B_c-meson decays into J/ψ plus a light meson in the improved perturbative QCD formalism

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(Received 23 May 2023; accepted 9 October 2023; published 8 November 2023)

In the wake of measurements on $B_c^+ \to J/\psi K^+$, $B_c^+ \to J/\psi \pi^+\pi^-\pi^+$, and $B_c^+ \to J/\psi K^+K^-\pi^+$ at Large Hadron Collider experiments, we propose to study the decays $B_c^+ \to J/\psi M^+$ comprehensively, with M being the light charged pseudoscalar (P), vector (V), scalar (S), axial-vector (A), and tensor (T) mesons, within the improved perturbative QCD (iPQCD) formalism at leading order in the Standard Model. The theoretical predictions for experimental observables such as branching fractions, relative ratios, and longitudinal polarization fractions in the iPQCD formalism await near future examinations relying on the upgraded Large Hadron Collider, even the forthcoming Circular Electron-Positron Collider. We emphasize that the investigations on the factorizable-emission-suppressed or -forbidden decays like $B_c^+ \to J/\psi S^+$, $B_c^+ \to J/\psi A^+_{1^1P_1}$, and $B_c^+ \to J/\psi T^+$, should go definitely beyond naive factorization to explore the rich dynamics, which could, in turn, further help understand the QCD nature of B_c meson, as well as that of related hadrons. The future confirmations on those predictions about the relative ratios between the branching fractions of $B_c^+ \to J/\psi b_1(1235)^+ (a_0(980)^+, a_0(1450)^+, a_2(1320)^+)$ and $B_c^+ \to J/\psi \pi^+$ could further examine the reliability of this iPQCD formalism. Because of containing only tree-level $\bar{b} \to \bar{c}$ transitions, the CP asymmetries in the $B_c^+ \to J/\psi M^+$ decays exhibit naturally zero.

DOI: 10.1103/PhysRevD.108.096006

I. INTRODUCTION

In 1998, the first discovery of B_c meson at Tevatron [1,2] proclaimed the beginning of its experimental studies. After that, its properties, e.g., lifetime and mass, were measured combined through semileptonic decay $B_c^\pm \to J/\psi \ell^\pm \nu_\ell$ [3,4] and nonleptonic decay $B_c^\pm \to J/\psi \pi^\pm$, the so-called "golden channel" in B_c-meson decays [5,6]. Ever since the running of Large Hadron Collider (LHC) in 2009, the attention has always been be payed from the community of heavy flavor physics on the measurements of B_c-meson decays. The underlying reason is that a B_c meson is the ground state of a unique meson family containing two different kinds of heavy flavor, namely, b and c, in the Standard Model [7,8] and it is the only meson whose decays of both constituents compete with each other. The B_c-meson decays contain rich dynamics in the perturbative regimes, besides the nonperturbative nature. What is more, its decays might shed light on possible new physics beyond the Standard Model (for example, see very recent literature [9–12] and references therein).

The LHC experiments have measured several nonleptonic decay channels of B_c meson [13,14], however, unfortunately, the exact values of individual branching fractions ($\mathcal{B}$) for those observed decays are not available yet because of experimentally complicated background with proton-proton collisions at LHC. Among them, the B_c-meson decays into J/ψ plus a light meson such as $B_c^+ \to J/\psi a_1(1260)^+$ (In the following context, we will describe it as a_1^+ for convenience, unless otherwise stated.) and $B_c^+ \to J/\psi K^+$ were observed through the relative ratios of branching fractions between the related B_c^+ decays. Explicitly,

(i) In 2012, the decay $B_c^+ \to J/\psi \pi^+\pi^-\pi^+$ was reported for the first time by the Large Hadron Collider-beauty (LHCb) Collaboration [15]. The ratio between the branching fractions of $B_c^+ \to J/\psi \pi^+\pi^-\pi^+$ and $B_c^+ \to J/\psi \pi^+$ was measured to be

$$R^{\rm Exp}_{3\pi/\pi} \equiv \frac{\mathcal{B}(B_c^+ \to J/\psi \pi^+\pi^-\pi^+)}{\mathcal{B}(B_c^+ \to J/\psi \pi^+)} = 2.41 \pm 0.45, \qquad (1)$$

where "the background-subtracted distribution of the $M(\pi^+\pi^-\pi^+)$ mass for the $B_c^+ \to J/\psi \pi^+\pi^-\pi^+$ data exhibits an a_1 peak" [15]. In 2015, the CMS Collaboration at LHC confirmed this ratio with newly measured value as [16]

$$R'^{\text{Exp}}_{3\pi/\pi} \equiv \frac{\mathcal{B}(B_c^+ \to J/\psi\pi^+\pi^-\pi^+)}{\mathcal{B}(B_c^+ \to J/\psi\pi^+)} = 2.55 \pm 0.87. \quad (2)$$

In the above two ratios, the statistic and systematic errors have been added in quadrature. By combining these two measurements, the Heavy Flavor Averaging Group gave the averaged ratio as 2.45 ± 0.40 [14]. Theoretically, the decay $B_c^+ \to J/\psi\pi^+\pi^-\pi^+$ has been investigated via definition $R_{3\pi/\pi}$ in Refs. [17–21] with different values, however, which are basically consistent with the current data, except for a smaller value 1.5 estimated in [18].

Moreover, in 2013, the decay $B_c^+ \to J/\psi K^+K^-\pi^+$ was also detected for the first time by the LHCb Collaboration [22], and the ratio between $\mathcal{B}(B_c^+ \to J/\psi K^+K^-\pi^+)$ and $\mathcal{B}(B_c^+ \to J/\psi\pi^+)$ was measured to be

$$R^{\text{Exp}}_{2K\pi/\pi} \equiv \frac{\mathcal{B}(B_c^+ \to J/\psi K^+K^-\pi^+)}{\mathcal{B}(B_c^+ \to J/\psi\pi^+)} = 0.53 \pm 0.11, \quad (3)$$

where various errors have been added in quadrature too. This measurement agrees well with the available theoretical predictions 0.49 and 0.47 [23] corresponding to the resonance approximation with contribution of a_1.

Very recently, the LHCb Collaboration studied the B_c-meson decaying to charmonia plus multihadron final states and reported the ratio between the branching fractions of $B_c^+ \to J/\psi K^+K^-\pi^+$ and $B_c^+ \to J/\psi\pi^+\pi^-\pi^+$, largely proceeding via $a_1^+ \to K^+\bar{K}^{*0} \to K^+K^-\pi^+$ and $a_1^+ \to \rho^0\pi^+ \to \pi^+\pi^-\pi^+$, respectively, as follows [24],

$$R^{\text{Exp}}_{2K\pi/3\pi} \equiv \frac{\mathcal{B}(B_c^+ \to J/\psi K^+K^-\pi^+)}{\mathcal{B}(B_c^+ \to J/\psi\pi^+\pi^-\pi^+)} = 0.185 \pm 0.014, \quad (4)$$

with the statistic and systematic uncertainties being added in quadrature. This ratio deviates slightly from 0.22 ± 0.06 deduced by the previous LHCb measurements as presented in Eqs. (1) and (3).

(ii) In 2013, the decay $B_c^+ \to J/\psi K^+$ was observed for the first time by the LHCb Collaboration [25]. The ratio between the branching fractions of $B_c^+ \to J/\psi K^+$ and $B_c^+ \to J/\psi\pi^+$ was measured to be

$$R^{\text{Exp}}_{K/\pi} \equiv \frac{\mathcal{B}(B_c^+ \to J/\psi K^+)}{\mathcal{B}(B_c^+ \to J/\psi\pi^+)} = 0.069 \pm 0.020, \quad (5)$$

and, subsequently, this ratio was updated in 2016 with a good precision as [26],

$$R'^{\text{Exp}}_{K/\pi} \equiv \frac{\mathcal{B}(B_c^+ \to J/\psi K^+)}{\mathcal{B}(B_c^+ \to J/\psi\pi^+)} = 0.079 \pm 0.008. \quad (6)$$

The ratio $R'^{\text{Exp}}_{K/\pi}$ supersedes the previous $R^{\text{Exp}}_{K/\pi}$ however still agrees well with each other. Notice that the errors from different sources in the above two ratios have been added in quadrature. Interestingly, the theoretical predictions for this ratio locate at a broad region of [0.052, 0.088] (See them in Table I), which implies different understanding on the dynamics in different formalisms for these two B_c-meson decays. That is to say, the adequate and complementary studies are still demanded to further clarify the involved dynamics, in particular, based on certain frameworks of QCD-inspired factorization.

Certainly, along with the successful upgrade of LHC, around 10^{10} B_c-meson events could be accessed per year. Thus more varieties of B_c-meson decays would be measured by the upcoming experiments at the ongoing LHC, even the forthcoming Circular Electron-Positron Collider (CEPC).

Motivated by the above-mentioned observations and near future rich measurements on the B_c-meson decays, we propose to study the decays $B_c^+ \to J/\psi M^+$, in which, M denotes the pseudoscalar (P), vector (V), axial-vector (A), scalar (S), and tensor (T) mesons composed of light quarks, in a comprehensive manner within the improved perturbative QCD (iPQCD) formalism at leading order [41,42]. It is worth emphasizing that the resummation formula adopted in the conventional PQCD approach to B_c-meson decays [38,43,44] is not appropriate. The conventional PQCD formalism has recently been improved by taking into account the finite charm quark mass effects

TABLE I. Relative ratios from the branching fractions for the decays $B_c^+ \to J/\psi P^+$, $B_c^+ \to J/\psi V^+$, and $B_c^+ \to J/\psi a_1^+$ in the literature at both aspects of theory and experiment.

Ratios	[27]	[28]	[29]	[30]	[31]	[32]	[33]	[34]	[35]	[36]	[21]	[37]	[38]	[39]	[40]	Data
$R_{K/\pi}$	0.077	0.076	0.052	0.074	0.049	0.082	0.076	0.079	0.088	⋯	0.075	0.079	0.082	$0.076^{+0.015}_{-0.015}$	$0.075^{+0.005}_{-0.005}$	$0.079^{+0.008}_{-0.008}$
$R_{K^*/\rho}$	0.054	0.054	0.054	0.057	0.038	0.063	0.057	0.058	0.050	⋯	0.053	⋯	0.057	0.059	0.056	⋯
$R_{\rho/\pi}$	3.01	3.22	2.85	2.85	19.31	2.62	2.88	3.16	⋯	5.29	2.77	⋯	3.31	3.52	5.65	⋯
$R_{K^*/\pi}$	0.16	0.17	0.15	0.16	0.73	0.16	0.16	0.18	⋯	0.26	0.15	0.16	0.19	0.21	0.32	⋯
$R_{a_1/\pi}$	⋯	⋯	4.0	⋯	⋯	⋯	⋯	⋯	⋯	⋯	5.5	⋯	⋯	⋯	⋯	⋯

through k_T resummation at the next-to-leading-logarithm accuracy [42]. The resultant Sudakov factor $s_c(Q, b)$ makes the framework for the B_c-meson and B-meson decays into charmonia plus light mesons really complete at leading order. So far, partial decay modes of $B_c^+ \to J/\psi M^+$ such as $B_c^+ \to J/\psi P^+$ and $B_c^+ \to J/\psi V^+$, even $B_c^+ \to J/\psi a_1^+$, have been investigated in many different models or methods based on factorization assumptions and the relative ratios of their branching fractions between the different B_c^+ decays are collected in Table I. But, it is indicated that different branching fractions with large discrepancy for these $B_c^+ \to J/\psi P^+$, $B_c^+ \to J/\psi V^+$, and $B_c^+ \to J/\psi a_1^+$ channels appear, though the ratios among these branching fractions are comparable to each other, even to data. As discussed in [45], the $B_c^+ \to J/\psi P^+$ decays, as well as the $B_c^+ \to J/\psi V^+$ ones, are predominated by the factorizable emission amplitudes, in association with the negligible nonfactorizable emission ones. It means that the theoretical predictions for $R_{K/\pi}$ in various kinds of models and methods should be consistent with each other, as naively anticipated in factorization ansatz. That is to say, if the $B_c^+ \to J/\psi V^+$ decays are basically governed by the longitudinal polarization contributions, then $R_{K^*/\rho}$ is expected to be close to $R_{K/\pi}$. Certainly, they cannot tell us more dynamics in the related decays, even if $R^{\rm Theo}_{K/\pi}$ and $R^{\rm Theo}_{K^*/\rho}$ agree well with those at experiments. The fact is that the decay amplitude in the decays like $B_c^+ \to J/\psi P^+$, $B_c^+ \to J/\psi V^+$, even $B_c^+ \to J/\psi a_1^+$ can be approximately written into the product of decay constant and transition form factor as described in naive factorization, then the above-mentioned ratios can be further written as the ratio of squared decay constants multiply by the ratio of squared Cabibbo-Kaboyashi-Maskawa (CKM) matrix elements. Then, for example, the relation of the branching factions between the $B_c^+ \to J/\psi K^+$ and $B_c^+ \to J/\psi \pi^+$ decays could be naively derived as,

$$R_{K/\pi} \equiv \frac{\mathcal{B}(B_c^+ \to J/\psi K^+)}{\mathcal{B}(B_c^+ \to J/\psi \pi^+)} \simeq \frac{|V_{us}|^2}{|V_{ud}|^2} \cdot \frac{f_K^2}{f_\pi^2} \sim 0.081, \qquad (7)$$

with $|V_{us}| = 0.2265$, $|V_{ud}| = 0.9740$, $f_K = 0.16$ GeV, and $f_\pi = 0.131$ GeV [13]. This naive expectation agrees perfectly with the latest measurements as shown in Eq. (6) indeed. Notice that, however, for the decays with suppressed or vanished factorizable-emission amplitudes while with enhanced nonfactorizable emission ones, e.g., $B_c^+ \to J/\psi S^+$, $B_c^+ \to J/\psi T^+$, etc., one should go beyond naive factorization to explore the rich but complicated dynamics within the factorization framework based on QCD. We can then understand deeply the perturbative and nonperturbative QCD dynamics involved in these B_c-meson decays.

The B_c meson is treated as a heavy-light system [41] and the related decays are analyzed in its rest frame with momentum $P_1 = m_{B_c}/\sqrt{2}(1, 1, \mathbf{0}_T)$ in the light-cone coordinates. Then, for $B_c^+ \to J/\psi M^+$ decays, M and J/ψ mesons are assumed to move correspondingly in the plus and minus z-directions carrying the momenta P_2 and P_3 as,

$$P_2 = \frac{m_{B_c}}{\sqrt{2}}(1 - r_3^2, r_2^2, \mathbf{0}_T), \qquad P_3 = \frac{m_{B_c}}{\sqrt{2}}(r_3^2, 1 - r_2^2, \mathbf{0}_T), \qquad (8)$$

associated with polarization vectors ϵ_2 and ϵ_3 in the longitudinal (L) and transverse (T) polarizations, if M is V or A, as,

$$\epsilon_2(L) = \frac{1}{\sqrt{2(1 - r_3^2)} r_2}(1 - r_3^2, -r_2^2, \mathbf{0}_T), \quad \epsilon_2(T) = (0, 0, \mathbf{1}_T), \qquad (9)$$

$$\epsilon_3(L) = \frac{1}{\sqrt{2(1 - r_2^2)} r_3}(-r_3^2, 1 - r_2^2, \mathbf{0}_T), \quad \epsilon_3(T) = (0, 0, \mathbf{1}_T), \qquad (10)$$

where the ratios $r_2 = m_M/m_{B_c}$ and $r_3 = m_{J/\psi}/m_{B_c}$, and those two polarization vectors (The capital L and T in the parentheses describe the longitudinal and transverse polarizations, respectively. Not to be confused with the abbreviation T for tensors.) satisfy $P \cdot \epsilon = 0$ and $\epsilon^2 = -1$.[1] Notice that, due to conservation of the angular momentum, only the longitudinal polarization vector ϵ_{3L} of J/ψ is required in the decays $B_c^+ \to J/\psi P^+$ and $B_c^+ \to J/\psi S^+$. We stress that, due to small contributions around 5% to the $B_c^+ \to J/\psi M^+$ branching fractions, the terms proportional to r_2^2 and r_2^4 will be safely neglected in the numerators of factorization formulas. The momenta of the spectator quarks in the involved hadrons are parametrized as

$$k_1 = (x_1 P_1^+, x_1 P_1^-, \mathbf{k}_{1T}), \quad k_2 = (x_2 P_2^+, x_2 P_2^-, \mathbf{k}_{2T}), \quad k_3 = (x_3 P_3^+, x_3 P_3^-, \mathbf{k}_{3T}), \qquad (11)$$

where x_i ($i = 1, 2, 3$) is the momentum fraction of valence quark in the involved mesons.

The $B_c^+ \to J/\psi M^+$ decay amplitude in the iPQCD formalism can therefore be conceptually written as follows,

[1] As described in Ref. [46], since only three helicities $\ell = 0, \pm 1$ contribute to the $B_c^+ \to J/\psi T^+$ modes, the involved light tensor meson can then be treated as a vectorlike meson with tensor meson mass. In other words, the polarization tensor of tensor meson can be constructed through the spin-1 polarization vector of vector meson [47]. A new polarization vector ϵ_T for tensor meson can then be read as $\epsilon_T(L) = \sqrt{\frac{2}{3}}\epsilon(L)$ and $\epsilon_T(T) = \sqrt{\frac{1}{2}}\epsilon(T)$ [48].

$$A(B_c^+ \to J/\psi M^+) \sim \int dx_1 dx_2 dx_3 b_1 db_1 b_2 db_2 b_3 db_3 \cdot \mathrm{Tr}\big[C(t)\Phi_{B_c}(x_1, b_1)\Phi_M(x_2, b_2) \times \Phi_{J/\psi}(x_3, b_3) H(x_i, b_i, t) e^{-S(t)}\big], \quad (12)$$

where b_i is the conjugate space coordinate of transverse momentum k_{iT}; t is the largest running energy scale in hard kernel $H(x_i, b_i, t)$; Tr denotes the trace over Dirac and SU(3) color indices; $C(t)$ stands for the Wilson coefficients including the large logarithms $\ln(m_W/t)$ [49]; and Φ is the wave function describing the hadronization of quarks and antiquarks to the meson. The Sudakov factor $e^{-S(t)}$ arises from k_T resummation, which provides a strong suppression on the long distance contributions in the small k_T (or large b) region [50] The detailed discussions for $e^{-S(t)}$ can be easily found in the original Refs. [41,42,50]. Thus, with Eq. (12), we can give the convoluted amplitudes of the decays $B_c^+ \to J/\psi M^+$ explicitly through the evaluations of the hard kernel $H(x_i, b_i, t)$ at leading order in the α_s expansion with the iPQCD formalism.

The wave function for B_c meson with a heavy-light structure can generally be defined as [41,49,51]

$$\Phi_{B_c}(x, k_T) = \frac{i}{\sqrt{2N_c}} \{(\not{P} + m_{B_c})\gamma_5 \phi_{B_c}(x, k_T)\}_{\alpha\beta}, \quad (13)$$

where α, β are the color indices; P is the momentum of B_c meson; $N_c = 3$ is the color factor; and x and k_T are the momentum fraction and intrinsic transverse momentum of charm quark in the B_c meson; $\phi_{B_c}(x, k_T)$ is the B_c-meson leading-twist distribution amplitude.

For the vector J/ψ meson, its wave function has been studied within the nonrelativistic QCD approach [52]. The longitudinal and transverse wave functions have been derived as,

$$\Phi^L_{J/\psi}(x) = \frac{1}{\sqrt{2N_c}} \{m_{J/\psi}\not{\epsilon}_L \phi^L_{J/\psi}(x) + \not{\epsilon}_L \not{P} \phi^t_{J/\psi}(x)\}_{\alpha\beta}, \quad (14)$$

$$\Phi^T_{J/\psi}(x) = \frac{1}{\sqrt{2N_c}} \{m_{J/\psi}\not{\epsilon}_T \phi^v_{J/\psi}(x) + \not{\epsilon}_T \not{P} \phi^T_{J/\psi}(x)\}_{\alpha\beta}. \quad (15)$$

Here, x describes the distribution of charm quark momentum in J/ψ meson, ϵ_L and ϵ_T are the two polarization vectors of J/ψ, and $\phi^L_{J/\psi}(x)$ and $\phi^T_{J/\psi}(x)$ are the twist-2 distribution amplitudes, while $\phi^t_{J/\psi}(x)$ and $\phi^v_{J/\psi}(x)$ are the twist-3 ones.

The light-cone wave functions including distribution amplitudes for light pseudoscalars, scalars, vectors, axial-vectors, and tensors have been given in the QCD sum rules up to twist-3. They are collected as follows:

(i) For P and S mesons [53–57],

$$\Phi_P(x) = \frac{i}{\sqrt{2N_c}} \gamma_5 \Big\{\not{P}\phi^A_P(x) + m^P_0 \phi^P_P(x) + m^P_0(\not{n}\not{v} - 1)\phi^T_P(x)\Big\}_{\alpha\beta}, \quad (16)$$

and

$$\Phi_S(x) = \frac{i}{\sqrt{2N_c}} \Big\{\not{P}\phi_S(x) + m_S \phi^S_S(x) + m_S(\not{n}\not{v} - 1)\phi^T_S(x)\Big\}_{\alpha\beta}, \quad (17)$$

with $m_S(m^P_0)$ denoting (chiral) mass of light (pseudoscalars) scalars, $n = (1, 0, \mathbf{0}_T)$ and $v = (0, 1, \mathbf{0}_T)$ being the light-like dimensionless vectors on the light-cone. And $\phi^A_P(x)$ and $\phi_S(x)$ are the leading-twist distribution amplitudes, while $\phi^{P,T}_P(x)$ and $\phi^{S,T}_S(x)$ are the twist-3 ones.

(ii) For V and A mesons with polarizations [58–61],

$$\Phi^L_V(x) = \frac{1}{\sqrt{2N_c}} \Big\{m_V \not{\epsilon}^L_V \phi_V(x) + \not{\epsilon}^L_V \not{P}\phi^t_V(x) + m_V \phi^s_V(x)\Big\}_{\alpha\beta}, \quad (18)$$

$$\Phi^T_V(x) = \frac{1}{\sqrt{2N_c}} \Big\{m_V \not{\epsilon}^T_V \phi^v_V(x) + \not{\epsilon}^T_V \not{P}\phi^T_V(x) + m_V i\epsilon_{\mu\nu\rho\sigma}\gamma_5\gamma^\mu \epsilon^{T\nu}_V n^\rho v^\sigma \phi^a_V(x)\Big\}_{\alpha\beta}, \quad (19)$$

and

$$\Phi^L_A(x) = \gamma_5 \Phi^L_V(x), \qquad \Phi^T_A(x) = \gamma_5 \Phi^T_V(x), \quad (20)$$

where m, P, and ϵ are the mass, the momentum and the polarization vector for (axial-)vector mesons, $\phi(x)$ and $\phi^T(x)$ are the leading-twist distribution amplitudes, and $\phi^{t,s}(x)$ and $\phi^{v,a}(x)$ are the twist-3 ones, and x is the momentum fraction of quark carrying in the (axial-)vector mesons.

(iii) For T mesons with polarizations [62,63],

$$\Phi^L_T(x) = \frac{1}{\sqrt{2N_c}} \Big\{m_T \not{\epsilon}^L_T \phi_T(x) + \not{\epsilon}^L_T \not{P}\phi^t_T(x) + m_T \phi^s_T(x)\Big\}_{\alpha\beta}, \quad (21)$$

$$\Phi^T_T(x) = \frac{1}{\sqrt{2N_c}} \Big\{m_T \not{\epsilon}^T_T \phi^v_T(x) + \not{\epsilon}^T_T \not{P}\phi^T_T(x) + m_T i\epsilon_{\mu\nu\rho\sigma}\gamma_5\gamma^\mu \epsilon^{T\nu}_T n^\rho v^\sigma \phi^a_T(x)\Big\}_{\alpha\beta}, \quad (22)$$

with the tensor meson mass m_T, the twist-2 distribution amplitudes $\phi_T(x)$ and $\phi_T^T(x)$, the twist-3 distribution amplitudes $\phi_T^{t,s}(x)$ and $\phi_T^{v,a}(x)$, the momentum P and polarization vector ϵ_T satisfying $P \cdot \epsilon_T = 0$, and the momentum fraction x carried by quark in the tensor meson.

In the above wave functions with polarizations, we adopt the convention $\epsilon^{0123} = 1$ for the Levi-Civitá tensor $\epsilon^{\mu\nu\alpha\beta}$. Notice that the explicit expressions for all the above-mentioned distribution amplitudes with their masses, decay constants and Gegenbauer moments can be found later in Appendix A.

For the $B_c^+ \to J/\psi M^+$ decays induced by the $\bar{b} \to \bar{c}$ transition at the quark level, the related weak effective Hamiltonian $H_{\rm eff}$ can be written as [64]

$$H_{\rm eff} = \frac{G_F}{\sqrt{2}} \left\{ V_{cb}^* V_{uq} [C_1(\mu) O_1(\mu) + C_2(\mu) O_2(\mu)] \right\} + {\rm H.c.}, \tag{23}$$

with the Fermi constant $G_F = 1.16639 \times 10^{-5}\ {\rm GeV}^{-2}$, the CKM matrix elements V, and the Wilson coefficients $C_i(\mu)$ at the renormalization scale μ. The local four-quark tree operators O_1 and O_2 are written as

$$O_1 = \bar{q}_\alpha \gamma_\mu (1-\gamma_5) u_\beta \bar{c}_\beta \gamma_\mu (1-\gamma_5) b_\alpha,$$
$$O_2 = \bar{q}_\alpha \gamma_\mu (1-\gamma_5) u_\alpha \bar{c}_\beta \gamma_\mu (1-\gamma_5) b_\beta, \tag{24}$$

where q denotes the light down quark $d(s)$ for the $\Delta S = 0(1)$, namely, CKM-favored (suppressed) processes with S being the strange number (Not to be confused with the abbreviation S for scalars).

The related Feynman diagrams for the decays $B_c^+ \to J/\psi M^+$ in the iPQCD formalism at leading order are illustrated in Fig. 1. As presented [see Eqs. (31)–(46) for details] in Ref. [41], we have given the factorization formulas and analytic $B_c^+ \to J/\psi \pi^+$ decay amplitudes with all elements. The similar calculations could be repeated for the rest $B_c^+ \to J/\psi M^+$ decay modes in this work. Hereafter, for the sake of simplicity, we will use F_e and M_e to describe the factorizable emission and the nonfactorizable emission amplitudes induced by the $(V-A)(V-A)$ operators in these types of $B_c^+ \to J/\psi M^+$ decays. Furthermore, in light of the successful clarification of most branching ratios and polarization fractions in the $B \to VV$ decays by keeping the terms proportional to $r_V^2 = m_V^2/m_B^2$ in the denominator of propagators for virtual quarks and gluons with the PQCD approach [65], we will follow this treatment in the present work. That is, we will retain the terms like r_2^2 and r_3^2 in dealing with the denominators of factorization formulas for the decays $B_c^+ \to J/\psi M^+$, which could be examined by future measurements to further clarify its universality. The related factorization formulas can be found in Appendix B.

The $B_c^+ \to J/\psi M^+$ decay amplitude can thus be decomposed into

$$A^{(\sigma)}(B_c^+ \to J/\psi M^+) = V_{cb}^* V_{uq} \left(F_e^{(\sigma)} \cdot f_M + M_e^{(\sigma)} \right), \tag{25}$$

with $\sigma = L$ for the modes $B_c^+ \to J/\psi P^+$ and $J/\psi S^+$ involving contributions from only longitudinal polarization while $\sigma = L, N, T$ for the channels $B_c^+ \to J/\psi V^+$, $J/\psi A^+$, and $J/\psi T^+$ containing contributions from longitudinal, normal, and transverse polarizations, which result in the formulas for calculating branching fractions of the decays $B_c^+ \to J/\psi M^+$ as follows,

(i) For the decays $B_c^+ \to J/\psi M^+$ with $M = P$ and S,

$$\begin{aligned} \mathcal{B}(B_c^+ \to J/\psi M^+) &\equiv \tau_{B_c} \cdot \Gamma(B_c^+ \to J/\psi M^+) \\ &= \tau_{B_c} \cdot \frac{G_F^2 m_{B_c}^3}{32\pi} \cdot \Phi(r_2, r_3) \cdot |A(B_c^+ \to J/\psi M^+)|^2, \end{aligned} \tag{26}$$

where τ_{B_c} is the lifetime of B_c meson and $\Phi(r_2, r_3)$ is the phase space factor of $B_c^+ \to J/\psi M^+$ decays with $\Phi(x,y) \equiv \{[1-(x+y)^2]\cdot[1-(x-y)^2]\}^{1/2}$ [66].

(ii) For the decays $B_c^+ \to J/\psi M^+$ with $M = V$, A, and T,

$$\begin{aligned} \mathcal{B}(B_c^+ \to J/\psi M^+) &\equiv \tau_{B_c} \cdot \Gamma(B_c^+ \to J/\psi M^+) \\ &= \tau_{B_c} \cdot \frac{G_F^2 |\mathbf{P_c}|}{16\pi m_{B_c}^2} \sum_{\sigma=L,N,T} A^{\sigma\dagger}(B_c^+ \to J/\psi M^+) \\ &\quad \times A^\sigma(B_c^+ \to J/\psi M^+), \end{aligned} \tag{27}$$

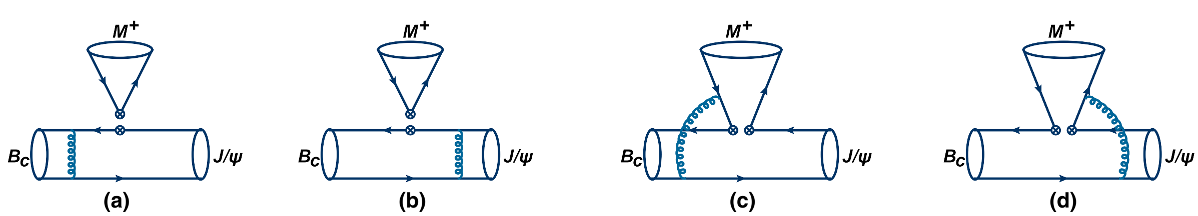

FIG. 1. Leading order Feynman diagrams for the decays $B_c^+ \to J/\psi M^+$ in the iPQCD formalism.

where $|\mathbf{P_c}| \equiv |\mathbf{P_{2z}}| = |\mathbf{P_{3z}}|$ is the momentum of either the outgoing M meson or J/ψ meson and $A^\sigma(B_c^+ \to J/\psi M^+)$ denotes the decay amplitudes with helicities for the decays $B_c^+ \to J/\psi V^+, J/\psi A^+$, and $J/\psi T^+$, respectively.

Now we turn to carry out the numerical calculations and make phenomenological analyses. In numerical calculations, central values of the input parameters will be used implicitly unless otherwise stated. The relevant QCD scale (GeV), masses (GeV), and B_c-meson lifetime (ps) are the following [13,49]

$$\Lambda_{\overline{\rm MS}}^{(f=4)} = 0.250, \quad m_W = 80.41, \quad m_{B_c} = 6.275, \quad m_{J/\psi} = 3.097, \quad \tau_{B_c} = 0.507, \quad m_b = 4.8, \quad m_c = 1.5. \tag{28}$$

For the CKM matrix elements, the Wolfenstein parametrization up to $\mathcal{O}(\lambda^4)$ is adopted [67],

$$V_{\rm CKM} = \begin{pmatrix} 1-\frac{1}{2}\lambda^2 & \lambda & A\lambda^3(\rho - i\eta) \\ -\lambda & 1-\frac{1}{2}\lambda^2 & A\lambda^2 \\ A\lambda^3(1-\rho-i\eta) & -A\lambda^2 & 1 \end{pmatrix} + \mathcal{O}(\lambda^4), \tag{29}$$

with the updated parameters $A = 0.790$ and $\lambda = 0.2265$ [13].

A. $B_c^+ \to J/\psi(P^+, V^+)$

As inferred from the literature, the decays $B_c^+ \to J/\psi P^+$ and $J/\psi V^+$ have been studied extensively in many different approaches and methods, though with different individual branching fractions, for example, see Refs. [21,27–40]. The CP-averaged branching fractions of the $B_c^+ \to J/\psi P^+$ decays in the iPQCD formalism are

$$\mathcal{B}(B_c^+ \to J/\psi\pi^+) = 1.17^{+0.31}_{-0.23}(\beta_{B_c})^{+0.08}_{-0.08}(f_M)^{+0.00}_{-0.00}(a_\pi) \times 10^{-3}, \tag{30}$$

$$\mathcal{B}(B_c^+ \to J/\psi K^+) = 8.68^{+2.32}_{-1.73}(\beta_{B_c})^{+0.64}_{-0.62}(f_M)^{+0.66}_{-0.63}(a_K) \times 10^{-5}, \tag{31}$$

where the dominant error arises from the shape parameter β_{B_c} in B_c-meson distribution amplitude. In the $B_c^+ \to J/\psi K^+$ mode, the uncertainties from the combined decay constants of B_c and J/ψ and from the SU(3)-flavor symmetry breaking factor a_1^K can compete with each other. These two branching ratios are around $\mathcal{O}(10^{-3})$ and $\mathcal{O}(10^{-4})$, respectively, within uncertainties. Although the individual $B_c^+ \to J/\psi\pi^+$ branching fraction is not definitely available yet, our iPQCD prediction for its value agrees generally with most of the predictions from various models and methods already presented in the literature, in particular, with that given in QCD factorization approach [68], and presented very recently in covariant confined quark model [39] and light-cone sum rules approach [40]. Furthermore, our predictions about the $B_c^+ \to J/\psi P^+$ branching fractions agree well with those presented in Refs. [39,40].

The ratio between the CP-averaged branching fractions of $B_c^+ \to J/\psi K^+$ and $B_c^+ \to J/\psi\pi^+$ in the iPQCD formalism is therefore given theoretically as,

$$R_{K/\pi}^{\rm Theo} \equiv \frac{\mathcal{B}(B_c^+ \to J/\psi K^+)}{\mathcal{B}(B_c^+ \to J/\psi\pi^+)} = 0.074^{+0.006}_{-0.005}, \tag{32}$$

which agrees well with the very recent predictions in Refs. [39,40] and the latest measurement as shown in Eq. (6) within errors. This natural agreement is trivial because these two $B_c^+ \to J/\psi P^+$ modes are absolutely dominated by the factorizable emission diagrams while with dramatically small nonfactorizable emission contributions due to the considerable cancelation under the isospin limit. It is clear to see that $R_{K/\pi}^{\rm Theo}$ predicted in iPQCD formalism is consistent with the naive expectation $R_{K/\pi}$ as described in Eq. (7), besides in good consistency with the measured one. The slight deviation between $R_{K/\pi}^{\rm Theo}$ and $R_{K/\pi}$ arises from the SU(3)-flavor symmetry breaking effects in the leading-twist distribution amplitude of kaon, due to the fact that the contributions induced by the nonfactorizable emission diagrams are proportional to $\phi_P^A(x)$ as shown in Eq. (33) of Ref. [41]. Therefore, as a by-product, it could be anticipated that the above naive expectation is also validated for B_c-meson decays into other charmonia, e.g., $\eta_c, \chi_{cJ}(J = 0, 1, 2), \ldots$, plus a pion or a kaon. However, it is worth stressing that the measurements on the ratio while without individual decay rate cannot help reveal the involved dynamics, even further constrain the related hadronic parameters, for example, the shape parameter β_{B_c} in the B_c-meson leading-twist distribution amplitude ϕ_{B_c}. In other words, the investigations on the related modes with large nonfactorizable contributions are of great necessity.

Now, we turn to analyze the $B_c^+ \to J/\psi V^+$ decays. The $B_c^+ \to J/\psi V^+$ branching fractions in the iPQCD formalism can be read as follows:

$$\mathcal{B}(B_c^+ \to J/\psi\rho^+) = 3.69^{+1.02}_{-0.76}(\beta_{B_c})^{+0.28}_{-0.27}(f_M)^{+0.01}_{-0.00}(a_\rho) \times 10^{-3}, \tag{33}$$

$$\mathcal{B}(B_c^+ \to J/\psi K^{*+}) = 2.23^{+0.62}_{-0.46}(\beta_{B_c})^{+0.20}_{-0.18}(f_M)^{+0.02}_{-0.01}(a_{K^*}) \times 10^{-4}, \tag{34}$$

where the errors are dominated by the B_c-meson shape parameter β_{B_c} and the combination of decay constants of the B_c-meson and related vectors. These predictions are well consistent with those in Ref. [39] within theoretical errors.

Based on the helicity amplitudes, we can define the transversity ones as follows:

$$\mathcal{A}_L = \xi m_B^2 A^L, \qquad \mathcal{A}_\parallel = \xi\sqrt{2} m_B^2 A^N, \\ \mathcal{A}_\perp = \xi m_V m_{J/\psi}\sqrt{2(r^2-1)}A^T, \tag{35}$$

for the longitudinal, parallel, and perpendicular polarizations, respectively, with the normalization factor $\xi = \sqrt{G_F^2|\mathbf{P_c}|/(16\pi m_{B_c}^2\Gamma)}$ and the ratio $r = P_2\cdot P_3/(m_V\cdot m_{J/\psi})$. These amplitudes satisfy the relation,

$$|\mathcal{A}_L|^2 + |\mathcal{A}_\parallel|^2 + |\mathcal{A}_\perp|^2 = 1. \tag{36}$$

following the summation in Eq. (27). Since the transverse-helicity contributions can manifest themselves through polarization observables, we therefore define CP-averaged longitudinal polarization fractions f_L as the following,

$$f_L \equiv \frac{|\mathcal{A}_L|^2}{|\mathcal{A}_L|^2 + |\mathcal{A}_\parallel|^2 + |\mathcal{A}_\perp|^2}(=|\mathcal{A}_L|^2). \tag{37}$$

Then the CP-averaged longitudinal polarization fractions of the $B_c^+ \to J/\psi V^+$ modes can be presented as follows,

$$f_L(B_c^+ \to J/\psi\rho^+) = (89.1^{+0.1}_{-0.1})\%, \\ f_L(B_c^+ \to J/\psi K^{*+}) = (85.6^{+0.2}_{-0.2})\%. \tag{38}$$

It is found that the decays $B_c^+ \to J/\psi V^+$ are generally governed by the longitudinal amplitudes in the iPQCD formalism but with slightly different fractions.

The ratio between the CP-averaged branching fractions of $B_c^+ \to J/\psi K^{*+}$ and $B_c^+ \to J/\psi\rho^+$ is then obtained as follows,

$$R^{\rm Theo}_{K^*/\rho} \equiv \frac{\mathcal{B}(B_c^+ \to J/\psi K^{*+})}{\mathcal{B}(B_c^+ \to J/\psi\rho^+)} = 0.060^{+0.002}_{-0.002}, \tag{39}$$

which roughly meets with the value $R_{K^*/\rho} \sim 0.066$ anticipated by naive factorization with $f_{K^*} = 0.118$ GeV, $f_\rho = 0.107$ GeV, $|V_{us}| = 0.2265$, and $|V_{ud}| = 0.9740$, and is indeed close to the ratio $R^{\rm Theo}_{K/\pi}$ presented in Eq. (32) within errors. The ratio $R^{\rm Theo}_{K^*/\rho}$ is expected to be measured at the near future LHC experiments.

With the normalization channel, i.e., $B_c^+ \to J/\psi\pi^+$, therefore the ratios between the branching fractions of $B_c^+ \to J/\psi V^+$ and $B_c^+ \to J/\psi\pi^+$ predicted for future examination are as follows,

$$R^{\rm Theo}_{\rho/\pi} \equiv \frac{\mathcal{B}(B_c^+ \to J/\psi\rho^+)}{\mathcal{B}(B_c^+ \to J/\psi\pi^+)} = 3.15^{+0.09}_{-0.10}, \\ R^{\rm Theo}_{K^*/\pi} \equiv \frac{\mathcal{B}(B_c^+ \to J/\psi K^{*+})}{\mathcal{B}(B_c^+ \to J/\psi\pi^+)} = 0.19^{+0.01}_{-0.01}. \tag{40}$$

B. $B_c^+ \to J/\psi A^+$

The p-wave light axial-vectors have been investigated at both experimental and theoretical aspects. However, the understanding about their internal structure is far from satisfactory [69]. As presented in the particle list by Particle Data Group, the nonstrange axial-vectors a_1 and $b_1(1235)$ (In the following context, we will use b_1 to denote this state for simplification.) belong to two different types of bound states in the constituent quark model with quantum numbers $1^3P_1(J^{PC} = 1^{++})$ and $1^1P_1(J^{PC} = 1^{+-})$, respectively. While for the strange $K_1(1270)$ and $K_1(1400)$ mesons (We will conveniently adopt K_1 and K_1' to denote these two states), these two physical states are considered as the mixtures of $K_{1A}(1^3P_1)$ and $K_{1B}(1^1P_1)$ with a single angle θ_{K_1} due to the mass difference of the strange and nonstrange light quarks [70],

$$\begin{pmatrix} |K_1\rangle \\ |K_1'\rangle \end{pmatrix} = \begin{pmatrix} \sin\theta_{K_1} & \cos\theta_{K_1} \\ \cos\theta_{K_1} & -\sin\theta_{K_1} \end{pmatrix}\begin{pmatrix} |K_{1A}\rangle \\ |K_{1B}\rangle \end{pmatrix}, \tag{41}$$

In analogy to the decays $B \to \phi K_1^{(\prime)}$ [71], we take both $\theta_{K_1} \approx 33°$ and $58°$ into account of the calculations to estimate the branching fractions because of the currently unknown nature.

Then, for the two $\Delta S = 0$ channels, $B_c^+ \to J/\psi a_1^+$ and $B_c^+ \to J/\psi b_1^+$, the predictions for their branching fractions in the iPQCD formalism can be read as,

$$\mathcal{B}(B_c^+ \to J/\psi a_1^+) \\ = 5.90^{+1.63}_{-1.24}(\beta_{B_c})^{+0.66}_{-0.64}(f_M)^{+0.00}_{-0.00}(a_{a_1}) \times 10^{-3}, \tag{42}$$

$$\mathcal{B}(B_c^+ \to J/\psi b_1^+) \\ = 7.93^{+2.43}_{-1.83}(\beta_{B_c})^{+0.93}_{-0.89}(f_M)^{+3.09}_{-2.59}(a_{b_1}) \times 10^{-4}. \tag{43}$$

One can easily find that the $B_c^+ \to J/\psi b_1^+$ branching ratio suffers from large errors induced by the uncertainties of Gegenbauer moments in the b_1-meson distribution amplitudes. Owing to the nearly vanished decay constant $f_{b_1^+} = f_{b_1}\cdot a^\parallel_{0b_1} \sim 0.0005$ (Here, f_{b_1} and $a^\parallel_{0b_1}$ are the "normalization" decay constant and the zeroth Gegenbauer moment for the meson b_1^+, respectively, and they could be found explicitly in Table IV of Appendix A. Note that $a^\parallel_{0b1} = 0$ under the isospin limit.), therefore the factorizable emission contributions are extremely suppressed, which results in the smaller $\mathcal{B}(B_c^+ \to J/\psi b_1^+)$ that comes almost from the nonfactorizable emission decay amplitudes. However, the antisymmetric behavior of the b_1-meson leading-twist distribution amplitude changes the destructive interferences between the two diagrams like Figs. 1(c) and 1(d) in the $B_c^+ \to J/\psi a_1^+$ channel into the constructive ones in the $B_c^+ \to J/\psi b_1^+$ mode, which lead to $\mathcal{B}(B_c^+ \to J/\psi b_1^+)$ around $\mathcal{O}(10^{-3})$ within large errors.

The ratios between the branching fractions $\mathcal{B}(B_c^+ \to J/\psi a_1^+/b_1^+)$ and $\mathcal{B}(B_c^+ \to J/\psi\pi^+)$ can be defined as follows,

$$R_{a_1/\pi}^{\rm Theo} \equiv \frac{\mathcal{B}(B_c^+ \to J/\psi a_1^+)}{\mathcal{B}(B_c^+ \to J/\psi\pi^+)} = 5.04_{-0.15}^{+0.10},$$
$$R_{b_1/\pi}^{\rm Theo} \equiv \frac{\mathcal{B}(B_c^+ \to J/\psi b_1^+)}{\mathcal{B}(B_c^+ \to J/\psi\pi^+)} = 0.68_{-0.22}^{+0.26}, \tag{44}$$

which would help probe these two channels experimentally in the near future. The large ratio $R_{b_1/\pi}^{\rm Theo}$ needs experimental tests as soon as possible to provide useful hints to test the reliability of iPQCD formalism utilized in this type of decays. If the information from experiments is positive, then these types of B_c-meson decay modes would offer good opportunities to help explore the shape parameter β_{B_c} in the B_c-meson distribution amplitude phenomenologically.

As presented in Eqs. (1) and (2), the decay $B_c^+ \to J/\psi a_1^+$ has been studied experimentally through the $B_c^+ \to J/\psi\pi^+\pi^-\pi^+$ channel via the invariant mass distributions corresponded to the favorite resonance state a_1^+ at the LHC experiments. When the relation of the decay rates $\mathcal{B}(a_1^+ \to \pi^+\pi^-\pi^+) \approx \mathcal{B}(a_1^+ \to \pi^+\pi^0\pi^0) \sim 50\%$ is adopted [72], then the branching fraction of $B_c^+ \to J/\psi\pi^+\pi^-\pi^+$ could be derived under narrow-width approximation as

$$\begin{aligned} &\mathcal{B}(B_c^+ \to J/\psi\pi^+\pi^-\pi^+)_{\rm iPQCD} \\ &\quad \equiv \mathcal{B}(B_c^+ \to J/\psi a_1^+) \cdot \mathcal{B}(a_1^+ \to \pi^+\pi^-\pi^+) \\ &\quad = (2.95_{-0.70}^{+0.88}) \times 10^{-3}, \end{aligned} \tag{45}$$

which is consistent surprisingly well with the predictions using $B_c \to J/\psi$ form factors in three different models [17] and would be tested by the experiments at LHC in the future. Subsequently, the ratio $R_{3\pi/\pi}^{\rm Theo}$ between the branching fractions of $B_c^+ \to J/\psi\pi^+\pi^-\pi^+$ and $B_c^+ \to J/\psi\pi^+$ could be obtained straightforwardly as $2.52_{-0.10}^{+0.05}$ in the iPQCD formalism, which is clearly in perfect consistency with data reported by the CMS and LHCb Collaborations and the Heavy Flavor Averaging Group. Theoretically, the $B_c^+ \to J/\psi\pi^+\pi^-\pi^+$ decay has been investigated in Refs. [17–21] with different ratios $R_{3\pi/\pi}$, however, which are basically consistent with the current measurements, except for that with the result 1.5 [18]. The future tests on $R_{a_1/\pi}^{\rm Theo}$ and $R_{3\pi/\pi}^{\rm Theo}$ in the iPQCD formalism with high precision at the LHC, even CEPC experiments could help understand the property of a_1 meson.

Additionally, with the ratio $R_{2K\pi/\pi}^{\rm Exp}$ in Eq. (3) and the branching fractions of $B_c^+ \to J/\psi a_1^+$ and $B_c^+ \to J/\psi\pi^+$ in the iPQCD formalism, thus the branching ratio of $a_1^+ \to K^+K^-\pi^+$ could be deduced under narrow-width approximation as

$$\begin{aligned} \mathcal{B}(a_1^+ \to K^+K^-\pi^+)_{\rm iPQCD} &\equiv R_{2K\pi/\pi}^{\rm Exp} \cdot \frac{\mathcal{B}(B_c^+ \to J/\psi\pi^+)}{\mathcal{B}(B_c^+ \to J/\psi a_1^+)} \\ &\approx (10.5_{-1.9}^{+2.0})\%. \end{aligned} \tag{46}$$

The detection on this $a_1^+ \to K^+K^-\pi^+$ branching ratio would help understand the nature of a_1 that is usually provided from the hadron physics side. Meanwhile, with the help of $R_{2K\pi/3\pi}^{\rm Exp}$ in Eq. (4) and $\mathcal{B}(B_c^+ \to J/\psi\pi^+\pi^-\pi^+)_{\rm iPQCD}$ in Eq. (45), we could derive the branching fraction of $B_c^+ \to J/\psi K^+K^-\pi^+$ in the iPQCD formalism as,

$$\begin{aligned} &\mathcal{B}(B_c^+ \to J/\psi K^+K^-\pi^+)_{\rm iPQCD} \\ &\quad \equiv R_{2K\pi/3\pi}^{\rm Exp} \cdot \mathcal{B}(B_c^+ \to J/\psi\pi^+\pi^-\pi^+)_{\rm iPQCD} \\ &\quad = (5.46_{-1.40}^{+1.68}) \times 10^{-4}, \end{aligned} \tag{47}$$

which is consistent with the predictions given in different form factors [23] within a bit large errors. These two values could be confronted with the future measurements.

In order to help investigate the behavior between the vector ρ meson and the axial-vector a_1 and b_1 ones, we also define the following two ratios with the $B_c^+ \to J/\psi\rho^+$ decay rate,

$$R_{a_1/\rho}^{\rm Theo} \equiv \frac{\mathcal{B}(B_c^+ \to J/\psi a_1^+)}{\mathcal{B}(B_c^+ \to J/\psi\rho^+)} = 1.60_{-0.11}^{+0.10},$$
$$R_{b_1/\rho}^{\rm Theo} \equiv \frac{\mathcal{B}(B_c^+ \to J/\psi b_1^+)}{\mathcal{B}(B_c^+ \to J/\psi\rho^+)} = 0.21_{-0.07}^{+0.09}. \tag{48}$$

These two ratios are expected to be measured in the near future.

Meantime, the CP-averaged longitudinal polarization fractions of the $B_c^+ \to J/\psi a_1^+$ and $J/\psi b_1^+$ channels are predicted in the iPQCD formalism theoretically as,

$$f_L(B_c^+ \to J/\psi a_1^+) = (74.8_{-0.3}^{+0.0})\%,$$
$$f_L(B_c^+ \to J/\psi b_1^+) = (98.9_{-0.0}^{+0.0})\%. \tag{49}$$

It is evident that the decay $B_c^+ \to J/\psi b_1^+$ is governed absolutely by the longitudinal contributions.

And, for the two $\Delta S = 1$ modes $B_c^+ \to J/\psi K_1^{(\prime)+}$, the branching fractions are predicted in the iPQCD formalism with two different mixing angles as follows,

$$\mathcal{B}(B_c^+ \to J/\psi K_1^+) = \begin{cases} 4.07_{-0.90}^{+1.26}(\beta_{B_c})_{-0.36}^{+0.35}(f_M)_{-0.66}^{+0.71}(B_{K_1}) \times 10^{-4} \\ 4.86_{-1.03}^{+1.31}(\beta_{B_c})_{-0.50}^{+0.52}(f_M)_{-0.51}^{+0.55}(B_{K_1}) \times 10^{-4} \end{cases}, \tag{50}$$

$$\mathcal{B}(B_c^+ \to J/\psi K_1^{\prime +}) = \begin{cases} 1.42_{-0.30}^{+0.43}(\beta_{B_c})_{-0.21}^{+0.22}(f_M)_{-0.24}^{+0.27}(B_{K_1}) \times 10^{-4} \\ 6.22_{-1.59}^{+2.83}(\beta_{B_c})_{-0.47}^{+0.52}(f_M)_{-0.88}^{+1.38}(B_{K_1}) \times 10^{-5} \end{cases}, \tag{51}$$

where the first (second) entry corresponds to the value obtained at $\theta_{K_1} = 33°$ (58°). The similar patterns also appear in the following observables for related modes. In the numerical results, the dominant errors arise from the uncertainties of the shape parameter β_{B_c} and from the combined Gegenbauer moment B_{K_1} of $a_{K_{1A}}$ and $a_{K_{1B}}$ (See Table V for detail.). For the decay $B_c^+ \to J/\psi K_1^+$, the iPQCD value for branching fraction at $\theta_{K_1} \sim 33°$ can compete with that at $\theta_{K_1} \sim 58°$, while, for the decay $B_c^+ \to J/\psi K_1'^+$, the result for $\mathcal{B}(B_c^+ \to J/\psi K_1'^+)$ at $\theta_{K_1} \sim 33°$ is roughly two times larger than that at $\theta_{K_1} \sim 58°$. As inferred from Refs. [13,24], the $B_c^+ \to J/\psi K_1'^+$ decay might be explored through the ratio $R^{\rm Exp}_{2\pi K/3\pi} \equiv \mathcal{B}(B_c^+ \to J/\psi K^+\pi^-\pi^+)/\mathcal{B}(B_c^+ \to J/\psi \pi^+\pi^-\pi^+)$ in the near future, where the $B_c^+ \to J/\psi K^+\pi^-\pi^+$ mode proceeds largely via $K^{*0} \to K^+\pi^-$. As a by-product, the branching fraction of $B_c^+ \to J/\psi K^+\pi^-\pi^+$ in the iPQCD formalism could be derived via the currently measured ratio $R^{\rm Exp}_{2\pi K/3\pi} = (6.4 \pm 1.0) \times 10^{-2}$ [24] and the iPQCD branching fraction $\mathcal{B}(B_c^+ \to J/\psi \pi^+\pi^-\pi^+)$ as the following,

$$\begin{aligned} \mathcal{B}(B_c^+ \to J/\psi K^+\pi^-\pi^+)_{\rm iPQCD} &= R^{\rm Exp}_{2\pi K/3\pi} \cdot \mathcal{B}(B_c^+ \to J/\psi \pi^+\pi^-\pi^+)_{\rm iPQCD} \\ &= (1.89^{+0.63}_{-0.54}) \times 10^{-4}, \end{aligned} \tag{52}$$

which is in good consistency with the predictions [23] within errors, and is also expected to be detected soon at experiments.

We also predict the longitudinal polarization fractions for the decays $B_c^+ \to J/\psi K_1^{(\prime)+}$ in the iPQCD formalism under two referenced angles as follows,

$$f_L(B_c^+ \to J/\psi K_1^+) = \begin{cases} (90.2^{+1.7}_{-1.4})\% \\ (82.7^{+1.2}_{-1.2})\% \end{cases},$$
$$f_L(B_c^+ \to J/\psi K_1'^+) = \begin{cases} (52.0^{+2.5}_{-1.9})\% \\ (62.1^{+14.9}_{-11.2})\% \end{cases}. \tag{53}$$

It is clear to observe that, within the theoretical errors, the longitudinal polarization decay amplitudes dominate the $B_c^+ \to J/\psi K_1^+$ mode, however, both the longitudinal and the transverse polarization decay amplitudes generally compete with each other in the $B_c^+ \to J/\psi K_1'^+$ channel. It means that the significantly constructive (destructive) interferences occur at the longitudinal polarization in the former (latter) mode. The future measurements on these two decays might reveal the information of the mixing angle θ_{K_1} between K_1^+ and $K_1'^+$.

Meanwhile, for the convenience of future probes to the decays $B_c^+ \to J/\psi K_1^+$ and $J/\psi K_1'^+$, the relative ratios of the branching fractions between $B_c^+ \to J/\psi K_1^{(\prime)+}$ and $B_c^+ \to J/\psi \pi^+$ could be derived in the iPQCD formalism as follows,

$$R^{\rm Theo}_{K_1/\pi} \equiv \frac{\mathcal{B}(B_c^+ \to J/\psi K_1^+)}{\mathcal{B}(B_c^+ \to J/\psi \pi^+)} = \begin{cases} 0.35^{+0.06}_{-0.06} \\ 0.42^{+0.04}_{-0.05} \end{cases},$$
$$R^{\rm Theo}_{K_1'/\pi} \equiv \frac{\mathcal{B}(B_c \to J/\psi K_1'^+)}{\mathcal{B}(B_c \to J/\psi \pi^+)} = \begin{cases} 0.12^{+0.02}_{-0.02} \\ 0.05^{+0.02}_{-0.01} \end{cases}. \tag{54}$$

We also present the ratios between the $\Delta S = 1$ and $\Delta S = 0$ decay rates in these $B_c^+ \to J/\psi A^+$ modes as the following,

$$R^{\rm Theo}_{K_1/a_1} \equiv \frac{\mathcal{B}(B_c^+ \to J/\psi K_1^+)}{\mathcal{B}(B_c^+ \to J/\psi a_1^+)} = \begin{cases} 0.07^{+0.01}_{-0.01} \\ 0.08^{+0.01}_{-0.01} \end{cases},$$
$$R^{\rm Theo}_{K_1'/a_1} \equiv \frac{\mathcal{B}(B_c^+ \to J/\psi K_1'^+)}{\mathcal{B}(B_c^+ \to J/\psi a_1^+)} = \begin{cases} 0.02^{+0.01}_{-0.00} \\ 0.01^{+0.00}_{-0.00} \end{cases}, \tag{55}$$

and

$$R^{\rm Theo}_{K_1/b_1} \equiv \frac{\mathcal{B}(B_c^+ \to J/\psi K_1^+)}{\mathcal{B}(B_c^+ \to J/\psi b_1^+)} = \begin{cases} 0.51^{+0.13}_{-0.08} \\ 0.61^{+0.20}_{-0.12} \end{cases},$$
$$R^{\rm Theo}_{K_1'/b_1} \equiv \frac{\mathcal{B}(B_c^+ \to J/\psi K_1'^+)}{\mathcal{B}(B_c^+ \to J/\psi b_1^+)} = \begin{cases} 0.18^{+0.04}_{-0.03} \\ 0.08^{+0.03}_{-0.01} \end{cases}. \tag{56}$$

These ratios would be helpful to explore the QCD dynamics in the considered axial-vector mesons, especially in K_1 and K_1'.

C. $B_c^+ \to J/\psi S^+$

As we currently known, our colleagues categorize the light scalars in the two-quark structure into two different scenarios [56], namely, the scalars below 1 GeV could be considered as the $q\bar{q}$-ground states and those around 1.5 GeV could be viewed as the first excited states correspondingly in scenario 1 (S1), while the scalars around 1.5 GeV might be the lowest-lying $q\bar{q}$-bound states and those below 1 GeV have to be the four-quark states in scenario 2 (S2). This means explicitly that $a_0(980)^+$ (It will be denoted as a_0^+ for convenience.) and κ^+ in S1 and $a_0(1450)^+$ and $K_0^*(1430)^+$ (They will be expressed as $a_0'^+$ and K_0^{*+} for simplicity.) in both S1 and S2 will be taken into this study, due to the availability of factorization approach. Different from the factorizable-emission-diagrams-dominated decays $B_c^+ \to J/\psi(P^+, V^+)$, due to the highly small vector decay constant f_S, the factorizable emission diagrams are strongly suppressed in the $B_c^+ \to J/\psi S^+$ modes. We present explicitly the decay amplitudes of $B_c^+ \to J/\psi S^+$ from the factorizable emission and nonfactorizable

TABLE II. The decay amplitudes (in units of 10^{-3} GeV^{-3}) from different diagrams of $B_c^+ \to J/\psi S^+$ in the iPQCD formalism. The upper (lower) entry corresponds to the scalars $a_0'^+$ and K_0^{*+} in scenario 1 (2) at every line. For comparison, the decay amplitudes of $B_c^+ \to J/\psi\pi^+$ are also provided in the last column. For the sake of simplicity, only the central values are quoted for clarifications.

Modes	$B_c^+ \to J/\psi a_0^+$	$B_c^+ \to J/\psi\kappa^+$	$B_c^+ \to J/\psi a_0'^+$	$B_c^+ \to J/\psi K_0^{*+}$	$B_c \to J/\psi\pi^+$
Decay amplitudes (fe)	$-0.11 - i0.73$	$-1.17 - i7.60$	$0.09 + i0.63$ $-0.15 - i0.97$	$1.07 + i7.06$ $-1.60 - i10.60$	$18.21 + i91.88$
Decay amplitudes (nfe)	$-48.13 + i52.70$	$-7.87 + i12.70$	$-68.32 + i33.43$ $-44.16 + i7.08$	$-11.39 + i6.47$ $-10.30 + i2.90$	$-2.04 + i3.22$

emission diagrams respectively in Table II.[2] Therefore, the estimations going beyond naive factorization are essential for us to find out the useful constraints on β_{B_c} with the help of large branching fractions arising almost from the nonfactorizable emission contributions. Moreover, different from the decay $B_c^+ \to J/\psi b_1^+$ with angular decomposition, the decays $B_c^+ \to J/\psi S^+$ have only longitudinal contributions because of conservation of the angular momentum.

The CP-averaged branching fractions of the $B_c^+ \to J/\psi S^+$ decays in the iPQCD formalism are given as,

$$\mathcal{B}(B_c^+ \to J/\psi a_0^+) = 5.98^{+1.77}_{-1.39}(\beta_{B_c})^{+0.80}_{-0.78}(f_M)^{+1.31}_{-1.22}(B_i) \times 10^{-4}, \quad (57)$$

$$\mathcal{B}(B_c^+ \to J/\psi\kappa^+) = 1.31^{+0.50}_{-0.36}(\beta_{B_c})^{+0.18}_{-0.18}(f_M)^{+0.41}_{-0.35}(B_i) \times 10^{-5}, \quad (58)$$

and

$$\mathcal{B}(B_c^+ \to J/\psi a_0'^+) = \begin{cases} 6.39^{+2.09}_{-1.56}(\beta_{B_c})^{+1.52}_{-1.37}(f_M)^{+2.19}_{-1.89}(B_i) \times 10^{-4} \\ 2.20^{+0.71}_{-0.54}(\beta_{B_c})^{+0.54}_{-0.48}(f_M)^{+1.37}_{-1.02}(B_i) \times 10^{-4} \end{cases}, \quad (59)$$

$$\mathcal{B}(B_c^+ \to J/\psi K_0^{*+}) = \begin{cases} 3.22^{+0.81}_{-0.66}(\beta_{B_c})^{+0.74}_{-0.64}(f_M)^{+0.37}_{-0.35}(B_i) \times 10^{-5} \\ 2.23^{+0.87}_{-0.70}(\beta_{B_c})^{+0.55}_{-0.51}(f_M)^{+0.65}_{-0.45}(B_i) \times 10^{-5} \end{cases}, \quad (60)$$

where the first (second) entry corresponds to the value obtained in S1 (S2). The similar patterns also appear in the following ratios for related modes. We stress that the less experimental constraints on the shape parameter β_{B_c} of B_c meson, and on the scalar decay constant $\bar{f}_S$ and the Gegenbauer moments B_i in the light-cone distribution amplitudes of scalars result in the remarkably large errors in theory.

Generally speaking, the theoretical uncertainties induced by the input parameters are usually canceled to a great extent in the relative ratios of the branching fractions. The ratios between the corresponding branching fractions of the $\Delta S = 1$ and $\Delta S = 0$ modes in the $B_c^+ \to J/\psi S^+$ decays can then be read as,

$$R^{\rm Theo}_{\kappa/a_0} \equiv \frac{\mathcal{B}(B_c^+ \to J/\psi\kappa^+)}{\mathcal{B}(B_c^+ \to J/\psi a_0^+)} = 0.022^{+0.002}_{-0.002},$$
$$R^{\rm Theo}_{K_0^*/a_0'} \equiv \frac{\mathcal{B}(B_c^+ \to J/\psi K_0^{*+})}{\mathcal{B}(B_c^+ \to J/\psi a_0'^+)} = \begin{cases} 0.050^{+0.014}_{-0.008} \\ 0.101^{+0.050}_{-0.022} \end{cases}. \quad (61)$$

And, the relative ratios of $\mathcal{B}(B_c^+ \to J/\psi S^+)$ over $\mathcal{B}(B_c^+ \to J/\psi\pi^+)$ are presented for future detections as follows,

$$R^{\rm Theo}_{a_0/\pi} \equiv \frac{\mathcal{B}(B_c^+ \to J/\psi a_0^+)}{\mathcal{B}(B_c^+ \to J/\psi\pi^+)} = 0.51^{+0.13}_{-0.12},$$
$$R^{\rm Theo}_{\kappa/\pi} \equiv \frac{\mathcal{B}(B_c^+ \to J/\psi\kappa^+)}{\mathcal{B}(B_c^+ \to J/\psi\pi^+)} = 0.011^{+0.005}_{-0.004}, \quad (62)$$

and

$$R^{\rm Theo}_{a_0'/\pi} \equiv \frac{\mathcal{B}(B_c^+ \to J/\psi a_0'^+)}{\mathcal{B}(B_c^+ \to J/\psi\pi^+)} = \begin{cases} 0.55^{+0.19}_{-0.17} \\ 0.19^{+0.12}_{-0.09} \end{cases},$$
$$R^{\rm Theo}_{K_0^*/\pi} \equiv \frac{\mathcal{B}(B_c^+ \to J/\psi K_0{}^{*+})}{\mathcal{B}(B_c^+ \to J/\psi\pi^+)} = \begin{cases} 0.028^{+0.003}_{-0.003} \\ 0.019^{+0.006}_{-0.005} \end{cases}. \quad (63)$$

Strictly speaking, the errors induced by the hadronic parameters of light scalars are hard to be effectively canceled due to their unknown nature. It is found that the errors are still large in the above ratios generally. Nevertheless, these decay modes can provide chances to understand the QCD dynamics because they must be studied in the factorization framework of QCD going beyond naive factorization hypothesis.

[2] Due to the SU(3) flavor symmetry breaking effects, specifically, the strange quark mass and the up or down quark mass satisfying the relation of $m_s \gg m_{u,d}$, then the considerable contributions induced by the vector decay constants f_{κ^+} and $f_{K_0^{*+}}$ according to Eq. (A11) appear from the factorizable emission diagrams in the $B_c^+ \to J/\psi S^+(\Delta S = 1)$ channels, relative to those in the $B_c^+ \to J/\psi S^+(\Delta S = 0)$ modes with tiny isospin symmetry breaking.

D. $B_c^+ \to J/\psi T^+$

As discussed in footnote 1, the factorization formulas analogous to the $B_c^+ \to J/\psi V^+$ decays in the modes $B_c^+ \to J/\psi T^+$ could then be easily got because of conservation of the angular momentum. But, extremely different from the $B_c^+ \to J/\psi V^+$ decays, the tensor mesons cannot be produced via the vector current. Hence, the factorizable contributions associated with T-emission in these $B_c^+ \to J/\psi T^+$ modes are forbidden intuitively. It means that the decays $B_c^+ \to J/\psi T^+$ must be explored beyond the naive factorization approach. Their branching fractions are contributed completely by the nonfactorizable emission diagrams. It is emphasized that, due to few studies on the light-cone distribution amplitudes of tensor meson, only the available asymptotic forms of tensor meson's distribution amplitudes are adopted tentatively in this work.

Therefore, the branching fractions of the $B_c^+ \to J/\psi T^+$ decays predicted in the iPQCD formalism are presented as follows,

$$\mathcal{B}(B_c^+ \to J/\psi a_2^+) = 1.39^{+0.45}_{-0.33}(\beta_{B_c})^{+0.19}_{-0.18}(f_M) \times 10^{-4}, \quad (64)$$

$$\mathcal{B}(B_c^+ \to J/\psi K_2^{*+}) = 9.05^{+2.91}_{-2.22}(\beta_{B_c})^{+1.03}_{-0.98}(f_M) \times 10^{-6}, \quad (65)$$

where the dominant errors come from the shape parameter β_{B_c} and the less constrained decay constant f_T, respectively. Based on the assumptions of $\mathcal{B}(a_2^+ \to \pi^+\pi^-\pi^+) \approx \mathcal{B}(a_2^+ \to \pi^+\pi^0\pi^0)$ and the validity of narrow-width approximation associated with the branching fractions $\mathcal{B}(a_2 \to 3\pi) = (70.1 \pm 2.7)\%$ [13] and $\mathcal{B}(B_c^+ \to J/\psi a_2^+) = (1.39^{+0.49}_{-0.38}) \times 10^{-4}$, the large branching fraction $\mathcal{B}(B_c^+ \to J/\psi a_2^+(\to \pi^+\pi^-\pi^+))_{\text{iPQCD}} = (0.49^{+0.17}_{-0.14}) \times 10^{-4}$ will be tested at the relevant experiments in the near future. The measurements on this value will help examine the reliability of iPQCD formalism and further obtain the information of β_{B_c}, even the tensor mesons' QCD behavior from the related observables.

The longitudinal polarization fractions of the $B_c^+ \to J/\psi T^+$ modes are also predicted in the iPQCD formalism as the following,

$$f_L(B_c^+ \to J/\psi a_2^+) = (96.1^{+0.1}_{-0.1})\%,$$
$$f_L(B_c^+ \to J/\psi K_2^{*+}) = (95.3^{+0.0}_{-0.2})\%, \quad (66)$$

which meets the naively expected hierarchy, i.e., $f_L \sim 1$, in the tree-level $\bar{b} \to \bar{c}$ transitions based on quark-helicity conservation [73,74].

Like $R_{K/\pi}$ in the $B_c^+ \to J/\psi P^+$ sector, we can define $R_{K_2^*/a_2}$ in the $B_c^+ \to J/\psi T^+$ decays and predict its value within the iPQCD formalism as follows,

$$R^{\text{Theo}}_{K_2^*/a_2} \equiv \frac{\mathcal{B}(B_c^+ \to J/\psi K_2^{*+})}{\mathcal{B}(B_c^+ \to J/\psi a_2^+)} = 0.065^{+0.002}_{-0.002}. \quad (67)$$

It is of great interest to find that this result agrees so well with $R_{K^*/\rho} \approx |f_{K^*}/f_\rho|^2 \cdot |V_{us}/V_{ud}|^2$ naively anticipated in factorization ansatz, and is also very close to the ratio $R^{\text{Theo}}_{K/\pi}$. The underlying reason is that, relative to the predominant contributions from factorizable emission diagrams while with negligible nonfactorizable emission contributions in $B_c^+ \to J/\psi V^+$, the $B_c^+ \to J/\psi T^+$ decays are absolutely contributed from the nonfactorizable emission ones. Therefore, the ratio in the latter decays could be cleanly written as $|f_{K_2^*}/f_{a_2}|^2 \cdot |V_{us}/V_{ud}|^2$ due to the SU(3)-flavor symmetry in the leading-twist distribution amplitude $\phi_T(x)$, however, that in the former decays could only be approximately expressed as $|f_{K^*}/f_\rho|^2 \cdot |V_{us}/V_{ud}|^2$ due to a bit destructive interferences from the nonfactorizable emission decay amplitudes in fact.

The ratios of $\mathcal{B}(B_c^+ \to J/\psi T^+)$ over $\mathcal{B}(B_c^+ \to J/\psi \pi^+)$ in the iPQCD formalism could be written as,

$$R^{\text{Theo}}_{a_2/\pi} \equiv \frac{\mathcal{B}(B_c^+ \to J/\psi a_2^+)}{\mathcal{B}(B_c^+ \to J/\psi \pi^+)} = 0.12^{+0.01}_{-0.01},$$
$$R^{\text{Theo}}_{K_2^*/\pi} \equiv \frac{\mathcal{B}(B_c^+ \to J/\psi K_2^{*+})}{\mathcal{B}(B_c^+ \to J/\psi \pi^+)} = 0.008^{+0.000}_{-0.001}, \quad (68)$$

which will be utilized to help explore these two $B_c^+ \to J/\psi T^+$ decay modes at LHC, even CEPC experiments in the future.

Because of no very rigorous constraints on the shape parameter β_{B_c} in the B_c-meson distribution amplitude and on the Gegenbauer moments in the light-cone distribution amplitudes of p-wave light hadrons from the aspects of current experiments, we suggest our experimental colleagues to make much more relevant measurements on the predictions, especially the relative ratios of the branching fractions, to further understand the involved perturbative and nonperturbative QCD dynamics in the related channels, despite no easy detections on the individual decay rates. The related measurements could also help differentiate the reliability of the adopted approaches and/or methods.

Finally, one more comment is that, within the framework of this iPQCD formalism in association with the newly derived Sudakov factor by including the charm quark mass effects, we could extend the related studies to the decays involving B_c to charmonia transitions, such as $B_c^+ \to (\eta_c, \chi_{cJ}(J = 0, 1, 2), \ldots)M^+$, which request detailed investigations on the related modes because of the involved complicated dynamics. These studies will be performed in the future and the numerical results will be presented elsewhere.

In summary, we have systematically studied the B_c-meson decays into J/ψ plus a light meson by completely including the charm quark mass effects in the iPQCD formalism at leading order. Several interesting predictions for the observables such as branching fractions, relative ratios, and longitudinal polarization fractions are presented explicitly. Based on the numerical results and phenomenological analyses, we find that

(i) The iPQCD predictions for the ratios between the branching fractions of $B_c^+ \to J/\psi K^+/B_c^+ \to J/\psi a_1^+(\to \pi^+\pi^-\pi^+)$ and $B_c^+ \to J/\psi \pi^+$ are highly consistent with the current data, though their individual decay rates are not yet available presently. The branching fractions of CKM-favored $B_c^+ \to J/\psi P^+$, $B_c^+ \to J/\psi V^+$, and $B_c^+ \to J/\psi a_1^+$ decays, which are predominated by the factorizable emission diagrams, around $\mathcal{O}(10^{-3})$ are predicted in the iPQCD formalism and await near future tests at the LHC experiments.

(ii) The iPQCD predictions for the branching fractions of the factorizable-emission-diagrams-suppressed/forbidden modes such as $B_c^+ \to J/\psi b_1^+$, $B_c^+ \to J/\psi S^+$, and $B_c^+ \to J/\psi T^+$ are provided theoretically for the first time in the literature and will be confronted with the future measurements at LHC, even CEPC experiments. Objectively speaking, the investigations on these decay modes with large nonfactorizable decay amplitudes should go beyond naive factorization to further help understand the involved perturbative and nonperturbative QCD dynamics. Phenomenologically, the precise measurements on this type of decays could help constrain the shape parameter β_{B_c} in $\phi_{B_c}(x)$, as well as explore the nature of light hadrons.

(iii) Almost all of the $B_c^+ \to J/\psi M^+$ decays with polarization contributions are governed by the longitudinal decay amplitudes, which are consistent with the naive expectation $f_L \sim 1$ in the tree-level $\bar{b} \to \bar{c}$ transitions, except for the $B_c^+ \to J/\psi K_1(1400)^+$ channel with possibly destructive interferences between $B_c^+ \to J/\psi K_{1A}^+$ and $B_c^+ \to J/\psi K_{1B}^+$ in longitudinal polarization. The related iPQCD predictions will be examined at relevant experiments in the future.

(iv) The model-independent ratio between the branching fractions of CKM-suppressed and CKM-favored $B_c^+ \to J/\psi T^+$ decays is obtained and expected to shed light on the information of β_{B_c} promisingly. By utilizing the golden channel $B_c^+ \to J/\psi \pi^+$ as normalization, the relative ratios between the branching fractions of $B_c^+ \to J/\psi M^+ (M = V, A, S, T)$ and $B_c^+ \to J/\psi \pi^+$ are predicted in the iPQCD formalism and would be helpful to search for these decay modes in near future examinations.

ACKNOWLEDGMENTS

X. L. thanks Professors H.-n. Li and Z. J. Xiao for their useful discussions and constant encouragements, and also thanks Professor H.-n. Li for reading the manuscript and Ivan Belyaev for a valuable suggestion. This work is supported by the National Natural Science Foundation of China under Grants No. 11875033 and No. 11205072, by the Qing Lan Project of Jiangsu Province (No. 9212218405), and by the Research Fund of Jiangsu Normal University under Grant No. HB2016004.

APPENDIX A: MESONS' DISTRIBUTION AMPLITUDES

As aforementioned, the distribution amplitudes of initial and final mesons have been presented in the literature. For the sake of simplicity, we here just collect them in this appendix.

For the initial B_c meson, the distribution amplitude $\phi_{B_c}(x, \mathbf{b})$ in the conjugate $\mathbf{b}$ space of transverse momentum $\mathbf{k}_T$ could be written as follows [41],

$$\phi_{B_c}(x, \mathbf{b}) = \frac{f_{B_c}}{2\sqrt{2N_c}} N_{B_c} x(1-x) \exp\left[-\frac{(1-x)m_c^2 + x m_b^2}{8\beta_{B_c}^2 x(1-x)}\right] \times \exp[-2\beta_{B_c}^2 x(1-x)\mathbf{b}^2], \quad \text{(A1)}$$

The shape parameter $\beta_{B_c} = 1.0 \pm 0.1$ GeV could enable the distribution of $\phi_{B_c}(x)$ to coincide with those proposed in Refs. [75,76]. The normalization constant N_{B_c} is fixed by the following relation,

$$\int_0^1 \phi_{B_c}(x, \mathbf{b} = 0)dx \equiv \int_0^1 \phi_{B_c}(x)dx = \frac{f_{B_c}}{2\sqrt{2N_c}}, \quad \text{(A2)}$$

where the decay constant $f_{B_c} = 0.489 \pm 0.005$ GeV has been obtained in lattice QCD by the TWQCD Collaboration [77].

For J/ψ meson, the explicit forms for the distribution amplitudes of twist-2 $\phi_{J/\psi}^{L,T}(x)$ and twist-3 $\phi_{J/\psi}^{t,v}(x)$ could be read as [52],

$$\begin{aligned}
\phi_{J/\psi}^L(x) &= \phi_{J/\psi}^T(x) \\
&= 9.58 \frac{f_{J/\psi}}{2\sqrt{2N_c}} x(1-x) \left[\frac{x(1-x)}{1-2.8x(1-x)}\right]^{0.7}, \\
\phi_{J/\psi}^t(x) &= 10.94 \frac{f_{J/\psi}}{2\sqrt{2N_c}} (1-2x)^2 \left[\frac{x(1-x)}{1-2.8x(1-x)}\right]^{0.7}, \\
\phi_{J/\psi}^v(x) &= 1.67 \frac{f_{J/\psi}}{2\sqrt{2N_c}} [1+(2x-1)^2] \left[\frac{x(1-x)}{1-2.8x(1-x)}\right]^{0.7},
\end{aligned} \quad \text{(A3)}$$

where $f_{J/\psi}$ is the decay constant with value 0.405 ± 0.014 GeV.

For light mesons, the distribution amplitudes for pseudoscalars, vectors, axial-vectors, scalars, and tensors have been obtained in the QCD sum rules [62].

(i) For light pseudoscalars (P): pion and kaon

The light-cone distribution amplitudes ϕ_P^A (twist-2), and ϕ_P^P and ϕ_P^T (twist-3) have been parametrized as [53–55]

$$\phi_P^A(x) = \frac{f_P}{2\sqrt{2N_c}} 6x(1-x)\Big[1 + a_1^P C_1^{3/2}(2x-1) + a_2^P C_2^{3/2}(2x-1) + a_4^P C_4^{3/2}(2x-1)\Big],$$
$$\phi_P^P(x) = \frac{f_P}{2\sqrt{2N_c}}\left[1 + \left(30\eta_3 - \frac{5}{2}\rho_P^2\right) C_2^{1/2}(2x-1) - 3\left(\eta_3\omega_3 + \frac{9}{20}\rho_\pi^2(1+6a_2^P)\right) C_4^{1/2}(2x-1)\right],$$
$$\phi_P^T(x) = \frac{f_P}{2\sqrt{2N_c}}(1-2x)\left[1 + 6\left(5\eta_3 - \frac{1}{2}\eta_3\omega_3 - \frac{7}{20}\rho_P^2 - \frac{3}{5}\rho_P^2 a_2^P\right)(1 - 10x + 10x^2)\right], \tag{A4}$$

with the decay constants $f_\pi = 0.131$ GeV and $f_K = 0.16$ GeV; the Gegenbauer moments $a_1^\pi = 0$, $a_1^K = 0.17 \pm 0.17$, $a_2^P = 0.115 \pm 0.115$, $a_4^P = -0.015$; the mass ratio $\rho_{\pi(K)} = m_{\pi(K)}/m_0^{\pi(K)}$; $m_0^\pi = 1.4$ GeV and $m_0^K = 1.6$ GeV being the chiral masses; and the Gegenbauer polynomials $C_n^\nu(t)$,

$$C_1^{3/2}(t) = 3t,$$
$$C_2^{1/2}(t) = \frac{1}{2}(3t^2-1), \qquad C_2^{3/2}(t) = \frac{3}{2}(5t^2-1),$$
$$C_4^{1/2}(t) = \frac{1}{8}(3 - 30t^2 + 35t^4),$$
$$C_4^{3/2}(t) = \frac{15}{8}(1 - 14t^2 + 21t^4). \tag{A5}$$

In the above distribution amplitudes for kaon, the momentum fraction x is carried by the s quark (This definition is same for the strange mesons in the following items). We choose the parameters $\eta_3 = 0.015$ and $\omega_3 = -3$ [53,54] for both pion and kaon.

(ii) For light vectors (V): ρ and K^*

The twist-2 distribution amplitudes for the longitudinally and transversely polarized vector meson can be parametrized as [58]:

$$\phi_V(x) = \frac{3f_V}{\sqrt{2N_c}} x(1-x)\Big[1 + a_{1V}^{\parallel} C_1^{3/2}(2x-1) + a_{2V}^{\parallel} C_2^{3/2}(2x-1)\Big], \tag{A6}$$

$$\phi_V^T(x) = \frac{3f_V^T}{\sqrt{2N_c}} x(1-x)\Big[1 + a_{1V}^{\perp} C_1^{3/2}(2x-1) + a_{2V}^{\perp} C_2^{3/2}(2x-1)\Big], \tag{A7}$$

Here f_V and f_V^T are the decay constants of the vector meson in longitudinal and transverse polarizations, respectively. The decay constants and Gegenbauer moments for light vectors at scale $\mu = 1$ GeV have been studied extensively in the literature [58,59] and could be found in the following Table III. The masses of ρ and K^* are taken as 0.775 GeV and 0.892 GeV, respectively.

The asymptotic forms of the twist-3 distribution amplitudes $\phi_V^{t,s}$ and $\phi_V^{v,a}$ are [78]:

$$\phi_V^t(x) = \frac{3f_V^T}{2\sqrt{2N_c}}(2x-1)^2,$$
$$\phi_V^s(x) = -\frac{3f_V^T}{2\sqrt{2N_c}}(2x-1), \tag{A8}$$

$$\phi_V^v(x) = \frac{3f_V}{8\sqrt{2N_c}}(1 + (2x-1)^2),$$
$$\phi_V^a(x) = -\frac{3f_V}{4\sqrt{2N_c}}(2x-1). \tag{A9}$$

(iii) For light scalars (S): a_0, κ, $a_0(1450)$ and $K_0^*(1430)$

In general, the leading-twist light-cone distribution amplitude $\phi_S(x,\mu)$ can be expanded as the Gegenbauer polynomials [56,57]:

TABLE III. Decay constants (in GeV) and Gegenbauer moments for light vectors.

f_ρ	f_ρ^T	$a_{1\rho}^{\parallel}$	$a_{2\rho}^{\parallel}$	$a_{1\rho}^{\perp}$	$a_{2\rho}^{\perp}$
0.107 ± 0.006	0.105 ± 0.021	⋯	0.15 ± 0.07	⋯	0.14 ± 0.06
f_{K^*}	$f_{K^*}^T$	$a_{1K^*}^{\parallel}$	$a_{2K^*}^{\parallel}$	$a_{1K^*}^{\perp}$	$a_{2K^*}^{\perp}$
0.118 ± 0.005	0.077 ± 0.014	0.03 ± 0.02	0.11 ± 0.09	0.04 ± 0.03	0.10 ± 0.08

$$\phi_S(x,\mu) = \frac{3}{\sqrt{2N_c}} x(1-x)\left\{ f_S(\mu) + \bar{f}_S(\mu) \times \sum_{m=1}^{\infty} B_m(\mu) C_m^{3/2}(2x-1) \right\}, \quad \text{(A10)}$$

where $f_S(\mu)$ and $\bar{f}_S(\mu)$, $B_m(\mu)$, and $C_m^{3/2}(t)$ are the vector and scalar decay constants, Gegenbauer moments, and Gegenbauer polynomials for the scalars, respectively.

For charged scalar mesons, there exists a relation between the vector and the scalar decay constants,

$$\bar{f}_S = \mu_S f_S \quad \text{and} \quad \mu_S = \frac{m_S}{m_2(\mu) - m_1(\mu)}, \quad \text{(A11)}$$

where m_1 and m_2 are the running current quark masses in the related scalars.

The values for scalar decay constants and Gegenbauer moments in the scalar meson distribution amplitudes have been investigated at scale $\mu = 1$ GeV in Ref. [56] and are collected in Table IV. The masses of the considered scalars are $m_{a_0} = 0.98$ GeV, $m_\kappa = 0.845$ GeV, $m_{a_0(1450)} = 1.474$ GeV, and $m_{K_0^*(1430)} = 1.425$ GeV, respectively.

As for the twist-3 distribution amplitudes ϕ_S^S and ϕ_S^T, we adopt the following asymptotic forms:

$$\phi_S^S = \frac{1}{2\sqrt{2N_c}} \bar{f}_S, \qquad \phi_S^T = \frac{1}{2\sqrt{2N_c}} \bar{f}_S (1-2x). \quad \text{(A12)}$$

TABLE IV. Scalar decay constant $\bar{f}_S$ (in GeV) and Gegenbauer moments $B_{1,3}$ for light scalars.

Scalars	$\bar{f}_S$	B_1	B_3
a_0	0.365 ± 0.020	-0.93 ± 0.10	0.14 ± 0.08
κ	0.340 ± 0.020	-0.92 ± 0.11	0.15 ± 0.09
$a_0(1450)$	-0.280 ± 0.030	0.89 ± 0.20	-1.38 ± 0.18
	0.460 ± 0.050	-0.58 ± 0.12	-0.49 ± 0.15
$K_0^*(1430)$	-0.300 ± 0.030	0.58 ± 0.07	-1.20 ± 0.08
	0.445 ± 0.050	-0.57 ± 0.13	-0.42 ± 0.22

Notice that, as inferred from Ref. [79], the Gegenbauer polynomials of twist-3 distribution amplitudes for light scalars are only available in S2 [80] and could mainly modify CP asymmetries in the B-meson decays. Because of no CP violations in the considered B_c-meson decays, we will left this issue for future investigations.

(iv) for light axial-vectors (A): a_1, b_1, K_{1A}, and K_{1B}

More discussions on light-cone distribution amplitudes of the light axial-vectors have been made in the literature [60,61]. Here, we just simply collect the expressions adopted in this work. The details about these distribution amplitudes could be found, e.g., in Ref. [60]. The twist-2 distribution amplitudes for the longitudinally and transversely polarized axial-vector 1^3P_1 and 1^1P_1 mesons can be parametrized as [61],

$$\phi_A(x) = \frac{3f}{\sqrt{2N_c}} x(1-x)\left[a_{0A}^{\parallel} + a_{1A}^{\parallel} C_1^{3/2}(2x-1) + a_{2A}^{\parallel} C_2^{3/2}(2x-1)\right], \quad \text{(A13)}$$

$$\phi_A^T(x) = \frac{3f}{\sqrt{2N_c}} x(1-x)\left[a_{0A}^{\perp} + a_{1A}^{\perp} C_1^{3/2}(2x-1) + a_{2A}^{\perp} C_2^{3/2}(2x-1)\right], \quad \text{(A14)}$$

As for twist-3 distribution amplitudes for axial-vector meson, we use the following form [61]:

$$\phi_A^t(x) = \frac{3f}{2\sqrt{2N_c}}\left\{a_{0A}^{\perp}(2x-1)^2 + \frac{1}{2} a_{1A}^{\perp}(2x-1)(3(2x-1)^2-1)\right\}, \quad \text{(A15)}$$

$$\phi_A^s(x) = \frac{3f}{2\sqrt{2N_c}} \frac{d}{dx}\{x(1-x)(a_{0A}^{\perp} + a_{1A}^{\perp}(2x-1))\}. \quad \text{(A16)}$$

$$\phi_A^v(x) = \frac{3f}{4\sqrt{2N_c}}\left\{\frac{1}{2} a_{0A}^{\parallel}(1+(2x-1)^2) + a_{1A}^{\parallel}(2x-1)^3\right\}, \quad \text{(A17)}$$

$$\phi_A^a(x) = \frac{3f}{4\sqrt{2N_c}} \frac{d}{dx}\left\{x(1-x)(a_{0A}^{\parallel} + a_{1A}^{\parallel}(2x-1))\right\}. \quad \text{(A18)}$$

where f is the "normalization" decay constant (More related discussions could be found in [61]).

The decay constants and Gegenbauer moments have been studied extensively in the literature (see, e.g., Ref. [60] and references therein), here we adopt the values at scale $\mu = 1$ GeV as collected in Table V. Moreover, the masses of

TABLE V. Decay constants (in GeV) and Gegenbauer moments for light axial-vectors.

f_{a_1}	$a^{\parallel}_{0a_1}$	$a^{\parallel}_{1a_1}$	$a^{\parallel}_{2a_1}$	$a^{\perp}_{0a_1}$	$a^{\perp}_{1a_1}$	$a^{\perp}_{2a_1}$
0.238 ± 0.010	1	$\cdots$	-0.02 ± 0.02	0	-1.04 ± 0.34	$\cdots$
f_{b_1}	$a^{\parallel}_{0b_1}$	$a^{\parallel}_{1b_1}$	$a^{\parallel}_{2b_1}$	$a^{\perp}_{0b1}$	$a^{\perp}_{1b_1}$	$a^{\perp}_{2b_1}$
0.180 ± 0.008	0.0028 ± 0.0026	-1.95 ± 0.35	$\cdots$	1	$\cdots$	0.03 ± 0.19
$f_{K_{1A}}$	$a^{\parallel}_{0K_{1A}}$	$a^{\parallel}_{1K_{1A}}$	$a^{\parallel}_{2K_{1A}}$	$a^{\perp}_{0K_{1A}}$	$a^{\perp}_{1K_{1A}}$	$a^{\perp}_{2K_{1A}}$
0.250 ± 0.013	1	0.00 ± 0.26	-0.05 ± 0.03	0.08 ± 0.09	-1.08 ± 0.48	0.02 ± 0.20
$f_{K_{1B}}$	$a^{\parallel}_{0K_{1B}}$	$a^{\parallel}_{1K_{1B}}$	$a^{\parallel}_{2K_{1B}}$	$a^{\perp}_{0K_{1B}}$	$a^{\perp}_{1K_{1B}}$	$a^{\perp}_{2K_{1B}}$
0.190 ± 0.010	0.14 ± 0.15	-1.95 ± 0.45	0.02 ± 0.10	1	0.17 ± 0.22	-0.02 ± 0.22

TABLE VI. Decay constants (in GeV) for light tensors.

f_{a_2}	$f^T_{a_2}$	$f_{K_2^*}$	$f^T_{K_2^*}$
0.107 ± 0.006	0.105 ± 0.021	0.118 ± 0.005	0.077 ± 0.014

related axial-vectors are $m_{a_1} = 1.23$ GeV, $m_{b_1} = 1.23$ GeV, $m_{K_{1A}} = 1.32$ GeV, and $m_{K_{1B}} = 1.34$ GeV, respectively.

(v) For light tensors (T): a_2 and K_2^*

Here, we present the light-cone distribution amplitudes of light tensor mesons following Refs. [62,63]:

$$\phi_T(x) = \frac{3f_T}{\sqrt{2N_c}}\phi_\parallel(x), \quad \phi_T^T(x) = \frac{3f_T^T}{\sqrt{2N_c}}\phi_\perp(x),$$
$$\phi_T^t(x) = \frac{f_T^T}{2\sqrt{2N_c}}h_\parallel^t(x), \quad \phi_T^s(x) = \frac{f_T^T}{4\sqrt{2N_c}}\frac{d}{dx}h_\parallel^s(x),$$
$$\phi_T^v(x) = \frac{f_T}{2\sqrt{2N_c}}g_\parallel^v(x), \quad \phi_T^a(x) = \frac{f_T}{8\sqrt{2N_c}}\frac{d}{dx}g_\perp^a(x), \tag{A19}$$

with

$$\phi_\parallel(x) = \phi_\perp(x) = x(1-x)[a_1 C_1^{3/2}(t)],$$
$$h_\parallel^t(x) = \frac{15}{2}(1-6x+6x^2)t, \qquad h_\parallel^s(x) = 15x(1-x)t,$$
$$g_\perp^v(x) = 5t^3, \qquad g_\perp^a(x) = 20x(1-x)t. \tag{A20}$$

with the Gegenbauer moment $a_1 = \frac{5}{3}$ for the first rough estimates. It is worth commenting that, in principle, the Gegenbauer moments for a_2 and K_2^* should usually be different due to the expected SU(3)-flavor symmetry breaking effects. Therefore, the larger Gegenbauer moment a_1 adopted here will demand further improvements through precise measurements. The decay constants for a_2 and K_2^* are presented in Table VI. Moreover, the masses for a_2 and K_2^* are adopted as 1.318 and 1.427 GeV, respectively.

APPENDIX B: FACTORIZATION FORMULAS FOR $B_c^+ \to J/\psi M^+$ DECAYS

In this section, we present the factorization formulas explicitly for the $B_c^+ \to J/\psi M^+$ decays calculated in the iPQCD formalism. First of all, the expressions for $B_c^+ \to J/\psi P^+$ decays could be referred to Ref. [41] for details, and are no longer presented here. For $B_c^+ \to J/\psi S^+$ decays, the factorization formulas are presented as follows,

(i) For factorizable emission diagrams,

$$F_e(S) = -8\pi C_F m_{B_c}^4 \int_0^1 dx_1 dx_3 \int_0^\infty b_1 db_1 b_3 db_3 \phi_{B_c}(x_1, b_1)(r_3^2 - 1)$$
$$\times \Big\{ \Big[r_3(r_b + 2x_3 - 2)\phi^t_{J/\psi}(x_3) - (2r_b + x_3 - 1)\phi^L_{J/\psi}(x_3) \Big] h_a(x_1, x_3, b_1, b_3)E_f(t_a)$$
$$+ [r_3^2(x_1 - 1) - r_c]\phi^L_{J/\psi}(x_3)h_b(x_1, x_3, b_1, b_3)E_f(t_b) \Big\}, \tag{B1}$$

where the ratios $r_b = m_b/m_{B_c}$ and $r_c = m_c/m_{B_c}$. The hard function $h_i(x_i, b_i)$ and the evolution function $E_f(t_i)$ could refer to those expressions in Ref. [41] and are collected in Appendix C.

(ii) For nonfactorizable emission diagrams,

$$M_e(S)=\frac{32}{\sqrt{6}}\pi C_F m_{B_c}^4\int_0^1 dx_1dx_2dx_3\int_0^\infty b_1db_1b_2db_2\phi_{B_c}(x_1,b_1)\phi_S(x_2)(r_3^2-1)$$
$$\times\Big\{\Big[(r_3^2-1)(x_1+x_2-1)\phi_{J/\psi}^L(x_3)+r_3(x_3-x_1)\phi_{J/\psi}^t(x_3)\Big]E_f(t_c)h_c(x_1,x_2,x_3,b_1,b_2)$$
$$+\Big[(2x_1-(x_2+x_3)+r_3^2(x_2-x_3))\phi_{J/\psi}^L(x_3)+r_3(x_3-x_1)\phi_{J/\psi}^t(x_3)\Big]h_d(x_1,x_2,x_3,b_1,b_2)E_f(t_d)\Big\}. \quad \text{(B2)}$$

Then, for $B_c^+\to J/\psi V^+$ decays, the factorization formulas with polarization contributions are collected as the following,

(i) For factorizable emission diagrams,

$$F_e^L(V)=8\pi C_F m_{B_c}^4\int_0^1 dx_1dx_3\int_0^\infty b_1db_1b_3db_3\phi_{B_c}(x_1,b_1)\sqrt{1-r_3^2}$$
$$\times\Big\{\Big[r_3(r_b+2x_3-2)\phi_{J/\psi}^t(x_3)-(2r_b+x_3-1)\phi_{J/\psi}^L(x_3)\Big]h_a(x_1,x_3,b_1,b_3)E_f(t_a)$$
$$+\Big[r_3^2(x_1-1)-r_c\Big]\phi_{J/\psi}^L(x_3)h_b(x_1,x_2,b_1,b_2)E_f(t_b)\Big\}, \quad \text{(B3)}$$

$$F_e^N(V)=8\pi C_F m_{B_c}^4 r_2\int_0^1 dx_1dx_3\int_0^\infty b_1db_1b_3db_3\phi_{B_c}(x_1,b_1)$$
$$\times\Big\{\Big[(r_3^2(r_b+4x_3-2)+r_b-2)\phi_{J/\psi}^T(x_3)-r_3((4r_b+x_3(1+r_3^2)-2))\phi_{J/\psi}^v(x_3)\Big]h_a(x_1,x_3,b_1,b_3)E_f(t_a)$$
$$-r_3\big[r_3^2+2r_c-2x_1+1\big]\phi_{J/\psi}^v(x_3)h_b(x_1,x_2,b_1,b_2)E_f(t_b)\Big\}, \quad \text{(B4)}$$

$$F_e^T(V)=16\pi C_F m_{B_c}^4 r_2\int_0^1 dx_1dx_3\int_0^\infty b_1db_1b_3db_3\phi_{B_c}(x_1,b_1)$$
$$\times\Big\{\Big[(r_b-2)\phi_{J/\psi}^T(x_3)+r_3x_3\phi_{J/\psi}^v(x_3)\Big]h_a(x_1,x_3,b_1,b_3)E_f(t_a)-r_3\phi_{J/\psi}^v(x_3)h_b(x_1,x_2,b_1,b_2)E_f(t_b)\Big\}, \quad \text{(B5)}$$

(ii) For nonfactorizable emission diagrams,

$$M_e^L(V)=\frac{32}{\sqrt{6}}\pi C_F m_{B_c}^4\int_0^1 dx_1dx_2dx_3\int_0^\infty b_1db_1b_2db_2\phi_{B_c}(x_1,b_1)\phi_V(x_2)\sqrt{1-r_3^2}$$
$$\times\Big\{\Big[(r_3^2-1)(x_1+x_2-1)\phi_{J/\psi}^L(x_3)+r_3(x_3-x_1)\phi_{J/\psi}^t(x_3)\Big]E_f(t_c)h_c(x_1,x_2,x_3,b_1,b_2)$$
$$+\Big[(2x_1-(x_2+x_3)+r_3^2(x_2-x_3))\phi_{J/\psi}^L(x_3)+r_3(x_3-x_1)\phi_{J/\psi}^t(x_3)\Big]h_d(x_1,x_2,x_3,b_1,b_2)E_f(t_d)\Big\}, \quad \text{(B6)}$$

$$M_e^N(V)=-\frac{32}{\sqrt{6}}\pi C_F m_{B_c}^4\int_0^1 dx_1dx_2dx_3\int_0^\infty b_1db_1b_2db_2\phi_{B_c}(x_1,b_1)r_2$$
$$\times\Big\{\Big[(r_3^2(x_1-x_2-2x_3+1)+x_1+x_2-1)\phi_V^v(x_2)+(1-r_3^2)^2(x_1+x_2-1)\phi_V^a(x_2)\Big]$$
$$\times\phi_{J/\psi}^T(x_3)E_f(t_c)h_c(x_1,x_2,x_3,b_1,b_2)+\Big[((r_3^2(x_1+x_2-2x_3)+x_1-x_2)\phi_V^v(x_2)$$
$$+(1-r_3^2)^2(x_1-x_2)\phi_V^a(x_2))\phi_{J/\psi}^T(x_3)+2r_3((x_3-x_2)r_3^2+x_2+x_3-2x_1)\phi_V^v(x_2)\phi_{J/\psi}^v(x_3)\Big]$$
$$\times h_d(x_1,x_2,x_3,b_1,b_2)E_f(t_d)\Big\}, \quad \text{(B7)}$$

$$
\begin{aligned}
M_e^T(V) = &-\frac{64}{\sqrt{6}}\pi C_F m_{B_c}^4 \int_0^1 dx_1 dx_2 dx_3 \int_0^\infty b_1 db_1 b_2 db_2 \phi_{B_c}(x_1, b_1) r_2 \\
&\times \Big\{ \Big[(r_3^2(x_1 - x_2 - 2x_3 + 1) + x_1 + x_2 - 1)\phi_V^a(x_2) + (x_1 + x_2 - 1)\phi_V^v(x_2) \Big] \\
&\times \phi_{J/\psi}^T(x_3) E_f(t_c) h_c(x_1, x_2, x_3, b_1, b_2) + \Big[((x_1 - x_2)\phi_V^v(x_2) + (x_1(1 + r_3^2)^2 + x_2(r_3^2 - 1) - 2r_3^2 x_3) \\
&\times \phi_V^a(x_2))\phi_{J/\psi}^T(x_3) + 2r_3(x_3(r_3^2 + 1) - x_2(r_3^2 - 1) - 2x_1)\phi_V^v(x_2)\phi_{J/\psi}^v(x_3) \Big] h_d(x_1, x_2, x_3, b_1, b_2) E_f(t_d) \Big\}.
\end{aligned} \tag{B8}
$$

For $B_c^+ \to J/\psi A^+$ decays, the factorization formulas could be easily obtained as

$$F_A^h = -F_V^h, \qquad M_A^h = -M_V^h, \tag{B9}$$

associated with the replacements of $\phi_V \to \phi_A$ and $r_V \to r_A$ correspondingly.

And, for $B_c^+ \to J/\psi T^+$ decays, as discussed in Ref. [46], the related factorization formulas could be straightforwardly obtained through those for $B_c^+ \to J/\psi V^+$ decays as follows,

$$M_T^L = \sqrt{\frac{2}{3}} M_V^L, \qquad M_T^{N,T} = \sqrt{\frac{1}{2}} M_V^{N,T}. \tag{B10}$$

in association with the corresponding replacements of $\phi_V \to \phi_T$ and $r_V \to r_T$.

APPENDIX C: RELATED FUNCTIONS

We here collect the related functions, i.e., hard functions $h_i(x_i, b_i)$ and evolution functions $E_f(t_i)$, in the factorization formulas.

The general form of $h_i(x_i, b_i)$ in the factorization formulas could be written as follows,

$$h_{a,b}(x_1, x_3, b_1, b_3) = \Big[\theta(b_3 - b_1) I_0(\sqrt{\beta_{a,b}} b_1) K_0(\sqrt{\beta_{a,b}} b_3) + (b_1 \leftrightarrow b_3)\Big] K_0(\sqrt{\alpha} b_1), \tag{C1}$$

$$h_{c,d}(x_1, x_2, x_3, b_1, b_2) = [\theta(b_2 - b_1) I_0(\sqrt{\alpha} b_1) K_0(\sqrt{\alpha} b_2) + (b_1 \leftrightarrow b_2)] K_0(\sqrt{\beta_{c,d}} b_2), \tag{C2}$$

with the factors α and $\beta_{a,b,c,d}$ and the hard scales $t_{a,b,c,d}$,

$$\alpha = -[(x_1 - x_3(1 - r_2^2))(x_1 - x_3 r_3^2)] m_{B_c}^2, \tag{C3}$$

$$\beta_a = -[(1 - x_3(1 - r_2^2))(1 - x_3 r_3^2) - r_b^2] m_{B_c}^2, \qquad \beta_b = -[(1 - x_1 - r_2^2)(r_3^2 - x_1) - r_c^2] m_{B_c}^2, \tag{C4}$$

$$\beta_c = -[(x_3 r_3^2 + (1 - x_2)(1 - r_3^2) - x_1)(x_3(1 - r_2^2) + (1 - x_2) r_2^2 - x_1)] m_{B_c}^2, \tag{C5}$$

$$\beta_d = -[(x_2 r_2^2 + x_3(1 - r_2^2) - x_1)(x_3 r_3^2 + x_2(1 - r_3^2) - x_1)] m_{B_c}^2, \tag{C6}$$

$$t_a = \max(\sqrt{|\alpha|}, \sqrt{|\beta_a|}, 1/b_1, 1/b_3), \qquad t_b = \max(\sqrt{|\alpha|}, \sqrt{|\beta_b|}, 1/b_1, 1/b_3), \tag{C7}$$

$$t_c = \max(\sqrt{|\alpha|}, \sqrt{|\beta_c|}, 1/b_1, 1/b_2), \qquad t_d = \max(\sqrt{|\alpha|}, \sqrt{|\beta_d|}, 1/b_1, 1/b_2). \tag{C8}$$

Note that, as α and $\beta_{a,b,c,d}$ are negative, the associated Bessel functions transform as

$$K_0(\sqrt{y}) = K_0(-i\sqrt{|y|}) = \frac{i\pi}{2}[J_0(\sqrt{|y|}) + iN_0(\sqrt{|y|})], \qquad I_0(\sqrt{y}) = J_0(\sqrt{|y|}), \tag{C9}$$

for $y < 0$.

The evolution functions $E_f(t) \equiv \alpha_s(t) C_i(t) S_i(t)$ contain the Wilson coefficients

$$C_{ab}(t) = \frac{1}{3} C_1(t) + C_2(t), \qquad C_{cd}(t) = C_1(t), \tag{C10}$$

and the Sudakov factors

$$S_{ab}(t) = s_c(x_1P_1^+, b_1) + s_c(x_3P_3^-, b_3) + s_c((1-x_3)P_3^-, b_3) - \frac{1}{\beta_1}\left[\frac{11}{6}\ln\frac{\ln(t/\Lambda)}{\ln(m_c/\Lambda)}\right], \tag{C11}$$

$$S_{cd}(t) = s_c(x_1P_1^+, b_1) + s(x_2P_2^+, b_2) + s((1-x_2)P_2^+, b_2) + s_c(x_3P_3^-, b_1) + s_c((1-x_3)P_3^-, b_1) - \frac{1}{\beta_1}\left[\frac{11}{6}\ln\frac{\ln(t/\Lambda)}{\ln(m_c/\Lambda)} + \ln\frac{\ln(t/\Lambda)}{-\ln(b_2\Lambda)}\right], \tag{C12}$$

where the explicit expression of the Sudakov exponent $s(Q,b)$ for an energetic light quark is referred to Ref. [49] and that of the other Sudakov factor $s_c(Q,b)$ with inclusion of finite charm quark mass effects can be found in Appendix D.

APPENDIX D: SUDAKOV FACTOR $S_c(Q,b)$ FOR B_c-MESON DECAYS

Here, we show the explicit form of newly derived Sudakov factor at next-to-leading logarithm accuracy with two-loop running coupling constant α_s that could make the framework of iPQCD formalism for B_c-meson decays more self-consistent.

$$\begin{aligned}
S_c(Q,b) = {} & \frac{a_1}{2\beta_1}\{\hat{Q}\ln\hat{Q} - \hat{c}\ln\hat{c} - (\hat{Q}-\hat{c})(1+\ln\hat{b})\} + \frac{a_2}{4\beta_1^2}\left\{-\ln\frac{\hat{Q}}{\hat{c}} + \frac{\hat{Q}-\hat{c}}{\hat{b}}\right\} + \frac{a_1}{4\beta_1}\ln\frac{\hat{Q}}{\hat{c}}\ln\frac{e^{2\gamma_E-1}}{2} \\
& + \frac{a_1\beta_2}{4\beta_1^3}\left\{\ln\frac{\hat{Q}}{\hat{c}} + \frac{1}{2}(\ln^2 2\hat{Q} - \ln^2 2\hat{c}) - \frac{1}{\hat{b}}(1+\ln 2\hat{b})(\hat{Q}-\hat{c})\right\} \\
& - \frac{a_2\beta_2}{4\beta_1^4}\left\{\frac{1}{4\hat{b}^2}(1+2\ln 2\hat{b})(\hat{Q}-\hat{c}) + \frac{3}{4}\left(\frac{1}{\hat{Q}} - \frac{1}{\hat{c}}\right) + \frac{1}{2}\left(\frac{\ln 2\hat{Q}}{\hat{Q}} - \frac{\ln 2\hat{c}}{\hat{c}}\right)\right\} \\
& + \frac{a_2\beta_2^2}{16\beta_1^6}\left\{\left(\frac{2}{27} + \frac{2}{9}\ln 2\hat{b} + \frac{1}{3}\ln^2 2\hat{b}\right)\frac{\hat{Q}-\hat{c}}{\hat{b}} + \frac{19}{108}\left(\frac{1}{\hat{Q}^2} - \frac{1}{\hat{c}^2}\right) + \frac{5}{18}\left(\frac{\ln 2\hat{Q}}{\hat{Q}^2} - \frac{\ln 2\hat{c}}{\hat{c}^2}\right) + \frac{1}{6}\left(\frac{\ln^2 2\hat{Q}}{\hat{Q}^2} - \frac{\ln^2 2\hat{c}}{\hat{c}^2}\right)\right\} \\
& + \frac{a_1\beta_2}{8\beta_1^3}\left\{\frac{1}{\hat{Q}} - \frac{1}{\hat{c}} + \frac{\ln 2\hat{Q}}{\hat{Q}} - \frac{\ln 2\hat{c}}{\hat{c}}\right\}\ln\frac{e^{2\gamma_E-1}}{2},
\end{aligned} \tag{D1}$$

with definitions: $\hat{Q} \equiv \ln[xP^+/\Lambda]$, $\hat{c} \equiv \ln[m_c/(\sqrt{2}\Lambda)]$, and $\hat{b} \equiv \ln[1/(b\Lambda)]$, and the constants $a_1 = C_F = \frac{4}{3}$, $a_2 = \frac{67}{9} - \frac{\pi^2}{3} - \frac{10}{27}n_f$, $\beta_1 = \frac{33-2n_f}{12}$, and $\beta_2 = \frac{153-19n_f}{24}$ with n_f being the flavor number. Notice that, when the replacement $\hat{c} \to \hat{b}$ is adopted, then the formula presented in Eq. (D1) will recover the Sudakov factor for B-meson decays with strong running coupling constant α_s at two-loop level [49]. And furthermore, when the term β_2 in Eq. (D1) is turned off, the equation will then return to the Sudakov factor with one-loop running coupling constant α_s that has been adopted in this work,

$$S_c(Q,b) = \frac{a_1}{2\beta_1}\{\hat{Q}\ln\hat{Q} - \hat{c}\ln\hat{c} - (\hat{Q}-\hat{c})(1+\ln\hat{b})\} + \frac{a_2}{4\beta_1^2}\left\{-\ln\frac{\hat{Q}}{\hat{c}} + \frac{\hat{Q}-\hat{c}}{\hat{b}}\right\} + \frac{a_1}{4\beta_1}\ln\frac{\hat{Q}}{\hat{c}}\ln\frac{e^{2\gamma_E-1}}{2}. \tag{D2}$$

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