The boundary of the gravitational standard-model extension

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A modification of general relativity that is based on the gravitational standard-model extension and incorporates nondynamical background fields has recently been studied via the ADM formalism. Our objective in this paper is to develop a better understanding of the additional contributions that arise on the spacetime boundary $\partial\mathcal{M}$. An extension of the previously introduced boundary terms, which are relevant in the context of asymptotically flat spacetimes, follows from the decomposition of $\partial\mathcal{M}$ into timelike and spacelike hypersurfaces. Furthermore, we present an alternative method of deriving the field equations satisfied by the induced metric on the purely spacelike hypersurfaces of the foliated spacetime. This leads to the dynamical part of the Einstein equations modified by the background fields. Our results have the potential to be applicable in various contexts such as modified black holes and cosmology.

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I. INTRODUCTION

The abundance of experimental tests of general relativity (GR) [1–6] carried out for more than 100 years has demonstrated that GR provides a description of gravitational phenomena that works astoundingly well. The geometrization of gravity that Einstein envisioned also has a certain undeniable aesthetics to it and, so far, it has been impossible to adopt this description to the other fundamental interactions of nature. Despite the vast experimental support as well as the beauty of Einstein's gravity theory, even if one has the viewpoint that gravity stays classical at all length scales, GR exhibits at least some unsatisfactory properties.

For example, many physicists would agree that the occurrence of spacetime singularities in some solutions of GR, e.g., black-hole spacetimes is an issue that cannot simply be ignored. Both indirect observations of black-hole mergers via gravitational-wave detection by LIGO [7–10] and the impressive photographs of black-hole accretion disks in M87 and the Milky Way made by the EHT [11,12] undoubtedly demonstrate that black holes are not merely mathematical vacuum solutions of the Einstein equations, but part of our reality. Thus, the proper understanding and treatment of black-hole singularities is paramount.

Cosmology reveals another possible issue of GR. Cosmological time evolution is largely affected by the gravitational pull of the matter content of our Universe. Therefore, GR forms the theoretical foundation of the current cosmological standard model, ΛCDM. Measurements of the large-scale structure of our Universe [13–15] and the precise mapping of the cosmological microwave background radiation [16–21] hint toward the existence of a completely mysterious entity known as dark energy [22], which is needed to account for the accelerated expansion of the Universe. Nothing whatsoever is known about the nature of dark energy and its physical properties, e.g., its negative pressure contradict the characteristics of any form of matter or energy that can be investigated in the laboratory. So it is needless to say that its introduction into cosmology is unsatisfactory. However, it could be the case that dark energy is only needed to make contact with measurements, since GR suffers severe alterations at the very large length scales that dominate cosmological late-time evolution.

These and other arguments suggest a refinement of GR, let it be at microscopic and/or cosmological scales. While a large number of modified-gravity theories have been proposed in the literature [23–26], which are more or less well motivated, our article will be dedicated to a specific class of such theories. Our intention is to respect coordinate invariance as well as the full nonlinear structure of GR. Moreover, we will be working in a classical setting, i.e., no attempt is made to quantize gravity. Although extensions such as Finsler geometry [27–29] could be considered, in principle, Riemannian geometry is maintained as the underlying geometrical foundation.

Instead, we give up one of the defining characteristics of Einstein's gravity, which is diffeomorphism invariance. The violation of the latter is parameterized by particular nondynamical background fields that are contained in the gravitational sector of the standard-model extension

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(SME) [30–42]. This effective approach is comprehensive and parameterizes violations of diffeomorphism symmetry and local Lorentz invariance in gravity in a model-independent way. The SME is understood as a field theory framework that enables broad experimental tests of nonstandard gravitational physics such as diffeomorphism violation. The yearly updated data tables [43] provide an extensive compilation of experimental constraints on symmetry violation in gravity—among the even larger set of bounds on Lorentz violation in a nongravitational setting.

Recently, the Hamiltonian formulation [44–50] has been developed for extensions [51–53] of GR that exhibit diffeomorphism invariance breaking. Analyses of this kind rest upon the $(3+1)$ decomposition, which is often also referred to as the ADM decomposition (formulation) according to the names of the physicists [44] that introduced this technique into GR. The latter is a formidable theoretical toolset being the base of advanced black-hole physics [54,55] as well as of numerical relativity [56–59]. It is one of the cornerstones of powerful computer codes such as the *Einstein Toolkit* [60] and *GRHydro* [61] that solve highly complicated problems in GR numerically.

The ADM formulation has also proven to be a valuable technique to analyze modified-gravity theories from a formal perspective. For this reason, it forms the technical foundation of the papers [51–53]. In our current work, emphasis will be put on the behavior of the theory on the spacetime boundary. We intend to avoid integrations by parts, as these may imply essential contributions on the spacetime boundary that cannot simply be discarded. Furthermore, we will carry out a proper treatment of boundary terms that are of relevance in such an analysis.

One of the principal motivations to implement the ADM formulation in the context of the SME was to explore diffeomorphism violation in a strong-gravity regime complementing the studies within linearized modified gravity [62–76], in particular, on gravitational-wave physics. The ADM formulation has also been fruitful to stimulate a new branch of research, which could be coined SME cosmology [51,77–79]. Moreover, this formalism enables the definition of a slew of important physical quantities such as the ADM mass [80] or the ADM momentum [56,59], which are useful in, e.g., black-hole physics. So having the ADM-decomposed gravitational SME at someone's disposal, brings them into a position to study modified black holes. Finally, the canonical formulation of SME gravity could shed light on the possible issues related to the Bianchi identity of pseudo-Riemannian geometry in the context of explicit symmetry violation in gravity [30,38,39,81–86] such as in Hořava-Lifshitz gravity [87,88] (see also Refs. [51,53,89,90]) and dRGT massive gravity [91–94].

The modified-gravity theory under consideration in Ref. [52] was shown to require an extended Gibbons-Hawking-York (GHY) boundary term [95,96] involving the nondynamical background fields. The introduction of such boundary terms [52] prevents higher-order time derivatives of the metric from occurring and, thus, they are crucial to ensure a well-defined principle of stationary action. By doing so, the Hamiltonian of the modified-gravity theory was constructed and one set of the Hamilton equations was shown to be equivalent to the modified Einstein equations in the covariant approach [31], when these are projected onto spacelike hypersurfaces Σ_t of the spacetime foliation [53].

The modern research program on spacetime boundaries in gravity was established by the pioneering works of Arnowitt, Deser, and Misner [44] as well as Choquet-Bruhat [97]. These papers laid the foundations for research on noncompact and asymptotically flat spacetimes $\mathcal{M}$, which play a significant role, in particular, in the study of stars and black holes. Furthermore, the works of Gibbons, Hawking, and York [95,96] demonstrated the importance and peculiarities of the variational formulation in gravity, which brought with it a powerful approach for analyzing the physics on spacetime boundaries.

Other contexts that provide motivation for understanding boundary terms in gravity include the dynamics of binary systems and the gravitational waves they emit [7,98,99], open inflation [100], and the search for a theory of quantum gravity [101,102]. Moreover, in the setting of the AdS/CFT correspondence it is worthwhile to mention the regularization of the action in AdS spacetimes [103,104], extended regularization methods [105] for the physical notion of mass and angular momentum [54,55], black-hole physics [106], the formal derivation of the ADM energy in the limit of asymptotically flat spacetimes [107], and extensions to nonorthogonal boundaries [108]. In general, a definition of physically meaningful conserved charges in (asymptotically flat) spacetimes requires an averaging process over spatial and temporal regions at infinity [80,109,110]. Hence, these quantities involve surface integrals demonstrating how the properties of the gravitational system on spacetime boundaries contain essential information.

In the current paper, we focus on a specific form of the spacetime boundary $\partial\mathcal{M}$, which allows us to derive the dynamical field equations and to acquire an even better knowledge of the true role of the extended GHY boundary term. We will be obtaining a new set of boundary terms depending on the extrinsic curvature k of two-dimensional hypersurfaces that give rise to a foliation of the timelike part of $\partial\mathcal{M}$. The results are applicable in the context of black-hole physics modified by the presence of SME background fields. A substantial amount of research [111–116] has already been performed in this subarea, which highlights that our approach and findings have the potential to be taken up by researchers in the future.

The paper is organized as follows. In Sec. II we introduce the modified-gravity theory focused on, recapitulate some of its properties and define the notation to be used throughout the remainder of the article. Here, we also analyze the additional contributions on the spacetime boundary that emerge due to the presence of the SME background fields. Section III is dedicated to deriving the dynamical field equations based on the findings in Sec. II. A nontrivial shift vector will be included, which generalizes previous results. Finally, our findings will be concluded on in Sec. IV. Our metric signature is $(-,+,+,+)$ and we will employ natural units with $c=1$ unless otherwise stated. As in our previous articles [52,53], the MATHEMATICA package *xTensor* [117] provides significant computational support.

II. THE EXTENDED ACTION

Consider the following modified Einstein-Hilbert (EH) action that involves a subset of coefficients of the minimal gravitational SME [30,38]:

$$S_G = S_b + S_{\substack{\text{ext}\\ \text{GHY}}}, \tag{1a}$$

with the bulk action

$$S_b = \int_{\mathcal{M}} \mathrm{d}^4x \frac{\sqrt{-g}}{2\kappa}\left[(1-u)^{(4)}R + s^{\mu\nu\,(4)}R_{\mu\nu}\right], \tag{1b}$$

and the boundary action

$$S_{\substack{\text{ext}\\ \text{GHY}}} = \oint_{\partial\mathcal{M}} \mathrm{d}^3y \frac{\varepsilon\sqrt{q}}{2\kappa}\left[2(1-u)K - s^{\mathbf{nn}}K + K_{ab}s^{ab}\right], \tag{1c}$$

with $\kappa = 8\pi G_N$. We cover the four-dimensional spacetime manifold $\mathcal{M}$ with coordinates x^μ carrying Greek indices. As customary, $g_{\mu\nu}$ is the spacetime metric and $g \coloneqq \det(g_{\mu\nu})$ its determinant. Furthermore, ${}^{(4)}R_{\mu\nu}$ denotes the Ricci tensor and ${}^{(4)}R \coloneqq {}^{(4)}R^\mu{}_\mu$ the Ricci scalar on $\mathcal{M}$. The EH action is modified by a scalar background field u and a tensor-valued one, which is called $s^{\mu\nu}$. The latter are nondynamical and lead to a breakdown of diffeomorphism invariance [52,53].

We note in passing that in the literature on the gravitational SME, the Ricci tensor ${}^{(4)}R_{\mu\nu}$ contracted with the tensor-valued background field in Eq. (1b) is usually replaced by its trace-free version, which amounts to assuming that $s^{\mu\nu}$ is traceless. This is done to avoid an ambiguity, as one may choose $s^{\mu\nu} = ug^{\mu\nu}$ having a nonvanishing trace, which will give rise to another contribution in the u sector. However, the latter choice is only reasonable in the context of dynamical background fields, as the (inverse) metric is, by definition, dynamical. Since our setting involves nondynamical background fields, we avoid such redefinitions, i.e., the u and $s^{\mu\nu}$ sectors are taken as completely independent of each other; see also Ref. [52] for a more elaborate discussion.

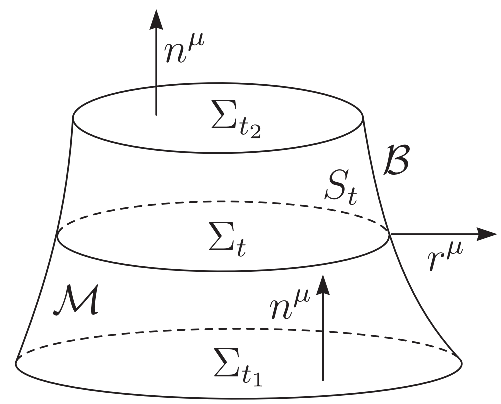

FIG. 1. Foliation of four-dimensional spacetime $\mathcal{M}$ in terms of embedded three-dimensional spacelike hypersurfaces Σ_t. The caps are formed by Σ_{t_1} and Σ_{t_2}, respectively. The mantle $\mathcal{B}$ is foliated in terms of two-dimensional hypersurfaces S_t. Also, n^μ is normal to the caps and r^μ is orthogonal to the mantle.

To render Hamilton's principle well defined, we included an extended GHY boundary term [52] where $q \coloneqq \det(q_{ab})$ is the determinant of the induced metric q_{ab} on the boundary $\partial\mathcal{M}$ of $\mathcal{M}$. Generic coordinates and indices are employed in Eq. (1c), which will be made more explicit after decomposing $\partial\mathcal{M}$ into substantially different parts below. The GHY action involves the extrinsic-curvature tensor K_{ab} and the trace of the latter, $K \coloneqq K^a{}_a = q^{ab}K_{ab}$. Moreover, $\varepsilon = \mp 1$ depending on whether $\partial\mathcal{M}$ is spacelike (timelike). Lightlike regions on $\partial\mathcal{M}$ are sets of measure zero, which do not contribute to the surface integral in Eq. (1c). Moreover, setting all SME coefficients to zero, Eq. (1) reproduces the EH action with the GHY boundary term, as expected.

Our first objective is to derive an ADM-decomposed action from Eq. (1), which will be given by Eq. (21) toward the end of the current section. The machinery and procedure employed to arrive at the latter are to be developed as follows. First of all, we focus on a spacetime $\mathcal{M}$ whose boundary $\partial\mathcal{M}$ is topologically a 3-cylinder, $\mathbb{R}\times S^2$, see Fig. 1. Let us foliate $\mathcal{M}$ in terms of spacelike hypersurfaces Σ_t such that the boundary is expressed as $\partial\mathcal{M} = \Sigma_{t_1} \cup \Sigma_{t_2} \cup \mathcal{B}$ with purely spacelike caps $\Sigma_{t_1}, \Sigma_{t_2}$ and a timelike mantle $\mathcal{B}$, according to Fig. 1. For $\Sigma_t \subset \mathcal{M}$, which also includes Σ_{t_1} and Σ_{t_2}, we consider coordinates y^a with Latin indices $a, b, c, \ldots$.

The foliation leads to a natural decomposition of the tensor-valued background field $s^{\mu\nu}$ into three independent components:

$$s^{\alpha\beta} = q^\alpha{}_\mu q^\beta{}_\nu s^{\mu\nu} - (q^\alpha{}_\nu n^\beta + q^\beta{}_\nu n^\alpha)s^{\nu\mathbf{n}} + n^\alpha n^\beta s^{\mathbf{nn}}, \tag{2}$$

where $q^\mu{}_\nu = \delta^\mu{}_\nu + n^\mu n_\nu$ projects a part of a spacetime tensor described by a single Lorentz index onto Σ_t and n^μ is a unit normal vector orthogonal to Σ_t. To define the components of this decomposition in a manner convenient for us, we introduce the set of vectors e^{μ}_{a}, which is given by

$$e^{\mu}_{a} \coloneqq \frac{\partial x^{\mu}}{\partial y^{a}}, \tag{3}$$

where the spacetime coordinates are understood to be parameterized as $x^{\mu}(y^{a})$. Note that the objects e^{μ}_{a} govern pullback operations of covariant tensor fields [56,59,118] that exist due to the embedding of Σ_t into $\mathcal{M}$. With this in mind, we define the tensor-valued purely spacelike part s^{ab} of the background field through the relation

$$q^{\alpha}{}_{\mu}q^{\beta}{}_{\nu}s^{\mu\nu} \eqqcolon e^{\alpha}_{a}e^{\beta}_{b}s^{ab}, \tag{4a}$$

and the scalar purely timelike contribution by

$$s^{\mathbf{nn}} \coloneqq s^{\mu\nu}n_{\mu}n_{\nu}. \tag{4b}$$

Since $s^{\mu\nu}$ and s^{ab} are contravariant, by construction, Eq. (4a) cannot simply be solved for s^{ab}. Hence, s^{ab} is defined implicitly by Eq. (4a) and the right-hand side of this relation can be interpreted as the pushforward of s^{ab} from Σ_t into $\mathcal{M}$; see Eq. (16.10) in Ref. [118]. Then, s^{ab} is understood as $s^{\mu\nu}$ suitably restricted to Σ_t by the application of two vectors of Eq. (3). It is also helpful to recall that $q^{\alpha\beta} = e^{\alpha}_{a}e^{\beta}_{b}q^{ab}$, i.e., q^{ab} can be lifted to $\mathcal{M}$ by a pushforward operation.

In principle, Eq. (2) also contains a vector-valued mixed piece given by $s^{\mu\mathbf{n}} \eqqcolon e^{\mu}_{a}s^{a\nu}n_{\nu}$, but the latter can be gauged away at first order in the coefficients [52], which is why we will discard $s^{\mu\mathbf{n}}$ in the following. Note also that in Ref. [52] we did not find any GHY-like boundary term associated with $s^{\mu\mathbf{n}}$, cf. Eq. (1c).

Moreover, it is reasonable to distinguish between quantities defined on spacelike and timelike hypersurfaces, respectively, via different sets of indices. Therefore, let us introduce the following different submanifolds with their corresponding coordinates. For the mantle $\mathcal{B} \subset \partial\mathcal{M}$ we use coordinates z^{i} and Latin indices $i, j, k, \ldots$. For the closed two-surface $S_t \subset \Sigma_t$, which is the boundary of Σ_t, we employ coordinates θ^{A} and capital Latin indices $A, B, C, \ldots$.

The boundary action of Eq. (1c) is then decomposed as

$$S_{\underset{\mathrm{GHY}}{\mathrm{ext}}} = S_{-\Sigma_{t_1}} + S_{\Sigma_{t_2}} + S_{\mathcal{B}}, \tag{5a}$$

with the contributions on the two caps and the mantle,

$$S_{-\Sigma_{t_1}} = \int_{\Sigma_{t_1}} \mathrm{d}^3y \frac{\sqrt{q}}{2\kappa}\left[2(1-u)K - s^{\mathbf{nn}}K + K_{ab}s^{ab}\right], \tag{5b}$$

$$S_{\Sigma_{t_2}} = -\int_{\Sigma_{t_2}} \mathrm{d}^3y \frac{\sqrt{q}}{2\kappa}\left[2(1-u)K - s^{\mathbf{nn}}K + K_{ab}s^{ab}\right], \tag{5c}$$

$$S_{\mathcal{B}} = \int_{\mathcal{B}} \mathrm{d}^3z \frac{\sqrt{-\gamma}}{2\kappa}\left[2(1-u)\mathcal{K} + \mathcal{K}_{ij}s^{ij}\right], \tag{5d}$$

where we follow the conventions of Refs. [54,55], i.e., the normal n^{μ} of Σ_{t_1} is chosen as future-directed, by definition, as is the normal of Σ_{t_2}, cf. Fig. 1. Thus, to have an outward-directed normal for Σ_{t_1}, which is the generic situation for the remainder of the boundary, an additional relative sign must be considered in comparison to Σ_{t_2}, which eliminates the factor $\varepsilon = -1$ from Eq. (1c).

Also, the extrinsic curvature is defined appropriately on each hypersurface. In particular, on $\mathcal{B}$ we define the induced metric $\gamma_{ij} \coloneqq g_{\alpha\beta}e^{\alpha}_{i}e^{\beta}_{j}$ with $e^{\alpha}_{i} \coloneqq \partial x^{\alpha}/\partial z^{i}$. We choose r^{a} to be the unit normal to S_t with associated four-vector $r^{\alpha} = r^{a}e^{\alpha}_{a}$ and the set of vectors e^{α}_{a} introduced previously in Eq. (3). Note that $r^{\alpha}n_{\alpha} = 0$, as r^{α} is understood to live in Σ_t; cf. Fig. 1. Furthermore, we define the extrinsic-curvature tensor on $\mathcal{B}$ as $\mathcal{K}_{ij} \coloneqq e^{\alpha}_{i}e^{\beta}_{j}\nabla_{\beta}r_{\alpha}$ where $\mathcal{K} \coloneqq \mathcal{K}^{i}_{i} = \gamma_{ij}\mathcal{K}^{ij}$ is its corresponding trace. The covariant derivative ∇_{μ} is compatible with the metric $g_{\mu\nu}$ of $\mathcal{M}$.

Note also that, in principle, Eq. (5d) would contain a term proportional to $s^{\mathbf{rr}}\mathcal{K}$ with $s^{\mathbf{rr}} \coloneqq s^{\mu\nu}r_{\mu}r_{\nu}$. However, since $s^{\mathbf{nn}}$ provides a nonvanishing contribution for a timelike normal vector n^{μ} by its definition via Eq. (4b), it must hold that $s^{\mathbf{rr}} = 0$ for a spacelike normal vector r^{μ} due to $r^{\mu}n_{\mu} = 0$. The purely spacelike components of $s^{\mu\nu}$ are already contained in the term $\mathcal{K}_{ij}s^{ij}$ in Eq. (5d), which is an implication of the way how $s^{\mu\nu}$ is decomposed in the foliation according to Eq. (2).

The key part of the forthcoming analysis is to focus on contributions providing total derivatives compatible with $g_{\mu\nu}$ and q_{ab}, respectively. We will find that the latter only occur for s^{ab}, which makes sense, as these coefficients result from restricting $s^{\mu\nu}$ to the purely spacelike hypersurfaces Σ_t.

Let us now consider the decompositions (see, e.g., Refs. [56,59])

$${}^{(4)}R = R + K^2 - K_{ab}K^{ab} - 2R_{\mathbf{nn}}, \tag{6a}$$

$$R_{\mathbf{nn}} = K^2 - K_{ab}K^{ab} + \nabla_{\mu}\zeta^{\mu}, \tag{6b}$$

$$\begin{aligned} q^{\beta}{}_{\nu}q^{\delta}{}_{\sigma}{}^{(4)}R_{\beta\delta} &= R_{\nu\sigma} + \nabla_{\mu}\psi^{\mu}{}_{\nu\sigma} - a^{\beta}n_{\nu}K_{\beta\sigma} \\ &\quad - a^{\delta}n_{\sigma}K_{\nu\delta} - a_{\nu}a_{\sigma} - D_{\nu}a_{\sigma}, \end{aligned} \tag{6c}$$

where $R \coloneqq R^{a}{}_{a} = q^{ab}R_{ab}$ is the Ricci scalar obtained from the trace of the Ricci tensor R_{ab} on Σ_t. Moreover, D_{μ} is the covariant derivative compatible with the induced metric $q_{\mu\nu}$ and $a_{\alpha} \coloneqq n^{\mu}\nabla_{\mu}n_{\alpha}$ denotes the ADM acceleration. In Eq. (6c) these quantities have been lifted to $\mathcal{M}$, but a pullback onto Σ_t can be performed via $D_{a}v_{b} = e^{\alpha}_{a}e^{\beta}_{b}D_{\alpha}v_{\beta}$ and $a_{c} = e^{\alpha}_{c}a_{\alpha}$ with the vectors of Eq. (3). We also defined the vector

$$\zeta^\mu := n^\nu \nabla_\nu n^\mu - K n^\mu, \tag{7a}$$

convenient to be used in Eq. (6b), as well as the third-rank tensor

$$\psi^\mu{}_{\nu\sigma} := n^\mu K_{\nu\sigma}, \tag{7b}$$

which occurs in Eq. (6c). We emphasize that four-divergences of the latter quantities with $g_{\mu\nu}$-compatible derivatives can be found in Eq. (6). These will play an important role below.

By applying the $(3+1)$ decomposition of the intrinsic curvature encoded in Eq. (6) as well as the decomposition of $s^{\mu\nu}$ in Eq. (2)—with the mixed coefficients omitted—to Eq. (1b) the bulk action can be ADM-decomposed as

$$S_b = \int_{\mathcal{M}} \mathrm{d}^4x \frac{\sqrt{-g}}{2\kappa} \Big[(1-u)(R - K^2 + K_{ab}K^{ab} - 2\nabla_\mu \zeta^\mu) + s^{\mathbf{nn}} n^\mu n^{\nu\,(4)}R_{\mu\nu} + s^{\mu\nu} q^\beta{}_\mu q^\delta{}_\nu{}^{(4)}R_{\beta\delta}\Big]. \tag{8}$$

Hence, the ADM decomposition of the EH action is scaled by the factor of $1-u$. Furthermore, the decomposition of $s^{\mu\nu}$ into purely timelike and spacelike parts, respectively, is evident. To rewrite the last two terms, we benefit from Eqs. (4), (6b), and (6c) leading to

$$S_b = \int_{\mathcal{M}} \mathrm{d}^4x \frac{\sqrt{-g}}{2\kappa} \Big[(1-u)(R - K^2 + K_{ab}K^{ab} - 2\nabla_\mu \zeta^\mu) + s^{\mathbf{nn}}(K^2 - K_{ab}K^{ab} + \nabla_\mu \zeta^\mu) + s^{ab}(R_{ab} + e^\nu_a e^\sigma_b \nabla_\mu \psi^\mu{}_{\nu\sigma} - a_a a_b - D_a a_b)\Big], \tag{9}$$

which has now been expressed completely in terms of the components of $s^{\mu\nu}$ defined in Eqs. (4a), (4b). Then, the key terms giving rise to total derivatives in the bulk action (9) are given by

$$S_b \supset \int_{\mathcal{M}} \mathrm{d}^4x \frac{\sqrt{-g}}{2\kappa} \Big\{[-2(1-u) + s^{\mathbf{nn}}]\nabla_\mu \zeta^\mu + s^{\nu\sigma}\nabla_\mu \psi^\mu{}_{\nu\sigma} - s^{ab} D_a a_b\Big\}. \tag{10}$$

Note that the last term even contains a q_{ab}-compatible covariant derivative, which is a property not to be encountered in the EH action. Expressing these contributions as total covariant derivatives plus suitable correction terms amounts to

$$S_b \supset \int_{\mathcal{M}} \mathrm{d}^4x \frac{\sqrt{-g}}{2\kappa} \Big\{\nabla_\mu[-2(1-u)\zeta^\mu + s^{\mathbf{nn}}\zeta^\mu] - \zeta^\mu \nabla_\mu(2u + s^{\mathbf{nn}}) + \nabla_\mu(s^{ab}\psi^\mu{}_{ab}) - \psi^\mu{}_{\nu\sigma}\nabla_\mu s^{\nu\sigma}\Big\} + \int_{\mathcal{M}} \mathrm{d}t\mathrm{d}^3y \frac{\sqrt{q}}{2\kappa}\big[-D_a(N s^{ab} a_b) + a_b D_a(N s^{ab})\big], \tag{11}$$

where the lapse function $N =: 1/\sqrt{-g^{00}}$ occurs in the second line. Here, we have used that $s^{\nu\sigma}K_{\nu\sigma} = s^{ab}K_{ab}$, which can be proven from Eq. (4a). The first and third terms of Eq. (11) now involve $g_{\mu\nu}$-compatible total derivatives, whereas the second line contains a q_{ab}-compatible total derivative. These are complemented by correction terms such that Eq. (10) can be reproduced neatly. Furthermore, the integral measure of the second line has been ADM-decomposed, since the integrand only depends on properly ADM-decomposed quantities.

Gauss' theorem transforms the total derivatives in the first and third terms of Eq. (11) into boundary terms:

$$\begin{aligned}
\int_{\mathcal{M}} \mathrm{d}^4x \frac{\sqrt{-g}}{2\kappa} \nabla_\mu \Big\{[-2(1-u) + s^{\mathbf{nn}}]\zeta^\mu + s^{ab}\psi^\mu{}_{ab}\Big\} &= \int_{(-\Sigma_{t_1})\cup\Sigma_{t_2}} \mathrm{d}^3y \frac{\sqrt{q}}{2\kappa}(-n_\mu)\Big\{[-2(1-u) + s^{\mathbf{nn}}]\zeta^\mu + s^{ab}\psi^\mu{}_{ab}\Big\} \\
&\quad + \int_{\mathcal{B}} \mathrm{d}^3z \frac{\sqrt{-\gamma}}{2\kappa} r_\mu[-2(1-u)\zeta^\mu + s^{ab}\psi^\mu{}_{ab}] \\
&= \int_{(-\Sigma_{t_1})\cup\Sigma_{t_2}} \mathrm{d}^3y \frac{\sqrt{q}}{2\kappa}\Big\{[2(1-u) - s^{\mathbf{nn}}]K + s^{ab}K_{ab}\Big\} \\
&\quad + \int_{\mathcal{B}} \mathrm{d}^3z \frac{\sqrt{-\gamma}}{2\kappa}[-2(1-u) r_\mu n^\nu \nabla_\nu n^\mu],
\end{aligned} \tag{12}$$

where we employed the definitions of Eq. (7). We also benefited from the basic properties $n^2 = -1$, $a_\mu n^\mu = 0$, and $r_\mu n^\mu = 0$ as well as $s^{\mathbf{rr}} = 0$ on $\mathcal{B}$. Note that the general directed volume element on $\partial\mathcal{M}$ reads $\mathrm{d}\Sigma_\mu = \varepsilon n_\mu \mathrm{d}\Sigma$ where $\mathrm{d}\Sigma$ is the volume element of the hypersurface, see Eq. (3.16) in Refs. [54,55]. Therefore, as n^μ is a future-directed vector according to the conventions, cf. Fig. 1, proper care has to be taken to compensate the additional sign for Σ_{t_1} that we introduced in Eq. (5b). Since Eq. (7a) depends on a^μ and $r_\mu a^\mu \neq 0$, the contribution on $\mathcal{B}$ stated in the last line of Eq. (12) survives.

Now, the parts of Eq. (12) given by the two terms on the spacelike caps Σ_{t_1} and Σ_{t_2} cancel Eqs. (5b) and (5c), respectively, whereas the contributions on the mantle $\mathcal{B}$ will be treated later. As the term $D_a(Ns^{ab}a_b)$ involves a q_{ab}-compatible total derivative, it provides a boundary term on $\mathcal{B}$, which is already foliated in terms of two-dimensional hypersurfaces S_t as follows:

$$\int_{\mathcal{M}} \mathrm{d}t\mathrm{d}^3y \frac{\sqrt{q}}{2\kappa}[-D_a(Ns^{ab}a_b)] = \int_{t_1}^{t_2} \mathrm{d}t \oint_{S_t} \mathrm{d}^2\theta \frac{N\sqrt{\sigma}}{2\kappa}(-r_i s^{ij}a_j). \tag{13}$$

Note that via a coordinate transformation, the indices a, b are replaced by i, j to represent coordinates on the foliated hypersurface $\mathcal{B}$. Here, $\sqrt{\sigma}$ is the integration measure on S_t that depends on the induced metric σ_{AB} on S_t. The latter will be considered in more detail below.

The total action is then of the form

$$\begin{aligned} S_G = {}& \int_{\mathcal{M}} \mathrm{d}t\mathrm{d}^3y \frac{N\sqrt{q}}{2\kappa}\Big[(1-u)(R-K^2+K_{ab}K^{ab}) + s^{\mathbf{nn}}(K^2-K_{ab}K^{ab}) + s^{ab}R_{ab} - s^{ab}a_a a_b - \zeta^\mu\nabla_\mu(2u+s^{\mathbf{nn}}) \\ & - \psi^\mu_{\nu\sigma}\nabla_\mu s^{\nu\sigma} + \frac{1}{N}a_b D_a(Ns^{ab})\Big] + \int_{\mathcal{B}} \mathrm{d}^3z \frac{\sqrt{-\gamma}}{2\kappa}\Big[-2(1-u)r_\mu n^\nu\nabla_\nu n^\mu\Big] - \int_{t_1}^{t_2}\mathrm{d}t \oint_{S_t} \mathrm{d}^2\theta \frac{N\sqrt{\sigma}}{2\kappa} r_i a_j s^{ij} \\ & + \int_{\mathcal{B}} \mathrm{d}^3z \frac{\sqrt{-\gamma}}{2\kappa}\big[2(1-u)\mathcal{K} + \mathcal{K}_{ij}s^{ij}\big], \end{aligned} \tag{14}$$

where the second and third to last integrals contain the terms on the mantle $\mathcal{B}$ that remain after applying Gauss' theorem to each of the total derivatives in Eq. (11). As mentioned before, the penultimate contribution is already foliated properly in terms of S_t, which will be helpful in the following. Note that the terms on $\mathcal{B}$ do not simply cancel with the original boundary action $S_{\mathcal{B}}$ of Eq. (5d), which has been reinstated explicitly as the last term of Eq. (14). A more sophisticated treatment of these contributions is indispensable, though.

Moreover, the bulk of Eq. (14) now depends on $g_{\mu\nu}$-compatible directional derivatives of the SME coefficients. It is beneficial to express these in terms of the ADM acceleration a_c defined on Σ_t and Lie derivatives [119] with respect to the vector $m^\mu \coloneqq Nn^\mu$:

$$\zeta^\mu\nabla_\mu u = a^c D_c u - \frac{K}{N}\mathcal{L}_m u, \tag{15a}$$

$$\zeta^\mu\nabla_\mu s^{\mathbf{nn}} = a^c D_c s^{\mathbf{nn}} - \frac{K}{N}\mathcal{L}_m s^{\mathbf{nn}}, \tag{15b}$$

$$\psi^\mu{}_{\nu\sigma}\nabla_\mu s^{\nu\sigma} = \frac{1}{N}K_{ab}\mathcal{L}_m s^{ab} + 2K_{ab}K^a{}_c s^{cb}. \tag{15c}$$

The occurrence of Lie derivatives $\mathcal{L}_m$ of the background fields is characteristic when the ADM formalism is applied to sectors of the gravitational SME [52,53]. For Eq. (15c) it is important to take into account that the Lie derivative along m^μ of a quantity living in Σ_t remains in Σ_t.

The next step is to investigate the contributions on $\mathcal{B}$. We intend to combine the second and third to last terms of Eq. (14) with the last one. To accomplish this endeavor, we introduce the quantity $\Upsilon \coloneqq \mathcal{K} - r_\mu n^\nu\nabla_\nu n^\mu$ that is to be suitably reformulated by the following chain of reasoning [54,55]:

$$\begin{aligned} \Upsilon &= \mathcal{K} + (\nabla_\nu r_\mu)n^\mu n^\nu \\ &= \nabla_\nu r_\mu(g^{\mu\nu} - r^\mu r^\nu + n^\mu n^\nu). \end{aligned} \tag{16}$$

To arrive at this result, several ingredients are valuable. First, we benefit from the identity $r_\mu\nabla_\nu n^\mu = -n^\mu\nabla_\nu r_\mu$, which follows from $r^\mu n_\mu = 0$. Second, the trace $\mathcal{K}$ of the extrinsic curvature on $\mathcal{B}$ is expressed in terms of the metric on $\mathcal{M}$ and r^μ, which is the unit normal of $\mathcal{B}$. This is possible, as $\mathcal{B}$ is a timelike hypersurface embedded into $\mathcal{M}$:

$$\begin{aligned} \mathcal{K} &= \gamma^{ij}\mathcal{K}_{ij} = \gamma^{ij}(\nabla_\nu r_\mu e^\mu_i e^\nu_j) = (\nabla_\nu r_\mu)\gamma^{ij}e^\mu_i e^\nu_j \\ &= \nabla_\nu r_\mu(g^{\mu\nu} - r^\mu r^\nu). \end{aligned} \tag{17}$$

We continue by interpreting S_t as being embedded into $\mathcal{M}$, which implies the induced metric $\sigma_{AB} = g_{\alpha\beta}e^\alpha_A e^\beta_B$ on S_t with the set of vectors $e^\alpha_A \coloneqq \partial x^\alpha/\partial\theta^A$. Then,

$$\Upsilon = \nabla_\nu r_\mu(\sigma^{AB}e^\mu_A e^\nu_B) = \sigma^{AB}(\nabla_\nu r_\mu e^\mu_A e^\nu_B). \tag{18}$$

Last but not least, due to their embedding into $\mathcal{M}$, the two-dimensional hypersurfaces S_t also have an extrinsic curvature associated with them as do Σ_t and $\mathcal{B}$. The latter is frequently denoted as k_{AB} in the literature [54,55] where $k \coloneqq k_{AB}\sigma^{AB}$ is its corresponding trace. Making use of that, we finally obtain

$$\Upsilon = \sigma^{AB} k_{AB} = k. \tag{19}$$

To handle the terms in Eq. (14) on $\mathcal{B}$ containing s^{ij}, consider $k_{AB} \coloneqq e^i_A e^j_B k_{ij}$ with $e^i_A \coloneqq \partial z^i/\partial\theta^A$ based on the embedding of S_t into $\mathcal{B}$. So,

$$\begin{aligned} k_{ij} &= \tilde{D}_i r_j = \sigma^k{}_i \sigma^l{}_j D_k r_l = \sigma^k{}_i D_k r_j \\ &= (\delta^k{}_i - r_i r^k) D_k r_j = \mathcal{K}_{ij} - r_i a_j, \end{aligned} \tag{20}$$

where $\tilde{D}_i$ is the covariant derivative compatible with σ_{ij} and $\sigma^k{}_i = \delta^k{}_i - r^k r_i$ projects a part of a tensor living in Σ_t onto S_t. This object is analogous to the projector $q^\mu{}_\nu$ introduced in Eq. (2) that is responsible for projections from $\mathcal{M}$ onto Σ_t. Note the sign difference in the second terms on the right-hand sides of the definitions of $q^\mu{}_\nu$ and $\sigma^k{}_i$, which is due to n^μ being timelike and r^μ being spacelike. Moreover, $r^l D_k r_l = 0$ is employed in the first line of Eq. (20). So the latter equation means that the extrinsic curvature k_{AB} of S_t lifted to $\mathcal{B}$ is expressed through the extrinsic curvature $\mathcal{K}_{ij}$ of $\mathcal{B}$.

Finally, we foliate $\mathcal{B}$ in terms of S_t and recast the corresponding integral measure into the following form: $\mathrm{d}^3z\sqrt{-\gamma} = \mathrm{d}t\mathrm{d}^2\theta N\sqrt{\sigma}$. Now it makes sense to define the tensor field s^{AB}, which can be interpreted as s^{ij} restricted to S_t. The former exists because of the embedding of S_t into $\mathcal{B}$. We introduce s^{AB} in a manner analogous to how we defined s^{ab} implicitly as $s^{\mu\nu}$ restricted to Σ_t via Eq. (4a). The defining relationship is $\sigma^i{}_m \sigma^j{}_n s^{mn} \eqqcolon e^i_A e^j_B s^{AB}$ with the vectors defined directly above Eq. (20). As a consequence, $k_{ij}s^{ij} = k_{AB}s^{AB}$ can be deduced on S_t. We then arrive at the final form of the ADM-decomposed action, which is one of the central results of the current work:

$$S_G = \int_{t_1}^{t_2} \mathrm{d}t (L_b + B_S), \tag{21a}$$

with the Lagrangian in the bulk,

$$L_b = \int_{\Sigma_t} \mathrm{d}^3y \mathcal{L}_b, \tag{21b}$$

$$\begin{aligned} \mathcal{L}_b = \frac{N\sqrt{q}}{2\kappa}\bigg[&(1-u)(R - K^2 + K_{ab}K^{ab}) \\ &+ s^{\mathbf{nn}}(K^2 - K_{ab}K^{ab}) + s^{ab}R_{ab} \\ &- \frac{1}{N}K_{ab}\mathcal{L}_m s^{ab} - 2K_{ab}K^a{}_c s^{cb} + a_b D_a s^{ab} \\ &- a^c D_c(2u + s^{\mathbf{nn}}) + \frac{K}{N}(2\mathcal{L}_m u + \mathcal{L}_m s^{\mathbf{nn}})\bigg], \end{aligned} \tag{21c}$$

and the boundary term

$$B_S = \oint_{S_t} \mathrm{d}^2\theta \frac{N\sqrt{\sigma}}{2\kappa}[2(1-u)k + k_{AB}s^{AB}]. \tag{21d}$$

Let us summarize what we did. We ADM-decomposed the modified EH action stated in Eq. (1b) including the extended GHY boundary term of Eq. (1c). The latter was shown to partially cancel with boundary terms arising from total covariant derivatives in the bulk action. A piece of the extended GHY boundary term evaluated on the two-dimensional hypersurfaces S_t remained. This part, which is given by Eq. (21d), was expressed completely in terms of quantities living in S_t.

Now, the resulting bulk Lagrangian of Eq. (21b) involves four classes of terms. First, there are contributions depending on R and R_{ab}, i.e., they encode the intrinsic geometry of Σ_t. Such terms occur in GR, the u, and the s^{ab} sectors, but not for $s^{\mathbf{nn}}$. Second, terms quadratic in the extrinsic curvature or its trace are found for all sectors. Third, each sector comes with a Lie derivative of the corresponding SME coefficients along m^μ. Last but not least, there are three contributions involving the ADM acceleration. Note that the surface term of Eq. (21d) does not depend on the background field $s^{\mathbf{nn}}$, which is closely related to the observation of there being no term of the form $s^{\mathbf{nn}}R$ in L_b. This property is to be explained in more detail below.

III. PALATINI METHOD OF VARIATION

Our recent work [53] is dedicated to a derivation of the modified Einstein equations based on Eq. (1) by resorting to the Hamiltonian formulation of this theory. Our incentive was to verify whether or not the Hamiltonian approach gives rise to the same dynamics as does the covariant formulation. The reply to this question was found to be in the affirmative, i.e., both approaches can be neatly connected to each other.

In the following, we intend to derive the dynamical field equations again, but this time by using a different approach that incorporates a detailed analysis of the boundary terms. Such a treatment can be beneficial in the future to explore the limit of asymptotic flatness. Besides, as an extension of Ref. [53], we now allow for a nonzero shift vector N^a where $N_a \eqqcolon g_{0a}$. Doing so poses a natural next step, as the shift vector is needed to change coordinates when going from one spatial hypersurface Σ_t to the next.

Studying a dynamical process in numerical relativity, e.g., the frame dragging effect of a Kerr black hole or the collapse of a star into a black hole, one finds that coordinates can get twisted such that coordinate singularities and even physical singularities may arise. There exists a gauge known as minimal distortion [56,59,120] that relies on the shift vector as a means to compensate the twisting of coordinate lines. Thus, to be able to treat gravity systems numerically, a nonzero shift vector seems indispensable.

Now, we will dedicate ourselves to the dynamics of the modified-gravity theory stated in Eq. (1). To do so, we consider the Palatini method of variation [45], in which coordinate and momentum variables are treated as independent. In our case, the Palatini action is expressed in terms of a generic induced metric q_{ab} and the corresponding canonical momentum Π^{ab} as

$$S_G = \int_{t_2}^{t_1} \mathrm{d}t \left[\int_{\Sigma_t} \Pi^{ab} \dot{q}_{ab} \mathrm{d}^3y - H_G(\Pi^{ab}, q_{ab}) \right]. \tag{22}$$

Here, H_G is the total Hamiltonian containing a boundary term that may be of the form of Eq. (21d). By considering

$$\delta H_G = \int_{\Sigma_t} \mathrm{d}^3y (\mathcal{P}^{ab} \delta q_{ab} + \mathcal{F}_{ab} \delta \Pi^{ab}), \tag{23}$$

a variation of the action in Eq. (22) leads to

$$\delta S_G = \int_{t_1}^{t_2} \mathrm{d}t \int_{\Sigma_t} \mathrm{d}^3y \Big[(\dot{q}_{ab} - \mathcal{F}_{ab}) \delta \Pi^{ab} - \left(\dot{\Pi}^{ab} + \mathcal{P}^{ab} \right) \delta q_{ab} \Big], \tag{24}$$

where we have discarded a contribution that arises from an integration by parts, since $\delta q_{ab} = 0$ on the boundary, by definition. In principle, the above variation may lead to nonvanishing boundary terms depending on covariant derivatives of δq_{ab} along the normal direction r^c of S_t. Also, one may have to include boundary terms already contained in the action. In a rigorous treatment, each of these contributions should be kept track of in the derivation.

Now, the requirement that the action be stationary implies the following field equations in the Palatini formalism:

$$\dot{q}_{ab} = \mathcal{F}_{ab}, \tag{25a}$$

$$\dot{\Pi}^{ab} = -\mathcal{P}^{ab}. \tag{25b}$$

It is challenging to invert the extrinsic curvature for the canonical momentum, i.e., to compute the Hamiltonian H_G when all SME coefficients $u, s^{\mathbf{nn}}$, and s^{ab} are present simultaneously. Therefore, we will be restricting ourselves to three separate analyses below, as we already did in previous works [52,53,79].

On the one hand, in each of these cases, Eq. (25a) gives rise to the generic geometric identity [45,48,50]

$$\dot{q}_{ab} = 2NK_{ab} + D_a N_b + D_b N_a, \tag{26}$$

which, in principle, corresponds to the definition of the extrinsic curvature. This relation remains unmodified, even in the presence of u and $s^{\mu\nu}$, since the geometric setting is still pseudo-Riemannian geometry.

On the other hand, Eq. (25b) encodes the dynamics of the modified-gravity theory under study. Thus, we will focus on the latter, as it describes how gravitational dynamics is affected by diffeomorphism violation. However, if we were working with Eq. (25b) directly, there would be no chance of taking into account possible boundary terms. Instead, in what follows, we will cast the action of Eq. (21) into the form of Eq. (24) and compute the variation of the Hamiltonian H_G for q_{ab}. After taking proper care of boundary terms, the integral over Σ_t can be dropped, which leads us automatically to Eq. (25b), evaluated for the specific sector explored.

A. Dynamics in the u sector

First, we focus on the u sector, i.e., let $\mathcal{L}_u$ be the Lagrange density following from Eq. (21c) such that $\mathcal{L}_u \coloneqq \mathcal{L}_b|_{s^{\mathbf{nn}}=s^{ab}=0}$. The canonical momentum then reads

$$\pi^{ab} \coloneqq \frac{\partial \mathcal{L}_u}{\partial \dot{q}_{ab}} = \frac{\sqrt{q}}{2\kappa} \left[(1-u)(K^{ab} - q^{ab}K) + \frac{1}{N} q^{ab} \mathcal{L}_m u \right]. \tag{27}$$

Recall that the extrinsic curvature proper is the standard quantity of pseudo-Riemannian geometry; cf. Eq. (26). However, relationships between canonical variables and geometrical quantities are affected by diffeomorphism violation, which is observed here.

The ADM-decomposed action $S_{G,u}$ is

$$S_{G,u} = \int_{t_1}^{t_2} \mathrm{d}t \int_{\Sigma_t} \mathrm{d}^3y \mathcal{L}_u. \tag{28}$$

To apply the Palatini formalism, the latter must be expressed in terms of a Hamiltonian via an inverse Legendre transformation:

$$S_{G,u} = \int_{t_1}^{t_2} \mathrm{d}t \left(\int_{\Sigma_t} \pi^{ab} \dot{q}_{ab} \mathrm{d}^3y - H_{\Sigma,u} + B_{S,u} \right), \tag{29a}$$

with the bulk Hamiltonian

$$H_{\Sigma,u} = \int_{\Sigma_t} \mathrm{d}^3y \Bigg\{ -\frac{N\sqrt{q}}{2\kappa} [(1-u)R - 2a^c D_c u] + \frac{2\kappa N}{\sqrt{q}(1-u)} \left(\pi^{ab}\pi_{ab} - \frac{\pi^2}{2} \right) + \frac{\mathcal{L}_m u}{1-u} \left(\pi - \frac{3}{4} \frac{\sqrt{q}}{\kappa N} \mathcal{L}_m u \right) + 2\pi^{ab} D_a N_b \Bigg\}, \tag{29b}$$

and the boundary term of Eq. (21d) restricted to u:

$$B_{S,u} = \oint_{S_t} \mathrm{d}^2\theta \frac{N\sqrt{\sigma}}{\kappa} (1-u)k. \tag{29c}$$

Computational details on how to actually obtain a bulk Hamiltonian are provided by Ref. [52]. In general, the

Hamilton density follows from the Lagrange density via a Legendre transformation where the generalized velocities are expressed in terms of the canonical momenta. In particular, for the Hamiltonian stated in Eq. (29b), we refer to Sec. IV. B. 5 of the latter paper. The crucial difference between both derivations is that we avoided integrations by parts in our current article. This becomes evident in the second and the last terms of Eq. (29b) when compared to the Hamiltonian in Eq. (68) of Ref. [52].

In the following, we intend to evaluate the variations for q_{ab} of each term in $H_{\Sigma,u}$ that contributes to the action of Eq. (29). The variation of the contribution including the Ricci tensor requires an integration by parts, which generates a nonvanishing boundary term on S_t. We proceed to present the calculation with some detail. Although the latter bears many similarities with the corresponding computation done in GR, its exposition is still expected to be worthwhile for the reader, as it may serve as a foundation to understand the more intricate analysis in the s^{ab} sector to be done later. As a warm-up, it is useful to consider the variation

$$\delta(N\sqrt{q}R) = -N\sqrt{q}G^{ab}\delta q_{ab} - N\sqrt{q}(D_cN)\delta V^c + \sqrt{q}D_c(N\delta V^c), \tag{30}$$

where we defined the contravariant Einstein tensor on Σ_t by $G^{ab} \coloneqq R^{ab} - (R/2)q^{ab}$. Moreover, we introduced the quantity

$$\delta V^c = q^{ab}\delta\Gamma^c{}_{ab} - q^{ac}\delta\Gamma^b{}_{ab}, \tag{31}$$

which includes variations of the Christoffel symbols $\Gamma^c{}_{ab}$ of Σ_t. They can be expressed in terms of variations of the corresponding induced metric:

$$\delta\Gamma^c{}_{ab} = \frac{1}{2}q^{cf}(D_a\delta q_{fb} + D_b\delta q_{fa} - D_f\delta q_{ab}). \tag{32}$$

By taking into account that tangential derivatives of δq_{ab} vanish on S_t, an integration by parts provides

$$\delta(N\sqrt{q}R) = \sqrt{q}(-NG^{ab} + D^aD^bN - q^{ab}D_cD^cN)\delta q_{ab} + \sqrt{q}D_c(N\delta V^c). \tag{33}$$

Applying these ingredients to the u sector leads to

$$\begin{aligned}\delta\int_{\Sigma_t}\mathrm{d}^3y\,\frac{N\sqrt{q}}{2\kappa}(1-u)R &= \int_{\Sigma_t}\mathrm{d}^3y\,\frac{\sqrt{q}}{2\kappa}\Big(-N(1-u)G^{ab} + D^aD^b[(1-u)N] \\ &\quad - q^{ab}D_cD^c[(1-u)N]\Big)\delta q_{ab} \\ &\quad + \int_{\Sigma_t}\mathrm{d}^3y\,\frac{\sqrt{q}}{2\kappa}D_c[N(1-u)\delta V^c].\end{aligned} \tag{34}$$

The last integral on the right-hand side can be evaluated with Gauss' theorem to provide

$$\int_{\Sigma_t}\mathrm{d}^3y\,\frac{\sqrt{q}}{2\kappa}D_c[N(1-u)\delta V^c] = \oint_{S_t}\mathrm{d}^2\theta\,\frac{N\sqrt{\sigma}}{2\kappa}(1-u)r_c\delta V^c. \tag{35}$$

A derivative of δq_{ab} in the normal direction, which is nonzero, occurs on the right-hand side of the latter relationship. In particular,

$$r_c\delta V^c = -\sigma^{ab}r^cD_c\delta q_{ab}, \tag{36}$$

such that we arrive at

$$\begin{aligned}\delta\int_{\Sigma_t}\mathrm{d}^3y\,\frac{N\sqrt{q}}{2\kappa}(1-u)R &= \int_{\Sigma_t}\mathrm{d}^3y\,\frac{\sqrt{q}}{2\kappa}\Big\{-N(1-u)G^{ab} + D^aD^b[(1-u)N] - q^{ab}D_cD^c[(1-u)N]\Big\}\delta q_{ab} - \delta B_{R,u} \\ &= \int_{\Sigma_t}\mathrm{d}^3y\,\frac{N\sqrt{q}}{2\kappa}\Big\{-(1-u)G^{ab} + (D^a+a^a)[(D^b+a^b)(1-u)] \\ &\quad - q^{ab}(D_c+a_c)[(D^c+a^c)(1-u)]\Big\}\delta q_{ab} - \delta B_{R,u},\end{aligned} \tag{37a}$$

with a boundary term denoted as $\delta B_{R,u}$, which reads

$$\delta B_{R,u} = \oint_{S_t}\mathrm{d}^2\theta\,\frac{N\sqrt{\sigma}}{2\kappa}(1-u)\sigma^{ab}r^cD_c\delta q_{ab}. \tag{37b}$$

Here, we employed the valuable relationship $Na_c = D_cN$ relating the ADM acceleration a_c to the lapse function N, see, e.g., Ref. [56,59]. Thus, terms of the form $D_c + a_c$ arise when covariant derivatives act on products of the lapse function N and functions of the background field such as the combination $1-u$.

The variation of the term in Eq. (29b) depending only on the ADM acceleration provides

$$\delta\int_{\Sigma_t}\mathrm{d}^3y\,\frac{N\sqrt{q}}{\kappa}(-a^cD_cu)=\int_{\Sigma_t}\mathrm{d}^3y\,\frac{N\sqrt{q}}{2\kappa}(-q^{ab}a_cD^cu+2a^{(a}D^{b)}u)\delta q_{ab}. \tag{38}$$

Whenever a pair of indices in a tensor is enclosed by parentheses, the object is understood to be symmetrized in these indices.

We continue with the remaining variations necessary. Recall that the Lie derivative of a generic tensor field with respect to a vector field is a measure of how the tensor field changes infinitesimally due to the flux defined by the vector field. Therefore, the Lie derivative of a generic tensor field can be defined without resorting to any (pseudo-)Riemannian metric [119]. So the variation of the Lie derivative of u along m^μ vanishes:

$$\delta\int_{\Sigma_t}\mathrm{d}^3y\,\mathcal{L}_mu=0. \tag{39}$$

Thus, we infer that

$$\delta\int_{\Sigma_t}\mathrm{d}^3y\,\frac{\mathcal{L}_mu}{1-u}\left(\pi-\frac{3}{4}\frac{\sqrt{q}}{\kappa N}\mathcal{L}_mu\right)=\int_{\Sigma_t}\mathrm{d}^3y\,\frac{\mathcal{L}_mu}{1-u}\left(\pi^{ab}-\frac{3}{8}\frac{\sqrt{q}}{\kappa N}q^{ab}\mathcal{L}_mu\right)\delta q_{ab}. \tag{40}$$

Furthermore, for the canonical-momentum terms,

$$\delta\int_{\Sigma_t}\mathrm{d}^3y\,\frac{2\kappa N}{\sqrt{q}(1-u)}\left(\pi^{cd}\pi_{cd}-\frac{\pi^2}{2}\right)=\int_{\Sigma_t}\mathrm{d}^3y\,\frac{2\kappa N}{\sqrt{q}(1-u)}\left[2\pi^{ac}\pi^b{}_c-\pi\pi^{ab}-\frac{1}{2}\left(\pi_{cd}\pi^{cd}-\frac{\pi^2}{2}\right)q^{ab}\right]\delta q_{ab}. \tag{41}$$

Next, the contribution depending on the components of the shift vector provides

$$\delta\int_{\Sigma_t}\mathrm{d}^3y\,\pi^{cd}D_cN_d=\int_{\Sigma_t}\mathrm{d}^3y\,\frac{\sqrt{q}}{2}D_c\left(\frac{2N^{(a}\pi^{b)c}-\pi^{ab}N^c}{\sqrt{q}}\right)\delta q_{ab}. \tag{42}$$

Finally, the variation of the boundary term in the action, Eq. (29c), remains to be computed:

$$\delta B_{S,u}=\oint_{S_t}\mathrm{d}^2\theta\,\frac{N\sqrt{\sigma}}{\kappa}(1-u)\delta k, \tag{43a}$$

with the variation of the extrinsic-curvature scalar k on S_t, which can be expressed as

$$\delta k=\delta(\sigma^{ab}D_ar_b)=\frac{1}{2}\sigma^{ab}r^cD_c\delta q_{ab}. \tag{43b}$$

We see that the latter cancels the boundary term of Eq. (37b), which results from varying the contribution proportional to the Ricci scalar: $\delta B_{S,u}-\delta B_{R,u}=0$.

After canceling the boundary terms, putting together the individual pieces of Eqs. (37a) (with $\delta B_{R,u}$ discarded), (38), (40), (41), and (42) and inserting those into the second line of Eq. (24) leads to an integral of a second-rank tensor over Σ_t, which must be equal to zero. The foliation and, therefore, Σ_t is arbitrary, so is δq_{ab}. Thus, the integral is equal to zero if and only if the integrand vanishes. This line of reasoning implies the field equations:

$$\begin{aligned}\dot{\pi}^{ab}&=\frac{\sqrt{q}}{2\kappa}\Big(-N(1-u)G^{ab}+D^aD^b[(1-u)N]-q^{ab}D_cD^c[(1-u)N]\Big)\\&\quad-\frac{2\kappa N}{\sqrt{q}(1-u)}\left[2\pi^{ac}\pi^b{}_c-\pi\pi^{ab}-\frac{1}{2}\left(\pi_{cd}\pi^{cd}-\frac{\pi^2}{2}\right)q^{ab}\right]+\frac{N\sqrt{q}}{2\kappa}(2a^{(a}D^{b)}u-q^{ab}a_cD^cu)\\&\quad-\frac{\mathcal{L}_mu}{1-u}\left(\pi^{ab}-\frac{3}{8}\frac{\sqrt{q}}{\kappa N}q^{ab}\mathcal{L}_mu\right)-\sqrt{q}D_c\left(\frac{2N^{(a}\pi^{b)c}-\pi^{ab}N^c}{\sqrt{q}}\right).\end{aligned} \tag{44}$$

The reader can check that the latter correctly reduce to $(\vec{q}^*\mathbf{Q})^{ij}=0$ with Eq. (30a) in Ref. [53] in the limit of $N^a=0$. These field equations are the physical part of the modified Einstein equations on $\mathcal{M}$ [31] and encode the dynamical information of the theory described by Eq. (1) with $s^{\mu\nu}=0$. Modifications of the Hamiltonian and momentum constraints from GR have been separated from the latter.

B. Dynamics in the $s^{\mathbf{nn}}$ sector

The computations are similar for the $s^{\mathbf{nn}}$ sector. We define the Lagrange density from Eq. (21c) by $\mathcal{L}_1 \coloneqq \mathcal{L}_b|_{u=s^{ab}=0}$. The canonical momentum follows from the latter as before:

$$p^{ab} \coloneqq \frac{\partial\mathcal{L}_1}{\partial\dot{q}_{ab}} = \frac{\sqrt{q}}{2\kappa}\left[(1-s^{\mathbf{nn}})(K^{ab}-q^{ab}K)+\frac{1}{2N}q^{ab}\mathcal{L}_m s^{\mathbf{nn}}\right], \tag{45}$$

i.e., the relationship between the canonical momentum and the extrinsic curvature is modified in a way quite similar to Eq. (27) for u. Then, the ADM-decomposed action is

$$S_{G,1} = \int_{t_1}^{t_2}\mathrm{d}t\int_{\Sigma_t}\mathrm{d}^3y\,\mathcal{L}_1. \tag{46}$$

To apply the Palatini formalism, the latter is cast into the following more suitable form:

$$S_{G,1} = \int_{t_1}^{t_2}\mathrm{d}t\left(\int_{\Sigma_t}p^{ab}\dot{q}_{ab}\mathrm{d}^3y - H_{\Sigma,1} + B_S\right), \tag{47a}$$

with the Hamiltonian in the bulk,

$$\begin{aligned}H_{\Sigma,1} = \int_{\Sigma_t}\mathrm{d}^3y\bigg\{&-\frac{N\sqrt{q}}{2\kappa}(R-a^cD_cs^{\mathbf{nn}}) \\ &+\frac{2\kappa N}{\sqrt{q}(1-s^{\mathbf{nn}})}\left(p^{ab}p_{ab}-\frac{p^2}{2}\right) \\ &+\frac{\mathcal{L}_m s^{\mathbf{nn}}}{2(1-s^{\mathbf{nn}})}\left(p-\frac{3}{8}\frac{\sqrt{q}}{\kappa N}\mathcal{L}_m s^{\mathbf{nn}}\right)+2p^{ab}D_aN_b\bigg\},\end{aligned} \tag{47b}$$

and the boundary term

$$B_S = \oint_{S_t}\mathrm{d}^2\theta\frac{N\sqrt{\sigma}}{\kappa}k, \tag{47c}$$

corresponding to that of GR. The principal derivation of the bulk Hamiltonian of Eq. (47b) agrees with that presented in Sec. IV. B. 4 of Ref. [52]. The canonical momentum is now denoted as p^{ab}, which is the notation taken over from Ref. [53]. As for the u sector, integrations by parts are not performed for the current analysis. Consequently, the second and last terms of Eq. (47b) differ from the corresponding contributions in Eq. (62) of Ref. [52].

Computing the variations works such as it does for u. It is even a bit simpler, since a term that multiplies the curvature scalar R with $s^{\mathbf{nn}}$ is absent. From the point of view established until this moment, this property makes perfect sense. The boundary term of Eq. (47c) does not contain a piece proportional to $s^{\mathbf{nn}}$, which would have to be canceled against the boundary term arising from the variation of R, cf. Eqs. (37b), (43) for the u sector.

The remaining variations can be computed in a manner analogous to how we did it in Eqs. (40), (41), and (42) for u. The Ricci scalar term does not involve the coefficient $s^{\mathbf{nn}}$. Therefore, the variation of this term gives rise to the same boundary term on S_t that must also be considered for the EH action. We will denote the latter as $-\delta B_R$. The total variation then reads

$$\begin{aligned}\delta\int_{\Sigma_t}\mathrm{d}^3y\frac{N\sqrt{q}}{2\kappa}R = \int_{\Sigma_t}\mathrm{d}^3y\frac{\sqrt{q}}{2\kappa}(&D^aD^bN-q^{ab}D_cD^cN \\ &-NG^{ab})\delta q_{ab}-\delta B_R,\end{aligned} \tag{48a}$$

with

$$\delta B_R = \oint_{S_t}\mathrm{d}^2\theta\frac{N\sqrt{\sigma}}{2\kappa}\sigma^{ab}r^cD_c\delta q_{ab}. \tag{48b}$$

Here, we have used the result of Eq. (37b) for $u=0$. We now immediately consider the variation of the boundary term in the action, i.e., Eq. (47c), which amounts to

$$\delta B_S = \oint_{S_t}\mathrm{d}^2\theta\frac{N\sqrt{\sigma}}{\kappa}\delta k. \tag{49}$$

By benefiting from Eq. (43b), the boundary term of Eq. (48b) compensates the variation of Eq. (49), as expected: $\delta B_S - \delta B_R = 0$.

For the contribution in Eq. (47b) depending on the ADM acceleration we have

$$\begin{aligned}\delta\int_{\Sigma_t}\mathrm{d}^3y\frac{N\sqrt{q}}{2\kappa}(-a^cD_cs^{\mathbf{nn}}) = \int_{\Sigma_t}\mathrm{d}^3y\frac{N\sqrt{q}}{4\kappa}(&-q^{ab}a_cD^cs^{\mathbf{nn}} \\ &+2a^{(a}D^{b)}s^{\mathbf{nn}})\delta q_{ab},\end{aligned} \tag{50}$$

cf. Eq. (38). Furthermore,

$$\delta\int_{\Sigma_t}\mathrm{d}^3y\frac{\mathcal{L}_m s^{\mathbf{nn}}}{2(1-s^{\mathbf{nn}})}\left(p-\frac{3}{8}\frac{\sqrt{q}}{\kappa N}\mathcal{L}_m s^{\mathbf{nn}}\right) = \int_{\Sigma_t}\mathrm{d}^3y\frac{\mathcal{L}_m s^{\mathbf{nn}}}{2(1-s^{\mathbf{nn}})}\left(p^{ab}-\frac{3}{16}\frac{\sqrt{q}}{\kappa N}q^{ab}\mathcal{L}_m s^{\mathbf{nn}}\right)\delta q_{ab}, \tag{51a}$$

$$\delta\int_{\Sigma_t}\mathrm{d}^3y\frac{2\kappa N}{\sqrt{q}(1-s^{\mathbf{nn}})}\left(p^{cd}p_{cd}-\frac{p^2}{2}\right) = \int_{\Sigma_t}\mathrm{d}^3y\frac{2\kappa N}{\sqrt{q}(1-s^{\mathbf{nn}})}\left[2p^{ac}p^b{}_c-pp^{ab}-\frac{1}{2}\left(p_{cd}p^{cd}-\frac{p^2}{2}\right)q^{ab}\right]\delta q_{ab}. \tag{51b}$$

Compiling the variations of Eqs. (48a) (with δB_R dropped), (50), (51) as well as the analog of Eq. (42) and inserting them into the second line of Eq. (24) implies another second-rank tensor integrated over Σ_t, which has to vanish. The same argument that we previously employed for u results in the dynamical part of the modified Einstein equations:

$$\dot{p}^{ab} = \frac{\sqrt{q}}{2\kappa}(-NG^{ab} + D^aD^bN - q^{ab}D_cD^cN) - \frac{2\kappa N}{\sqrt{q}(1-s^{\mathbf{nn}})}\left[2p^{ac}p^b{}_c - pp^{ab} - \frac{1}{2}\left(p_{cd}p^{cd} - \frac{p^2}{2}\right)q^{ab}\right]$$
$$+ \frac{N\sqrt{q}}{4\kappa}(2a^{(a}D^{b)}s^{\mathbf{nn}} - q^{ab}a_cD^cs^{\mathbf{nn}}) - \frac{\mathcal{L}_m s^{\mathbf{nn}}}{2(1-s^{\mathbf{nn}})}\left(p^{ab} - \frac{3}{16}\frac{\sqrt{q}}{\kappa N}q^{ab}\mathcal{L}_m s^{\mathbf{nn}}\right) - \sqrt{q}D_c\left(\frac{2N^{(a}p^{b)c} - p^{ab}N^c}{\sqrt{q}}\right). \quad (52)$$

The validity of $(\vec{q}^*\mathbf{J}_1)^{ij} = 0$ based on Eq. (35a) in Ref. [53] is confirmed for $N^a = 0$. Similarly, Eq. (52) describes the dynamics of the modified-gravity theory governed by $\mathcal{L}_1$.

C. Dynamics in the s^{ab} sector

Last but not least, let $\mathcal{L}_2$ be the Lagrange density based on Eq. (21c) restricted to a nonzero s^{ab} only, i.e., $\mathcal{L}_2 \coloneqq \mathcal{L}_b|_{u=s^{\mathbf{nn}}=0}$. Then, the canonical momentum is given by

$$P^{ab} \coloneqq \frac{\partial\mathcal{L}_2}{\partial\dot{q}_{ab}} = \frac{\sqrt{q}}{2\kappa}\left[K^{ab} - q^{ab}K - (s^{ac}K_c{}^b + s^{bc}K_c{}^a) - \frac{1}{2N}\mathcal{L}_m s^{ab}\right]. \quad (53)$$

Due to the tensorial nature of s^{ab}, the latter relation has a more complicated structure as did Eqs. (27), (45) for u and $s^{\mathbf{nn}}$, respectively. Therefore, it does not come as a surprise that the s^{ab} sector is involved from a calculational perspective. After all, it contains six independent coefficients, which makes it challenging to invert Eq. (53) for the extrinsic curvature in a closed form. Therefore, as we did before in Refs. [52,53], we will be working at first order in s^{ab} and derivatives thereof.

Applying the $(3+1)$ decomposition to the action then implies

$$S_{G,2} = \int_{t_1}^{t_2}\mathrm{d}t\int_{\Sigma_t}\mathrm{d}^3y\,\mathcal{L}_2. \quad (54)$$

Again, the latter is expressed in a form adequate for the Palatini formalism:

$$S_{G,2} = \int_{t_1}^{t_2}\mathrm{d}t\left(\int_{\Sigma_t}P^{ab}\dot{q}_{ab}\mathrm{d}^3y - H_{\Sigma,2} + B_{S,2}\right), \quad (55a)$$

with the bulk Hamiltonian

$$H_{\Sigma,2} = \int_{\Sigma_t}\mathrm{d}^3y\left\{-\frac{N\sqrt{q}}{2\kappa}(R + s^{ab}R_{ab} + a_bD_as^{ab}) + \left(P_{ab} - \frac{P}{2}q_{ab}\right)\mathcal{L}_m s^{ab} + \frac{2\kappa N}{\sqrt{q}}\left[P^{ab}P_{ab} - (1-s^a{}_a)\frac{P^2}{2}\right.\right.$$
$$\left.\left. - 2s^{ab}(P_{ab}P - P_a{}^cP_{cb})\right] + 2P^{ab}D_aN_b\right\} + \mathcal{O}[(s^{ab})^2]. \quad (55b)$$

The symbol $\mathcal{O}$ indicates that we have approximated the Hamiltonian to linear order in the background field and derivatives thereof when we expressed the extrinsic curvature in terms of momenta via Eq. (53). Moreover, the boundary term is

$$B_{S,2} = \oint_{S_t}\mathrm{d}^2\theta\frac{N\sqrt{\sigma}}{2\kappa}(2k + k_{AB}s^{AB}). \quad (55c)$$

Basically, major parts of Sec. IV. B. 2 of Ref. [52] can be taken over to the derivation of the bulk Hamiltonian in Eq. (55b), in particular, Eqs. (45), (46), and (49) of the latter paper. Now the canonical momentum is called P^{ab} to be consistent with the notation introduced in the follow-up work [53]. To arrive at Eq. (55b), the derivation must only be adapted in two respects. First, in contrast to what we did in Ref. [52], the Lie derivative $\mathcal{L}_m s^{ab}$ is now taken into account at first order. Second, as already pointed out for the u and $s^{\mathbf{nn}}$ sectors, integrations by parts are not carried out for the third and last terms of Eq. (55b). Hence, these terms are different from the corresponding ones in the Hamiltonian for the purely spacelike sector of $s^{\mu\nu}$, which is stated in our earlier article [52] on the ADM formalism of the minimal gravitational SME.

Obtaining the dynamical field equations through the variation of the action is tedious. However, the computational steps involved are similar to those of the u and $s^{\mathbf{nn}}$ sectors investigated before. We need the variation of the

Ricci scalar contribution, which was already computed in Eq. (48a). In a manner analogous to how the latter implies a boundary term on S_t—recall Eqs. (37b), (48b)—the variation of the contribution in Eq. (55b) that involves the Ricci tensor provides another boundary term on S_t. To compute this variation, it is convenient to consider

$$\delta(N\sqrt{q}s^{ab}R_{ab}) = \frac{N}{2}\sqrt{q}s^{ab}R_{ab}q^{cd}\delta q_{cd} + \sqrt{q}D_c(N\delta Q^c) - D_c(Ns^{ab})\delta\Gamma^c{}_{ab} + D_b(Ns^{ab})\delta\Gamma^c{}_{ac}, \tag{56a}$$

with

$$\delta Q^c = s^{ab}\delta\Gamma^c{}_{ab} - s^{ac}\delta\Gamma^b{}_{ab}. \tag{56b}$$

We also consult the following expression in 3 dimensions analogous to Eq. (D13) of Ref. [52], which is

$$r_c\delta Q^c = -\frac{1}{2}\sigma^d{}_a\sigma^e{}_b s^{ab}r^c D_c\delta q_{de}, \tag{56c}$$

where we discarded the term proportional to $s^{\mathbf{nn}}$, as the latter coefficient would have to be replaced by $s^{\mathbf{rr}} = 0$ in this sector. By doing so, we arrive at the variation

$$\begin{aligned}\delta\int_{\Sigma_t}\mathrm{d}^3y\frac{N\sqrt{q}}{2\kappa}s^{cd}R_{cd} &= \int_{\Sigma_t}\mathrm{d}^3y\frac{\sqrt{q}}{4\kappa}\{q^{ab}[NR_{cd}s^{cd} - D_cD_d(Ns^{cd})] + D_c[D^a(Ns^{bc}) + D^b(Ns^{ac}) - D^c(Ns^{ab})]\}\delta q_{ab} - \delta B_{R,s} \\ &= \int_{\Sigma_t}\mathrm{d}^3y\frac{N\sqrt{q}}{4\kappa}\{q^{ab}[R_{cd} - (D_c + a_c)(D_d + a_d)]s^{cd} + (D_c + a_c)[(D^a + a^a)s^{bc} + (D^b + a^b)s^{ac} \\ &\quad - (D^c + a^c)s^{ab}]\}\delta q_{ab} - \delta B_{R,s},\end{aligned} \tag{57a}$$

where

$$\delta B_{R,s} = \oint_{S_t}\mathrm{d}^2\theta\frac{N\sqrt{\sigma}}{2\kappa}\sigma^d{}_a\sigma^e{}_b s^{ab}r^c D_c\delta q_{de}. \tag{57b}$$

Note that *xTensor* is powerful when it comes to computing results like Eq. (57a), but it omits boundary terms such as that stated in Eq. (57b). Hence, these must be taken into account by hand. Now, the total boundary term corresponds to the sum of Eq. (48b), which results from the variation of the EH term, and of Eq. (57b), which we have just obtained. So we define

$$\delta B_{R,2} \coloneqq \delta B_R + \delta B_{R,s}. \tag{58}$$

Note also the compelling form of Eq. (57a) that depends only on combinations of the covariant derivative and the ADM acceleration, $D_c + a_c$.

Moreover, the variation of the term involving the ADM acceleration can also be cast into an appealing form as follows:

$$\delta\int_{\Sigma_t}\mathrm{d}^3y\frac{N\sqrt{q}}{2\kappa}a_cD_ds^{cd} = \int_{\Sigma_t}\mathrm{d}^3y\left(-D_c\Upsilon^{abc} + \frac{N\sqrt{q}}{4\kappa}q^{ab}a_cD_ds^{cd}\right)\delta q_{ab}, \tag{59a}$$

with

$$\Upsilon^{abc} = \frac{N\sqrt{q}}{4\kappa}(2a^{(a}s^{b)c} - s^{ab}a^c + q^{ab}s^{dc}a_d). \tag{59b}$$

The last term of Eq. (59a) results directly from varying $\sqrt{q}$. The remaining part can be written as a total covariant derivative of the third-rank tensor in Eq. (59b). In contrast, it is impossible to state the variations of the contributions $(\sqrt{q}/\kappa)a^cD_cu$ and $\sqrt{q}/(2\kappa)a^cD_cs^{\mathbf{nn}}$ in Eq. (29b) and Eq. (47b), respectively, in a similar fashion.

The Lie derivative of s^{ab} along m^μ is independent of the induced metric on Σ_t, as we argued around Eq. (39). So,

$$\delta\int_{\Sigma_t}\mathrm{d}^3y\mathcal{L}_ms^{ab} = 0. \tag{60}$$

Then, the variation of the terms depending on Lie derivatives of the background tensor is

$$\begin{aligned}\delta\int_{\Sigma_t}\mathrm{d}^3y\left(P_{cd} - \frac{P}{2}q_{cd}\right)\mathcal{L}_ms^{cd} &= \int_{\Sigma_t}\mathrm{d}^3y\bigg\{P^a{}_c\mathcal{L}_ms^{cb} + P^b{}_c\mathcal{L}_ms^{ca} \\ &\quad - \frac{1}{2}(P^{ab}q_{cd}\mathcal{L}_ms^{cd} + P\mathcal{L}_ms^{ab})\bigg\}\delta q_{ab}.\end{aligned} \tag{61}$$

Varying the contributions involving the canonical momentum is lengthy, but *xTensor* provides the result in a straightforward manner:

$$
\begin{aligned}
&\delta\int_{\Sigma_t}\mathrm{d}^3y\,\frac{2\kappa N}{\sqrt{q}}\left[P^{cd}P_{cd}-\frac{1}{2}(1-s^c{}_c)P^2-2s^{cd}(PP_{cd}-P_c{}^eP_{de})\right]\\
&=\int_{\Sigma_t}\mathrm{d}^3y\,\frac{\kappa N}{2\sqrt{q}}\Big(\big[(1-s^c{}_c)P^2-2P^{cd}P_{cd}+4s^{cd}(PP_{cd}-P_c{}^eP_{de})\big]q^{ab}\\
&\quad+8\Big[P^{ac}P_c{}^b+s^{ac}P_{cd}P^{db}+s^{bc}P_{cd}P^{da}-s^{ac}PP_c{}^b-s^{bc}PP_c{}^a\\
&\quad+s^{cd}(P^a{}_cP_d{}^b-P_{cd}P^{ab})\Big]+2P^2s^{ab}-4(1-s^c{}_c)PP^{ab}\Big)\delta q_{ab}.
\end{aligned}
\tag{62}
$$

Finally, we should not forget to vary the boundary term in the action, Eq. (55c):

$$
\begin{aligned}
\delta B_{S,2}=&\oint_{S_t}\mathrm{d}^2\theta\,\frac{N\sqrt{\sigma}}{2\kappa}\sigma^{ab}r^cD_c\delta q_{ab}\\
&+\oint_{S_t}\mathrm{d}^2\theta\,\frac{N\sqrt{\sigma}}{2\kappa}\sigma^m{}_i\sigma^n{}_js^{ij}r^kD_k\delta q_{mn},
\end{aligned}
\tag{63a}
$$

where we have used

$$
\delta k_{ij}=\sigma^m{}_i\sigma^n{}_js^{ij}r^kD_k\delta q_{mn}. \tag{63b}
$$

As we found for the u and $s^{\mathbf{nn}}$ sectors, Eq. (63) neatly cancels the sum in Eq. (58): $\delta B_{S,2}-\delta B_{R,2}=0$.

Now we are ready to compile Eqs. (48a) (with δB_R omitted), (57a) (with $\delta B_{R,s}$ discarded), (59), (61), and (62) as well as Eq. (42) adapted to the current sector. After inserting these variations into the second line of Eq. (24) and dropping the integral over Σ_t, we can cast the dynamical part of the modified Einstein equations into the following form:

$$
\begin{aligned}
\dot{P}^{ab}=&\frac{\sqrt{q}}{2\kappa}(-NG^{ab}+D^aD^bN-q^{ab}D_cD^cN)+\frac{N\sqrt{q}}{4\kappa}\Big(q^{ab}[R_{cd}-(D_c+a_c)(D_d+a_d)+a_cD_d]s^{cd}\\
&+(D_c+a_c)[(D^a+a^a)s^{bc}+(D^b+a^b)s^{ac}-(D^c+a^c)s^{ab}]\Big)-D_c\left(\frac{N\sqrt{q}}{4\kappa}\left[2a^{(a}s^{b)c}-s^{ab}a^c+q^{ab}s^{dc}a_d\right]\right)\\
&+\frac{1}{2}(P^{ab}q_{cd}\mathcal{L}_ms^{cd}+P\mathcal{L}_ms^{ab})-(P^a{}_c\mathcal{L}_ms^{cb}+P^b{}_c\mathcal{L}_ms^{ca})\\
&-\frac{\kappa N}{2\sqrt{q}}\Big(\big[(1-s^c{}_c)P^2-2P^{cd}P_{cd}+4s^{cd}(PP_{cd}-P_c{}^eP_{de})\big]q^{ab}+8\Big[P^{ac}P_c{}^b+s^{ac}P_{cd}P^{db}\\
&+s^{bc}P_{cd}P^{da}-s^{ac}PP_c{}^b-s^{bc}PP_c{}^a+s^{cd}(P^a{}_cP_d{}^b-P_{cd}P^{ab})\Big]+2P^2s^{ab}-4(1-s^c{}_c)PP^{ab}\Big)\\
&-\sqrt{q}D_c\left(\frac{2N^{(a}P^{b)c}-P^{ab}N^c}{\sqrt{q}}\right)+\mathcal{O}[(s^{ab})^2].
\end{aligned}
\tag{64}
$$

The latter is a generalization of $(\vec{\boldsymbol{q}}^*\mathbf{J}_2)^{ij}=0$ given in Eq. (38a) of Ref. [53] to a nonzero shift vector. Equations (44), (52), and (64) completely govern the dynamics of the u, $s^{\mathbf{nn}}$, and s^{ab} sectors of the modified-gravity theory based on Eq. (1). The dynamical field equations of GR in Eq. (28f) of Ref. [49] are reproduced when diffeomorphism violation is switched off, $u=s^{\mathbf{nn}}=s^{ab}=0$, as expected. The complexity of Eq. (64) illustrates the challenge of dealing with all sectors simultaneously, which is a manifestation of the profoundly nonlinear character of Eq. (1). At the moment the best strategy seems to separate the sectors from each other in phenomenological studies.

IV. FINAL REMARKS

In this work, we have investigated a modification of GR governed by the u- and $s^{\mu\nu}$-type background fields contained in the minimal gravitational SME. The background fields were assumed to be nondynamical, which implies diffeomorphism breaking. Having carried out the ADM decomposition of this theory in previous articles, our current focus was on a rigorous treatment of the gravitational boundary terms, which are unavoidable in this context.

To do so, we decomposed the spacetime boundary into two spacelike and one timelike hypersurface. As a consequence, the extended GHY boundary term split into three parts, each evaluated on one of the hypersurfaces previously referred to. By treating total-derivative terms in the action suitably, we were able to cancel corresponding contributions of the extended GHY boundary term on the spacelike hypersurfaces. Foliating the timelike part of the boundary properly into two-dimensional hypersurfaces S_t, the remaining boundary contributions neatly combined to give rise to boundary terms on S_t. This procedure led us to the ADM-decomposed action of Eq. (21), which is one of our central results.

Variations of the boundary term on S_t for the induced metric were demonstrated to compensate further boundary terms originating from varying the Ricci scalar and Ricci tensor, respectively. Compiling the variations of each contribution in the action for the induced metric implied the dynamical field equations stated in Eqs. (44), (52), and (64) for each of the three sectors of the ADM-decomposed modified-gravity theory. A bonus of this new analysis is that it generalizes some of the findings in our previous paper [53] to a nonzero shift vector.

The formalism presented and the results obtained are a well-suited starting point for phenomenology in black-hole physics affected by diffeomorphism violation. Moreover, from a theoretical viewpoint they show that explicit diffeomorphism violation in gravity does not necessarily imply internal inconsistencies—at least not at the level studied here and in our previous papers [52,53]. Time will show whether or not this conclusion can be upheld under different criteria.

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