Is expansion blind to the spatial curvature?

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In Q. Vigneron, Non-relativistic regime and topology: Topological term in the Einstein equation, we proposed and motivated a modification of the Einstein equation as a function of the topology of the Universe in the form of a biconnection theory. The new equation features an additional "topological term" related to a second nondynamical reference connection and chosen as a function of the spacetime topology. In the present paper, we analyze the consequences for cosmology of this modification. First, we show that expansion becomes blind to the spatial curvature in this new theory; i.e., the expansion laws do not feature the spatial curvature parameter anymore (i.e., $\Omega_{\neq K} = 1, \ \forall \ \Omega_K$), while this curvature is still present in the evaluation of distances. Second, we derive the first order perturbations of this homogeneous solution. Two additional gauge invariant variables coming from the reference connection are present compared with general relativity: a scalar and a vector mode, both sourced by the shear of the cosmic fluid. Finally, we confront this model with observations. The differences with the Lambda cold dark matter model are negligible; in particular, the Hubble and curvature tensions are still present. Nevertheless, since the main difference between the two models is the influence of the background spatial curvature on the dynamics, an increased precision on the measure of that parameter might allow us to observationally distinguish them.

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I. INTRODUCTION

In Ref. [1], we showed that the nonrelativistic limit of the Einstein equation is only possible if the spatial topology is Euclidean, i.e., for which the covering space is $\mathbb{E}^3$. We argued that this result can be interpreted as a signature of an inconsistency of general relativity in non-Euclidean topologies (see Sec. 4.3 in [1]). We then raised the following question: What relativistic equation admitting a nonrelativistic limit in any topology should we consider? The main requirements we drew for the new relativistic equation were the following: (i) It should reduce to the Einstein equation in a Euclidean topology, and (ii) it must be second order in the metric derivatives. In that same paper [1], we proposed an answer to the above question in the form of a biconnection theory similar to the one introduced by Rosen [2]. It is composed of one physical Lorentzian structure $(\boldsymbol{g}, \boldsymbol{\nabla})$ and one nondynamical reference connection $\bar{\boldsymbol{\nabla}}$. The equations in this theory are the same as in [2]; in particular, the Einstein equation is modified such that the physical spacetime Ricci curvature $R_{\mu\nu}$ is replaced by the difference between that curvature and the reference Ricci curvature $\bar{R}_{\mu\nu}$ arising from the reference connection [see Eq. (6)]. The fundamental difference between Rosen's theory and the approach of [1] is in the choice of reference connection which [1] takes to be related to the spacetime topology. This theory only differs from general relativity in the case of non-Euclidean topologies, for which $\bar{R}_{\mu\nu} \neq 0$, and should be considered instead of the latter if one wants to study a model universe compatible with the nonrelativistic regime in any topology.

The goal of the present paper is to derive the equations of the cosmological model that result from this biconnection theory (presented in Sec. II) and confront them with observational data. Within the Standard Model of cosmology, three main sets of equations are used:

(i) The homogeneous and isotropic solution of the Einstein equation to describe global expansion.

(ii) The weak field limit to describe the linear regime of inhomogeneities in the early Universe, and in the late Universe on large scales. These equations allows us to test the model using the cosmic microwave background (CMB) data, baryonic acoustic oscillation (BAO) data, and supernovae (SN1a) data in particular.

(iii) The nonrelativistic equations (cosmological Newton equations) to describe nonlinear structure formation in the late Universe. N-body simulations performed using these equations allow us to test the model by comparing mock catalogs with catalogs of galaxies.

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These sets of equations need to be derived within the framework of the biconnection theory for a complete cosmological model.

The nonrelativistic equations resulting from the biconnection theory were already derived in [3,4]: For Euclidean and non-Euclidean topologies, they correspond to the cosmological Newton equations and the non-Euclidean Newtonian equations, respectively. The latter theory describes Newtonian (i.e., nonrelativistic) gravitation on non-Euclidean topologies (e.g., spherical, hyperbolic, etc). It is shown in [4] how it could be used to study nonlinear structure formation in spherical topologies.

To complete the cosmological model related to the biconnection theory of [1], it remains to derive the homogeneous and isotropic solution (to describe expansion), along with the weak field limit of the biconnection theory (to describe the linear regime of inhomogeneities). The former is derived in Sec. III, where we show, in particular, that the curvature parameter Ω_K is not present anymore in the expansion law (i.e., $\Omega_{\neq K} = 1,\ \forall\ \Omega_K$) compared to the same solution derived from the Einstein equation (i.e., $\Omega_{\neq K} + \Omega_K = 1$): Expansion is blind to the spatial curvature. As a complementary result, we show in Appendix that this expansion law also holds for a general nonperturbative inhomogeneous solution in the nonrelativistic limit. The weak field limit of the biconnection theory is derived in Sec. IV where we show that, as for the background solution, the presence of the curvature parameter in the equations is significantly changed compared to the Standard Model.

On scales where nonlinearities are important, the effects of the background spatial curvature and topology are expected to be smaller than current observational precision. Since the difference between general relativity and the biconnection theory developed in [1] is related to these two parameters, we expect observational differences to appear only on large scales, i.e., on scales described by the linear approximation. Therefore, while an N-body simulation seems not relevant to test the cosmological model related to the biconnection theory,[1] a direct comparison with CMB data (in particular) using the weak field equations derived in Sec. IV would provide a first test of this new theory. This test is performed in Sec. V. We conclude in Sec. VI.

II. THE BICONNECTION THEORY OF [1]

The biconnection theory introduced in [1] is defined on a 4-manifold $\mathcal{M} = \mathbb{R}\times\Sigma$ where Σ is a closed 3-manifold, which we equip with

(i) a physical Lorentzian metric $\boldsymbol{g}$ and its connection $\boldsymbol{\nabla}$. It defines the physical (spacetime) Riemann tensor $R^{\mu}{}_{\alpha\beta\nu}$, the physical Ricci tensor $R_{\mu\nu} \coloneqq R^{\alpha}{}_{\mu\alpha\nu}$, and the physical scalar curvature $R \coloneqq g^{\mu\nu}R_{\mu\nu}$.

(ii) a nondynamical reference connection $\bar{\boldsymbol{\nabla}}$. It defines the reference (spacetime) Riemann tensor $\bar{R}^{\mu}{}_{\alpha\beta\nu}$ and the reference Ricci tensor $\bar{R}_{\mu\nu} \coloneqq \bar{R}^{\alpha}{}_{\mu\alpha\nu}$. No reference scalar curvature can be defined from $\bar{\boldsymbol{\nabla}}$ alone.

The reference connection $\bar{\boldsymbol{\nabla}}$ is nondynamical in the sense that it is the same for any physical metric and energy-momentum tensor. In the approach of [1], that connection depends on topological properties of $\mathcal{M}$ in the sense that it is chosen to be related to the universal cover $\tilde{\mathcal{M}} = \mathbb{R}\times\tilde{\Sigma}$ of $\mathcal{M}$, where $\tilde{\Sigma}$ is the universal cover of Σ. The universal cover does not determine the precise topology of $\mathcal{M}$, but only its class. Since we always consider globally hyperbolic spacetimes (i.e., $\mathcal{M} = \mathbb{R}\times\Sigma$), the choice of spacetime universal cover $\tilde{\mathcal{M}}$ is equivalent to the choice of spatial universal cover $\tilde{\Sigma}$.

The choice of $\bar{\boldsymbol{\nabla}}$ made in [1] is the following: We assume that there exists a coordinate system $\{x^0, x^i\}$ adapted to a foliation of Σ-hypersurfaces such that the reference Riemann tensor writes[2]

$$\bar{R}^{\mu}{}_{\alpha\nu\beta} = \delta^{\mu}_{a}\delta^{i}_{\alpha}\delta^{b}_{\nu}\delta^{j}_{\beta}\,{}^{\tilde{\Sigma}}\bar{\mathcal{R}}^{a}{}_{ibj}(x^k), \tag{1}$$

where ${}^{\tilde{\Sigma}}\bar{\mathcal{R}}^{a}{}_{ibj}$ is independent of x^0 and corresponds to the standard Riemann tensor of the covering space $\tilde{\Sigma}$. In the cases of interest for the present paper, $\tilde{\Sigma}$ will either be the Euclidean $\mathbb{E}^3$, the spherical $\mathbb{S}^3$, or the hyperbolic $\mathbb{H}^3$ covering spaces, but in general, five other types of topologies are possible, as described by the Thurston decomposition [5]. In these three cases, we, respectively, have

$$\begin{aligned} &{}^{\mathbb{E}^3}\bar{\mathcal{R}}^{a}{}_{ibj} = 0, \qquad {}^{\mathbb{S}^3}\bar{\mathcal{R}}^{a}{}_{ibj} = \delta^{a}_{b}\bar{h}^{\mathbb{S}^3}_{ij} - \delta^{a}_{j}\bar{h}^{\mathbb{S}^3}_{ib}, \\ &{}^{\mathbb{H}^3}\bar{\mathcal{R}}^{a}{}_{ibj} = \delta^{a}_{b}\bar{h}^{\mathbb{H}^3}_{ij} - \delta^{a}_{j}\bar{h}^{\mathbb{H}^3}_{ib}, \end{aligned} \tag{2}$$

where $\bar{h}^{\mathbb{S}^3}_{ij}$ (respectively, $\bar{h}^{\mathbb{H}^3}_{ij}$) are homogeneous and isotropic metrics on $\mathbb{S}^3$ (respectively, $\mathbb{H}^3$). Therefore, in this paper, the reference Ricci tensor has the form

$$\bar{R}_{\mu\nu} = 2K\delta^{i}_{\mu}\delta^{j}_{\nu}\bar{h}_{ij}(x^k), \tag{3}$$

with $\bar{h}_{ij}$ a homogeneous and isotropic metric, and $K = 0$, 1, and -1 for, respectively, Euclidean, spherical, and hyperbolic topologies. From this formula, $\dim[\ker\bar{R}_{\mu\nu}] = 3$ (in the spherical and hyperbolic cases), and therefore $\bar{R}_{\mu\nu}$ defines a reference observer of 4-velocity G^{μ} such that $G^{\mu}\bar{R}_{\mu\nu} = 0$. The presence of this vector will be important for Sec. III A. Furthermore, as will be shown in that section,

[1]For this same reason, a parametrized-post-Newtonian calculation aimed at testing modified gravity theories on solar system scales would not be relevant to test the biconnection theory.

[2]Throughout this paper, we denote indices running from 0 to 3 by greek letters and indices running from 1 to 3 by roman letters.

the normalization factor of the reference Ricci curvature is a gauge choice.

In this theory, the Einstein equation is modified to feature the reference curvature as follows:

$$G_{\alpha\beta} = \kappa T_{\alpha\beta} - \Lambda g_{\alpha\beta} + \mathcal{T}_{\alpha\beta}, \tag{4}$$

where $\kappa \coloneqq 8\pi G$, $T_{\alpha\beta}$ is the energy-momentum tensor, Λ the cosmological constant, and $\mathcal{T}_{\alpha\beta}$ is defined as

$$\mathcal{T}_{\alpha\beta} \coloneqq \bar{R}_{\alpha\beta} - \frac{\bar{R}_{\mu\nu} g^{\mu\nu}}{2} g_{\alpha\beta}. \tag{5}$$

Since the reference curvature directly depends on the spacetime topology by the choice (1), the term $\mathcal{T}_{\alpha\beta}$ can be considered a topological term. Equation (4) can be rewritten in the more convenient form

$$R_{\alpha\beta} - \bar{R}_{\alpha\beta} = \kappa\left(T_{\alpha\beta} - \frac{T^{\mu}{}_{\mu}}{2} g_{\alpha\beta}\right) + \Lambda g_{\alpha\beta}. \tag{6}$$

We see from this equation that the difference with general relativity is to replace the physical spacetime Ricci tensor with the difference between that tensor and the reference spacetime Ricci tensor. The main interpretation of that equation is that matter does not curve spacetime anymore, as in general relativity, but only induces a departure of the physical Ricci curvature from the reference, topological, Ricci curvature.

The additional term $\mathcal{T}_{\alpha\beta}$ in the Einstein equation is conserved:

$$g^{\mu\nu}\left(\nabla_{\mu}\bar{R}_{\nu\alpha} - \frac{1}{2}\nabla_{\alpha}\bar{R}_{\mu\nu}\right) = 0. \tag{7}$$

This equation, called the biconnection condition, constrains the diffeomorphism freedom in the definition of $\bar{R}_{\mu\nu}$ with respect to $g_{\mu\nu}$.

Equations (6) and (7) are equivalent to the ones of the biconnection theory proposed by Rosen [2]. The only, but fundamental, difference is the choice and motivation for the reference connection: Rosen chose a reference connection related to a de Sitter metric in order to remove singularities from general relativity, while in our case, the reference connection is topology dependent as it is related to the universal cover of the spacetime manifold $\mathcal{M}$.

In the case of a Euclidean topology, i.e., $\tilde{\Sigma} = \mathbb{E}^3$, we have $\bar{R}_{\mu\nu} = 0$, implying that Eq. (6) is equivalent to the Einstein equation and that Eq. (7) is trivial. Therefore, general relativity and the biconnection theory of [1] coincide for Euclidean topologies, and differ for any other type of topology. In terms of the cosmological model, this will imply that the two theories will differ only if $\Omega_K \neq 0$.

III. HOMOGENEOUS AND ISOTROPIC SOLUTION OF THE BICONNECTION THEORY

A. Derivation

We assume $\boldsymbol{g}$ to be the Friedmann-Lemaître-Robertson-Walker (FLRW) metric. Therefore, the covering space $\tilde{\Sigma}$ is either $\mathbb{E}^3$, $\mathbb{S}^3$, or $\mathbb{H}^3$, and the reference Ricci curvature has the form (3). We define $\boldsymbol{n}$ to be the 4-velocity of the observer relative to the homogeneous foliation of $\boldsymbol{g}$, for which the spatial metric is denoted $\boldsymbol{h}$. Note that at this stage, $\bar{h}_{ij}$ present in (3) is *a priori* different from h_{ij}, and these two tensors are not necessarily related to the same foliation. The relation between them will be constrained by Eq. (4).

Because of homogeneity and isotropy, we can write both $T_{\mu\nu}$ and $\mathcal{T}_{\mu\nu}$ as[3]

$$T_{\alpha\beta} = \rho n_{\alpha} n_{\beta} + p h_{\alpha\beta}, \tag{8}$$

$$\mathcal{T}_{\alpha\beta} = \bar{\rho} n_{\alpha} n_{\beta} + \bar{p} h_{\alpha\beta}, \tag{9}$$

where ρ and p are, respectively, the energy density and pressure of matter, and

$$\bar{\rho} \coloneqq \frac{1}{2}\left(n^{\mu} n^{\nu} \bar{R}_{\mu\nu} + h^{\mu\nu}\bar{R}_{\mu\nu}\right), \tag{10}$$

$$3\bar{p} \coloneqq \frac{1}{2}\left(3 n^{\mu} n^{\nu} \bar{R}_{\mu\nu} - h^{\mu\nu}\bar{R}_{\mu\nu}\right) \tag{11}$$

are the effective energy density and pressure coming from the topological term. Then the expansion laws take the form

$$3H^2 = \kappa\rho + \bar{\rho} + \Lambda - \mathcal{R}/2, \tag{12}$$

$$3\ddot{a}/a = -\frac{\kappa}{2}(\rho + 3p) - \frac{1}{2}(\bar{\rho} + 3\bar{p}) + \Lambda, \tag{13}$$

where $\mathcal{R} = 6K/a^2$ is the scalar spatial curvature related to the physical spatial metric $\boldsymbol{h}$, with $a(t)$ the scale factor and $H = \dot{a}/a$ the expansion rate. It remains to find a more explicit formula for $\bar{\rho}$ and $\bar{p}$.

The heat flux relative to the term $\boldsymbol{\mathcal{T}}$ being zero implies $n^{\mu} h^{\alpha\nu} \bar{R}_{\mu\nu} = 0$. Coupled with the fact that $\dim[\ker \bar{R}_{\mu\nu}] = 3$ from relation (3), then $n^{\mu}\bar{R}_{\mu\nu} = 0$. This implies that the observer related to the homogeneity foliation induced by the FLRW metric corresponds to the reference observer induced by the reference spacetime curvature, i.e., $G^{\mu} \propto n^{\mu}$. Then, using (9)–(11) along with $\mathcal{T}_{\alpha\beta} \coloneqq \bar{R}_{\alpha\beta} - \frac{\bar{R}_{\mu\nu} g^{\mu\nu}}{2} g_{\alpha\beta}$, we get

$$\bar{\rho} = -3\bar{p} = \frac{1}{2} h^{\mu\nu}\bar{R}_{\mu\nu}, \tag{14}$$

[3]The most general solution *a priori* features heat fluxes q_{α} and $\bar{q}_{\alpha}$ from, respectively, $T_{\alpha\beta}$ and $\mathcal{T}_{\alpha\beta}$ constrained to be $\bar{q}_{\alpha} = -q_{\alpha}$ by Eq. (4). This corresponds to a tilted cosmological model: Both the fluid and the reference observer defined by G^{α} are tilted with respect to the homogeneous foliation.

$$\bar{R}_{\alpha\beta} = -2\bar{p}h_{\alpha\beta}. \tag{15}$$

In coordinates adapted to the foliation of homogeneity, the second relation, along with (3), leads to

$$2K\bar{h}_{ij} = -2\bar{p}h_{ij}. \tag{16}$$

Both K and $\bar{p}$ being spatial constants, the above equation implies

$$\bar{\mathcal{R}}_{ij} = \mathcal{R}_{ij}, \tag{17}$$

where $\bar{\mathcal{R}}_{ij} = 2K\bar{h}_{ij}$ is the Ricci tensor associated with $\bar{h}_{ij}$. Furthermore, for $K \neq 0$, the inverse of (16) leads to

$$\bar{h}^{ij} = -\frac{K}{\bar{p}}h^{ij}, \tag{18}$$

where $\bar{h}^{ij}$ is the inverse of $\bar{h}_{ij}$ (i.e., $\bar{h}^{ij} \neq \bar{h}_{cd}h^{ci}h^{dj}$). Then, using Eq. (17) we get $\mathcal{R} \coloneqq \mathcal{R}_{ij}h^{ij} = \bar{\mathcal{R}}_{ij}h^{ij}$, which, along with relation (18), leads to

$$6\bar{p} = -\mathcal{R}. \tag{19}$$

This implies $\bar{\rho} = \mathcal{R}/2$. Finally, the expansion laws (12) and (13) of an exact homogeneous and isotropic solution of the biconnection theory are

$$3H^2 = \kappa\rho + \Lambda, \quad \forall \ K, \tag{20}$$

$$3\ddot{a}/a = -\frac{\kappa}{2}(\rho + 3p) + \Lambda. \tag{21}$$

These expansion laws are the ones of a flat homogeneous and isotropic model as derived with the Einstein equation, but here they hold even in the nonflat cases, i.e., for all K.

While the biconnection theory has the additional field $\bar{R}_{\mu\nu}$ with respect to general relativity, we see that the exact homogeneous and isotropic solution does not have an additional parameter linked to this field. The only role of $\bar{R}_{\alpha\beta}$ is to set the topology, the equations being independent of the value chosen for the reference scalar curvature $\bar{\mathcal{R}} \coloneqq \bar{h}^{ij}\bar{\mathcal{R}}_{ij}$. Indeed, rescaling the choice (3) by a constant factor, which would rescale $\bar{\mathcal{R}}$ by the same factor, only results in a rescaling of the scale factor $a(t)$. Therefore, the value of $\bar{\mathcal{R}}$ is just a gauge choice. This was not the case with Rosen's choice of reference curvature [2], where a reference cosmological constant was introduced.

B. Why is it expected?

Equation (20) shows that within the framework of the present biconnection theory, the expansion scenario is the same for a Euclidean, spherical, or hyperbolic universe. While we discuss the consequences of this result in more detail in Sec. III C, in the present section we explain why it is expected from any relativistic theory which we require to have a nonrelativistic limit in any topology.

Let us consider the first Friedmann equation resulting from the Einstein equation, where we reintroduce the speed of light $c \neq 1$:

$$3H^2 - \kappa\rho - \Lambda + c^2\mathcal{R}/2 = 0. \tag{22}$$

We see that the curvature term appears as a -1 order in $1/c^2$, while the other terms are all zeroth order terms. Requiring the nonrelativistic limit to exist corresponds to requiring that this equation be written as a Taylor series of $1/c^2$, and therefore that each order needs to be independently zero. This implies

$$\frac{\mathcal{R}(t)}{2} = 0, \qquad (\text{order } -1), \tag{23}$$

$$3H^2 = \kappa\rho + \Lambda, \qquad (\text{order } 0). \tag{24}$$

Therefore, the solution necessarily needs to describe a flat universe. This is a rough derivation of the result in [1] for the specific case of a homogeneous and isotropic solution, stating that no nonrelativistic limit of the Einstein equation exists for a solution describing a non-Euclidean spatial topology.

The role of the reference spacetime curvature added in the Einstein equation is to allow for this limit to be possible. In the present case of a homogeneous solution, the term $\mathcal{T}$ adds an effective density in the Friedmann equations, which, as shown in Sec. III A, cancels the spatial curvature term. The consequence is that the expansion law does not feature the negative order in $1/c^2$ anymore, and therefore, from the Taylor series, we only obtain (24) without the zero curvature constraint (23), i.e., without constraining the topology to be Euclidean. For this reason, we expect the expansion law (20) to hold for any relativistic theory admitting a nonrelativistic limit in any topology, i.e., not only with the biconnection theory of [1].

C. Expansion is blind to the spatial curvature

The expansion laws in our cosmological model [Eqs. (20) and (21)] are a flat Lambda cold dark matter (ΛCDM) model, regardless of the spatial curvature:

$$\Omega = 1, \quad \forall \ \Omega_K, \tag{25}$$

where $\Omega \coloneqq \Omega_{\rm m} + \Omega_{\rm r} + \Omega_\Lambda$, with $\Omega_{\rm m} \coloneqq \kappa\rho_{\rm m}/(3H^2)$ the matter parameter, $\Omega_{\rm r} \coloneqq \kappa\rho_{\rm r}/(3H^2)$ the radiation parameter, $\Omega_K \coloneqq -K/(a^2H^2)$ the curvature parameter, and $\Omega_\Lambda \coloneqq \Lambda/(3H^2)$ the cosmological constant parameter. This result also holds in the presence of inhomogeneities and nonlinearities if these are nonrelativistic (see Appendix). Therefore, the expansion as predicted by the biconnection theory is blind to the spatial curvature, while this curvature still affects the measure of distances. Therefore, in the biconnection theory, spatial curvature has smaller effects on the dynamics than in general relativity; the effect remains essentially geometrical. This is a strong difference between the two theories, which leads to two main questions:

(i) What is the value of the curvature parameter resulting from a reanalysis of the cosmological data with relation (25)?

(ii) In the case where the reevaluated curvature parameter is not negligible anymore, are the values of other cosmological parameters changed such that recent observational tensions within the ΛCDM model can be solved?

The first question is especially interesting in light of a rising debate on the value of the spatial curvature that should be inferred from the CMB data of the *Planck* space observatory. As shown in, e.g., [6–12], the best fit of the Planck CMB power spectrum at all scales seems to prefer[4] a value at current time of $\Omega_{K,0} \simeq -0.045$, which differs from the standard constraint $|\Omega_{K,0}| \lesssim 10^{-3}$ obtained when BAO data are taken into account[5] [14]. The main issue with this result is that it leads to a value of the Hubble constant $H_0^{\rm CMB}$ inferred from the CMB that is particularly low, strongly increasing the tension with the local supernovae measurement $H_0^{\rm SN1a}$ [15]: from $H_0^{\rm SN1a} - H_0^{\rm CMB} \sim 5$ km/s/Mpc to $H_0^{\rm SN1a} - H_0^{\rm CMB} \sim 20$ km/s/Mpc (with large error bars). Since expansion is not directly affected by spatial curvature in our model, one may expect that the preference for nonzero Ω_K when fitting the CMB alone could stay without changing the Hubble constant, and thus not increasing the Hubble tension.

The reason why H_0 must change when $\Omega_K \neq 0$ in an analysis of CMB data comes mainly from the angular diameter distance

$$d_{\rm A}(z) = \frac{1}{\sqrt{|\Omega_{K,0}|}H_0(1+z)} S_K\left(\sqrt{|\Omega_{K,0}|}H_0 \int_0^z \frac{{\rm d}z'}{H(z')}\right), \tag{26}$$

where

$$S_K(x) = \begin{cases} \sin x, & \Omega_K < 0, \\ x, & \Omega_K = 0, \\ \sinh x, & \Omega_K > 0. \end{cases} \tag{27}$$

When evaluated at the recombination redshift, the angular diameter distance sets the typical distance under which we observe CMB angular anisotropies. The typical angular scale of CMB anisotropies (the "sound horizon" θ_s) is very well determined by Planck to subpercent precision [14]. As Ω_K scales like $(1+z)^2$, it does not affect the early Universe, but can affect the angular diameter distance, hereby changing θ_s. The degeneracy with H_0 allows us to compensate for the effect of Ω_K at the background level.

In fact, this formula is common to both the ΛCDM model and our model, but with the $H(z)$ solution of either $\Omega + \Omega_K = 1$ (ΛCDM model) or $\Omega = 1\ \forall\ \Omega_K$ (our model). Therefore, for the Standard Model, spatial curvature appears in two places in formula (26): as a geometrical effect in the function $S_K(x)$, and as a dynamical effect in $H(z)$; while in our model, it only appears as a geometrical effect in $S_K(x)$. Consequently, if the shift from $H_0^{\rm SN1a} - H_0^{\rm CMB} \sim 5$ km/s/Mpc to $H_0^{\rm SN1a} - H_0^{\rm CMB} \sim 20$ km/s/Mpc obtained in, e.g., [7,8,10], comes mainly from the presence of Ω_K in $S_K(x)$, this shift should still be present in our model. However, if it comes from the presence of Ω_K in $H(z)$, this shift should disappear in our model, and the curvature tension might be solved.

To properly determine which case we are in, and to answer the above two questions, a full analysis of the CMB data using a Boltzmann code is necessary. It is performed in Sec. V. This requires the derivation of the first order perturbation equations of the biconnection theory, which are presented in the next section.

IV. WEAK FIELD LIMIT

A. Gauge invariant variables and equations

In this section, we recall the definitions and the gauge invariant equations used in the weak field limit. We follow the notation of [16]. The limit is a first order perturbation of a FLRW metric $g_{\mu\nu} = g_{\mu\nu}^{\rm FLRW} + \delta g_{\mu\nu}$ with

$$g_{\mu\nu}^{\rm FLRW} = a^2 \begin{pmatrix} -1 & 0 \\ 0 & h_{ij} \end{pmatrix} \tag{28}$$

and

$$\delta g_{\mu\nu} = a^2 \begin{pmatrix} -2\phi & D_iB - S_i \\ D_iB - S_i & -2\psi h_{ij} + 2D_iD_jE + 2D_{(i}F_{j)} + 2f_{ij} \end{pmatrix}, \tag{29}$$

[4]Although, this depends on the likelihood used. With the Planck "Camspec" (and the new NPIPE) [9,13], there is less preference for a curved universe than the conventional "plik" likelihood [14].

[5]Let us mention that because of the tension between Planck and BAO data regarding curvature, it has been argued that it may not be statistically consistent to combine both datasets [10].

where $D_c S^c \coloneqq 0$, $D_c F^c \coloneqq 0$, $f_c{}^c \coloneqq 0$, and $D_c f^{ci} \coloneqq 0$. The FLRW metric is written in conformal time τ. The expansion rate is denoted $\mathcal{H} = a'/a$ where the prime derivative is with respect to conformal time. We stress that in the present convention (which we consider for all of Sec. IV), h_{ij} is the comoving spatial metric with D_i its connection, and all the spatial indices are raised and lowered with that metric.

Under a gauge transformation, which can be described by an infinitesimal change of coordinates $x^\mu \to x^\mu + \xi^\mu$, the gauge invariant variables related to the metric are

$$\Psi \coloneqq \psi + \mathcal{H}\sigma, \tag{30}$$

$$\Phi \coloneqq \phi - \mathcal{H}\sigma - \sigma', \tag{31}$$

$$Q_i \coloneqq S_i + F'_i, \tag{32}$$

with $\sigma \coloneqq E' - B$. The tensor mode f_{ij} is already gauge invariant.

The components of the energy-momentum tensor take the form

$$T^0{}_0 = -(\rho + \delta\rho), \tag{33}$$

$$T^0{}_i = (\rho + p)(D_i v + v_i + D_i B - S_i), \tag{34}$$

$$T^i{}_j = (p + \delta p)\delta^i_j + \left(D^i D_j - \frac{1}{3}\delta^i_j \Delta\right)\Pi + \frac{1}{2}\left(D^i \Pi_j + D_j \Pi^i\right) + \Pi^i{}_j, \tag{35}$$

where ρ and p are the homogeneous energy density and pressure, Π, Π_i, and Π_{ij} are, respectively, the scalar, vector, and tensor parts of the anisotropic stress. The gauge invariant quantities related to the energy-momentum tensor are

$$\delta\rho_\sigma \coloneqq \delta\rho - \rho'\sigma, \tag{36}$$

$$V \coloneqq v + E', \tag{37}$$

$$\delta p_{\neq \mathrm{ad}} \coloneqq \delta p - c_\mathrm{s}^2 \delta\rho, \tag{38}$$

$$q_i \coloneqq (\rho + p)(v_i - S_i), \tag{39}$$

with $c_\mathrm{s}^2 \coloneqq p'/\rho'$. The anisotropic stress variables are already gauge invariant. We also define $\delta \coloneqq \delta\rho_\sigma/\rho$.

Then, the first order gauge invariant equations from the Einstein equation $G_{\mu\nu} = \kappa T_{\mu\nu}$, in which we include Λ in $T_{\mu\nu}$, are

(i) for scalar modes

$$(\Delta + 3K)\Psi = a^2 \frac{\kappa}{2}\rho\delta + 3\mathcal{H}(\Psi' + \mathcal{H}\Phi), \tag{40}$$

$$\Psi' + \mathcal{H}\Phi = -a^2 \frac{\kappa}{2}(\rho + p)V, \tag{41}$$

$$\Psi'' + 2\mathcal{H}\Psi' + \mathcal{H}\Phi' + (2\mathcal{H}' + \mathcal{H}^2)\Phi - K\Psi = a^2 \frac{\kappa}{2}\left(c_\mathrm{s}^2 \rho\delta + \delta p_{\neq \mathrm{ad}} + \frac{2}{3}\Delta\Pi\right), \tag{42}$$

$$\Psi - \Phi = a^2 \kappa \Pi, \tag{43}$$

(ii) for vector modes

$$(\Delta + 2K)Q_i = -a^2 2\kappa(\rho + p)(V_i - Q_i), \tag{44}$$

$$Q'_i + 2\mathcal{H}Q_i = a^2 \kappa \Pi_i, \tag{45}$$

(iii) for tensor modes

$$f''_{ij} + 2\mathcal{H}f'_{ij} + (2K - \Delta)f_{ij} = a^2 \kappa \Pi_{ij}. \tag{46}$$

The first order gauge invariant equations from the conservation law $\nabla_\nu T^{\mu\nu} = 0$ are

(i) for scalar modes

$$\delta' + 3\mathcal{H}\left[(c_\mathrm{s}^2 - w)\delta + \delta p_{\neq \mathrm{ad}}/\rho\right] + (1 + w)(\Delta V - 3\Psi') = 0, \tag{47}$$

$$V' + \mathcal{H}(1 - 3c_\mathrm{s}^2)V + \Phi + \frac{1}{\rho + p}\left[c_\mathrm{s}^2 \rho\delta + \delta p_{\neq \mathrm{ad}} + \frac{2}{3}(\Delta + 3K)\Pi\right] = 0, \tag{48}$$

(ii) for vector modes

$$q'_i + 4\mathcal{H}q_i = -\frac{1}{2}(\Delta + 2K)\Pi_i, \tag{49}$$

where we defined $w \coloneqq p/\rho$. In a noninteracting multifluid approach, these last three equations are fulfilled for each fluid component.

The goal of the next sections is to derive the same gauge invariant equations as (40)–(49) in the case where the Einstein equation features the topological term $\mathcal{T}_{\mu\nu} \coloneqq \bar{R}_{\alpha\beta} - \frac{\bar{R}_{\mu\nu} g^{\mu\nu}}{2} g_{\alpha\beta}$. For this, we can treat $\mathcal{T}_{\mu\nu}$ as an additional effective energy-momentum tensor with zeroth order quantities $\bar{\rho}$ and $\bar{p}$, and first order gauge invariant quantities $\delta\bar{\rho}_\sigma$, $\bar{V}$, $\delta\bar{p}_{\neq \mathrm{ad}}$, $\bar{q}_i$, $\bar{\Pi}$, $\bar{\Pi}_i$, and $\bar{\Pi}_{ij}$. Then, Eqs. (40)–(46) are changed as follows:

$$\delta\rho_\sigma \to \delta\rho_\sigma + \frac{1}{\kappa}\delta\bar{\rho}_\sigma, \qquad (\rho+p)V \to (\rho+p)V + \frac{1}{\kappa}(\bar{\rho}+\bar{p})\bar{V}, \qquad \delta p_{\neq \mathrm{ad}} \to \delta\bar{p}_{\neq \mathrm{ad}} + \frac{1}{\kappa}\delta\bar{p}_{\neq \mathrm{ad}}, \qquad q_i \to q_i + \frac{1}{\kappa}\bar{q}_i,$$
$$\Pi \to \Pi + \frac{1}{\kappa}\bar{\Pi}, \qquad \Pi_i \to \Pi_i + \frac{1}{\kappa}\bar{\Pi}_i, \qquad \Pi_{ij} \to \Pi_{ij} + \frac{1}{\kappa}\bar{\Pi}_{ij}. \tag{50}$$

It remains to find the first order quantities associated with the topological term.

B. Gauge invariant quantities of the topological term

As shown in Sec. III, for a homogeneous and isotropic solution, in a coordinate system where the physical spacetime metric can be written as (28), the reference spacetime curvature takes the form $\bar{R}_{\mu\nu} = 2K\delta^i_\mu \delta^j_\nu h_{ij}$ (i.e., the reference spatial metric $\bar{h}_{ij}$ corresponds to the comoving spatial metric). In the framework of the weak field limit, this formula corresponds to the zeroth order of $\bar{R}_{\mu\nu}$. As presented in Sec. II, that tensor is nondynamical; i.e., it is fixed for a given topology and is not affected by the physics behind $T_{\mu\nu}$ and $g_{\mu\nu}$. Nevertheless, this does not mean that $\bar{R}_{\mu\nu}$ is only a zeroth order term within the weak field limit. Indeed, the first order of the physical metric (29) and the first order of the energy-momentum tensor (33)–(35) not only come from physics, but also from gauge freedom. This implies that $\bar{R}_{\mu\nu}$ has, in general, a nonzero first order term solely coming from gauge freedom, i.e.,

$$\bar{R}_{\mu\nu} = \overset{0}{\bar{R}}_{\mu\nu} + \mathcal{L}_{\boldsymbol{X}} \overset{0}{\bar{R}}_{\mu\nu}, \tag{51}$$

where X^μ is a first order 4-vector, $\mathcal{L}_{\boldsymbol{X}}$ is the Lie derivative along $\boldsymbol{X}$, and $\overset{0}{\bar{R}}_{\mu\nu} = 2K\delta^i_\mu\delta^j_\nu h_{ij}$. We direct the reader to Appendix F of [1] for a more detailed justification of (51). This is done within the framework of the nonrelativistic limit, but the derivation is equivalent to that with the weak field limit.

Because $\overset{0}{\bar{R}}_{\mu\nu}$ is purely spatial and its time derivative is zero, only the spatial components of X^μ remain in the first order. Therefore, we have

$$\bar{R}_{\mu\nu} = 2K \begin{pmatrix} 0 & D_i\chi' + \chi'_i \\ D_i\chi' + \chi'_i & h_{ij} + 2D_iD_j\chi + 2D_{(i}\chi_{j)} \end{pmatrix}, \tag{52}$$

where $\delta^i_\mu X^\mu =: D^i\chi + \chi^i$ with $D_c\chi^c := 0$. Under a gauge transformation, we have

$$\begin{aligned} \bar{R}_{\mu\nu} &\overset{\xi}{\to} \bar{R}_{\mu\nu} + \mathcal{L}_{\boldsymbol{\xi}}\bar{R}_{\mu\nu} \\ &\overset{\xi}{\to} \overset{0}{\bar{R}}_{\mu\nu} + \mathcal{L}_{\boldsymbol{X}+\boldsymbol{\xi}}\overset{0}{\bar{R}}_{\mu\nu} \\ &\overset{\xi}{\to} 2K \begin{pmatrix} 0 & D_i(\chi'+\xi') + \chi'_i + \xi'_i \\ D_i(\chi'+\xi') + \chi'_i + \xi'_i & h_{ij} + 2D_iD_j(\chi+\xi) + 2D_{(i}[\chi_{j)} + \xi_{j)}] \end{pmatrix}, \end{aligned} \tag{53}$$

where $\xi^\mu \delta^i_\mu =: D^i\xi + \xi^i$ with $D_i\xi^i := 0$. So, χ and χ^i are not gauge invariant but transform as, respectively, $\chi \to \chi + \xi$ and $\chi^i \to \chi^i + \xi^i$. Therefore, the following quantities defined from the reference Ricci curvature are gauge invariant:

$$\mathcal{C} := \chi - E, \tag{54}$$

$$\mathcal{C}^i := \chi^i - F^i. \tag{55}$$

These variables are interpreted in Sec. IV D. We have

$$a^2\mathcal{T}^0{}_0 = -3K - 2K\Delta\mathcal{C} - 6K\psi, \tag{56}$$

$$a^2\mathcal{T}^0{}_i = -2K\left(D_i\mathcal{C}' + \mathcal{C}'_i + D_i\sigma + Q_i\right), \tag{57}$$

$$\begin{aligned} a^2\mathcal{T}^i{}_j = {}& -K\delta^i_j - 2K\delta^i_j\psi + 4KD_iD^j\mathcal{C} \\ & + 2K\left(D^i\mathcal{C}_j + D_j\mathcal{C}^i\right) - 4Kf^i_j. \end{aligned} \tag{58}$$

Using the above equations along with (33)–(39), the first order gauge invariant quantities (which we denote $\delta\bar{\rho}_\sigma$, $\bar{V}$, $\delta\bar{p}_{\neq \mathrm{ad}}$, $\bar{q}_i$, $\bar{\Pi}$, $\bar{\Pi}_i$, $\bar{\Pi}_{ij}$) defined from $\mathcal{T}_{\mu\nu}$ are

$$\delta\bar{\rho}_\sigma = \frac{2K}{a^2}(\Delta\mathcal{C} + 3\Psi), \qquad \bar{V} = -\mathcal{C}',$$

$$\delta\bar{p}_{\neq\mathrm{ad}} = 0, \qquad \bar{\Pi} = \frac{4K}{a^2}\mathcal{C}, \tag{59}$$

$$\bar{q}_i = -\frac{2K}{a^2}\mathcal{C}_i', \qquad \bar{\Pi}_i = \frac{4K}{a^2}\mathcal{C}_i, \tag{60}$$

$$\bar{\Pi}_{ij} = \frac{4K}{a^2}f_{ij}. \tag{61}$$

We recall that the zeroth order quantities are derived in Sec. III A and are $\bar{\rho} = 3K/a^2$ and $\bar{p} = -K/a^2$.

C. Gauge invariant equations of the biconnection theory

Introducing the gauge invariant quantities (59)–(61) of the topological term in Eqs. (40)–(49) as presented in (50), the first order gauge invariant equations of the biconnection equation (4), in which we include Λ in $T_{\mu\nu}$, are

(i) for scalar modes

$$\Delta\Psi = a^2\frac{\kappa}{2}\rho\delta + 3\mathcal{H}(\Psi' + \mathcal{H}\Phi) + K\Delta\mathcal{C}, \tag{62}$$

$$\Psi' + \mathcal{H}\Phi = -a^2\frac{\kappa}{2}(\rho + p)V + K\mathcal{C}', \tag{63}$$

$$\Psi'' + 2\mathcal{H}\Psi' + \mathcal{H}\Phi' + (2\mathcal{H}' + \mathcal{H}^2)\Phi = a^2\frac{\kappa}{2}\left(c_s^2\rho\delta + \delta p_{\neq\mathrm{ad}} + \frac{2}{3}\Delta\Pi\right) + K\Delta\mathcal{C}, \tag{64}$$

$$\Psi - \Phi = a^2\kappa\Pi + 4K\mathcal{C}, \tag{65}$$

(ii) for vector modes

$$(\Delta - 2K)Q_i = -a^2 2\kappa q_i + 4K\mathcal{C}_i', \tag{66}$$

$$Q_i' + 2\mathcal{H}Q_i = a^2\kappa\Pi_i + 4K\mathcal{C}_i, \tag{67}$$

(iii) for tensor modes

$$f_{ij}'' + 2\mathcal{H}f_{ij}' + (6K - \Delta)f_{ij} = a^2\kappa\Pi_{ij}. \tag{68}$$

The conservation of the topological term $\nabla_\nu\mathcal{T}^{\nu\mu} = 0$ leads to

$$\mathcal{C}'' + 2\mathcal{H}\mathcal{C}' - (\Delta + 4K)\mathcal{C} = \Phi - \Psi, \tag{69}$$

$$\mathcal{C}_i'' + 2\mathcal{H}\mathcal{C}_i' - (2\Delta + 4K)\mathcal{C}_i = -(Q_i' + 2\mathcal{H}Q_i). \tag{70}$$

Combining these last two equations with, respectively, (65) and (67), we obtain

$$\mathcal{C}'' + 2\mathcal{H}\mathcal{C}' - \Delta\mathcal{C} = -a^2\kappa\Pi, \tag{71}$$

$$\mathcal{C}_i'' + 2\mathcal{H}\mathcal{C}_i' - 2\Delta\mathcal{C}_i = -a^2\kappa\Pi_i. \tag{72}$$

These are wave equations for $\mathcal{C}$ and $\mathcal{C}_i$ that are sourced, respectively, by the scalar and vector parts of the anisotropic stress. Finally, since the conservation equations (47)–(49) do not depend on the Einstein equation, they are unchanged in the biconnection theory.

D. Interpretation of $\mathcal{C}$ and $\mathcal{C}^i$

Compared to the Einstein equation, the weak field limit of the biconnection theory features two additional variables: the scalar mode $\mathcal{C}$ and the vector mode $\mathcal{C}^i$, which are constrained by the wave equations (71) and (72). These variables quantify the additional degrees of freedom appearing with the introduction of the reference curvature $\bar{R}_{\mu\nu}$. In particular, they can be related to properties of the reference observer induced by $\bar{R}_{\mu\nu}$, in the cases $K \neq 0$. We recall that this observer is defined by a 4-velocity G^μ such that[6] $G^\nu\bar{R}_{\mu\nu} \coloneqq 0$ and $G^\mu G^\nu g_{\mu\nu} = -1$. We have

$$G^\mu = \frac{1}{a}(1 - \phi; -D^i\chi' - \chi^{i\prime}). \tag{73}$$

To have a better view of the link between $\mathcal{C}$, $\mathcal{C}_i$, and G^μ, let us also introduce the vector normal to the foliation of constant time n^μ, i.e., the foliation relative to the coordinates in which (28) and (29) hold, along with the 4-velocity u^μ of the fluid described by the energy-momentum tensor (33)–(35):

$$n^\mu \coloneqq \frac{1}{a}(1 - \phi; -D^iB + S^i), \tag{74}$$

$$u^\mu \coloneqq \frac{1}{a}(1 - \phi; D^iv + v^i). \tag{75}$$

The tilt between these three vectors is

$$u^\mu - n^\mu = \frac{1}{a}(0; D^i(V - \sigma) + V^i - Q^i), \tag{76}$$

$$n^\mu - G^\mu = \frac{1}{a}(0; D^i(\mathcal{C}' + \sigma) + \mathcal{C}^{i\prime} + Q^i), \tag{77}$$

$$u^\mu - G^\mu = \frac{1}{a}(0; D^i(\mathcal{C}' + V) + \mathcal{C}^{i\prime} + V^i), \tag{78}$$

where we introduced the gauge invariant variable $V_i \coloneqq v_i + F_i'$. We see that $\mathcal{C}$ and $\mathcal{C}_i$ quantify part of the tilt between the constant time, the fluid, and the reference observers. The 4-acceleration of the latter is given by

[6]While the property $\dim(\ker(\bar{R}_{\mu\nu})) = 1$ defines, up to a factor, a reference vector field G^μ, we are unsure if this vector can be normalized to be timelike with respect to the physical spacetime metric everywhere. However, the fact that it is possible at zeroth order, as $G^\mu \propto n^\nu$, suggests that it is not unphysical to consider that property to hold at first order. In any case, the weak field equations of Sec. IV C do not depend on the existence of such a normalization.

$$\begin{aligned} {}^{G}a^{\mu} &\coloneqq G^{\nu}\nabla_{\nu}G^{\mu} \\ &= \frac{1}{a^2}\delta^{\mu}_{i}\big[D^i(\Phi - \mathcal{C}'' - \mathcal{H}\mathcal{C}') - (\mathcal{C}^{i\prime} + Q^i)' \\ &\quad - \mathcal{H}(\mathcal{C}^{i\prime} + Q^i)\big], \end{aligned} \tag{79}$$

and its vorticity is

$$\begin{aligned} \Omega_{\mu\nu} &\coloneqq b_{\alpha[\mu}b^{\beta}{}_{\nu]}\nabla_{\beta}G^{\alpha} \\ &= -a\delta^{i}_{[\mu}\delta^{j}_{\nu]}D_i(\mathcal{C}'_j + Q_j), \end{aligned} \tag{80}$$

with $b_{\mu\nu} \coloneqq g_{\mu\nu} + G_{\mu}G_{\nu}$. We can also compute the curvature perturbation $\delta\mathcal{R}^{(G)}$ of the reference observer. It is defined as the first order of the scalar spatial curvature in a gauge choice where the scalar part of the tilt between n^{μ} and G^{μ} is zero, i.e., $\sigma = -\mathcal{C}'$. We have

$$\delta\mathcal{R}^{(G)} = \frac{4}{a^2}(\Delta + 3K)(\Psi + \mathcal{H}\mathcal{C}'). \tag{81}$$

Therefore, $\Psi + \mathcal{H}\mathcal{C}'$ quantifies the curvature perturbation of the reference observer rest frames.

As seen with Eq. (6), the main difference between general relativity and the biconnection theory is that matter only induces a departure of the physical curvature from the reference curvature. Therefore, it seems more relevant to consider not directly the perturbation of the spatial scalar curvature, i.e., $\delta\mathcal{R}_{|_{\sigma=-\mathcal{C}'}}$, but rather the perturbation of the spatial scalar curvature departure, i.e., $\delta(\mathcal{R} - b^{\mu\nu}\bar{R}_{\mu\nu})_{|_{\sigma=-\mathcal{C}'}}$, which is

$$\delta(\mathcal{R} - b^{\mu\nu}\bar{R}_{\mu\nu})_{|_{\sigma=-\mathcal{C}'}} = \frac{4}{a^2}\Delta(\Psi + \mathcal{H}\mathcal{C}' - K\mathcal{C}). \tag{82}$$

Therefore, Eqs. (78), (79), and (82) suggest the introduction of the following gauge invariant variables:

(i) $\tilde{\Phi} \coloneqq \Phi - \mathcal{C}'' - \mathcal{H}\mathcal{C}'$, i.e., the scalar mode of the acceleration of the reference observer.

(ii) $\tilde{\Psi} \coloneqq \Psi + \mathcal{H}\mathcal{C}' - K\mathcal{C}$, i.e., the perturbation of the spatial scalar curvature departure of the reference observer rest frames.

(iii) $\tilde{\delta} \coloneqq \delta - 3\mathcal{H}(1+w)\mathcal{C}'$, i.e., the density perturbation in the reference observer rest frames.

(iv) $\tilde{V} \coloneqq V + \mathcal{C}'$, i.e., the scalar mode of the tilt between the reference observer and the fluid.

(v) $\tilde{V}_i \coloneqq V_i + \mathcal{C}'_i$, i.e., the vector mode of the tilt between the reference observer and the fluid.

(vi) $\tilde{Q}_i \coloneqq Q_i + \mathcal{C}'_i$ quantifying the vorticity of the reference observer.

What is remarkable is that by introducing these variables in the system (62)–(67), the scalar mode $\mathcal{C}$, which is still sourced by Π with Eq. (71), becomes the only source for the reference gravitational slip $\tilde{\Psi} - \tilde{\Phi}$, and disappears from the rest of the scalar mode equations

$$\Delta\tilde{\Psi} = a^2\frac{\kappa}{2}\rho\tilde{\delta} + \mathcal{H}(4\tilde{\Psi}' - \tilde{\Phi}' + 3\mathcal{H}\tilde{\Phi}), \tag{83}$$

$$\tilde{\Psi}' + \mathcal{H}\tilde{\Phi} = -a^2\frac{\kappa}{2}(\rho + p)\tilde{V}, \tag{84}$$

$$\begin{aligned} &\tilde{\Psi}'' + 2\mathcal{H}\tilde{\Psi}' + \mathcal{H}\tilde{\Phi}' + (2\mathcal{H}' + \mathcal{H}^2)\tilde{\Phi} \\ &\quad = a^2\frac{\kappa}{2}\left(c_s^2\rho\tilde{\delta} + \delta p_{\neq \mathrm{ad}} + \frac{2}{3}\Delta\Pi\right), \end{aligned} \tag{85}$$

$$\tilde{\Psi} - \tilde{\Phi} = (\Delta + 3K)\mathcal{C}. \tag{86}$$

The vector modes equations become

$$(\Delta - 2K)\tilde{Q}_i = -2a^2\kappa(\rho + p)(\tilde{V}_i - \tilde{Q}_i) + (\Delta + 2K)\mathcal{C}'_i, \tag{87}$$

$$\tilde{Q}'_i + 2\mathcal{H}\tilde{Q}_i = 2(\Delta + 2K)\mathcal{C}_i. \tag{88}$$

The conservation equations for matter become

$$\begin{aligned} &\tilde{\delta}' + 3\mathcal{H}\big[(c_s^2 - w)\tilde{\delta} + \delta p_{\neq \mathrm{ad}}/\rho\big] \\ &\quad + (1+w)(\Delta\tilde{V} - 4\tilde{\Psi}' + \tilde{\Phi}') = 0, \end{aligned} \tag{89}$$

$$\begin{aligned} &\tilde{V}' + \mathcal{H}(1 - 3c_s^2)\tilde{V} + \tilde{\Phi} + \frac{1}{\rho + p}\bigg[c_s^2\rho\tilde{\delta} + \delta p_{\neq \mathrm{ad}} \\ &\quad + \frac{2}{3}(\Delta + 3K)\Pi\bigg] = 0. \end{aligned} \tag{90}$$

In a noninteracting multifluid approach, these last two equations are fulfilled for each fluid component.

Unfortunately, *a priori*, $\mathcal{C}$ and $\mathcal{C}_i$ cannot be totally removed from the equations by a change of gauge invariant variables. Therefore, in the general case of our model, there necessarily are two additional variables with respect to the weak field limit of the Einstein equation.

Remark 1. The weak field equations (62)–(67), i.e., written as functions of Φ, Ψ, V, and V^i, reduce to the standard weak field equations of general relativity in the case $K = 0$. However, the tilde variables $\tilde{\Phi}$, $\tilde{\Psi}$, $\tilde{V}$, and $\tilde{V}^i$ do not reduce to the nontilde ones. This means that if the usual scale invariant initial conditions taken for Ψ are shifted to $\tilde{\Psi}$, then even the case $K = 0$ can lead to a different prediction on the CMB power spectrum from with the Standard Model. However, it is not clear to us if a proper justification of this change of initial conditions from Ψ to $\tilde{\Psi}$ can be found.

Remark 2. Reference [1] showed that G^{μ} corresponds to the 4-velocity of a Galilean observer, i.e., defining the Newtonian notion of inertial frames in the nonrelativistic limit. Therefore, in the case $\mathcal{C} = 0 = \mathcal{C}^i$, the usual gauge invariant variables have an elegant interpretation: Φ describes the acceleration of inertial frames, Ψ their curvature perturbation, Q^i their vorticity, and V and V^i the tilt of the fluid with respect to these frames.

V. BLIND CURVATURE AND COSMOLOGICAL DATA

In this section, we fit our model with CMB, BAO, and SN1a data. Throughout the section, the "0" subscript for current time values of the Ω-cosmological parameters will be omitted.

A. Methods

We make use of a modified[7] version of the public CLASS[8] code [18] and run Markov-chain Monte Carlo runs using the Metropolis-Hasting algorithm implemented in MONTE PYTHON v3[9] [19,20]. We consider various combinations of the Planck TT/TE/EE and "conservative" lensing potential power spectra [14], measurements of the BAO from the CMASS and LOWZ galaxy samples of BOSS DR12 at $z = 0.38$, 0.51, and 0.61 [21], and the BAO measurements from 6dFGS at $z = 0.106$ and SDSS DR7 at $z = 0.15$ [22,23]; the Pantheon + SNIa catalog compiles information about the luminosity distance to over 1600 SN1a in the redshift range $0.01 < z < 2.3$ [24]. In all runs, we use large flat priors on H_0, the baryon and cold dark matter energy density ω_b and ω_cdm, respectively, and vary the curvature density fraction $\Omega_K \in [-0.5, 0.5]$. When considering Planck, we also include the amplitude and tilt of the scalar perturbations A_s and n_s, respectively (see next section for a proper definition), and the reionization optical depth τ_reio. We model free-streaming neutrinos as two massless species and one massive with $m_\nu = 0.06$ eV. We use HALOFIT to estimate the nonlinear matter clustering [25,26]. We consider chains to be converged using the conventional Gelman-Rubin criterion $|R-1| \lesssim 0.01$ [27]. To analyze the chains and produce our figures, we use GetDist [28].

B. Initial conditions

From the wave equations (71) and (72), we see that $\mathcal{C}$ and $\mathcal{C}^i$ are sourced by the scalar and vector parts of the fluid anisotropic stress. This leads to three possible situations.

(i) ($\Pi = 0$; $\Pi_i = 0$) and ($\mathcal{C} = 0$; $\mathcal{C}_i = 0$): This is the simplest case. The cosmological model defined via this system along with the expansion laws (20) and (21) has the same number of variables as the ΛCDM model with curvature, the only difference being the presence or not of the coupling terms with that curvature.

(ii) ($\Pi = 0$ and/or $\Pi_i = 0$) and ($\mathcal{C} \neq 0$ and/or $\mathcal{C}_i \neq 0$): Choosing $\mathcal{C} = 0 = \mathcal{C}_i$ without anisotropic stress is a restriction to the generality of the weak field equations. In particular, the gravitational slip, i.e., $\Psi - \Phi$, is not necessarily zero but sourced by $\mathcal{C}$. It is not clear to us if this choice is physical, especially since $\mathcal{C}$ and $\mathcal{C}_i$ vanish in the nonrelativistic limit, as shown in Appendix F of [1].

(iii) ($\Pi \neq 0$; $\Pi_i \neq 0$) and ($\mathcal{C} \neq 0$; $\mathcal{C}_i \neq 0$): The presence of anisotropic stress necessarily implies the presence of $\mathcal{C}$ and $\mathcal{C}_i$, as shown by the wave equations (71) and (72), and therefore implies the presence of additional parameters with respect to the weak field equations of general relativity.

Since anisotropic stress plays a non-negligible role in the CMB power spectrum due to the presence of free-streaming neutrinos (see, e.g., [29]), we will consider this third case when fitting Planck data with this cosmological model. Furthermore, with anisotropic stress being zero initially, we will consider a zero initial condition for $\mathcal{C}$ and $\mathcal{C}^i$. A proper justification for a more complex initial condition on these variables remains to be given, and is left for a future work.

In the ΛCDM model, the parametrizations of the primordial power spectrum for nonflat cases is debated (e.g., [30–33]), essentially because there is no consensus on a nonflat inflationary scenario. Given the lack of such a scenario in the context of our model, the issue remains. Therefore, we assume for simplicity the standard parametrization used by CLASS:

$$\Delta(k) = A_\mathrm{s}\left(\frac{k}{k_\star}\right)^{n_\mathrm{s}-1}, \tag{92}$$

with $k_\star = 0.05h/\mathrm{Mpc}^{-1}$ the conventional pivot scale. As mentioned previously, we vary A_s and n_s within broad flat priors in analyses that include Planck data.

C. Results

We perform three sets of analyses: Planck TT/TE/EE (no lensing), BAO+SN1a on their own, and finally the combination of Planck TT/TE/EE+lensing+BAO+SN1a. For each of these combinations of datasets, the fits are performed with the ΛCDM model (dubbed "normal curvature") for comparison and with our model (dubbed "blind curvature"). We provide the mean (best fit) $\pm 1\sigma$ error reconstructed for each parameter in Tables I and II.

Figure 1 presents the posterior distributions of the Planck alone (filled) and BAO + SN1a (empty) fits for ΛCDM

[7]Note that the correspondence between the Newtonian gauge variables used in CLASS, defined in Ma and Bertschinger [17] and our notation is

$$\Psi^{\mathrm{Ma}} = \Phi, \qquad \Phi^{\mathrm{Ma}} = \Psi, \qquad \delta^{\mathrm{Ma}} = \delta,$$
$$\theta^{\mathrm{Ma}} = -k^2 V, \qquad \sigma^{\mathrm{Ma}} = \frac{2}{3}\frac{k^2\Pi}{\rho + p}, \tag{91}$$

where k is the wave number of the harmonic decomposition.

[8]https://lesgourg.github.io/class_public/class.html.

[9]https://github.com/brinckmann/montepython_public.

TABLE I. Mean (best fit) $\pm 1\sigma$ errors in the ΛCDM + Ω_k model with normal or blind curvature, reconstructed from either Planck or Planck+BAO+SN1a.

	Planck		Planck+BAO+SN1a	
Curvature?	Normal	Blind	Normal	Blind
h	$0.543(0.544)^{+0.035}_{-0.041}$	$0.499(0.466)\pm 0.048$	$0.6758(0.6738)\pm 0.0062$	$0.6756(0.6744)\pm 0.0069$
Ω_K	$-0.044(-0.042)^{+0.020}_{-0.015}$	$-0.057(-0.072)^{+0.029}_{-0.019}$	$0.0005(0.0002)\pm 0.0019$	$0.0003(-0.0001)^{+0.0017}_{-0.0016}$
Ω_m	$0.442(0.563)^{+0.043}_{-0.053}$	$0.528(0.424)^{+0.071}_{-0.099}$	$0.3137(0.3158)\pm 0.0055$	$0.3135(0.3143)\pm 0.0056$
ω_cdm	$0.1182(0.1183)\pm 0.0015$	$0.1182(0.1169)\pm 0.0015$	$0.1200(0.1202)\pm 0.0013$	$0.1199(0.1199)\pm 0.0013$
$10^2\omega_\mathrm{b}$	$2.258(2.258)\pm 0.017$	$2.257(2.274)\pm 0.017$	$2.235(2.243)\pm 0.015$	$2.235(2.245)\pm 0.015$
$10^9 A_\mathrm{s}$	$2.067(2.089)\pm 0.035$	$2.065(2.048)\pm 0.035$	$2.101(2.089)\pm 0.030$	$2.101(2.109)\pm 0.029$
n_s	$0.9701(0.9702)\pm 0.0048$	$0.9698(0.9728)\pm 0.0047$	$0.9643(0.9636)\pm 0.0044$	$0.9646(0.9649)\pm 0.0043$
τ_reio	$0.0486(0.0539)\pm 0.0083$	$0.0482(0.047)\pm 0.0081$	$0.0545(0.0507)\pm 0.0072$	$0.0546(-0.0001)\pm 0.0072$

TABLE II. Mean (best fit) $\pm 1\sigma$ errors in the $\Lambda\mathrm{CDM} + \Omega_k$ model with normal or blind curvature, reconstructed from BAO + SN1a.

	BAO+SN1a	
Curvature?	Normal	Blind
h	$0.715(0.697)^{+0.03}_{-0.046}$	$0.770(0.851)^{+0.079}_{-0.071}$
Ω_K	$0.0199(0.085)^{+0.067}_{-0.068}$	$0.086(0.150)^{+0.17}_{-0.1}$
Ω_m	$0.353(0.389)^{+0.046}_{-0.049}$	$0.406(0.457)^{+0.13}_{-0.073}$

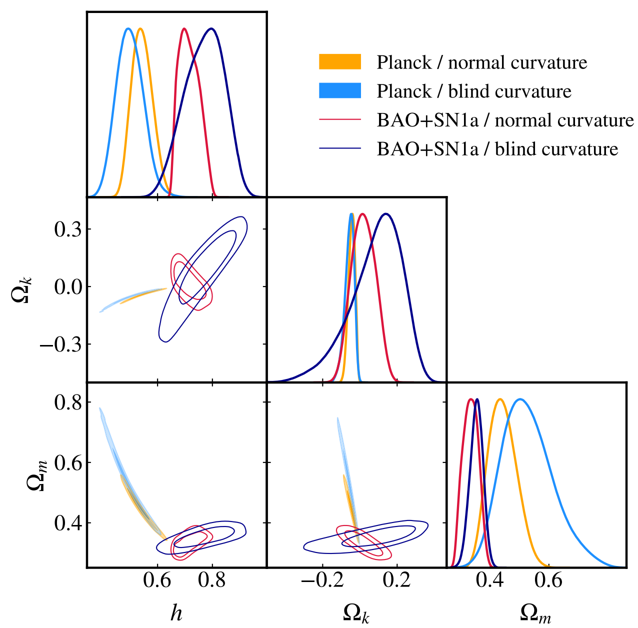

FIG. 1. 2D posterior distributions of $\{h, \Omega_K, \Omega_\mathrm{m}\}$ in the standard case (red/orange) and in the blind curvature (blue) model. We compare constraints from Planck (filled) alone to that obtained from BAO+SN1a (empty).

(orange/red) and the blind model (light/dark blue). The main difference brought by the blind model is an increase of the uncertainty, while the average values of the parameters remain approximately the same. In particular, the curvature tension mentioned earlier between the Planck and the BAO+SN1a datasets is still present in the blind model, even slightly enhanced. Interestingly, the degeneracy directions between H_0 and Ω_K in the BAO + SN1a analysis are opposite between the two models. This can be understood from the first order in Ω_K of the angular diameter distance formula (26) (which is the main formula governing the BAO + SN1a fit), assuming Ω_m and Ω_Λ as fixed variables independent of H_0 and Ω_K, which leads to

$$\Omega_K^{\Lambda\mathrm{CDM}} = -\frac{6\mathcal{I}_1(z)}{\mathcal{I}_1^3(z) - 6\mathcal{I}_2(z)} + \frac{6(1+z)d_\mathrm{A}(z)}{\mathcal{I}_1^3(z) - 6\mathcal{I}_2(z)} H_0 + \mathcal{O}\Omega_K^2, \tag{93}$$

$$\Omega_K^{\mathrm{blind}} = -\frac{6}{\mathcal{I}_1^2(z)} + \frac{6(1+z)d_\mathrm{A}(z)}{\mathcal{I}_1^3(z)} H_0 + \mathcal{O}\Omega_K^2, \tag{94}$$

where

$$\mathcal{I}_1(z) \coloneqq \int_0^z \frac{1}{[\Omega_\mathrm{m}(1+y)^3 + \Omega_\Lambda]^{1/2}} \mathrm{d}y > 0, \tag{95}$$

$$\mathcal{I}_2(z) \coloneqq \int_0^z \frac{(z+1)^2/2}{[\Omega_\mathrm{m}(1+y)^3 + \Omega_\Lambda]^{3/2}} \mathrm{d}y > 0. \tag{96}$$

In our model, the degeneracy (of Ω_K as a function of H_0) has a positive slope with factor $\frac{6(1+z)d_\mathrm{A}(z)}{\mathcal{I}_1^3(z)}$. In the ΛCDM model, this slope is $\frac{6(1+z)d_\mathrm{A}(z)}{\mathcal{I}_1^3(z) - 6\mathcal{I}_2(z)}$ and can be negative if $\mathcal{I}_1^3(z) < 6\mathcal{I}_2(z)$, as is the case with BAO + SN1a data. The slope of the degeneracy depends on the redshift, which

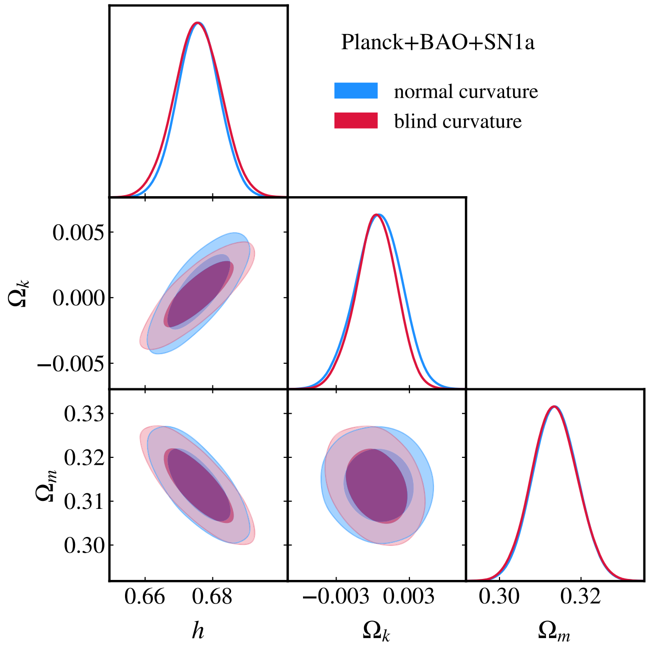

FIG. 2. 2D posterior distributions of $\{h, \Omega_K, \Omega_\mathrm{m}\}$ in the standard case and in the blind curvature model for the combined analysis Planck + BAO + SN1a.

explains why it is different between the Planck data and the BAO + SN1a data sets.

Figure 2 presents the posterior distributions for the combined Planck + BAO + SN1a fit in the ΛCDM normal curvature model (red) and the blind model (blue). No significant difference can be found between the two models. In particular, the curvature is still tightly constrained around zero.

In all the fits, we find that the additional gauge invariant variables present in our model at the level of perturbations play a negligible role, the difference with ΛCDM coming mainly from the modified background expansion laws (25).

Overall, the difference with respect to the best fit with the ΛCDM model and our model is not significant. In particular, while we still have a preference for a spherical universe when Planck data alone are used, the increase in the Hubble tension is still present. With respect to the discussion in Sec. III C, this shows that the main constraints given by cosmological data on the value of the curvature parameter come from the geometrical effects of that curvature.

VI. CONCLUSION

We derived the homogeneous and isotropic solution of the biconnection theory developed in [1], in which a term related to topology is added in the Einstein equation. The new expansion laws do not feature the curvature parameter anymore, regardless of its value, i.e., $\Omega_{\neq K} = 1, \;\; \forall \;\; \Omega_K$. In other words, in this cosmological model, the expansion scenario is equivalent for a Euclidean, spherical, or hyperbolic universe: I.e., expansion is blind to the spatial curvature. The first order perturbations around this homogeneous solution features two new gauge invariant variables compared to the Standard Model. A scalar and a vector mode, both sourced by the anisotropic stress of the fluid, and which can be related to a reference observer.

We tested our model against observations. In the different combinations of datasets, no significant difference was found between our model and the ΛCDM model. In particular, the curvature tension between the Planck and the BAO datasets remains present within our model, and is even slightly increased, with Planck preferring a closed universe with $\Omega_{K,0} = -0.057 \pm 0.025$ and $H_0 = 50 \pm 5$ km/s/Mpc.

Overall, these results show that removing the curvature parameter from the expansion laws does not significantly change its estimation from current cosmological data. Since the presence or not of spatial curvature is the main difference between our cosmological model and the ΛCDM model, only a better precision of the measurement of that parameter might enable us to distinguish between these two models, and therefore distinguish between general relativity and the biconnection theory of [1]. The study of a nonflat inflationary scenario under the framework of this theory is also an interesting perspective, especially since the flatness problem is not present with the blind expansion law anymore.

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APPENDIX: NO EFFECTIVE SPATIAL CURVATURE IN THE NEWTONIAN EXPANSION LAW

1. Motivation

In Newtonian gravitation, expansion is constrained by the averaged second Friedmann equation

$$3\ddot{a}/a = -\frac{\kappa}{2}\langle\rho\rangle_\Sigma + \Lambda - \Xi_{cd}\Xi^{cd}, \tag{A1}$$

where $\langle\cdot\rangle_\Sigma$ is the spatial average over the whole (spatial) volume of Σ, and Ξ_{cd} is traceless-transverse and represents anisotropic expansion. We will not consider Ξ_{ij} further. That law is valid both for Newtonian gravitation on a Euclidean topology [34–36] or for Newtonian gravitation on a non-Euclidean topology [3] and consequently was shown to derive from the nonrelativistic limit of, respectively, general relativity and the biconnection theory. However, while the second Friedman equation is explicitly obtained from the equations of Newtonian gravitation, the first Friedmann equation (i.e., featuring H^2) is only retrieved after integrating the former equation, as is well known in Newtonian cosmology, leading to

$$3H^2 = \kappa\langle\rho\rangle_\Sigma + \Lambda + C/a^2, \tag{A2}$$

where C is an integration constant mimicking a spatial curvature term. To our knowledge, textbooks and references talking about Newtonian cosmology always assume that C is free (even though it is generally taken to be zero).

Since the main result of the present paper is the fact that spatial curvature should not be present anymore in the expansion law once we consider a theory compatible with the nonrelativistic limit in any topology, then there seems to be a contradiction with (A2). The goal of this section is to show that this is not the case. We will show that a "hidden condition" can be found from the first order in the nonrelativistic limit of either the Einstein equation or the biconnection theory, which will constrain the integration constant C to be zero (in either the Euclidean, spherical, or hyperbolic cases), thus retrieving the law (20). In other words, the expansion laws of nonrelativistic (i.e., Newtonian) gravitation, if required to be compatible with either general relativity (Euclidean case) or the biconnection theory (non-Euclidean case) must not feature an effective spatial curvature term.

2. Derivation

The derivation of that result requires the nonrelativistic limit based on Galilean invariance that was developed by [37]. We will not reintroduce this limit in the present paper. Rather, we will directly use some formulas obtained in [1], which were derived from this limit. The Newtonian expansion law (A1) corresponds to the volume average of the zeroth order (in $1/c^2$) of the time-time components of the Einstein/biconnection equations (6) (in other words, the average of the zeroth order of the Raychaudhuri equation). The first Friedmann law is obtained with the volume average of the first order of the spatial trace of (6) [in other words, the average of the first order of the trace of the (3 + 1)-Ricci equation]. That first order equation is given by the trace of Eq. (135) in Appendix E of [1], which gives

$$3(\dot{H} + 3H^2) + D_iP^i = 3\left(\frac{\kappa}{2}\rho + \Lambda\right), \tag{A3}$$

where P^i is a vector depending on (post)-Newtonian terms that we do not need to detail, and D_i is the spatial connection related to a constant curvature metric equivalent to h_{ij} of Sec. III.

Equation (A3) is valid for Newtonian gravitation in either a Euclidean topology (i.e., from the nonrelativistic limit of the Einstein equation) or in a spherical/hyperbolic topology (i.e., from the nonrelativistic limit of the biconnection theory). As a local equation, it does not constrain the Newtonian dynamics more even if the density is present; i.e., it does not need to be considered on top of the Poisson equation. Rather, it is a dictionary to calculate P^i, which can be related to the first order of the spatial curvature tensor [1]. However, because the divergence D_iP^i vanishes with an averaging procedure, the global property of this equation gives additional constraints on the global dynamics. Taking the spatial average of (A3) and using the acceleration law (A1), we obtain

$$3H^2 = \kappa\langle\rho\rangle_\Sigma + \Lambda. \tag{A4}$$

This is the first Friedmann equation without a curvature term or integration constant, and, again, is obtain for Newtonian gravitation in any topology. This shows that, as for the homogenous solution of the biconnection theory, the expansion law of any inhomogeneous (non-inear) solution of Newtonian gravitation is also blind to the spatial curvature.

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