Gravitational memory of Casimir effect

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We investigate the influence of a time-varying spacetime background on the vacuum polarization of a massless quantum field confined to a Casimir cavity. The background is modeled as an anisotropic Bianchi-I spacetime, in which small time-dependent perturbations around the flat spacetime are vanishing in the far past and future. The spacetime admits asymptotic Minkowskian regions, thus allowing for an unambiguous definition of the in- and out-field vacua. Following Schwinger's proper-time approach, we evaluate the vacuum polarization inside the Casimir cavity. We show the presence of a small shift in the field vacuum energy, once the perturbation is over. The time-dependent background has distorted the field modes, causing a permanent change in the zero-point energy of the field confined to the Casimir apparatus. As an example, we briefly consider the case of a weak gravitational wave background, which can be locally identified with the previously employed Bianchi-I spacetime model. The present effect appears as a sort of gravitational memory of the Casimir effect.

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I. INTRODUCTION

One of the nontrivial features of quantum field theory is the occurrence of an infinite value of its zero-point energy—the vacuum energy. In flat spacetime such problem can be easily circumvented by subtracting the infinite amount of energy by a suitable renormalizing counterterm, thus giving the theory a physical meaning. However, in a curved, time-dependent background this is not generally possible, since the construction of a well-defined vacuum state becomes ambiguous. Rigorously speaking, this is because the Poincaré group is no longer a symmetry group of the spacetime. As a consequence, also the particle concept becomes an observer-dependent quantity, being related to the particular choice of the vacuum state [1].

When the background spacetime admits asymptotically flat regions in the far past and future, it is still possible to use the Minkowskian vacuum as the state characterized by the absence of particles according to all the inertial observers in those asymptotic regions. With respect to these asymptotic vacua, it is then possible to explore the influence of such a time-dependent gravitational background as regards particle creation and vacuum polarization effects.

Influence of gravitation on vacuum energy undoubtedly represents an interesting topic, ranging from subatomic to cosmological scales. It may also be that a deeper understanding of the physics at the microscale could help to shed light on some unresolved issues plaguing current models of our Universe, such as, for example, the cosmological constant problem [2–4].

In that respect, an interesting arena is offered by the Casimir effect [5–8], a purely quantum effect, experimentally verified, consisting in a tiny attractive force between two uncharged conducting plates, placed a short distance L from each other. The explanation of such effect relies on a small shift of the vacuum energy of the quantum field. Roughly speaking, such a shift originates from a distortion in the modes of a quantum field constrained in a finite region of space by some boundaries. The latter can be material as well as due to the geometrical properties of the background spacetime.

Influence of static as well as stationary gravitoinertial fields on the vacuum energy in the Casimir effect is indeed a theoretically relevant issue, which has been extensively investigated by several authors through the years [9–23].

Generally speaking, a time-varying background geometry can affect the dynamics of a given quantum field, altering both the vacuum persistence amplitude and the vacuum polarization. In the first case, we have particle creation. In the second case, the field modes suffer a distortion leading to a shift in the vacuum energy.

In the present paper, we will consider in detail the influence of a time-dependent background spacetime on the vacuum energy of a massless scalar field confined to a Casimir cavity. The gravitational background will be described by a slightly anisotropic Bianchi-I spacetime model [1,24], admitting asymptotic Minkowskian regions in the remote past and future.

Being time-dependent, the background is expected to give rise also to particle creation out of the quantum field

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vacuum inside the cavity [25–31]. Here, however, we will focus on the gravitational corrections to the vacuum polarization.

When computing the field vacuum energy, both particle creation and the polarization effect contribute to the field stress-energy tensor $T_{\mu\nu}$, which—on the other hand—couples to gravity. In order to disentangle the two contributions, we will follow Schwinger's proper-time approach [30,32–34]. We will compute the *real* part of the effective action W, related to the vacuum polarization (leading to the static Casimir effect), discarding the *imaginary* part of W, describing the particle creation (not to be confused with the dynamic Casimir effect as such, in which particle creation is induced by a rapid motion of the boundaries [35]).

We will manage the divergences arising during the calculations by means of a renormalization procedure based on the analytic continuation technique (see, e.g., [36]).

We will show that, once the gravitational perturbation is over (i.e., in the far future), a small shift in the vacuum energy is found as a consequence of the interaction. The time-dependent background has distorted the field modes, causing a permanent change in the zero-point energy of the quantum field confined to the cavity. This appears as a sort of a "gravitational memory" of the Casimir effect.

The paper is organized as follows. In Sec. II we introduce the time-dependent gravitational background. Then we solve the Klein-Gordon equation for a massless scalar field, minimally coupled to the gravitational field and confined to a Casimir cavity, represented by two large, perfectly reflecting parallel plates, separated by a small (proper) distance L. In Sec. III we follow Schwinger's proper-time approach, computing the effective action W for the quantum field. In Sec. IV we first check our computations, finding the Casimir energy density and the attractive force between the plates in the flat spacetime case. Subsequently, we adopt the same procedure in the case of a slightly perturbed background described by a suitable model of time-varying Bianchi-I spacetime. In Sec. V we discuss the emerging divergences, hence obtaining the Casimir vacuum energy density as a finite, physical quantity. In Sec. VI we adapt the present model to a specific case, in which the background spacetime describes a weak gravitational wave interacting with the Casimir cavity. In Sec. VII we discuss the results, also in connection with the weak energy conditions and the quantum energy inequalities, and give some concluding remarks.

In Appendix A we generalize our results, briefly discussing the case of a confined electromagnetic field. Appendix B is devoted to the analysis of the interaction with a gravitational wave propagating at an arbitrary direction in the reference frame of the Casimir cavity.

The approach followed in Appendix B can be straightforwardly applied to more general spacetime backgrounds such as, e.g., Bianchi type-IX models.

Throughout the paper, unless otherwise specified, use has been made of natural geometrized units. Greek indices take values from 0 to 3; latin ones take values from 1 to 3. The metric signature is -2, with determinant g.

II. THE GRAVITATIONAL BACKGROUND AND THE CASIMIR CAVITY

We are interested in a time-dependent background spacetime, admitting asymptotic Minkowskian regions in the far past and future, so that the definitions of in and out vacua are not ambiguous. On the other hand, time dependence will allow for particle creation as well as vacuum polarization effects. Considering also the possibility of anisotropies, we will focus on a Bianchi-I spacetime background. The Bianchi-I universe has zero intrinsic curvature but nonzero extrinsic curvature. The general line element is [1,24]

$$ds^2 = dt^2 - \sum_{i=1}^{3} a_i^2(t) dx_i^2, \tag{1}$$

namely, the simplest generalization of the homogeneous spatially flat Friedmann-Lemaître-Robertson-Walker (FLRW) universe. The functions $a_i(t)$ represent the directional scale factors along the axes $\{x, y, z\}$ in a matter-comoving coordinate frame (with 4-velocity $u^\mu = \delta_0^\mu$). In what follows we will suppose that the anisotropies are *small*, so that (1) reads

$$ds^2 = dt^2 - \sum_{i=1}^{3} [1 + h_i(t)](dx^i)^2, \tag{2}$$

where the condition

$$\lim_{t\to\pm\infty} h_i(t) = 0 \tag{3}$$

accounts for the asymptotic Minkowskian behavior. For simplicity, we will also impose the following constraints:

$$\max |h_i(t)| \ll 1, \tag{4}$$

$$\sum_{i=1}^{3} h_i(t) = 0, \tag{5}$$

$$h_3(t) \equiv h_z(t) = 0, \tag{6}$$

(for Bianchi-I type, coordinates can be chosen such that the spatial metric is diagonal and traceless [37,38]). Assume that the Casimir cavity is oriented in space so that the plates, each of (proper) area A, are orthogonal to the z axis, placed at $z = 0$ and $z = L$, respectively, with L representing the (proper) plate separation. The constraint (6) guarantees that the proper and the coordinate distance between the plates will coincide at any time. This will allow us to avoid possible complications due to "tidal" effects [the more general case in which (6) is relaxed will be briefly considered in Appendix B and discussed in Sec. VII]. We will consider a massless scalar field $\phi(x)$, satisfying the Klein-Gordon equation

$$\frac{1}{\sqrt{-g}}\partial_\mu[\sqrt{-g}g^{\mu\nu}\partial_\nu\phi(x)] + \xi R(x)\phi(x) = 0, \qquad (7)$$

where ξ is a parameter describing the coupling between the matter field and the background gravitational field and $R(x)$ is the scalar curvature. In what follows we will suppose minimal coupling, so that $\xi = 0$. We will also assume Dirichlet boundary conditions at the confining plates. To the lowest order, (7) reads

$$(\Box + \hat{V})\phi = 0, \qquad (8)$$

where

$$\hat{V} = h_x(t)\partial_x^2 + h_y(x)\partial_y^2 = h(t)(\partial_x^2 - \partial_y^2) \qquad (9)$$

(throughout the text, a caret will mean that the corresponding quantity has to be regarded as an operator) with $h(t) \equiv h_x(t) = -h_y(t)$ [see (5) and (6)].

Spatial translation invariance of Bianchi-I spacetime (1) is broken by the field confinement along the z direction. Nevertheless, it is still assured along the transverse direction x and y, so that we can search for solutions of (8) in the form

$$\phi(x) = Ne^{i\vec{p}_\perp\cdot\vec{x}_\perp}\sin\left(\frac{n\pi z}{L}\right)\eta(t), \qquad (10)$$

where $N = (\frac{2}{(2\pi)^3 L})^{1/2}$ is an overall normalization constant, $\vec{p}_\perp = (p_x, p_y)$, $\vec{x}_\perp = (x, y)$, and the function $\eta(t)$ satisfies

$$\eta(t) \to e^{-i\omega t}, \qquad h(t) \to 0. \qquad (11)$$

In the remote past ($t \to -\infty$), i.e., on lack of gravitational perturbation, the spacetime is Minkowskian and (10) reduces to the usual mode solution inside the cavity, namely,

$$\phi^{(0)}(x) = Ne^{i\vec{p}_\perp\cdot\vec{x}_\perp}\sin\left(\frac{n\pi z}{L}\right)e^{-i\omega t}. \qquad (12)$$

Using (10) in (8) yields, to the lowest order in h,

$$\partial_t^2\eta + \omega^2\eta - (h(t)p_\perp^2\cos 2\theta)e^{-i\omega t} = 0, \qquad (13)$$

where $\tan\theta = p_y/p_x$, and $\omega^2 = p_\perp^2 + (n\pi/L)^2$. The solution of (13) is

$$\eta(t) = e^{-i\omega t} + \int_{-\infty}^{t} dt' \frac{\sin(\omega(t-t'))}{\omega} h(t')p_\perp^2\cos 2\theta e^{-i\omega t'} = \alpha_p(t)e^{-i\omega t} + \beta_p(t)e^{i\omega t}, \qquad (14)$$

where

$$\alpha_p(t) = 1 + \frac{i}{2\omega}\int_{-\infty}^{t} dt'\, h(t')p_\perp^2\cos 2\theta, \qquad (15)$$

$$\beta_p(t) = -\frac{i}{2\omega}\int_{-\infty}^{t} dt'\, h(t')p_\perp^2\cos 2\theta e^{-2i\omega t'}, \qquad (16)$$

and

$$\phi(x) = N(\alpha_p(t)e^{-i\omega t} + \beta_p(t)e^{i\omega t})e^{i\vec{p}_\perp\cdot\vec{x}_\perp}\sin\left(\frac{n\pi z}{L}\right). \qquad (17)$$

In the limit $t \to +\infty$ we also have

$$\eta(t) = \alpha_p e^{-i\omega t} + \beta_p e^{i\omega t}, \qquad (18)$$

where α_p and β_p can be regarded as the Bogoliubov coefficients, connecting the in and out vacua, satisfying the condition $|\alpha_p|^2 - |\beta_p|^2 = 1$. To the present order of approximation, we have [1]

$$\alpha_p = 1 + \frac{i}{2\omega}\int_{-\infty}^{+\infty} dt\, h(t)p_\perp^2\cos 2\theta, \qquad (19)$$

$$\beta_p = -\frac{i}{2\omega}\int_{-\infty}^{+\infty} dt\, h(t)p_\perp^2\cos 2\theta e^{-2i\omega t}. \qquad (20)$$

III. SCHWINGER'S PROPER-TIME APPROACH

In this section we will follow Schwinger's proper-time approach [32–34] in order to derive an expression of the effective action W for the scalar field inside the Casimir cavity. In the presence of a time-dependent gravitational background, the effective action may become complex. In such case the real part of W describes phenomena related to the vacuum polarization, as the (static) Casimir effect; meanwhile the imaginary part is responsible for particle production. Actually, in the so-called "in-out" formalism the imaginary part of the effective action is related to the vacuum persistence amplitude

$$\langle 0\,\mathrm{out}|0\,\mathrm{in}\rangle = e^{iW}. \qquad (21)$$

Let us start writing the effective action W,

$$W = \lim_{\nu\to 0} W(\nu), \qquad (22)$$

where [30,32–34]

$$W(\nu) = -\frac{i}{2}\int_0^\infty ds\, s^{\nu-1}\mathrm{Tr}e^{-is\hat{H}} + \text{c.t.}, \tag{23}$$

and the limit $\nu \to 0$ has to be taken at the end of calculations. The additional counterterms (c.t.) are introduced to subtract divergent terms, hence recovering the required physical normalization. In (23),

$$\begin{aligned}\hat{H} &= \hat{H}_0 + \hat{V} \\ &= -\hat{p}_0^2 + \hat{p}_\perp^2 + \left(\frac{n\pi}{L}\right)^2 - h(t)p_\perp^2\cos 2\theta,\end{aligned} \tag{24}$$

where $\hat{p}_0 = i\partial_t$, $\hat{p}_\perp = -i\vec{\nabla}_\perp$ and the total trace

$$\mathrm{Tr}e^{-is\hat{H}} = \sum\!\!\!\!\!\!\int d^4x\langle x|e^{-is\hat{H}}|x\rangle \tag{25}$$

has to be evaluated all over the continuous as well the discrete degrees of freedom, including those of spacetime. Expanding the trace in terms of the eigenvectors $|\alpha\rangle \equiv |p_0, p_\perp, n\rangle$ of $\hat{H}$, we write

$$\mathrm{Tr}e^{-is\hat{H}} = \int d^4x \sum\!\!\!\!\!\!\int d\alpha\langle x|\alpha\rangle\langle\alpha|e^{-is(\hat{H}_0+\hat{V})}|\alpha'\rangle\langle\alpha'|x\rangle, \tag{26}$$

where $|x\rangle \equiv |t, x_\perp, z\rangle$ and

$$\sum\!\!\!\!\!\!\int d\alpha \equiv \sum_{n,n'}\int dp_0\, dp_0' dp_\perp dp_\perp'. \tag{27}$$

Since $[\hat{\vec{p}}, \hat{V}] = 0$, we have

$$\begin{aligned}\langle\alpha|e^{-is(\hat{H}_0+\hat{V})}|\alpha'\rangle &= e^{-isp_\perp^2}e^{-is(n\pi/L)^2} \\ &\quad\times\langle p_0|e^{-is(-\hat{p}_0^2 - h(t)p_\perp^2\cos 2\theta}|p_0'\rangle,\end{aligned} \tag{28}$$

and, taking into account (13),

$$\begin{aligned}\langle\alpha|e^{-is(\hat{H}_0+\hat{V})}|\alpha'\rangle &= e^{-isp_\perp^2}e^{-is(n\pi/L)^2}e^{is\omega^2} \\ &\quad\times\delta_{n,n'}\delta^{(2)}(p_\perp - p_\perp')\delta(\omega-\omega')\end{aligned} \tag{29}$$

[notice that the states $|\alpha\rangle$ are normalized according to the standard Dirac prescription: $\langle\alpha|\alpha'\rangle = \delta(\alpha, \alpha')$, where $\delta(\alpha, \alpha')$ is the Kronecker symbol $\delta_{\alpha,\alpha'}$ if $\{|\alpha\rangle\}$ is a discrete set and the Dirac delta function $\delta(\alpha - \alpha')$ if it is continuous].

Replacing (29) in (26) and using $|\langle x|\alpha\rangle|^2 = |\phi(x)|^2$ we get [see (17)]

$$\begin{aligned}\mathrm{Tr}e^{-is\hat{H}} &= N^2\int d^4x\int d^2p_\perp\, d\omega\sum_n\Big(|\alpha_p(t)|^2 + |\beta_p(t)|^2 \\ &\quad + 2\Re\mathrm{e}(\alpha_p(t)\beta_p(t)^* e^{-2i\omega t})\Big) \\ &\quad\times\sin^2\left(\frac{n\pi z}{L}\right)e^{-isp_\perp^2}e^{-is(n\pi/L)^2}e^{is\omega^2}.\end{aligned} \tag{30}$$

The rapidly oscillating term $\propto e^{-2i\omega t}$ appearing in the last term can be discarded in the evaluation of the action, since the involved time integration gives a vanishing mean value (this is known as the rotating-wave approximation). Performing the integration over the cavity volume ($=AL$) we find

$$\begin{aligned}\mathrm{Tr}e^{-is\hat{H}} &= \frac{A}{(2\pi)^3}\int dt\int_0^{2\pi}d\theta\int_0^{+\infty}p_\perp\, dp_\perp\int_{-\infty}^{+\infty}d\omega \\ &\quad\times\sum_n(|\alpha_p(t)|^2 + |\beta_p(t)|^2)e^{-isp_\perp^2}e^{-is(n\pi/L)^2}e^{is\omega^2}.\end{aligned} \tag{31}$$

IV. THE STATIC CASIMIR EFFECT

In this section we will discuss the *static* Casimir effect, deriving it from the real part of the effective action W. One could wonder if we are using the word static while considering a time-varying background. Actually, as we will see in the following, we are interested in the zero-point energy of the quantum field in the Casimir apparatus in the far *future*, namely, when the time-dependent gravitational perturbation is over. It is just in this limit that we can recover the static Casimir effect, thus evaluating possible shift induced by the gravitational interaction. A naive reasoning might lead one to expect no shift in the vacuum energy. As we will see, however, this is not the case.

Following Schwinger's proper-time approach, we have

$$\langle\epsilon_{\mathrm{Cas}}\rangle = -\frac{1}{AL}\lim_{\nu\to 0}\left[\lim_{t\to+\infty}\frac{\partial}{\partial t}\Re\mathrm{e}W(\nu)\right]. \tag{32}$$

From (23) and (31) we obtain

$$\langle\epsilon_{\mathrm{Cas}}\rangle = \frac{1}{2(2\pi)^3L}\lim_{\nu\to 0}\Re\mathrm{e}\left\{i\int_0^{+\infty}ds s^{\nu-1}\left[\int_0^{2\pi}d\theta\int_0^{+\infty}p_\perp dp_\perp\int_{-\infty}^{+\infty}d\omega\sum_n(1+2|\beta_p|^2)e^{-isp_\perp^2}e^{-is(n\pi/L)^2}e^{is\omega^2}\right]\right\}, \tag{33}$$

where use has also been made of the relation $|\alpha_p|^2 - |\beta_p|^2 = 1$, and $\beta_p = \lim_{t\to+\infty}\beta_p(t)$ is given by (20).

A. Flat spacetime background

As a consistence check, let us evaluate (33) in the flat spacetime background, i.e., $\beta_p = 0$. The integrations in the square brackets can be readily performed, giving

$$\langle\epsilon_{\rm Cas}\rangle_0 = \lim_{\nu\to 0}\Re{\rm e}\left\{\frac{\sqrt{i}}{16\pi^{3/2}L}\sum_n\int_0^{+\infty} ds s^{\nu-\frac{3}{2}-1}e^{-is(\frac{n\pi}{L})^2}\right\}. \tag{34}$$

The remaining integral can be converted into a Gamma function and the infinite sum yields a Riemann ζ-function

$$\langle\epsilon_{\rm Cas}\rangle_0 = \lim_{\nu\to 0}\Re{\rm e}\left\{\frac{-(i)^{-\nu}}{16\pi^{3/2}L}\left(\frac{\pi}{L}\right)^{2\nu-3}\zeta(2\nu-3)\Gamma(\nu-3/2)\right\}. \tag{35}$$

Performing the analytic continuation ($\nu\to 0$) we finally get [34]

$$\langle\epsilon_{\rm Cas}\rangle_0 = -\frac{\pi^2}{1440L^4}, \tag{36}$$

the well-known result for the Casimir energy density of a massless scalar field. We point out that, when $\beta_p = 0$, the quantity in curly brackets appearing in (33) is in itself real. In other words, the effective action W has no imaginary part. This means that—as previously recalled—we do not expect particle creation in a flat spacetime background.

We can also find the *attractive* force between the plates (per unit surface), obtaining

$$f^{(0)}_{\rm Cas} = \frac{F^{(0)}_{\rm Cas}}{A} = -\frac{1}{A}\frac{\partial E^{(0)}_{\rm Cas}}{\partial L} = -\frac{1}{A}\frac{\partial AL\langle\epsilon_{\rm Cas}\rangle_0}{\partial L} = -\frac{\pi^2}{480L^4}. \tag{37}$$

B. Bianchi-I spacetime background

Let us now consider the case $\beta_p \neq 0$, corresponding to the anisotropic background we have introduced in Sec. II. Looking at (33) we see that the correction to the flat Casimir result (36) reads

$$\langle\delta\epsilon_{\rm Cas}\rangle = \frac{1}{2(2\pi)^3L}\lim_{\nu\to 0}\Re{\rm e}\left\{i\int_0^{+\infty} ds s^{\nu-1}\left[\int_0^{2\pi}d\theta\int_0^{+\infty}p_\perp dp_\perp\int_{-\infty}^{+\infty}d\omega\sum_n 2|\beta_p|^2 e^{-isp_\perp^2}e^{-is(n\pi/L)^2}e^{is\omega^2}\right]\right\}. \tag{38}$$

Notice that, in (38), n, ω, and $p_\perp$ have to be considered as *independent* degrees of freedom. Before we go on, we need an explicit expression for the quantity β_p. For computational convenience let us assume the metric perturbation to have a Gaussian profile

$$h(t) = He^{-\sigma^2t^2}, \tag{39}$$

where H (with $|H| \ll 1$) represents the amplitude of the perturbation and $1/\sigma$ gives a rough estimate of the time duration of the perturbation. Notice that $\lim_{t\to\pm\infty} h(t) = 0$, thus guaranteeing that the in and out spacetime regions are Minkowskian, so that the asymptotic definition of the corresponding vacuum state is unambiguous. Using (20) we have

$$\beta_p = -\frac{iH\sqrt{\pi}}{2\omega\sigma}p_\perp^2 e^{-(\omega/\sigma)^2}\cos 2\theta. \tag{40}$$

Combining (38) and (40), and performing the integration over the variables s and θ, we obtain

$$\langle\delta\epsilon_{\rm Cas}\rangle = \frac{H^2}{16\pi\sigma^2L}\lim_{\nu\to 0}\Re{\rm e}\left\{i^{1-\nu}\Gamma(\nu)\times\sum_n\int_0^{+\infty}\frac{d\omega}{\omega^2}\int_0^{+\infty}dp_\perp p_\perp^5\frac{e^{-2\omega^2/\sigma^2}}{(\Omega^2-\omega^2)^\nu}\right\}, \tag{41}$$

where $\Omega^2 = p_\perp^2 + (\frac{n\pi}{L})^2$. The last integral in (41) can be solved recalling that [39,40]

$$\int_0^{+\infty}\frac{q^\mu dq}{(q^2+C^2)^\nu} = \frac{\Gamma(\frac{1+\mu}{2})\Gamma(\nu-\frac{1+\mu}{2})}{2\Gamma(\nu)C^{2\nu-1-\mu}}. \tag{42}$$

Hence

$$\langle\delta\epsilon_{\rm Cas}\rangle = \frac{H^2}{16\pi\sigma^2L}\lim_{\nu\to 0}\Re{\rm e}\left\{i^{1-\nu}\Gamma(\nu-3)\left(\frac{L}{\pi}\right)^{2\nu-6}I(\nu)\right\}, \tag{43}$$

where we have put

$$I(\nu) = \int_0^{+\infty}\frac{d\omega}{\omega^2}e^{-2\omega^2/\sigma^2}\sum_n\frac{1}{(n^2-\frac{\omega^2L^2}{\pi^2})^{\nu-3}}. \tag{44}$$

$I(\nu)$ is potentially plagued by singularities. However, in (43) we may replace $I(\nu)$ with

$$\tilde{I}(\nu) = \int_0^{+\infty}\frac{d\omega}{\omega^2}e^{-2|\omega^2|/\sigma^2}\sum_n\frac{1}{(n^2-\frac{\omega^2L^2}{\pi^2})^{\nu-3}}, \tag{45}$$

which obviously coincides with $I(\nu)$, since $\omega\in\mathbb{R}$. The following change of variable

$$\omega = -iu, \qquad u \in \mathbb{C}, \tag{46}$$

yields

$$\tilde{I}(\nu) = \int_0^{+i\infty} \frac{idu}{(u+\epsilon)^2} e^{-\frac{2|-(u+\epsilon)^2|}{\sigma^2}} \sum_n \frac{1}{\left(n^2 + \frac{(u+\epsilon)^2 L^2}{\pi^2}\right)^{\nu-3}}. \tag{47}$$

Notice that in (47) u is a purely imaginary quantity, ranging from 0 through $+i\infty$. In order to get rid of the poles of (47), we have also introduced—as usual—a small quantity $\epsilon > 0$ whose limit $\epsilon \to 0$ we will take at the end of calculations. We see that the poles of (47) lie at $u = -\epsilon$ and (as long as $\nu > 3$) at $u = -\epsilon \pm \frac{in\pi}{L}, n \in N - \{0\}$, i.e., to the left of the imaginary axis in the u complex plane. Hence, none of those poles is encountered in the $(0; +i\infty)$ integration involved in (47). Consider now the integral $\tilde{I}(\nu)$ extended to the whole u complex plane along a closed path $\Gamma = (0; +i\infty) \cup \gamma_\infty \cup (-\infty; 0)$, with γ_∞ being a curve placed at infinity in the u plane. There being no poles enclosed by Γ, the integrand is a holomorphic function in the considered domain. Taking into account that the contribution along γ_∞ is vanishing, we can write

$$\tilde{I}(\nu) = i \int_0^{+\infty} \frac{du}{(u+\epsilon)^2} e^{-\frac{2(u+\epsilon)^2}{\sigma^2}} \sum_n \frac{1}{\left(n^2 + \frac{(u+\epsilon)^2 L^2}{\pi^2}\right)^{\nu-3}}, \tag{48}$$

where we have made the replacement $|-(u+\epsilon)^2| \to (u+\epsilon)^2$, since u is now running along the real axis. We also recognize that the infinite sum in (48) represents a inhomogeneous Epstein-Hurwitz ζ-function $\zeta_{\mathrm{EH}}(s, q^2)$, which can be analytically continued to give [36,41]

$$\begin{aligned}\zeta_{\mathrm{EH}}(s, q^2) &= \sum_{n=1}^{\infty} (n^2 + q^2)^{-s} \\ &= -\frac{q^{2s}}{2} + \frac{\sqrt{\pi}}{2\Gamma(s)} \Gamma(s - 1/2) q^{-2s+1} \\ &\quad + \frac{2\pi^s}{\Gamma(s)} \; q^{1/2-s} \sum_{n=1}^{\infty} n^{s-1/2} K_{s-1/2}(2\pi n q),\end{aligned} \tag{49}$$

where $s = \nu - 3$, $q = \frac{(u+\epsilon)L}{\pi}$, and the rapidly converging sum involves the modified Bessel functions $K_\mu(z)$. Use of (49) allows to explicitly perform the u integration in (48). After a tedious but straightforward calculation we find that $I(\nu)$ is made of three contributions, stemming from the three pieces composing (49). Correspondingly, the correction to the Casimir energy density (43) can be written as

$$\langle \delta\epsilon_{\mathrm{Cas}} \rangle = \langle \delta\epsilon_{\mathrm{Cas}} \rangle_1 + \langle \delta\epsilon_{\mathrm{Cas}} \rangle_2 + \langle \delta\epsilon_{\mathrm{Cas}} \rangle_3, \tag{50}$$

where, in the limit $\epsilon \to 0$,

$$\langle \delta\epsilon_{\mathrm{Cas}} \rangle_1 = \frac{H^2}{32\pi\sigma^2 L} \lim_{\nu\to 0} \Re\mathrm{e}\left\{ i^{-\nu} \Gamma(\nu - 3) \frac{\sigma^{-2\nu+5}}{2^{-\nu+7/2}} \Gamma\left(-\nu + \frac{5}{2}\right) \right\}, \tag{51}$$

$$\langle \delta\epsilon_{\mathrm{Cas}} \rangle_2 = -\frac{H^2}{16\pi\sigma^2} \lim_{\nu\to 0} \Re\mathrm{e}\left\{ i^{-\nu} \frac{\sqrt{\pi}}{2} \Gamma\left(\nu - \frac{7}{2}\right) \frac{\sigma^{-2\nu+6}}{2^{-\nu+4}} \Gamma(-\nu + 3) \right\}, \tag{52}$$

$$\langle \delta\epsilon_{\mathrm{Cas}} \rangle_3 = -\frac{H^2}{16\pi\sigma^2 L} \lim_{\nu\to 0} \Re\mathrm{e}\left\{ i^{-\nu} 2\pi^{\nu-3} \left(\frac{L}{\pi}\right)^{\nu-5/2} \sum_n n^{\nu-7/2} J(\nu) \right\}, \tag{53}$$

having defined in (53)

$$J(\nu) = \int_0^{+\infty} du e^{-\frac{2u^2}{\sigma^2}} u^{-\nu+3/2} K_{\nu-\frac{7}{2}}(2nLu). \tag{54}$$

V. HANDLING THE DIVERGENCES

When working in a flat spacetime background, analytic continuation [36] often allows one to get rid of the divergences usually appearing in the evaluation of the vacuum energy, thus straightforwardly leading to the physical result one is looking for. This is just what happened in Sec. IV B, when computing the Casimir energy density. However, in the presence of a time-dependent background, such a mathematical tool is generally not enough, and further physical considerations are required in order to remove the emerging infinities.

Let us consider in some detail the various contributions (51)–(53) to the Casimir energy density in the $\nu \to 0$ limit. The first one is manifestly divergent, due to the $\Gamma(\nu - 3)$ pole. Such a term gives an infinite contribution to the Casimir energy, $E_{\mathrm{Cas}} = AL\langle \delta\epsilon_{\mathrm{Cas}} \rangle$ which is proportional to A, without any reference to the plate separation L. Following Schwinger's argument [33], such energy has to be normalized to zero, so there must be a term in the additional counterterm appearing in (23) that removes it.

The second contribution (52) represents a uniform spatial density of vacuum energy, independent of L. Since we are interested in vacuum energy dependence on the plate separation, we can discard this term, again absorbing it in the counterterms appearing in (23).

All we are left with is the last term in (50). Handling this term requires some care. Consider the following slightly modified form of the integral appearing in (53):

$$J(\nu,\nu') = \int_0^{+\infty} du e^{-\frac{2u^2}{\sigma^2}} u^{-\nu+\nu'+3/2} K_{\nu-7/2}(2nLu), \quad (55)$$

which obviously reduces to $J(\nu)$ in the $\nu' \to 0$ limit. The above integral converges provided the following inequality holds:

$$\Re\mathrm{e}(-\nu+\nu'+3/2) > |\Re\mathrm{e}(\nu-7/2)| - 1 \Rightarrow \nu' > \nu, \quad (56)$$

and can be solved [40] in terms of Whittaker functions $W_{\mu,\lambda}(z)$. We perform the integral (55) assuming that (56) is satisfied. Subsequently, we exploit analytic continuation, placing $J(\nu,\nu')$ in (53) and taking both $\nu \to 0$ and $\nu' \to 0$, thus obtaining

$$\begin{aligned}\langle\delta\epsilon_{\mathrm{Cas}}\rangle &\equiv \langle\delta\epsilon_{\mathrm{Cas}}\rangle_3 \\ &= -\frac{H^2}{16\pi\sigma^2 L} \lim_{\{\nu,\nu'\}\to 0} \Re\mathrm{e}\left\{ i^{-\nu} 2\pi^{\nu-3}\left(\frac{L}{\pi}\right)^{\nu-5/2} \sum_{n=1}^{\infty} n^{\nu-7/2} J(\nu,\nu')\right\} \\ &= \frac{H^2}{2^{15/4}\pi\sigma^{1/2}L^{9/2}} \sum_{n=1}^{\infty} n^{-9/2} e^{\frac{(\sigma L n)^2}{4}} W_{-\frac{3}{4},-\frac{7}{4}}\left(\frac{(\sigma L n)^2}{2}\right).\end{aligned} \quad (57)$$

Equation (57) is our main result. Being related to the real part of the effective action W, $\langle\delta\epsilon_{\mathrm{Cas}}\rangle$ represents a correction to the static Casimir energy density. In other words, it is a correction to the so-called "vacuum polarization."

Inspection of (57) shows that $\langle\delta\epsilon_{\mathrm{Cas}}\rangle$, induced by the chosen time-dependent perturbation (39) of the spacetime background is *positive*, while the Casimir energy density is (usually) *negative*.

We are thus in presence of a sort of "memory effect" in the Casimir energy, since the vacuum polarization retains trace of the gravitational perturbation at $t \to +\infty$, when the perturbation has left the cavity. Furthermore, the correction $\langle\delta\epsilon_{\mathrm{Cas}}\rangle$ gives rise to a reduction of the absolute value of the Casimir energy. Hence, we expect a tiny *reduction* of the Casimir force (37) acting between the plates once the gravitational perturbation is over.

Equations (36) and (37) differ by a factor of 2 from the results obtained by Casimir [5] considering an electromagnetic field. This is usually ascribed to the presence of two polarization photon states. We will briefly analyze the electromagnetic field case in Appendix A, finding that, indeed, also the above discussed memory effect comes with a factor of 2, as naively expected.

VI. THE GRAVITATIONAL WAVE CASE

Equation (57) can find application in the interesting case in which the background spacetime is that of an incoming gravitational wave, depicted as a short perturbation propagating along the z direction. In such a case $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, and once the transverse traceless gauge has been employed, the spacetime line element reads

$$\begin{aligned} ds^2 = dt^2 &- (1+h_+(u))dx^2 - (1-h_+(u))dy^2 \\ &- 2h_\times(u)dxdy - dz^2, \end{aligned} \quad (58)$$

with $h_+(u)$ and $h_\times(u)$ being the two physical states of polarization of the wave and $u = t - z$. Let us assume, for the sake of simplicity, that the wave has the form of a *linearly* polarized, short Gaussian pulse (the more general case of a gravitational pulse, propagating at an arbitrary direction and with both polarization states will be discussed in Appendix B), so that $h_\times(u) = 0$ and

$$h_+(u) \equiv h(t-z) = He^{-\sigma^2(t-z)^2}, \quad (59)$$

where H can now be interpreted as the gravitational "strain," while σ gives a rough estimate of the time duration of the pulse. If $\sigma L \ll 1$, we may expand $h(t-z)$ around $z = 0$ (one of the plate locations), considering $h(t-z)$ as a function of time t only. So we put

$$h(t-z) \simeq h(t) = He^{-\sigma^2 t^2} \quad (60)$$

inside the Casimir cavity. This implies that we can study the interaction of the confined quantum field with the gravitational wave just employing the Bianchi-I spacetime model we introduced in Sec. II [42].

Exploiting the rapid convergence of the sum appearing in (57), we expand $\langle\delta\epsilon_{\mathrm{Cas}}\rangle$ around the small parameter σL obtaining, to the leading order in $(\sigma L n)$,

$$\langle\delta\epsilon_{\mathrm{Cas}}\rangle \simeq \frac{15H^2}{64\sqrt{2\pi}\sigma^3 L^7}. \quad (61)$$

So, once the perturbation due to the gravitational wave pulse is over (at $t \to +\infty$), the total Casimir energy inside the cavity can be written as [recall (36)]

$$\langle E_{\mathrm{Cas}} \rangle = -\frac{A\hbar c \pi^2}{1440 L^3}\left(1 - \frac{675 c^3 H^2}{2\sqrt{2}\pi^{5/2}\sigma^3 L^3}\right), \quad (62)$$

where we have restored, for clarity, SI units. Notice that $\sigma \simeq \frac{1}{\Delta t_{\mathrm{pert}}}$, with Δt_{pert} being the typical time duration of the gravitational pulse. Looking at (62) it might seem that a sufficiently long gravitational pulse could cause the complete vanishing of the Casimir energy, or even a change in its sign, turning the Casimir force in a *repulsive* one. Although suggestive, such an occurrence cannot be considered too seriously, since our calculations have been carried on following a perturbative approach [see, e.g., the solution of the field equations, (14) and (17)], based on the smallness of the perturbation $h(t)$ in the background spacetime. So, in order our results can be considered predictive, it is likely that

$$\frac{675 c^3 H^2}{2\sqrt{2}\pi^{5/2}\sigma^3 L^3} \ll 1, \quad (63)$$

which, in turn, implies

$$\Delta t_{\mathrm{pert}} \ll H^{-\frac{2}{3}} L \ \mathrm{ns}. \quad (64)$$

For example, considering a Casimir cavity whose plate separation is $L = 10^{-6}$ m and a gravitational wave pulse having a strain $H = 10^{-21}$, the above constraint would give $\Delta t_{\mathrm{pert}} \ll 10^{-1}$ s.

VII. CONCLUDING REMARKS

In this paper we have studied the corrections to the vacuum energy density of a massless scalar field confined to a Casimir cavity in an anisotropic, time-dependent Bianchi-I background spacetime. We have modeled the background introducing small perturbations around flat spacetime, requiring them to vanish in the remote past and future, thus allowing for asymptotically Minkowskian regions. This, in turn, has guaranteed an unambiguous definition of the in- and out-field vacua.

Following Schwinger's proper-time approach, we have computed the real part of the effective action W for the field in the cavity, evaluating $\Re\mathrm{e}(W)$ in the far future, once the gravitational interaction was over. After a renormalization procedure, required to get rid of unphysical divergences, we have recovered the zero-point energy of the field, finding a permanent shift with respect to the vacuum energy density obtained in the flat spacetime case.

As an application of our result, in Sec. VI we have considered the case of the interaction with a weak gravitational wave, whose metric, in the small region occupied by the Casimir apparatus, looks just like that of the Bianchi-I spacetime model previously employed.

The main outcome of the present analysis is that, generally speaking, the gravitationally induced shift in the vacuum energy acts in order to *reduce* the absolute value of the negative Casimir energy density. Hence, also a tiny reduction in the attractive force between the Casimir plates is expected.

At a first sight, the result could sound odd. Actually, while (nonlocal) particle creation out of the field vacuum is expected as a result of the past interaction with a time-varying background geometry (and such latter effect is indeed encoded in the imaginary part of the action we have computed), distortions in the (local) vacuum polarization are usually considered to appear *during* the interaction.

However, we have shown that vacuum polarization effects can also exhibit a nonlocal behavior, thus yielding a permanent shift in the vacuum energy, once the spacetime perturbation is over. In that respect, we can consider the present result as a sort of gravitational memory of the Casimir effect.

Indeed, when computing the field vacuum energy, any renormalization procedure implies—more or less explicitly—a mode cutoff, which eventually captures the global spacetime structure through the long wavelength field modes [30]. So, also local quantities, such as those related to the vacuum polarization, after renormalization, can carry physically measurable information on the whole story of the gravitational interaction.

Some further comments are in order.

(1) While the present analysis has been carried out taking into account a massless scalar field, a more realistic study should involve the electromagnetic field. It is often stated that in this latter case the Casimir energy is doubled with respect to the scalar case, due to the presence of two polarization states. In Appendix A we have briefly considered the electromagnetic case, showing how the Schwinger approach offers a clear, beautiful explanation of such a naively expected result. Indeed, the evaluation of the total trace picks up all the degrees of freedom of the physical system, including the polarization states. A similar result is also found in Appendix B, where the interaction with a gravitational wave pulse carrying both the polarization states is considered.

(2) The constraints (4) and (5) adopted in Sec. II are those usually employed in cosmology when describing a small departure from the isotropic, homogeneous FLRW universe (see, e.g., [1,38]). Here, they have been adopted in order to keep as clear as possible the analysis of the gravitational influence on the Casimir effect. Relaxing such conditions would require a specific choice of the gravitational background in order to obtain definite results.

(3) As regards the constraint (6), this has been imposed to avoid further complications stemming from tidal

effects induced by the time-varying gravitational field on the boundaries (i.e., the cavity plates). Up to the present order of approximation, it is likely that tidal contributions simply add to the correction we found in the Casimir energy. On the other hand, tidal effects can also be analyzed (basically by virtue of the equivalence principle) considering a cavity with moving plates. In that respect, this mimics the dynamical Casimir effect [35], whose analysis suggests that the shift in the vacuum energy is proportional to $(v/c)^2$, with v being the plate velocity (see, e.g., [43] and references cited therein). In the present case the gravitationally induced fluctuation between the plates is roughly $\Delta L = h(t)L$. Using, e.g., (60) we find $(v/c)^2 \sim (\dot{h}(t)L/c)^2 = (\frac{2L\sigma^2 t}{c} H e^{-\sigma^2 t^2})^2$. Hence, tidal effects are expected to become negligible when compared to the correction $\langle \delta\epsilon_{\rm Cas}\rangle$ we found, as the latter scales with an inverse power of L (in a typical cavity we could have $L \sim 10^{-6}$ m, or less).

(4) Removal of the constraint (6) becomes nevertheless unavoidable in the case of a gravitational wave propagating at an arbitrary direction with respect to the reference frame of the Casimir cavity. Now, the polarization states of the wave involve the z direction too, and we need to transform the gravitational pulse to the cavity frame. A thorough analysis of such a case has been presented in Appendix B. The proposed approach can obviously be employed also when considering more general Bianchi spacetimes (e.g., type IX).

(5) As pointed out in Sec. VI, the shift in the vacuum energy shows a divergence in the $\sigma \to 0$ limit [see (57)]. Namely, $\langle \delta\epsilon_{\rm Cas}\rangle$ seems to increase without any upper bound, as the typical duration of the background perturbation increases. However, as discussed at the end of Sec. VI, this behavior must be considered with care, mainly because our calculations rely upon a *perturbative* approach.

Notice also that, in some respect, the present divergence represents the analog, in the time domain, of the divergence appearing in the flat Casimir effect [see (36)], as $L \to 0$. Consequently, the correction $\langle \delta\epsilon_{\rm Cas}\rangle$ to the polarization effect can be considered reasonably meaningful as far as the ratio $\langle \delta\epsilon_{\rm Cas}\rangle / \langle \epsilon_{\rm Cas}\rangle \ll 1$.

(6) On the other hand, the existence of an upper bound on the time duration of the perturbation, related to the discussed memory effect, could also have a different, deeper origin. Leaving aside the issues related to the perturbative approach, the increase of $\langle \delta\epsilon_{\rm Cas}\rangle$ as the time duration Δt of the gravitational perturbation increases, is suggestive of an upper limit for Δt which, in the case of the gravitational pulse discussed in Sec. VI, is about $\Delta t \sim L H^{-2/3}$ [see (63) and (64)]. In such a limiting case, we should observe the Casimir energy to vanish.

In any case, the effect of the gravitational perturbation is to reduce the absolute value of the (negative) Casimir energy. Such a behavior could recall (or even represent) a manifestation of the quantum energy inequalities (QEIs), first pioneered by Ford *et al.* [44], which dictate bounds on the duration of negative energy, hence *almost* preserving the weak energy conditions (WECs), which are violated in the Casimir effect. Differently stated, QEIs require that WEC violations are either small in magnitude or (as in our case) short-lived [45].

The discussed effect, being a correction to the Casimir effect (already tiny in itself), is too small to be detected in the case of any realistic gravitational wave amplitude. However, it could become relevant in some (maybe astrophysical) scenarios, in which the gravitational amplitude, although small, is not *so* small.

The employed weak field approximation seems to suggest that—under extreme conditions—the gravitational background could make the vacuum energy to vanish or even change its sign, turning the Casimir force into an attractive one. All this is probably only a signal of the breakdown of the perturbative approach. In that respect, investigation of higher-order corrections and/or use of nonperturbative techniques could shed further light on the effect.

Such a deeper analysis is clearly beyond the scope of the present paper. We hope this will be the subject of a future study.

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APPENDIX A: THE ELECTROMAGNETIC CASE

Although in most of the literature the study of the Casimir effect (and its possible modifications) is usually performed considering—for sake of simplicity—a scalar field, just as we did in the present paper, in his original seminal work (1948) Casimir considered an *electromagnetic* field confined to a cavity [5,6]. For completeness, in this appendix we propose a short analysis of the studied memory effect in the electromagnetic case, also comparing the results to the scalar one.

As in Sec. II, we will take the Casimir cavity oriented in space so that the confining plates, orthogonal to the z axis, are placed at $z = 0$ and $z = L$.

In the linearized theory, the Lagrangian of a given physical system is usually written as a sum of various contributions stemming from the possible couplings among the matter fields *and* between matter fields and gravity.

Here, we are interested in the coupling between gravity and the electromagnetic field confined to a small cavity.

The Lagrangian density can be split into two contributions [hereafter, a super- or subscript (0) will mean that the corresponding quantity is evaluated in a *flat* spacetime background]

$$\mathcal{L} = \mathcal{L}^{(0)} + \mathcal{L}_g, \tag{A1}$$

where $\mathcal{L}^{(0)}$ is the electromagnetic Lagrangian density in flat spacetime and $\mathcal{L}_g$ is the $O(h)$ gravitational contribution. We will use the following Lagrangian density:

$$\mathcal{L}^{(0)} = -\frac{1}{2}\partial_\mu A^\nu \partial^\mu A_\nu, \tag{A2}$$

first proposed by Fermi [46]. It is straightforward to check that (A2) yields indeed the same equations of motion for the 4-potential ($\Box A^\mu = 0$), provided the Lorentz gauge $\partial_\mu A^\mu = 0$ is imposed as a subsidiary condition.

The gravitational contribution $\mathcal{L}_g$ reads

$$\mathcal{L}_g = -\frac{1}{2}h^{\mu\nu}T^{(0)}_{\mu\nu}, \tag{A3}$$

with $T^{(0)}_{\mu\nu}$ being the electromagnetic stress-energy tensor evaluated in the flat spacetime

$$T^{(0)}_{\mu\nu} = \frac{\partial\mathcal{L}^{(0)}}{\partial(\partial^\mu A_\rho)}\partial_\nu A_\rho - \eta_{\mu\nu}\mathcal{L}^{(0)}. \tag{A4}$$

We point out that, at the present $O(h)$ order of approximation, $T^{(0)}_{\mu\nu}$ is independent of $h_{\mu\nu}$, so that it satisfies the ordinary conservation conditions, namely, $\partial_\mu T^{\mu\nu}_{(0)} = 0$. Furthermore, (A3) is gauge invariant with respect to gauge transformations of the coordinates

$$x^\mu \to x'^\mu = x^\mu + \xi^\mu(x^\nu), \tag{A5}$$

causing a change in the gravitational potentials

$$h_{\mu\nu} \to h'_{\mu\nu} = h_{\mu\nu} - \partial_\mu\xi_\nu - \partial_\nu\xi_\mu. \tag{A6}$$

Actually, the corresponding variation in $\mathcal{L}_g$ reads, up to first order in h,

$$\begin{aligned}\delta\mathcal{L}_g &= -\frac{1}{2}(\delta h_{\mu\nu})T^{\mu\nu}_{(0)} \\ &= \partial_\mu\xi_\nu T^{\mu\nu}_{(0)} = -\xi_\nu\partial_\mu T^{\mu\nu}_{(0)} + \partial_\mu(\xi_\nu T^{\mu\nu}_{(0)}) \\ &= \partial_\mu(\xi_\nu T^{\mu\nu}_{(0)}),\end{aligned} \tag{A7}$$

being $\partial_\mu T^{\mu\nu}_{(0)} = 0$, as stated. $\mathcal{L}_g$ is gauge invariant, since adding a 4-divergence does not alter the action. From (A2) we get the (flat spacetime) stress-energy tensor

$$T_{\mu\nu} = -\partial_\mu A_\rho\partial_\nu A^\rho + \frac{1}{2}\eta_{\mu\nu}\partial_\alpha A_\rho\partial^\alpha A^\rho. \tag{A8}$$

Plugging (A8) in (A3) yields

$$\begin{aligned}\mathcal{L}_g &= -\frac{1}{2}\left[\left(-h^{\mu\nu} + \frac{1}{2}h\eta^{\mu\nu}\right)\partial_\mu A_\rho\partial_\nu A^\rho\right] \\ &= \frac{1}{2}\bar{h}^{\mu\nu}\partial_\mu A_\rho\partial_\nu A^\rho,\end{aligned} \tag{A9}$$

where we have introduced the reduced potentials

$$\bar{h}^{\mu\nu} = h^{\mu\nu} - \frac{1}{2}h\eta^{\mu\nu}. \tag{A10}$$

The full electromagnetic Lagrangian (A1) thus reads

$$\mathcal{L} = -\frac{1}{2}\partial_\mu A^\nu\partial^\mu A_\nu + \frac{1}{2}\bar{h}^{\mu\nu}\partial_\mu A_\rho\partial_\nu A^\rho, \tag{A11}$$

and the corresponding equations of motion for A^μ are derived from the Euler-Lagrange equations

$$\Box A^\rho - \bar{h}^{\mu\nu}\partial_\mu\partial_\nu A^\rho - (\partial_\mu\bar{h}^{\mu\nu})\partial_\nu A^\rho = 0. \tag{A12}$$

Exploiting the above recalled gauge invariance, we may impose—as usual—the "harmonic" gauge $\partial_\mu\bar{h}^{\mu\nu} = 0$. Besides, from (2) and (5) we see that $\bar{h}^{\mu\nu} = h^{\mu\nu}$, so hereafter we will drop the overline. We finally obtain the (gravitationally corrected) Maxwell equations for the 4-potential A^μ,

$$\Box A^\rho - \bar{h}^{\mu\nu}\partial_\mu\partial_\nu A^\rho = 0. \tag{A13}$$

Notice that (A13) is formally equivalent to the scalar case (8) discussed in the text. In particular, given the metric (2) and the constraints (5) and (6), we have

$$(\Box + \hat{V})A^\rho(x) = 0, \quad r = 1, 2, \tag{A14}$$

where

$$\hat{V} = h_x(t)\partial_x^2 + h_y(x)\partial_y^2 = h(t)(\partial_x^2 - \partial_y^2), \tag{A15}$$

just as in the scalar case [see (9)].

It is well known that the Lorentz gauge $\partial_\mu A^\mu = 0$ does not fix uniquely A_μ. There remains a residual gauge freedom so that, if A_μ satisfies the Lorentz condition, so will $A'_\mu = A_\mu + \partial_\mu\Lambda(x)$ as long as $\Box\Lambda(x) = 0$. Choosing $\Lambda(x)$ to satisfy $\partial_t\Lambda = -A^0$, we get $A'^0 = 0$, hence (from $\partial_\mu A^\mu = 0$) $\vec{\nabla}\cdot\vec{A}' = 0$. Potentials satisfying this additional condition, namely, $A^0 = 0$, $\vec{\nabla}\cdot\vec{A} = 0$ are said to belong to the radiation (or Coulomb) gauge. So to say, the gauge shoots *twice*, hence reducing from four to two the number of independent components of A_μ. These two remaining degrees of freedom have to be identified with the two physical polarization states of the radiation field.

In such a gauge $\vec{E} = -\frac{\partial A^0}{\partial t}$ and $\vec{\nabla} \times \vec{A}$, being $\vec{E}$ and $\vec{B}$ the electric and magnetic fields which must satisfy the boundary conditions at the cavity plates

$$E_\parallel|_{z=0} = E_\parallel|_{z=L} = 0, \tag{A16}$$

$$B_\perp|_{z=0} = B_\perp|_{z=L} = 0, \tag{A17}$$

where $E_\parallel$ is the electric field component parallel to the cavity plates, while $B_\perp$ is the magnetic field component normal to the plates.

In the flat spacetime case ($h^{\mu\nu} = 0$) the corresponding modes read

$$\overset{(0)}{A}{}^j{}_r(x) = N_r f^{\hat{j}}(n,z)\epsilon_r^{\hat{j}} e^{i\vec{k}_\perp\cdot\vec{x}_\perp} e^{-i\omega t}, \quad r = 1, 2, \tag{A18}$$

where repeated hatted indices ($j = \{x, y, z\}$) are meant to be *not* summed and

$$f^j(n,z) = \begin{pmatrix} \sin(n\pi z/L) \\ \sin(n\pi z/L) \\ -i\cos(n\pi z/L) \end{pmatrix}. \tag{A19}$$

Also, N_r are normalization constants, $\vec{k}_\perp = (k_x, k_y)$, $\vec{x}_\perp = (x, y)$, and $\vec{\epsilon}_r$ ($r = 1$, 2) represent the two physical transverse polarization states of the electromagnetic field, obeying $\vec{\epsilon}_r \cdot \vec{\epsilon}_s = \delta_{rs}$ and $\vec{\epsilon}_r \cdot \vec{k} = 0$ ($r, s = 1, 2$).

In the time-varying spacetime (2) the corresponding field modes are similar to (10),

$$A_r^j(x) = N_r f^{\hat{j}}(n,z)\epsilon_r^{\hat{j}} e^{i\vec{k}_\perp\cdot\vec{x}_\perp}\eta(t), \quad r = 1, 2, \tag{A20}$$

where $N_r = (\frac{2}{(2\pi)^3 L})^{1/2}$, ($r = 1$, 2) is an overall normalization constant, and $\eta(t)$ is still given by (14) (with the $p \to k$ replacement). From here on, calculations are basically the same as those carried out in Sec. III, with a few obvious changes in the notation. The only relevant point to raise is that in the evaluation of the total trace (25) an extra summation over the two polarization states is now implied. In particular, (27) becomes

$$\sum\!\!\!\!\!\!\int d\alpha \equiv \sum_{n,n',r} \int dk_0 dk_0' dk_\perp dk_\perp', \tag{A21}$$

with $r = 1$, 2. The total trace reads

$$\mathrm{Tr} e^{-is\hat{H}} = \int dt \int d^3x \int_0^{2\pi} d\theta \int_0^{+\infty} k_\perp dk_\perp \int_{-\infty}^{+\infty} d\omega \times \sum_{n,r} |\langle x|\alpha\rangle_r|^2 e^{-isk_\perp^2} e^{-is(n\pi/L)^2} e^{is\omega^2}, \tag{A22}$$

where $|\langle x|\alpha\rangle_r|^2 = \sum_j |A_r^j(x)|^2$. Using (A20) we find

$$\begin{aligned} \sum_r \int d^3x |\langle x|\alpha\rangle_r|^2 &= \sum_r \int dx dy N_r^2 |\eta(t)|^2 \frac{L}{2} |\epsilon_r|^2 \\ &= \sum_r \frac{A}{(2\pi)^3} |\eta(t)|^2 \\ &= 2\frac{A}{(2\pi)^3} |\eta(t)|^2. \end{aligned} \tag{A23}$$

Recalling (14) and making use, as in the scalar field case, of the rotating-wave approximation, we get

$$\mathrm{Tr} e^{-is\hat{H}} = 2 \cdot \frac{A}{(2\pi)^3} \int dt \int_0^{2\pi} d\theta \int_0^{+\infty} k_\perp dk_\perp \int_{-\infty}^{+\infty} d\omega \times \sum_n (|\alpha_k(t)|^2 + |\beta_k(t)|^2) e^{-isk_\perp^2} e^{-is(n\pi/L)^2} e^{is\omega^2}. \tag{A24}$$

Comparing (A24) and (31) we see appearance of an extra factor of 2, ultimately due to the two polarization states of the electromagnetic field. Such a factor propagates to all the main results obtained in the scalar field case. So, for example, in the flat spacetime case, the Casimir energy and force [see (36) and (37)] now read

$$\langle \epsilon_{\mathrm{Cas}} \rangle_{em}^{(0)} = -\frac{\pi^2}{720 L^4}, \tag{A25}$$

$$f_{\mathrm{Cas}}^{em(0)} = -\frac{\pi^2}{240 L^4}, \tag{A26}$$

namely, the Casimir result for the electromagnetic field case.

The above results support (at least in the present case) the naive assumption that to obtain the energy density the force due to electromagnetic field fluctuations between two parallel conducting plates it suffices to multiply by a factor of 2 the corresponding scalar field result, accounting for the two photon polarization states.

The above extra factor of 2 obviously appears also in our main result, Eq. (57) and, e.g., in (62), as regards the gravitational case. So—at least to the present order of approximation—in the electromagnetic case we basically observe (for each polarization state) the very same effects discussed in the scalar case.

APPENDIX B: A MORE GENERAL CASE OF GRAVITATIONAL WAVE INTERACTION

It is possible, although a bit tricky, to examine the more general situation in which the gravitational perturbation does not propagate along the z direction of the cavity reference frame. Consider a gravitational (plane) wave, propagating along an arbitrary direction $\hat{\Omega} = (\cos\varphi \sin\vartheta, \sin\varphi \sin\vartheta, \cos\vartheta)$ with respect to the reference frame of the Casimir cavity, $\{x, y, z\}$. Let $\{\bar{x}, \bar{y}, \bar{z}\}$ be the "wave frame," namely, the reference frame with respect to which the gravitational wave is described [in the transverse traceless (TT) gauge] by the metric

$$ds^2 = g_{\mu\nu} d\bar{x}^\mu d\bar{x}^\nu = -d\bar{t}^2 + d\vec{\bar{x}}^2 + \bar{h}_{\mu\nu} d\bar{x}^\mu d\bar{x}^\nu, \qquad \text{(B1)}$$

where

$$\bar{h}_{\mu\nu} = \sum_{\lambda=+,\times} H_\lambda \bar{e}^\lambda_{\mu\nu} h(\bar{t} - \bar{z}). \qquad \text{(B2)}$$

In (B2) the index λ runs over the two physical polarization states (plus and cross) of the wave, H_λ are the amplitudes of the two polarizations, and

$$\bar{e}^+_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \bar{e}^\times_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad \text{(B3)}$$

are the corresponding polarization tensors. Let us define $R(\varphi, \vartheta) \equiv R_z(-\varphi) R_y(\vartheta)$, $(0 \le \varphi \le 2\pi,\ 0 \le \vartheta \le \pi)$, where

$$R_y(\vartheta) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\vartheta & 0 & \sin\vartheta \\ 0 & 0 & 1 & 0 \\ 0 & -\sin\vartheta & 0 & \cos\vartheta \end{pmatrix}, \qquad \text{(B4)}$$

$$R_z(-\varphi) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\varphi & -\sin\varphi & 0 \\ 0 & \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \qquad \text{(B5)}$$

Using $R(\phi, \vartheta)$, we transform (B2) from the TT frame to the Casimir cavity frame. Noticing that $e^\lambda = (R^T \bar{e}^\lambda R)$, we get (a superscript T means matrix transposition) [47]

$$h_{\mu\nu} = \sum_{\lambda=+,\times} H_\lambda {e^\lambda}_{\mu\nu} h(t - \hat{\Omega} \cdot \vec{x}). \qquad \text{(B6)}$$

Expanding (B6) near $\vec{x} = 0$ and neglecting the spatial dependence (thanks to the smallness of the cavity size, compared to the typical gravitational wavelength), we may approximate

$$h_{\mu\nu} = h(t) \sum_{\lambda=+,\times} H_\lambda {e^\lambda}_{\mu\nu}. \qquad \text{(B7)}$$

Using (B7) in the Klein-Gordon equation (7), we obtain $(\Box + \hat{V})\phi = 0$, where

$$\hat{V} = h(t) \sum_{\lambda=+,\times} H_\lambda e^{\mu\nu}_\lambda \partial_\mu \partial_\nu. \qquad \text{(B8)}$$

Searching for solutions in the form (10) leads to the following equation:

$$\partial_t^2 \eta + \omega^2 \eta - h(t) \sum_{\lambda=+,\times} H_\lambda e^{ij}_\lambda p_i p_j e^{-i\omega t} = 0, \qquad \text{(B9)}$$

where $\vec{p} = \{p^x, p^y, p^z\} = \{\vec{p}_\perp, (n\pi/L)\}$ and $\omega^2 = p_\perp^2 + (n\pi/L)^2$. Looking at (33) we see that all we need is $|\beta_p|^2$, namely, the squared modulus of the Bogoliubov coefficient

$$\beta_p = -\frac{i}{2\omega} \int_{-\infty}^{+\infty} dt h(t) e^{-2i\omega t} \sum_{\lambda=+,\times} H_\lambda e^{ij}_\lambda p_i p_j. \qquad \text{(B10)}$$

Using (60) and assuming, for simplicity, $H_+ = H_\times \equiv H$, we find, after some tedious algebra,

$$|\beta_p|^2 = \frac{\pi H^2}{4\omega^2 \sigma^2} e^{-2\omega^2/\sigma^2} F(\hat{\Omega}, p_\perp, n), \qquad \text{(B11)}$$

where

$$F(\hat{\Omega}, p_\perp, n) = \left(\sum_{\lambda=+,\times} e^{ij} p_i p_j \right)^2 = A(\hat{\Omega}) p_\perp^4 + B(\hat{\Omega}) \left(\frac{n\pi}{L}\right)^2 p_\perp^2 + C(\hat{\Omega}) \left(\frac{n\pi}{L}\right)^4, \qquad \text{(B12)}$$

and

$$A(\hat{\Omega}) = \frac{1}{32}(41 + 20\cos(2\vartheta) + 3\cos(4\vartheta) + 24\cos(4\varphi)\sin^4\vartheta), \qquad \text{(B13)}$$

$$B(\hat{\Omega}) = \frac{1}{2}[10 + 6\cos(2\vartheta) + 3\cos(2\vartheta - 4\varphi) - 6\cos(4\varphi) + 3\cos(2\vartheta + 4\varphi)]\sin^2\vartheta, \qquad \text{(B14)}$$

$$C(\hat{\Omega}) = 4\cos^2(2\varphi)\sin^4\vartheta. \qquad \text{(B15)}$$

Substituting (B11) in (38) (do not confuse ϑ with θ, the latter defining $p^x = p_\perp \cos\theta$ and $p^y = p_\perp \sin\theta$ in the reference frame of the cavity) we get, after integration over the variables s and θ, the correction to the Casimir effect due to the gravitational pulse propagating along the direction $\hat{\Omega}$,

$$\langle\delta\epsilon_{\rm Cas}\rangle = \frac{H^2}{16\pi\sigma^2 L}\lim_{\nu\to 0}\Re\mathrm{e}\left\{i^{1-\nu}\Gamma(\nu)\int_0^{+\infty}\frac{d\omega}{\omega^2}e^{-2\omega^2/\sigma^2}\times[A(\hat{\Omega})I_5 + B(\hat{\Omega})I_3 + C(\hat{\Omega})I_1]\right\}, \tag{B16}$$

where the quantities I_k $(k = \{1,3,5\})$ are defined as follows:

$$I_k = \sum_n \int_0^{+\infty} dp_\perp p_\perp^k \frac{(n\pi/L)^{5-k}}{(p_\perp^2 + (n\pi/L)^2 - \omega^2)^\nu}. \tag{B17}$$

Using ([39]) we find

$$I_5 = \frac{\Gamma(\nu-3)}{\Gamma(\nu)}\sum_n [(n\pi/L)^2 - \omega^2]^{3-\nu}, \tag{B18}$$

while it can be easily checked that

$$I_3 = -\frac{\Gamma(\nu-3)}{2\Gamma(\nu)}\left(\frac{\pi}{L}\right)^2\frac{\partial}{\partial(\frac{\pi}{L})^2}\sum_n\left[\left(\frac{n\pi}{L}\right)^2 - \omega^2\right]^{3-\nu}, \tag{B19}$$

$$I_1 = \frac{\Gamma(\nu-3)}{2\Gamma(\nu)}\left(\frac{\pi}{L}\right)^4\frac{\partial^2}{\partial((\frac{\pi}{L})^2)^2}\sum_n\left[\left(\frac{n\pi}{L}\right)^2 - \omega^2\right]^{3-\nu}. \tag{B20}$$

Using (B18)–(B20) we rewrite (B16) as

$$\langle\delta\epsilon_{\rm Cas}\rangle = \frac{H^2}{16\pi\sigma^2 L}\lim_{\nu\to 0}\Re\mathrm{e}\left\{i^{1-\nu}\Gamma(\nu-3)\times\int_0^{+\infty}\frac{d\omega}{\omega^2}e^{-\frac{2\omega^2}{\sigma^2}}\mathcal{D}_{(\frac{\pi}{L})}\sum_n\left(\frac{\pi}{L}\right)^{2\nu-6}\left[n^2 - \frac{\omega^2L^2}{\pi^2}\right]^{3-\nu}\right\}, \tag{B21}$$

where $\mathcal{D}_{(\pi/L)}$ represents the following operator:

$$\mathcal{D}_{(\pi/L)} = A(\hat{\Omega}) - \frac{1}{2}\left(\frac{\pi}{L}\right)^2 B(\hat{\Omega})\frac{\partial}{\partial(\frac{\pi}{L})^2} + \frac{1}{2}\left(\frac{\pi}{L}\right)^4 C(\hat{\Omega})\frac{\partial^2}{\partial((\frac{\pi}{L})^2)^2}. \tag{B22}$$

Carrying out the same procedure developed in Sec. IV B, we convert (B21) in an expression involving the Epstein-Hurwitz ζ-function,

$$\langle\delta\epsilon_{\rm Cas}\rangle = -\frac{H^2}{16\pi\sigma^2 L}\lim_{\nu\to 0}\Re\mathrm{e}\left\{i^{-\nu}\Gamma(\nu-3)\int_0^{+\infty}\frac{d\omega}{\omega^2}e^{-\frac{2\omega^2}{\sigma^2}}\times\mathcal{D}_{(\pi/L)}\sum_n\left(\frac{\pi}{L}\right)^{2\nu-6}\left[n^2 + \frac{\omega^2L^2}{\pi^2}\right]^{3-\nu}\right\}. \tag{B23}$$

By means of the expansion (49), we perform again the analysis of the emerging divergences, finding once more that the only surviving contribution is the one involving the last term in (49). We get [compare with (53)]

$$\langle\delta\epsilon_{\rm Cas}\rangle = -\frac{H^2}{16\pi\sigma^2 L}\lim_{\nu\to 0}\Re\mathrm{e}\left\{i^{-\nu}2\pi^{\nu-3}\times\mathcal{D}_{(\pi/L)}\sum_n\left(\frac{L}{\pi}\right)^{\nu-5/2}n^{\nu-7/2}J(\nu)\right\}, \tag{B24}$$

where $J(\nu)$ is given by (54). In the case of a gravitational pulse whose characteristic wavelength is large when compared with the size of the Casimir cavity, we can assume $\sigma L \ll 1$ and expand $J(\nu)$ to the lowest order, thus obtaining (in the limit $\nu \to 0$)

$$J \simeq -\frac{15\pi\sqrt{2}}{16\sigma}L^{-7/2}n^{-7/2}. \tag{B25}$$

Taking into account the rapid convergence of the sum in (B24), we retain the $n = 1$ term only, hence finding

$$\langle\delta\epsilon_{\rm Cas}\rangle \simeq \frac{15H^2}{64\sqrt{2\pi}\sigma^3 L}\mathcal{D}_{(\pi/L)}(L^{-6}) = \frac{15H^2}{64\sqrt{2\pi}\sigma^3L^7}\left(A(\hat{\Omega}) - \frac{3}{2}B(\hat{\Omega}) + 3C(\hat{\Omega})\right). \tag{B26}$$

The result (B26) represents the generalization of (61) to the case of a gravitational pulse propagating at an arbitrary direction $\hat{\Omega}$ with respect to the reference frame of the Casimir cavity. Notice that, in the case $\vartheta = \varphi = 0$, (B26) yields [see (B13)–(B15)]

$$\langle\delta\epsilon_{\rm Cas}\rangle = \frac{15H^2}{32\sqrt{2\pi}\sigma^3L^7}, \tag{B27}$$

e.g., just *twice* the result we found in (61). This is due to the fact that we are considering a gravitational wave characterized by two polarization states, H_+ and $H_\times$ (with equal amplitudes). This confirms once more what was pointed out in Appendix A. Namely, each polarization state makes its own contribution to the Casimir vacuum energy, just as in the electromagnetic case.

The result (B26) has been obtained assuming $H_+ = H_\times$. Such a constraint can be relaxed $(H_+ \neq H_\times)$ and the calculations are almost the same, although the final expression is rather cumbersome and not particularly appealing.

Finally, we point out that the present approach can be straightforwardly applied also in the case of more general background spacetimes as, e.g., in Bianchi type-IX models.

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