} = \{J_{ab}, H\} = 0$. As for classical mechanics, we introduce the evolving position observable:

$$B_a = X_a - \frac{t}{m} P_a, \quad \mathrm{d}_t B_a = \partial_t B_a + \{B_a, H\} = 0. \tag{26}$$

We compute the Poisson brackets between those observables,

$$\begin{aligned} \{B_a, P_b\} &= \delta_{ab} n, \\ \{J_{ab}, P_c\} &= \delta_{ac} P_b - \delta_{bc} P_a, \\ \{J_{ab}, B_c\} &= \delta_{ac} B_b - \delta_{bc} B_a, \\ \{J_{ab}, J_{cd}\} &= \delta_{ac} J_{bd} - \delta_{bc} J_{ad} - \delta_{ad} J_{bc} + \delta_{bd} J_{ac}, \end{aligned} \tag{27}$$

which form a centrally extended Galilean algebra, with the number of particles n as the central charge. We complete this set of conserved charges with the constants of motion encoding the evolution of the quadratic quantum uncertainty:

$$\begin{aligned} Q_+ &= mH, \\ 2Q_0 &= D - 2Ht, \\ 2mQ_- &= m\mathcal{X} - 2tD + 2t^2 H. \end{aligned} \tag{28}$$

The evolving constants of motion Q_0 and Q_- are the initial conditions at $t = 0$, respectively for the observable D and the position spread $\mathcal{X}$. Their explicit time dependence exactly compensates for their nonvanishing brackets with the Hamiltonian. As expected, these form a $\mathfrak{sl}(2,\mathbb{R})$ algebra,

$$\{Q_0, Q_\pm\} = \pm Q_\pm, \qquad \{Q_+, Q_-\} = -2Q_0, \tag{29}$$

whose Casimir is $\mathcal{C}_{\mathfrak{sl}} = Q_0^2 - Q_+ Q_-$. This is the quadratic uncertainty algebra of [16]. The remaining nonvanishing bracket is given by

$$\begin{aligned} \{Q_0, P_a\} &= +\frac{1}{2} P_a, \qquad \{Q_-, P_a\} = +B_a, \\ \{Q_0, B_a\} &= -\frac{1}{2} B_a, \qquad \{Q_+, B_a\} = -P_a. \end{aligned} \tag{30}$$

We recognize the same d-dimensional Schrödinger algebra $\mathfrak{sh}(d)$ as for classical mechanics,

$$\mathfrak{sh}(d) = (\mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{so}(d)) \oplus_s (\mathbb{R}^d \oplus \mathbb{R}^d). \tag{31}$$

The important difference with classical mechanics is that the Schrödinger Casimirs do not vanish anymore. This reveals a tower of extra degrees of freedom. Indeed, the Schrödinger charges for the classical particle could all be written as polynomials in the canonical position and momentum. This is no longer the case in quantum mechanics. The wave function Ψ contains infinitely more information than the classical position and momentum: the charges $(J_{ij}, Q_0, Q_\pm)$ are now independent from the linear observables (P_i, B_i) and encode the shape of the wave packet; they are legitimate degrees of freedom, representing the quantum fluctuations on top of the classical motion.

To be more precise, we can look into the $d=2$ case. A nonzero quadratic Casimir reveals an extra contribution to the angular momentum,

$$\mathcal{C}_2 = \langle \hat{x}_1 \rangle \langle \hat{p}_2 \rangle - \langle \hat{x}_2 \rangle \langle \hat{p}_1 \rangle - n \langle J_{12} \rangle \neq 0, \tag{32}$$

which actually means that the quantum state Ψ carries nontrivial correlation and entanglement between the two directions x_1 and x_2. Similarly, the cubic Casimir $\mathcal{C}_3$ relates the $\mathfrak{sl}_2$ Casimir for the conformal symmetry to the Galilean generators. The fact that it does not vanish anymore, and that it can take arbitrary values, reflects that the (quadratic) quantum uncertainty—the spread of the wave packet—measured by the Q's can evolve independently from the classical degrees of freedom X^a, P_a. From this perspective, nonzero values of the Schrödinger Casimirs, $\mathcal{C}_2 \neq 0$, $\mathcal{C}_3 \neq 0$, are witnesses of the quantumness of the system.

Once exponentiated, these conserved charges generate symmetries of the system according to Noether's theorem. This gives the Schrödinger group,

$$\mathrm{Sh}(d) = (\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SO}(d)) \ltimes (\mathbb{R}^d \times \mathbb{R}^d), \tag{33}$$

identified as the maximal symmetry group of the free Schrödinger equation by Niederer in [17]. We catalog, in Table I, the various symmetry transformations. While phase multiplication, translations and boosts are usual transformations, it is instructive to give a closer look at the conformal transformations. Indeed, these are not mere rescalings. They are nontrivial symmetry transformations, creating a complex phase factor, affecting the complex width of Gaussian wave packets, thus leading to physical effects. More precisely, these are given by time reparametrization, with a nontrivial rescaling of the space coordinates, following e.g., [18],

$$t \mapsto \tilde{t} = f(t), \qquad x_a \mapsto \tilde{x}_a = \dot{f}(t)^{\frac{1}{2}} x_a, \tag{34}$$

and both a conformal rescaling and a nontrivial phase for the wave function,

$$\Psi \mapsto \tilde{\Psi}(\tilde{t}, \tilde{x}_a) = \dot{f}(t)^{-\frac{d}{e}} e^{i \frac{m \ddot{f}}{4 \dot{f}} x_a x^a} \Psi(t, x_a), \tag{35}$$

which leads to the following transformation of the action,

$$S[\tilde{t}, \tilde{x}, \tilde{\Psi}] = S[t, x, \Psi] - \frac{m}{4} \int \mathrm{Sch}[f](t) x_a x^a \, \Psi \bar{\Psi}, \tag{36}$$

with the Schwarzian derivative of the reparametrization function,

$$\mathrm{Sch}[f] = \dot{h} - \frac{1}{2} h^2, \quad \text{with} \quad h = \ddot{f}/\dot{f}. \tag{37}$$

This is a symmetry as soon as the Schwarzian derivative vanishes, i.e. when f is a Moëbius transformation,

$$\mathrm{Sch}[f] = 0 \Leftrightarrow f(t) = \frac{\alpha t + \beta}{\gamma t + \delta}. \tag{38}$$

This is the $\mathrm{SL}(2,\mathbb{R})$ symmetry group generated by the three conserved charges Q_0 and $Q_\pm$, as can be directly checked by looking at infinitesimal Moëbius transformations.

The purpose of the present work is to show that this Schrödinger symmetry also controls black hole dynamics in general relativity. This underlines the universality of the Schrödinger charges, but also provides a direct bridge between black hole mechanics and quantum mechanics, which should shed clarifying light on the quantization of black holes.

TABLE I. Schrödinger conserved charges.

Charge	Symmetry
n	Phase transformation
P_a	Space translations
B_a	Galilean boosts
$Q_+ \propto H$	Time translation
Q_0	Time dilatation
Q_-	Special conformal

IV. SCHWARZSCHILD-(A)dS BLACK HOLE MECHANICS

We now turn to the main proof-of-concept model for general relativity, namely the eternal Schwarzschild-(A)dS black hole. The action driving the dynamics of the geometry of the black hole is obtained by symmetry reduction and gauge fixing from the vacuum Einstein-Hilbert-Λ action

$$S[g] = \frac{1}{\ell_P^2} \int_{\mathcal{M}} \mathrm{d}^4 x \sqrt{|\det g|} [\mathcal{R} - 2\Lambda], \tag{39}$$

where ℓ_P is the Planck length. Boundary terms do not play any relevant role in the present analysis. We consider a static spherically symmetric manifold $\mathcal{M} = \mathbb{R} \times \Sigma_\epsilon$ with line element

$$\mathrm{d}s^2 = \epsilon(-N^2(r)\mathrm{d}r^2 + \gamma_{tt}(r)\mathrm{d}t^2) + \gamma_{\theta\theta}(r)\mathrm{d}\Omega^2, \qquad (40)$$

where $\gamma_{ij}(r)$ is the induced metric on the constant r hypersurfaces Σ_ϵ, and $\mathrm{d}\Omega^2 = \mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\varphi^2$ is the standard 2-metric on the angular sector. The parameter $\epsilon = \pm 1$ allows one to deal with both the interior and exterior of the black hole using the same formalism. Our conventions are naturally adapted to the case $\epsilon = +$ corresponding to the black hole interior: the coordinate r is timelike, and the radial metric component $N(r)$ plays the role of the lapse between hypersurfaces. The case $\epsilon = -$ corresponds to the exterior region of the black hole where r is a spacelike coordinate and t is timelike.

We decompose the metric components as

$$\gamma_{tt} \coloneqq 2\beta(r)/\alpha(r), \qquad \gamma_{\theta\theta} \coloneqq \ell_s^2 \alpha(r), \qquad (41)$$

where we introduce a fiducial length scale ℓ_s defining the dimensionful unit for the 2-sphere radius. Evaluating the full Einstein-Hilbert-Λ action on this metric ansatz gives the reduced action encoding the dynamics of the black hole geometry [11,19]:

$$S_\epsilon[\alpha,\beta] = \epsilon c \ell_P \int \mathrm{d}\tau \left[\frac{\epsilon}{\ell_s^2} - \frac{\epsilon\alpha}{\ell_\Lambda^2} + \frac{\beta\dot{\alpha}^2 - 2\alpha\dot{\alpha}\dot{\beta}}{2\alpha^2}\right], \qquad (42)$$

where we have introduced a field-rescaled radial coordinate τ defined by

$$\mathrm{d}\tau = \sqrt{\frac{2\beta}{\alpha}} N(r)\mathrm{d}r, \qquad (43)$$

and the dot denotes the derivative with respect to τ. The length scale $\ell_\Lambda = 1/\sqrt{\Lambda}$ encodes the cosmological constant. The dimensionless constant c comes from restricting the range of spatial integration to a bounded region of the hypersurface Σ_ϵ. Indeed, the metric being homogeneous, the integration over the noncompact 3-manifold automatically yields an infinite result. This is naturally resolved by introducing an infrared cutoff ℓ_0 for the coordinate t. This gives

$$c = \frac{1}{\ell_P^3} \int_{t_i}^{t_f} \mathrm{d}t \oint \ell_s^2 \mathrm{d}\Omega = 4\pi \frac{\ell_0 \ell_s^2}{\ell_P^3}, \qquad (44)$$

as the ratio between the IR scale and the UV scale of the system.

The lapse $N(r)$ has been completely absorbed in the definition of the radial coordinate τ. We can safely proceed to describing the system's phase space and evolution with respect to this coordinate. This is equivalent to gauge fixing the lapse to $N = \sqrt{\alpha/2\beta}$. We must nevertheless retain the equation of motion corresponding to lapse variations δN, which implies that the Hamiltonian vanishes, as customary for relativistic systems. Solving the field equations gives the metric

$$\mathrm{d}s^2 = -\epsilon \frac{\alpha}{2\beta} \mathrm{d}\tau^2 + \epsilon \frac{2\beta}{\alpha} \mathrm{d}t^2 + \ell_s^2 \alpha \mathrm{d}\Omega^2, \qquad (45)$$

with $\alpha = k^2(\tau - \tau_0)^2$ and

$$-2\epsilon\beta = \frac{1}{\ell_s^2}(\tau - \tau_0)(\tau - \tau_1) - \frac{k^2}{3\ell_\Lambda^2}(\tau - \tau_0)^4, \qquad (46)$$

where τ_0, τ_1 and k are constants of integration. Rescaling the coordinates as $r = k\ell_s(\tau - \tau_0)$ and $\tilde{t} = t/k\ell_s$, we recover the Schwarzschild-(A)dS solutions,

$$\mathrm{d}s^2 = -f(r)\mathrm{d}\tilde{t}^2 + f(r)^{-1}\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2, \qquad (47)$$

with the metric component,

$$f(r) = 1 - \frac{\ell_M}{r} - \frac{r^2}{3\ell_\Lambda^2} \quad \text{with} \quad \ell_M = k\ell_s(\tau_1 - \tau_0). \qquad (48)$$

The constant of integration τ_0, τ_1, k and the IR regularization scale ℓ_s are combined together into the single physical parameter ℓ_M, which gives the Schwarzschild mass of the black hole.

In order to study the symmetries of black hole mechanics, it is convenient to switch to its phase space description. We compute the canonical momenta

$$p_\alpha = \frac{\epsilon c \ell_P}{\alpha^2}(\beta\dot{\alpha} - \alpha\dot{\beta}), \qquad p_\beta = -\epsilon c \ell_P \frac{\dot{\alpha}}{\alpha}, \qquad (49)$$

forming the canonical pairs $\{\alpha, p_\alpha\} = \{\beta, p_\beta\} = 1$. The Hamiltonian reads as

$$\mathcal{H} = \mathbf{H}^{(\Lambda)} - \frac{c\ell_P}{\ell_s^2} \quad \text{with} \quad \mathbf{H}^{(\Lambda)} = \mathbf{H}^{(0)} + \frac{c\ell_P}{\ell_\Lambda^2}\alpha, \qquad (50)$$

$$\text{and} \quad \mathbf{H}^{(0)} = -\frac{1}{\epsilon c \ell_P}\left[\alpha p_\alpha p_\beta + \frac{1}{2}\beta p_\beta^2\right]. \qquad (51)$$

Remember that we need to impose that the Hamiltonian vanishes $\mathcal{H} = 0$. This Hamiltonian constraint consists in a kinetic term $\mathbf{H}^{(0)}$, a potential term whose coupling is the cosmological constant and a constant shift. This constant shift depends on the IR/UV ratio c. It is crucial, since it changes the on shell value of $\mathbf{H}^{(\Lambda)}$. Now that the dynamics of black holes has been formulated as a mechanical system, let us show that it admits a symmetry group isomorphic to the Schrödinger group.

V. SCHRÖDINGER CHARGES FOR BLACK HOLES

As static spherically symmetric metrics in general relativity have been recast as a mechanical system with 2 degrees of freedom, we expect a symmetry under the $d = 2$ Schrödinger group, if it were a free system. The potential actually vanishes when the cosmological constant is set to 0, or equivalently when the cosmological scale is set to infinity, $\ell_\Lambda \to +\infty$. In that case, we naturally identify Schrödinger charges. Below, we further show that, surprisingly, the cosmological potential does not spoil this symmetry, and so the Schrödinger group still drives the black hole dynamics whatever the value of Λ.

Let us start with the case $\ell_\Lambda \to +\infty$, corresponding to a vanishing cosmological constant $\Lambda = 0$ and asymptotically flat Schwarzschild black holes. Symmetries are generated by conserved charges $\mathcal{O}$, here satisfying

$$\mathrm{d}_\tau \mathcal{O} = \partial_\tau \mathcal{O} + \{\mathcal{O}, \mathbf{H}^{(0)}\} = 0. \tag{52}$$

Time-independent charges, i.e. with $\partial_\tau \mathcal{O} = \{\mathcal{O}, \mathbf{H}^{(0)}\} = 0$, correspond to conformal Killing vectors in the field configuration space (α, β), while explicitly time-dependent charges, i.e. $\partial_\tau \mathcal{O} \neq 0$, correspond to conformal Killing vectors in an extended field configuration space given by the Eisenhart-Duval lift [20]. This general approach was pushed forward in [11] to investigate symmetries of gravitational minisuperspaces. Here, we identify translation and boost charges:

$$\begin{aligned} P_+ &= \sqrt{\alpha} p_\alpha + \frac{\beta p_\beta}{2\sqrt{\alpha}}, \qquad & c\ell_P B_+ &= \epsilon c \ell_P \frac{\beta}{\sqrt{\alpha}} + \tau P_+, \\ P_- &= \sqrt{\alpha} p_\beta, & c\ell_P B_- &= \epsilon c\ell_P 2\sqrt{\alpha} + \tau P_-. \end{aligned} \tag{53}$$

They form a closed Lie algebra with the charge $J = 2\alpha p_\alpha$:

$$\begin{aligned} \{P_-, P_+\} &= 0, \qquad & \{B_-, B_+\} &= 0, \\ \{B_\pm, P_\pm\} &= 0, & \{B_\pm, P_\mp\} &= \epsilon, \\ \{J, B_\pm\} &= \pm B_\pm, & \{J, P_\pm\} &= \pm P_\pm, \end{aligned} \tag{54}$$

where J generates $\mathfrak{so}(1,1)$ boosts. We recognize the algebra of Galilean symmetries in two dimensions. Further introducing the dilatation generator $D = (\alpha p_\alpha + \beta p_\beta)$, we complete this set of conserved charges with the following observables:

$$\begin{aligned} Q_+ &= c\ell_P \mathbf{H}^{(0)}, \qquad Q_0 = D - \tau \mathbf{H}^{(0)}, \\ c\ell_P Q_- &= -2\epsilon c\ell_P \beta - 2\tau D + \tau^2 \mathbf{H}^{(0)}, \end{aligned} \tag{55}$$

which form a $\mathfrak{sl}(2,\mathbb{R})$ Lie algebra,

$$\{Q_0, Q_\pm\} = \pm Q_\pm, \qquad \{Q_+, Q_-\} = -2Q_0. \tag{56}$$

The two sectors are coupled by nonvanishing Poisson brackets:

$$\begin{aligned} \{Q_0, P_\pm\} &= \frac{1}{2} P_\pm, \qquad & \{Q_0, B_\pm\} &= -\frac{1}{2} B_\pm, \\ \{Q_-, P_\pm\} &= -B_\pm, & \{Q_+, B_\pm\} &= P_\pm, \end{aligned} \tag{57}$$

leading to the 2D centrally extended Schrödinger algebra $\mathfrak{sh}(2) = (\mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{so}(1,1)) \oplus_s (\mathbb{R}^2 \oplus \mathbb{R}^2)$. The fact that we have the symmetry subalgebra $\mathfrak{so}(1,1)$ instead of $\mathfrak{so}(2)$ comes from working with a relativistic system, whose kinetic terms have a nonpositive signature. This will become clearer below when diagonalizing explicitly the kinetic terms. Its quadratic and cubic Casimir both vanish, as expected in classical mechanics:

$$\mathcal{C}_2 = P_+ B_- - P_- B_+ - \epsilon J = 0, \tag{58}$$

$$\begin{aligned} \mathcal{C}_3 = Q_0^2 - Q_+ Q_- - \frac{1}{4} J^2 - \epsilon B_+ B_- Q_+ - \epsilon P_+ P_- Q_- \\ - \epsilon (B_- P_+ + B_+ P_-) Q_0 + \frac{\epsilon}{2} (B_- P_+ - B_+ P_-) J = 0. \end{aligned}$$

The latter is the Schrödinger-invariant expression of the balance equation for the $\mathfrak{sl}_2$ Casimir,

$$Q_0^2 - Q_+ Q_- = \frac{1}{4} J^2. \tag{59}$$

It is interesting to notice that the evolving position observables $B_\pm$ allow one to define a canonical transformation to phase space coordinates that diagonalize the kinetic Hamiltonian. Indeed, we read position coordinates from $B_{pm}(\tau = 0)$:

$$X_+ = \beta/\sqrt{\alpha}, \qquad X_- = 2\sqrt{\alpha}, \qquad \{X_\mp, P_\pm\} = 1. \tag{60}$$

Now the Hamiltonian takes a very simple form,

$$\begin{aligned} \mathbf{H}^{(\Lambda)} &= -\frac{\epsilon}{c\ell_P} P_- P_+ + \frac{c\ell_P}{4\ell_\Lambda^2} X_-^2 \\ &= \frac{\epsilon}{2c\ell_P} (P_2^2 - P_1^2) + \frac{c\ell_P}{8\ell_\Lambda^2} (X_1 + X_2)^2, \end{aligned} \tag{61}$$

where we have introduced

$$P_\pm = \frac{P_1 \pm P_2}{\sqrt{2}}, \qquad X_\pm = \frac{X_1 \mp X_2}{\sqrt{2}}. \tag{62}$$

This clarifies the mapping of black hole mechanics onto the $d = 2$ particle, with the awkward sign switch in the kinetic term, here $(P_2^2 - P_1^2)$ instead of $(P_2^2 + P_1^2)$. This sign is a central feature of general relativity: we are working with a $1+1$-d relativistic particle. The nonpositive signature signals the gravitational instability (due to conformal factor) that leads to gravitational collapse, black holes and cosmological expansion. The black hole phase space IR/UV ratio c plays the role of the $1+1$-d particle mass. Keep in mind that the black hole mass is a variable in black hole mechanics. It is a property of the chosen classical solution. More precisely, it is actually a conserved quantity, which we express in terms of the Schrödinger charges below in (66). The cosmological constant creates a quadratic trapping potential for the center of mass of the system. As this is a quadratic potential, it seems that one could absorb it in a redefinition of the momenta. It is indeed what happens, as we show below. This is our main result.

Indeed, turning on the cosmological constant $\Lambda \neq 0$, we find, quite remarkably, that the Schrödinger algebra is preserved. The conserved charges are mildly modified. Explicitly, while P_- and B_- do not acquire corrections, the other translation and boost charges become

$$P_+^{(\Lambda)} = P_+ - \epsilon \frac{c^2 \ell_P^2}{\ell_\Lambda^2} \frac{\sqrt{\alpha}}{p_\beta},$$
$$B_+^{(\Lambda)} = \frac{\beta^{(\Lambda)}}{\sqrt{\alpha}} + \epsilon \frac{\tau}{c \ell_P} P_+^{(\Lambda)},$$
$$J^{(\Lambda)} = J - \epsilon \frac{4c^2 \ell_P^2}{3\ell_\Lambda^2} \frac{\alpha}{p_\beta}. \qquad (63)$$

The conformal sector is similarly modified,

$$Q_+^{(\Lambda)} = c\ell_P \mathbf{H}^{(\Lambda)}, \qquad Q_0^{(\Lambda)} = D^{(\Lambda)} - \tau \mathbf{H}^{(\Lambda)},$$
$$c\ell_P Q_-^{(\Lambda)} = -2\epsilon c \ell_P \beta^{(\Lambda)} - 2\tau D^{(\Lambda)} + \tau^2 \mathbf{H}^{(\Lambda)}, \qquad (64)$$

with the following Λ corrections:

$$\beta^{(\Lambda)} = \beta - \epsilon \frac{2c^2 \ell_P^2}{3\ell_\Lambda^2} \frac{\alpha}{p_\beta^2}, \quad D^{(\Lambda)} = D - \epsilon \frac{4c^2 \ell_P^2}{3\ell_\Lambda^2} \frac{\alpha}{p_\beta}. \qquad (65)$$

The new conserved charges satisfy $\partial_\tau \mathcal{O} + \{\mathcal{O}, \mathbf{H}^{(\Lambda)}\} = 0$, and the Hamiltonian simply reads as $c\ell_P \mathbf{H}^{(\Lambda)} = -\epsilon P_- P_+^{(\Lambda)}$. We get the same Lie algebra as for the $\Lambda = 0$ case. It follows that the mechanics of Schwarzschild-(A)dS black holes is also invariant under the nonrelativistic conformal Schrödinger symmetry.

This result parallels the fact that the Schrödinger symmetry for 1d classical mechanics is preserved for two specific potentials: the harmonic potential and the inverse square potential (whose quantization was studied in [21]). From that point of view, the Schwarzchild-(A)dS black hole mechanics can be viewed as an extension of the flat Schwarzchild black hole mechanics similar to the extension of the free particle to the harmonic oscillator (with a positive or negative pulsation).

We can compute the value of those observables on classical solutions. In particular, we get

$$J^{(\Lambda)} = \frac{c\ell_P}{\ell_s^2}(\tau_1 - \tau_0), \qquad P_- = -\epsilon 2c\ell_P k,$$

which allows one to identify the black hole mass as a conserved charge:

$$\ell_M = -\epsilon \frac{2\ell_s^3}{c^2 \ell_P^2} J^{(\Lambda)} P_-^{(\Lambda)}. \qquad (66)$$

It is definitely intriguing that the black hole mass is equal to the boost generator $J^{(\Lambda)}$ (up to the velocity factor P_-). This evocates recent discussions on black hole thermodynamics where Hawking's thermal radiation is derived from the identification of the (modular) Hamiltonian as a boost generator within a $\mathfrak{sl}(2,\mathbb{R})$ algebra of asymptotic symmetry generators (see e.g., [22]). Although the symmetry structure is very similar, the link between the present work and this framework is not obvious at all.

An important remark is that the cosmological constant ℓ_Λ never appears in the on shell values of the Schrödinger charges. For instance, the cosmological constant does not change the Schrödinger Casimirs $\mathcal{C}_2 = \mathcal{C}_3 = 0$. In fact, Λ shifts the definition of the conserved charges but does not affect at all the Schrödinger symmetry. Let us insist that these are not space-time isometries or diffeomorphisms, but nontrivial symmetry of general relativity under transformations acting on the space of metrics.

Here, we have found that the cosmological constant does not affect the symmetry of general relativity, at least in the spherically symmetric sector. From the point of view of symmetries, Λ will appear back when breaking the Schrödinger symmetry, for instance by introducing an "observer." This can be simply achieved by going beyond the gravitational sector and looking at the dynamics of matter fields coupled to the geometry, in case the cosmological constant will surely modify the dynamics and symmetries of the matter field evolution.

VI. DISCUSSION AND PROSPECTS

We have shown that the dynamics of stationary spherically symmetric metrics in general relativity can be formulated as a 2D mechanical system with a nontrivial potential whose coupling constant is the cosmological constant. We call this model black hole mechanics. Keep in mind that the evolution parameter here is the radial coordinate, which is spacelike outside the black hole and timelike in the interior region. This allowed us to show that the black hole mechanics is invariant under the $d = 2$ Schrödinger group. This invariance holds both for the interior and the exterior regions of the black hole.

Moreover, it holds whatever the value of the cosmological constant Λ is. The symmetry transformations act on the phase space of geometries and are not mere space-time transformations. They change the black hole mass ℓ_M and the singularity position τ_0, as well as the IR regularization scale ℓ_s, while leaving the equations of motion invariant.

Since the Schrödinger group is the key (maximal) symmetry of classical mechanics which is preserved under quantization, it is natural to expect quantum black holes to retain this symmetry. Breaking this symmetry when quantizing black holes would definitely signal a strong deviation with respect to the standard quantization logic and would reveal some important hidden physical ingredients in the description of black holes in general relativity.

Digging deeper in this direction, here we have taken the perspective of considering quantum mechanics as a field extension of classical mechanics. Putting aside conceptual issues (e.g., the measurement problem and collapse of the wave function), quantum mechanics is mathematically formulated as a description of the dynamics of the wave function: classical positions and momenta, evolving in time, are replaced by a wave function, considered as a space-time field, interpreted as a dressed classical object with classical positions and momenta, plus extra degrees of freedom representing the shape fluctuations of the wave packet.

From this viewpoint of quantization as field extension and turning to black holes, there are actually two natural field extensions of black hole mechanics:

(1) On the one hand, it is natural to quantize black hole mechanics and lift a classical black hole metric to a wave function with a fuzzy mass and a fuzzy singularity. Let us underline that this does not mean relaxing the hypothesis of stationarity or spherical symmetry: we describe quantum superposition of spherically symmetric metrics. This goes in the same direction as the line of research on effective black hole metrics taking into account quantum gravity corrections and attempting to solve the singularity problem without introducing anisotropy or leaving spherical symmetry, e.g., [23–25]. Our analysis means that preserving the Schrödinger symmetry should be crucial to this approach (see e.g., [26] using the conformal symmetry to constrain regularized black hole metrics in effective quantum gravity models).

(2) On the other hand, the natural field theory of black hole mechanics is general relativity, which reestablishes inhomogeneities and anisotropies on top of the spherically symmetric background and describes their dynamics. From this perspective, general relativity is to be interpreted more as the nonperturbative field theory of black hole excitations, instead of its usual interpretation as the field theory encoding the nonlinear properties of gravitational waves. More precisely, general relativity would lead to a nonperturbative hydrodynamic description of the black holes microstates with the black hole sector identified by the Schrödinger symmetry we have found here. Then it would be natural to understand if general relativity is invariant under an extension of the Schrödinger symmetry group. Let us underline that we expect that these symmetries would not be space-time diffeomorphisms, but nontrivial transformations on the phase space of geometries. Interestingly, it has been recently shown that the static perturbations of the Schwarzschild and Kerr black holes relevant to compute the Love numbers are also governed by a Schrödinger symmetry [27]. It would be interesting to further understand how this symmetry for perturbations can be related to the background symmetry discussed here. From a more general perspective, it would be enlightening to compare the Schrödinger charges derived here to the existing extended Bondi–Metzner–Sachs charges and $w_{1+\infty}$ charge algebra for asymptotically flat space-time as derived in e.g., [28–31].

For both field theory extensions of black hole mechanics, we expect the Schrödinger Casimirs, $\mathcal{C}_2$ and $\mathcal{C}_3$, not to vanish anymore, and to reflect the extra structures and degrees of freedom dressing the black hole evolution. If both quantum black holes and asymptotically flat general relativity turn out both to preserve the Schrödinger symmetry, it should definitely be a *key symmetry of quantum gravity.*

As a direct application of the present work, we would like to point out that the Schrödinger symmetry can be used to select quantum corrections to black hole dynamics. Indeed, now that the canonical analysis of the classical black hole dynamics has been clarified, it is straightforward to quantize the system. We define quantum states as wave functions of the metric components. Instead of using the original variables α, β, our analysis suggests that using the variable X_1, X_2, and thus considering wave functions $\Psi(X_1, X_2)$, is more convenient. Then this wave function is driven by a field action:

$$S[\Psi, \bar{\Psi}] = \int \mathrm{d}\tau \Big[i\hbar \bar{\Psi} \partial_\tau \Psi - \bar{\Psi}\, \widehat{\mathbf{H}^{(\Lambda)}}\, \Psi \Big], \tag{67}$$

where the Hamiltonian operator $\widehat{\mathbf{H}^{(\Lambda)}}$ consists of a kinetic operator, given by the $1+1$-dimensional Laplacian, plus a harmonic potential term whose coupling constant is given by the cosmological constant. The analysis of the symmetry of quantum mechanics, reviewed in Sec. III, shows that the Schrödinger symmetries are preserved by this standard quantization scheme. Just as atom-atom microscopic interactions in quantum mechanics can be taken into account by introducing a potential $\mathcal{V}[\Psi, \overline{\Psi}]$, which encodes the self-interaction of the wave-function fluctuations and excitations, e.g., [8,9], we similarly expect that quantum gravity will lead to a self-interaction between the quanta of geometry forming the black hole, thus leading to an effectively modified field action:

$$S_{QG} = \int d\tau \Big[i\hbar \bar{\Psi} \partial_\tau \Psi - \bar{\Psi}\, \widehat{\mathbf{H}^{(\Lambda)}}\, \Psi + \mathcal{V}[\Psi, \overline{\Psi}] \Big]. \tag{68}$$

The key point is that the Schrödinger charge algebra also holds for precise nonlinear extensions of the Schrödinger dynamics. Remarkably, depending on the spatial dimension d, one can show that the Schrödinger charge algebra is preserved for suitable self-interaction. Since such a potential is homogeneous, it does not affect the symmetry under phase transformations, translations and boosts. And one easily checks that the conformal symmetry (35) is preserved for $\mathcal{V} \propto |\Psi|^{2n}$ when $d(n-1) = 2$. For $d = 1$, this selects a self-interaction potential $\mathcal{V} \propto |\Psi|^6$ leading to the Tonks-Girardeau equation, which defines a quintic nonlinear extension of the Schrödinger equation. Here, the black hole minisuperspace model corresponds to $d = 2$ dimensions, and preserving the full Schrödinger symmetry selects a quadratic potential $\mathcal{V} \propto |\Psi|^4$:

$$S^{\kappa}_{QG} = \int d\tau \Big[i\hbar \bar{\Psi} \partial_\tau \Psi - \bar{\Psi}\, \widehat{\mathbf{H}^{(\Lambda)}}\, \Psi - \kappa |\Psi|^4 \Big]. \tag{69}$$

This shows that there exists a nontrivial UV corrected quantum dynamics protected by the Schrödinger symmetry. The evolution of the black hole wave function is then driven by a Gross-Pitaevskii equation,

$$i\hbar \partial_\tau \Psi = \widehat{\mathbf{H}^{(\Lambda)}} \Psi + 2\kappa |\Psi|^2 \Psi. \tag{70}$$

The new coupling κ controls the attraction or repulsion between wave packets depending on its sign. It will thus determine when and where one can have stable quantum superpositions of black hole states, and enter in a crucial fashion in the identification of the transitions between the semiclassical regime and the deep quantum regime.

Playing with this new parameter κ should lead to a new phenomenology for quantum black holes. It is very different from modified gravity theories, which usually propose modifications of the Hamiltonian operator but do not consider nonlinear self-interaction terms. We consider such symmetry-protected nonlinear extension of the Wheeler-de Witt equation as a universal template for the effective dynamics of quantum black holes. It would be enlightening to understand which quantum gravity models or scenarios generate such quantum black hole dynamics.

We would like to conclude with the remark that, using the phase-amplitude factorization of the wave function $\Psi = \sqrt{\rho} e^{i\theta}$, the Schrödinger equation and its nonlinear extension can be understood as Navier-Stokes's equation for compressible fluid dynamics leading to the hydrodynamics reformulation of quantum mechanics, e.g., see the original seminal work by Madelung [32] and the more recent analysis [33]. Since black hole mechanics is invariant under the $d = 2$ Schrödinger group, this means that we get an intriguing mapping between black hole quantum mechanics and 2D hydrodynamics. It is tempting to speculate that this could be related to a fluid dynamics for gravitational quanta on the black hole horizon considered as a 2D membrane (as in the corner dynamics for general relativity [34]). A more down-to-earth expectation is that this mapping surely provides a promising avenue to reformulate quantum black hole as a many-body Schrödinger system. In fact, it opens the door to the possibility of a new class of analog condensed matter models for black holes, cosmology and quantum gravity phenomenology, based on an exact mapping of symmetries, conserved charges and dynamics, instead of focusing on shaping and manufacturing equivalents of space-time metrics.

ACKNOWLEDGMENTS

The work of J. Ben Achour is supported by the Alexander von Humboldt foundation and the Sir John Templeton foundation.

[1] J. D. Bekenstein, Black holes and entropy, Phys. Rev. D **7**, 2333 (1973).
[2] J. M. Bardeen, B. Carter, and S. W. Hawking, The four laws of black hole mechanics, Commun. Math. Phys. **31**, 161 (1973).
[3] D. T. Son, Toward an AdS/cold atoms correspondence: A geometric realization of the Schrodinger symmetry, Phys. Rev. D **78**, 046003 (2008).
[4] K. Balasubramanian and J. McGreevy, Gravity duals for non-relativistic CFTs, Phys. Rev. Lett. **101**, 061601 (2008).
[5] M. Taylor, Non-relativistic holography, arXiv:0812.0530.
[6] W. D. Goldberger, AdS/CFT duality for non-relativistic field theory, J. High Energy Phys. 03 (2009) 069.
[7] M. Rangamani, S. F. Ross, D. T. Son, and E. G. Thompson, Conformal non-relativistic hydrodynamics from gravity, J. High Energy Phys. 01 (2009) 075.
[8] E. B. Kolomeisky, T. J. Newman, J. P. Straley, and X. Qi, Low-dimensional Bose liquids: Beyond the Gross-Pitaevskii approximation, Phys. Rev. Lett. **85**, 1146 (2000).
[9] P. K. Ghosh, Conformal symmetry and the nonlinear Schrodinger equation, Phys. Rev. A **65**, 012103 (2002).

[10] R. Banerjee, Deformed Schrodinger symmetry on noncommutative space, Eur. Phys. J. C **47**, 541 (2006).
[11] J. Ben Achour, E. R. Livine, D. Oriti, and G. Piani, Schrödinger symmetry in cosmology and black hole mechanics, arXiv:2207.07312.
[12] M. Geiller, E. R. Livine, and F. Sartini, Dynamical symmetries of homogeneous minisuperspace models, Phys. Rev. D **106**, 064013 (2022).
[13] J. Ben Achour, Proper time reparametrization in cosmology: Möbius symmetry and Kodama charges, J. Cosmol. Astropart. Phys. 12 (2021) 005.
[14] I. Bars, Conformal symmetry and duality between free particle, H—atom and harmonic oscillator, Phys. Rev. D **58**, 066006 (1998).
[15] F. Alshammari, P. S. Isaac, and I. Marquette, A differential operator realisation approach for constructing Casimir operators of non-semisimple Lie algebras, J. Phys. A **51**, 065206 (2018).
[16] E. R. Livine, Quantum uncertainty as an intrinsic clock, arXiv:2212.09442.
[17] U. Niederer, The maximal kinematical invariance group of the free Schrodinger equation, Helv. Phys. Acta **45**, 802 (1972).
[18] J. E. Lidsey, Inflationary cosmology, diffeomorphism group of the line and virasoro coadjoint orbits, arXiv:1802.09186.
[19] J. Ben Achour and E. R. Livine, Symmetries and conformal bridge in Schwarschild-(A)dS black hole mechanics, J. High Energy Phys. 12 (2021) 152.
[20] M. Cariglia, C. Duval, G. W. Gibbons, and P. A. Horvathy, Eisenhart lifts and symmetries of time-dependent systems, Ann. Phys. (Amsterdam) **373**, 631 (2016).
[21] V. de Alfaro, S. Fubini, and G. Furlan, Conformal invariance in quantum mechanics, Nuovo Cimento Soc. Ital. Fis. **34A**, 569 (1976).
[22] D. L. Jafferis and L. Lamprou, Inside the hologram: Reconstructing the bulk observer's experience, J. High Energy Phys. 03 (2022) 084.
[23] K. V. Kuchar, Geometrodynamics of Schwarzschild black holes, Phys. Rev. D **50**, 3961 (1994).
[24] M. Bojowald, H. A. Kastrup, F. Schramm, and T. Strobl, Group theoretical quantization of a phase space $\mathcal{S}^1 \times \mathbb{R}^+$ and the mass spectrum of Schwarzschild black holes in d space-time dimensions, Phys. Rev. D **62**, 044026 (2000).
[25] F. Sartini, Group quantization of the black hole minisuperspace, Phys. Rev. D **105**, 126003 (2022).
[26] J. Ben Achour and E. R. Livine, Polymer quantum cosmology: Lifting quantization ambiguities using a $SL(2,\mathbb{R})$ conformal symmetry, Phys. Rev. D **99**, 126013 (2019).
[27] J. Ben Achour, E. R. Livine, S. Mukohyama, and J.-P. Uzan, Hidden symmetry of the static response of black holes: applications to Love numbers, J. High Energy Phys. 07 (2022) 112.
[28] A. Strominger, w(1+infinity) and the celestial sphere, arXiv:2105.14346.
[29] L. Freidel, D. Pranzetti, and A.-M. Raclariu, Higher spin dynamics in gravity and $\mathrm{w}1+\infty$ celestial symmetries, Phys. Rev. D **106**, 086013 (2022).
[30] O. Fuentealba, M. Henneaux, P. Salgado-Rebolledo, and J. Salzer, Asymptotic structure of Carrollian limits of Einstein-Yang-Mills theory in four spacetime dimensions, Phys. Rev. D **106**, 104047 (2022).
[31] G. Barnich, K. Nguyen, and R. Ruzziconi, Geometric action for extended Bondi-Metzner-Sachs group in four dimensions, J. High Energy Phys. 12 (2022) 154.
[32] E. Madelung, Quantentheorie in hydrodynamischer form, Z. Phys. **40**, 322 (1927).
[33] P. A. Horvathy and P. M. Zhang, Non-relativistic conformal symmetries in fluid mechanics, Eur. Phys. J. C **65**, 607 (2010).
[34] W. Donnelly, L. Freidel, S. F. Moosavian, and A. J. Speranza, Gravitational edge modes, coadjoint orbits, and hydrodynamics, J. High Energy Phys. 09 (2021) 008.