Novel relations for twist-3 tensor-polarized fragmentation functions in spin-1 hadrons

Qin-Tao Song [*]

School of Physics and Microelectronics, Zhengzhou University, Zhengzhou, Henan 450001, China

(Received 12 September 2023; accepted 1 November 2023; published 29 November 2023)

There are three types of fragmentation functions (FFs) which are used to describe the twist-3 cross sections of the hard semi-inclusive processes under QCD collinear factorization, and they are called intrinsic, kinematical, and dynamical FFs. In this work, we investigate the theoretical relations among these FFs for a tensor-polarized spin-1 hadron. Three Lorentz-invariance relations are derived by using the identities between the nonlocal quark-quark and quark-gluon-quark operators, which guarantee the frame independence of the twist-3 spin observables. The QCD equation of motion relations are also presented for the tensor-polarized FFs. In addition, we show that the intrinsic and kinematical twist-3 FFs can be decomposed into the contributions of twist-2 FFs and twist-3 three-parton FFs, and the latter are also called dynamical FFs. If one neglects the dynamical FFs, we can obtain relations which are analogous to the Wandzura-Wilczek relation. Then, the intrinsic and kinematical twist-3 FFs are expressed in terms of the leading-twist ones. Since the FFs of a spin-1 hadron can be measured at various experimental facilities in the near future, these theoretical relations will play an important role in the analysis of the collinear tensor-polarized FFs.

DOI: 10.1103/PhysRevD.108.094041

I. INTRODUCTION

Parton distribution functions (PDFs) are key physical quantities in hadron spin physics, since they are used to solve the proton spin puzzle and to understand the inner structure of hadrons. For a spin-$1/2$ hadron, the theoretical relations of PDFs and fragmentation functions (FFs) have been well studied. Starting with the Wandzura-Wilczek (WW) relation, it is known that if one neglects the three-parton PDFs, the twist-3 PDF g_2 can be expressed in terms of the leading-twist one g_1 which has been well measured [1]. The violation of the WW relation comes from the three-parton PDFs, and it was shown that such violation can be as large as 15%–40% of the size of g_2 [2]. There also exist the so-called Lorentz-invariance relations (LIRs) for the PDFs in a spin-$1/2$ hadron, which were investigated in Refs. [2–9]. In addition to PDFs, LIRs were also derived for the quark FFs [9]. Recently, the authors of Ref. [10] performed a systematic study on the gluon PDFs and FFs, where the intrinsic and kinematical twist-3 gluon distributions are written in terms of the twist-2 distributions and the twist-3 dynamical distributions, and the latter are actually three-parton distributions; moreover, the complete LIRs are also listed for the gluon part. On the one hand, these interesting relations can be used as constraints for the analysis of twist-3 distributions. On the other hand, they are also crucial to describe the spin observables, for example, the LIRs can be used to guarantee the frame independence of the twist-3 cross sections, such as the single-spin asymmetries (SSAs) in the hadron production of lepton-nucleon collisions and the hadron production of hadronic collisions ($pp \to \Lambda^{\uparrow} X$) [9,11,12].

For a spin-1 hadron, there are unpolarized, vector-polarized and tensor-polarized distributions. The former two also exist for a spin-$1/2$ hadron, while the tensor-polarized distributions are the new ones. Among the tensor-polarized PDFs, $b_1(x)$ [or $f_{1LL}(x)$] [13,14] and the gluon transversity $\Delta_T g(x)$ [15,16] are the most interesting ones. The sum rule of $\int dx b_1(x) = 0$ was derived for an isoscalar object such as the deuteron, and the breaking of this sum rule is related to the contribution of a tensor-polarized component of the sea quarks and antiquarks [17]. In 2005, the HERMES collaboration performed the first measurement of $b_1(x)$ for deuteron with slightly large uncertainties [18], and it indicates that $b_1(x)$ is much larger than the theoretical prediction [19]. Since the theoretical estimate of $b_1(x)$ was given by considering deuteron as a weakly bound state of proton and neutron, the large $b_1(x)$ could indicate exotic components of deuteron such as a six-quark state and a hidden color state [20]. As for the gluon transversity $\Delta_T g(x)$, it is related to the helicity flipped amplitude, so it only exists in a hadron with spin more than or equal to 1 due to the angular momentum conservation.

[*]songqintao@zzu.edu.cn

In this case, one can infer that there are nonnucleonic components in the deuteron by the nonzero $\Delta_T g(x)$, which means that it is interesting to investigate the gluon transversity by experiment; for example, it can be extracted from the cross sections of deep-inelastic scattering [15,21] and Drell-Yan process [22,23] with a tensor-polarized deuteron target. In the near future, $b_1(x)$ and $\Delta_T g(x)$ will be measured at the Thomas Jefferson National Accelerator Facility (JLab) [24,25], Fermilab (Fermi National Accelerator Laboratory) [26–28], and Nuclotron-based Ion Collider fAcility (NICA) [29]. There are also interesting theoretical relations for the tensor-polarized PDFs; in Ref. [30] the twist-3 PDF $f_{LT}(x)$ was decomposed into the contributions of a twist-2 PDF $b_1(x)$ [$f_{1LL}(x)$] and the three-parton PDFs. Moreover, the WW-type relation was obtained by dropping the latter. The QCD equation of motion (e.o.m.) relations and LIR were derived in Ref. [31] for tensor-polarized PDFs. Recently, the gluon transversity generalized parton distribution was also investigated for a spin-1 hadron [32], which becomes the gluon transversity $\Delta_T g(x)$ in the forward limit. In addition to the collinear PDFs, one can find the tensor-polarized transverse-momentum dependent (TMD) PDFs up to twist 4 for a spin-1 hadron in Refs. [33–36].

The spin-1 hadrons are produced in the hard semi-inclusive processes, such as ρ, ϕ, K^* and a deuteron. In order to describe those processes, the tensor-polarized FFs are needed. The quark collinear FFs are defined in Ref. [37] up to twist 4 for a spin-1 hadron, and the tensor-polarized TMD FFs can be also found in Refs. [33,38]. In the future, the tensor-polarized FFs can be measured at BESIII and Belle II. Actually, such measurement is now in progress, for example, the FFs of ϕ in the process $e^+e^- \to \phi X$ by the BESIII Collaboration [39]. However, the theoretical relations of tensor-polarized FFs have not been completely investigated. In this work, we intend to derive the LIRs, QCD e.o.m., and WW-type relations for the tensor-polarized FFs in a spin-1 hadron, which can provide constraints for the future experimental and theoretical studies of these FFs.

This paper is organized as follows. In Sec. II, we define the intrinsic, kinematical, and dynamical twist-3 FFs, and general properties of them are discussed. We derive the theoretical relations among tensor-polarized FFs using QCD e.o.m. for quarks in Sec. III. The operator identities are obtained for the nonlocal quark-quark and quark-gluon-quark operators, then LIRs and WW-type relations are also given based on the matrix elements of the operator identities in Sec. IV. A brief summary of this work is presented in Sec. V.

II. TENSOR-POLARIZED FRAGMENTATION FUNCTIONS

The tensor polarization is often indicated by the matrix T for a spin-1 hadron, and the covariant form of $T^{\mu\nu}$ is expressed as [33,34]

$$
\begin{aligned}
T^{\mu\nu} = \frac{1}{2}\Bigg[&\frac{4}{3}S_{LL}\frac{(P_h^-)^2}{M^2}n^\mu n^\nu - \frac{2}{3}S_{LL}(n^{\{\mu}\bar{n}^{\nu\}} - g_T^{\mu\nu}) \\
&+ \frac{1}{3}S_{LL}\frac{M^2}{(P_h^-)^2}\bar{n}^\mu\bar{n}^\nu + \frac{P_h^-}{M}n^{\{\mu}S_{LT}^{\nu\}} - \frac{M}{2P_h^-}\bar{n}^{\{\mu}S_{LT}^{\nu\}} \\
&+ S_{TT}^{\mu\nu}\Bigg],
\end{aligned} \tag{1}
$$

where P_h and M are momentum and mass for the produced hadron, respectively. $a^{\{\mu}b^{\nu\}} = a^\mu b^\nu + a^\nu b^\mu$ denotes symmetrization of the indices. The light-cone vectors n and $\bar{n}$ are given by

$$
n^\mu = \frac{1}{\sqrt{2}}(1,0,0,-1), \qquad \bar{n}^\mu = \frac{1}{\sqrt{2}}(1,0,0,1), \tag{2}
$$

and P_h can be written as $P_h = P_h^- n + \frac{M^2}{2P_h^-}\bar{n}$. For a Lorentz vector a^μ, the light-cone components $a^\pm$ and transverse component a_T are defined by

$$
a^+ = a\cdot n, \qquad a^- = a\cdot\bar{n}, \qquad a_T^\mu = g_T^{\mu\nu}a_\nu \tag{3}
$$

with

$$
g_T^{\mu\nu} = g^{\mu\nu} - n^\mu\bar{n}^\nu - n^\nu\bar{n}^\mu. \tag{4}
$$

In Eq. (1), S_{LL}, S_{LT}^μ, and $S_{TT}^{\mu\nu}$ are the parameters which indicate different types of tensor polarization.

For a spin-1 hadron, the fragmentation correlator is defined as [33,36–38]

$$
\begin{aligned}
\Delta_{ij}(z) =& \frac{1}{N_c}\int\frac{d\xi^+}{2\pi}e^{i\frac{P_h^-\xi^+}{z}}\langle 0|\mathcal{W}[\infty^+;\xi^+]q_i(\xi^+)|P_h,T;X\rangle \\
&\times\langle P_h,T;X|\bar{q}_j(0)\mathcal{W}[0^+;\infty^+]|0\rangle, \\
=& \frac{1}{z}\Bigg\{S_{LL}\not{n}F_{1LL}(z) + \frac{M}{P_h^-}[\not{S}_{LT}F_{LT}(z) \\
&+ S_{LL}E_{LL}(z)] + \sigma^{i+}S_{LT,i}H_{1LT} \\
&+ \frac{M}{P_h^-}\Big[S_{LL}\sigma^{-+}H_{LL}(z) + \gamma_5\gamma_i\epsilon_T^{ij}S_{LT,j}G_{LT}\Big]\Bigg\},
\end{aligned} \tag{5}
$$

where z is the longitudinal momentum fraction carried by the produced hadron, N_c is the number of color, and $\mathcal{W}$ is a Wilson line which ensures color gauge invariance. The transverse tensor $\epsilon_T^{\alpha\beta}$ is given by

$$
\epsilon_T^{\alpha\beta} = \epsilon^{\alpha\beta\mu\nu}n_\mu\bar{n}_\nu \tag{6}
$$

with the convention $\epsilon^{0123} = 1$. In Eq. (5), the correlator is expressed in terms of six tensor-polarized FFs up to twist 3, and the FFs are real functions with the support region of $0 < z < 1$. $F_{1LL}(z)$ and $H_{1LT}(z)$ are leading-twist FFs, and the rest are also called intrinsic twist-3 FFs [9].

Since time-reversal invariance is not a necessary constraint for the fragmentation correlator, the last three FFs are actually time-reversal odd FFs. Note that there are also unpolarized and vector-polarized FFs in the correlator, which are neglected here since we are interested in the tensor-polarized ones.

The kinematical twist-3 FFs are related to the TMD FFs. In case of a tensor-polarized hadron, the TMD fragmentation correlator reads [40–44]

$$\Delta_{ij}(z,k_T) = \frac{1}{N_c}\int \frac{d\xi^+ d^2\xi_T}{(2\pi)^3} e^{i(k^-\xi^+ + k_T\cdot\xi_T)}\langle 0|\mathcal{W}_1[\infty;\xi]q_i(\xi) \times |P_h,T;X\rangle\langle P_h,T;X|\bar{q}_j(0)\mathcal{W}_2[0;\infty]|0\rangle_{\xi^-=0} \tag{7}$$

with

$$\mathcal{W}_1[\infty;\xi] = \mathcal{W}[\infty^+,\infty_T;\infty^+,\xi_T]\mathcal{W}[\infty^+,\xi_T;\xi^+,\xi_T],$$
$$\mathcal{W}_2[0;\infty] = \mathcal{W}[0^+,0_T;\infty^+,0_T]\mathcal{W}[\infty^+,0_T;\infty^+,\infty_T], \tag{8}$$

and the correlator can be written in terms of TMD FFs [33,38,45]. The k_T-weighted FFs are defined with the help of the TMD fragmentation correlator,

$$\Delta^\nu_{\partial,ij}(z) = \int d^2k_T k_T^\nu \Delta_{ij}(z,k_T), \tag{9}$$

which is parametrized by four k_T-weighted FFs at twist 3 [33],

$$\Delta^\nu_\partial(z) = \frac{M}{z}\left[-S^\nu_{LT}\not{n}F^{(1)}_{1LT}(z) - \epsilon_T^{\nu\rho}S_{LT\rho}\gamma_5\not{n}G^{(1)}_{1LT}(z) + S_{LL}\sigma^{\nu\alpha}n_\alpha H^{(1)}_{1LL}(z) - S^{\nu\alpha}_{TT}\sigma_{\alpha\beta}n^\beta H^{(1)}_{1TT}(z)\right], \tag{10}$$

and these FFs are also called kinematical twist-3 FFs in Ref. [9]. Due to Eq. (9), the kinematical twist-3 FFs are given by TMD FFs,

$$F^{(1)}(z) = -z^2\int d^2k_T \frac{k_T^2}{2M^2}F(z,z^2k_T^2), \tag{11}$$

where $F(z,z^2k_T^2)$ is a TMD FF.

Similarly, we define the collinear three-parton fragmentation correlator [30],

$$\Delta^\nu_{F,ij}(z,z_1) = \frac{1}{N_c}\int\frac{d\xi^+}{2\pi}\frac{d\xi_1^+}{2\pi}e^{iP_h^-\xi^+\frac{1}{z_1}+iP_h^-\xi_1^+(\frac{1}{z}-\frac{1}{z_1})} \times\langle 0|\mathcal{W}[\infty^+;\xi_1^+]igF^{-\nu}(\xi_1^+) \times\mathcal{W}[\xi_1^+;\xi^+]q_i(\xi^+)|P_h,T;X\rangle \times\langle P_h,T;X|\bar{q}_j(0)\mathcal{W}[0^+;\infty^+]|0\rangle. \tag{12}$$

By inserting a complete set of intermediate states, one can prove that

$$\Delta^\nu_F(z,z) = 0, \qquad \Delta^\nu_F(z,0) = 0, \tag{13}$$

and this corresponds to the vanishing partonic pole matrix elements which are important to understand the SSAs in the hard semi-inclusive processes [46]. Then, the support region of $\Delta^\nu_F(z,z_1)$ is

$$0\le z\le 1, \qquad 0<\frac{z}{z_1}<1. \tag{14}$$

Taking the derivative of this correlator with respect to $1/z_1$ and then setting $z_1 = z$, one can also obtain [9]

$$\left.\frac{\partial\Delta^\nu_F(z,z_1)}{\partial(1/z_1)}\right|_{z_1=z} = 0. \tag{15}$$

The parametrization of $\Delta^\nu_F(z,z_1)$ is just a copy of the corresponding three-parton distribution correlator [30], and it can be expressed in terms of four dynamical FFs at twist 3,

$$\Delta^\nu_{F,ij}(z,z_1) = \frac{M}{z}\left[-S^\nu_{LT}\not{n}\hat{F}_{LT}(z,z_1) - i\epsilon_T^{\nu\rho}S_{LT\rho}\gamma_5\not{n}\hat{G}_{LT}(z,z_1) - S_{LL}\gamma^\nu\not{n}\hat{H}^\perp_{LL}(z,z_1) - S^{\nu\rho}_{TT}\gamma_\rho\not{n}\hat{H}_{TT}(z,z_1)\right]. \tag{16}$$

Note that the dynamical FFs are complex functions which are different from the intrinsic and kinematical ones.

III. EQUATION OF MOTION RELATIONS FOR FFs

The intrinsic, kinematical, and dynamical FFs are not independent functions, since they can be related to each other by the e.o.m. relations. For a spin-1/2 hadron, the e.o.m. relations for FFs were derived in Refs. [9,47] based on the QCD e.o.m. for quarks, namely, $(i\not{D} - m_q)q(x) = 0$. In the following, we will investigate the e.o.m. relations for tensor-polarized FFs. After some algebra, the QCD e.o.m. for quarks becomes

$$(iD^\mu + \sigma^{\mu\nu}D_\nu + m_q\gamma^\mu)q(x) = 0, \tag{17}$$

where m_q is the mass of the quark. If we set $\mu = -$ and take the corresponding matrix element for Eq. (17), an e.o.m. relation can be obtained for the intrinsic, kinematical, and dynamical FFs,

$$\frac{E_{LL}(z)}{z}+\frac{iH_{LL}(z)}{z}-\frac{m_q}{M}F_{1LL}(z)$$
$$=2\left[-iH_{1LL}^{(1)}(z)+\mathcal{P}\int_z^\infty\frac{dz_1}{(z_1)^2}\frac{\hat{H}_{LL}^{\perp}(z,z_1)}{\frac{1}{z}-\frac{1}{z_1}}\right], \quad (18)$$

where $\mathcal{P}$ stands for the principal integral, and it can be neglected due to Eq. (13). All the FFs in Eq. (18) are related to the S_{LL}-type tensor polarization. Furthermore, this relation can be reexpressed in terms of the real and complex parts of the dynamical FF,

$$\frac{E_{LL}(z)}{z}=2\int_z^\infty\frac{dz_1}{(z_1)^2}\frac{\mathrm{Re}\left[\hat{H}_{LL}^{\perp}(z,z_1)\right]}{\frac{1}{z}-\frac{1}{z_1}}+\frac{m_q}{M}F_{1LL}(z), \quad (19)$$

$$\frac{H_{LL}(z)}{z}=2\int_z^\infty\frac{dz_1}{(z_1)^2}\frac{\mathrm{Im}\left[\hat{H}_{LL}^{\perp}(z,z_1)\right]}{\frac{1}{z}-\frac{1}{z_1}}-2H_{1LL}^{(1)}(z). \quad (20)$$

We can see that the time-reversal even and odd intrinsic FFs are related to the real and imaginary parts of the dynamical FFs, respectively. If we neglect the quark mass, the intrinsic twist-3 FFs $E_{LL}(z)$ and $H_{LL}(z)$ are given by the kinematical and dynamical twist-3 FFs.

Multiplying γ^ν on the left-hand side of Eq. (17), then antisymmetrizing μ and ν, we can obtain the identity as

$$[i(\gamma^\mu D^\nu-\gamma^\nu D^\mu)-\epsilon^{\mu\nu\rho\sigma}\gamma_\sigma\gamma_5 D_\rho+im_q\sigma^{\mu\nu}]q(x)=0. \quad (21)$$

Analogously, we set $\mu=-$ and consider ν as a transverse component, then the matrix element of Eq. (21) leads to

$$\frac{F_{LT}(z)}{z}+\frac{iG_{LT}(z)}{z}+\frac{im_q}{M}H_{1LT}(z)$$
$$=-iG_{1LT}^{(1)}(z)+\int_z^\infty\frac{dz_1}{(z_1)^2}\frac{\hat{G}_{LT}(z,z_1)}{\frac{1}{z}-\frac{1}{z_1}}$$
$$-\left[F_{1LT}^{(1)}(z)+\int_z^\infty\frac{dz_1}{(z_1)^2}\frac{\hat{F}_{LT}(z,z_1)}{\frac{1}{z}-\frac{1}{z_1}}\right], \quad (22)$$

and it indicates the relation among the intrinsic, kinematical, and dynamical FFs for the S_{LT}-type tensor polarization. Furthermore, Eq. (22) can be divided into two identities,

$$\frac{F_{LT}(z)}{z}=-\int_z^\infty\frac{dz_1}{(z_1)^2}\frac{\mathrm{Re}\left[\hat{F}_{LT}(z,z_1)-\hat{G}_{LT}(z,z_1)\right]}{\frac{1}{z}-\frac{1}{z_1}}$$
$$-F_{1LT}^{(1)}(z), \quad (23)$$

$$\frac{G_{LT}(z)}{z}=-\int_z^\infty\frac{dz_1}{(z_1)^2}\frac{\mathrm{Im}\left[\hat{F}_{LT}(z,z_1)-\hat{G}_{LT}(z,z_1)\right]}{\frac{1}{z}-\frac{1}{z_1}}$$
$$-G_{1LT}^{(1)}(z)-\frac{m_q}{M}H_{1LT}(z). \quad (24)$$

As indicated by Eq. (5), there are no intrinsic FFs for the S_{TT}-type tensor polarization. However, we can also derive the following identity using Eq. (21):

$$iH_{1TT}^{(1)}(z)+\int_z^\infty\frac{dz_1}{(z_1)^2}\frac{\hat{H}_{TT}(z,z_1)}{\frac{1}{z}-\frac{1}{z_1}}=0, \quad (25)$$

and it implies

$$\int_z^\infty\frac{dz_1}{(z_1)^2}\frac{\mathrm{Re}\left[\hat{H}_{TT}(z,z_1)\right]}{\frac{1}{z}-\frac{1}{z_1}}=0, \quad (26)$$

$$H_{1TT}^{(1)}(z)+\int_z^\infty\frac{dz_1}{(z_1)^2}\frac{\mathrm{Im}\left[\hat{H}_{TT}(z,z_1)\right]}{\frac{1}{z}-\frac{1}{z_1}}=0, \quad (27)$$

which complete the derivation of the QCD e.o.m. relations for tensor-polarized FFs.

IV. LORENTZ INVARIANCE AND WANDZURA-WILCZEK-TYPE RELATIONS

Taking the derivative of nonlocal quark-quark operators, one can obtain the identities where the quark-quark operators are expressed in terms of the quark-gluon-quark ones. The theoretical relations have been investigated for PDFs, FFs, and distribution amplitudes by using these identities of nonlocal operators, and this method was well explained in Refs. [9,48–56]. In this section, we adopt the same method to derive the theoretical relations for twist-3 tensor-polarized FFs such as LIRs and WW-type relations. We first consider the derivative of the nonlocal quark-quark operator [9],

$$
\begin{aligned}
&\frac{\partial}{\partial\xi_\alpha}\langle 0|\mathcal{W}[\infty\xi;-\xi]q(-\xi)|P_h,T;X\rangle\langle P_h,T;X|\bar{q}(\xi)\Gamma\mathcal{W}[\xi;\infty\xi]|0\rangle\\
&=-\langle 0|\mathcal{W}[\infty\xi;-\xi]\overrightarrow{D}^{\alpha}(-\xi)q(-\xi)|P_h,T;X\rangle\langle P_h,T;X|\bar{q}(\xi)\Gamma\mathcal{W}[\xi;\infty\xi]|0\rangle\\
&\quad+\langle 0|\mathcal{W}[\infty\xi;-\xi]q(-\xi)|P_h,T;X\rangle\langle P_h,T;X|\bar{q}(\xi)\overleftarrow{D}^{\alpha}(\xi)\Gamma\mathcal{W}[\xi;\infty\xi]|0\rangle\\
&\quad+i\int_{-1}^{\infty}dt t\langle 0|\mathcal{W}[\infty\xi;t\xi]gF^{\alpha\xi}(t\xi)\mathcal{W}[t\xi;-\xi]q(-\xi)|P_h,T;X\rangle\langle P_h,T;X|\bar{q}(\xi)\Gamma\mathcal{W}[\xi;\infty\xi]|0\rangle\\
&\quad+i\int_{\infty}^{1}dt t\langle 0|\mathcal{W}[\infty\xi;-\xi]q(-\xi)|P_h,T;X\rangle\langle P_h,T;X|\bar{q}(\xi)\Gamma\mathcal{W}[\xi;t\xi]gF^{\alpha\xi}(t\xi)\mathcal{W}[t\xi;\infty\xi]|0\rangle,
\end{aligned}
\tag{28}
$$

where ξ is not necessarily a light-cone vector and Γ is a gamma matrix. In Eq. (28), the terms with the covariant derivative D^α can be replaced by the total derivative of the nonlocal quark-quark operator, which is related to the translation of the operator, and its matrix element can be expressed as [9]

$$
\begin{aligned}
&\bar{\partial}^{\rho}\langle 0|\mathcal{W}[\infty\xi;-\xi]q(-\xi)|P_h,T;X\rangle\langle P_h,T;X|\bar{q}(\xi)\Gamma_1\mathcal{W}[\xi;\infty\xi]|0\rangle\\
&=\lim_{x_\rho\to 0}\frac{d}{dx_\rho}\langle 0|\mathcal{W}[\infty\xi+x;-\xi+x]q(-\xi+x)|P_h,T;X\rangle\langle P_h,T;X|\bar{q}(\xi+x)\Gamma_1\mathcal{W}[\xi+x;\infty\xi+x]|0\rangle\\
&=\langle 0|\mathcal{W}[\infty\xi;-\xi]\overrightarrow{D}^{\rho}(-\xi)q(-\xi)|P_h,T;X\rangle\langle P_h,T;X|\bar{q}(\xi)\Gamma_1\mathcal{W}[\xi;\infty\xi]|0\rangle\\
&\quad+\langle 0|\mathcal{W}[\infty\xi;-\xi]q(-\xi)|P_h,T;X\rangle\langle P_h,T;X|\bar{q}(\xi)\overleftarrow{D}^{\rho}(\xi)\Gamma_1\mathcal{W}[\xi;\infty\xi]|0\rangle\\
&\quad+\int_{-1}^{\infty}dt\langle 0|\mathcal{W}[\infty\xi;t\xi]igF^{\rho\xi}(t\xi)\mathcal{W}[t\xi;-\xi]q(-\xi)|P_h,T;X\rangle\langle P_h,T;X|\bar{q}(\xi)\Gamma_1\mathcal{W}[\xi;\infty\xi]|0\rangle\\
&\quad+\int_{\infty}^{1}dt\langle 0|\mathcal{W}[\infty\xi;-\xi]q(-\xi)|P_h,T;X\rangle\langle P_h,T;X|\bar{q}(\xi)\Gamma_1\mathcal{W}[\xi;t\xi]igF^{\rho\xi}(t\xi)\mathcal{W}[t\xi;\infty\xi]|0\rangle,
\end{aligned}
\tag{29}
$$

where Γ_1 stands for a gamma matrix such as γ^μ and $\sigma^{\mu\nu}$. Due to the translation invariance, the matrix element in Eq. (29) should vanish.

In the following, the Wilson lines are neglected in the operator identities, since this will not cause confusion. We derive a relation between quark-quark and quark-gluon-quark operators by choosing $\Gamma=(g^{\rho\alpha}g^{\lambda}{}_{\sigma}-g^{\alpha}{}_{\sigma}g^{\rho\lambda})\gamma_\lambda$ in Eq. (28) and $\Gamma_1=(\sigma^{\sigma\beta}\gamma^\rho+\gamma^\rho\sigma^{\sigma\beta})$ in Eq. (29),

$$
\begin{aligned}
&\xi_\alpha\left[\frac{\partial}{\partial\xi_\alpha}\langle 0|q(-\xi)|P_h,T;X\rangle\langle P_h,T;X|\bar{q}(\xi)\gamma^\sigma|0\rangle-\frac{\partial}{\partial\xi_\sigma}\langle 0|q(-\xi)|P_h,T;X\rangle\langle P_h,T;X|\bar{q}(\xi)\gamma^\alpha|0\rangle\right]\\
&=\left[\int_{-1}^{\infty}dt\langle 0|gF_{\rho\xi}(t\xi)q(-\xi)|P_h,T;X\rangle\langle P_h,T;X|\bar{q}(\xi)\gamma_\tau\gamma_5|0\rangle+\int_{\infty}^{1}dt\langle 0|q(-\xi)|P_h,T;X\rangle\langle P_h,T;X|\bar{q}(\xi)\gamma_\tau\gamma_5 gF_{\rho\xi}(t\xi)|0\rangle\right]\\
&\quad\times\epsilon^{\sigma\xi\rho\tau}-i\int_{-1}^{\infty}dt t\langle 0|gF^{\sigma\xi}(t\xi)q(-\xi)|P_h,T;X\rangle\langle P_h,T;X|\bar{q}(\xi)\not{\xi}|0\rangle-i\int_{\infty}^{1}dt t\langle 0|q(-\xi)|P_h,T;X\rangle\\
&\quad\times\langle P_h,T;X|\bar{q}(\xi)\not{\xi}gF^{\sigma\xi}(t\xi)|0\rangle,
\end{aligned}
\tag{30}
$$

where the matrix element of the total derivative operator is neglected, and the quark mass terms vanish. The quark-quark operator appears in the left-hand side of Eq. (30), which can be written in terms of the intrinsic tensor-polarized FFs as shown in Eq. (5). If the vector ξ is not necessarily on the light cone, the matrix element of the nonlocal quark-quark operator can be expressed as

$$
\langle 0|q(-\xi)|P_h,T;X\rangle\langle P_h,T;X|\bar{q}(\xi)\gamma^\sigma|0\rangle=8N_cM^2\int d\left(\frac{1}{z}\right)e^{\frac{2iP_\xi}{z}}\left[\frac{3}{4}A^\sigma\frac{F_{1LL}(z)}{z}+B^\sigma\frac{F_{LT}(z)}{z}\right]
\tag{31}
$$

with

$$A^\sigma = \frac{\xi\cdot T\cdot\xi}{(P_h\cdot\xi)^2}P_h^\sigma - M^2\frac{\xi\cdot T\cdot\xi}{(P_h\cdot\xi)^3}\xi^\sigma, \tag{32}$$

$$B^\sigma = \frac{T^{\sigma\mu}\xi_\mu}{P_h\cdot\xi} - \frac{\xi\cdot T\cdot\xi}{(P_h\cdot\xi)^2}P_h^\sigma + M^2\frac{\xi\cdot T\cdot\xi}{(P_h\cdot\xi)^3}\xi^\sigma. \tag{33}$$

Equation (31) is exact at twist 3 since the twist-4 FFs are not included. We substitute Eq. (31) into Eq. (30) to estimate the derivative, and take the light-cone limt of $\xi^2\to 0$, then, the left-hand side of Eq. (30) is given by $F_{1LL}(z)$ and $F_{LT}(z)$. The right-hand side of Eq. (30) can be directly calculated with the help of Eq. (16), and we obtain the following relation by combining the left- and right-hand sides,

$$\begin{aligned}\frac{3}{2}\tilde{F}_{1LL}(z) + \frac{1}{z}\frac{d\tilde{F}_{LT}(z)}{d(1/z)} = \int d\left(\frac{1}{z_1}\right)\mathcal{P}\left(\frac{1}{\frac{1}{z}-\frac{1}{z_1}}\right)\bigg\{&\left(\frac{\partial}{\partial(1/z)}+\frac{\partial}{\partial(1/z_1)}\right)\mathrm{Re}\left[\tilde{G}_{LT}(z,z_1)\right]\\ &-\left(\frac{\partial}{\partial(1/z)}-\frac{\partial}{\partial(1/z_1)}\right)\mathrm{Re}\left[\tilde{F}_{LT}(z,z_1)\right]\bigg\},\end{aligned} \tag{34}$$

where the convention of $\tilde{F}(z) = F(z)/z$ is used for a intrinsic or kinematical FF $F(z)$, and $\tilde{F}(z,z_1) = \hat{F}(z,z_1)/z$ for a dynamical one. Combining Eq. (34) with the e.o.m relation of Eq. (23), one can obtain

$$\frac{3}{2}\tilde{F}_{1LL}(z) - \tilde{F}_{LT}(z) - \left(1 - z\frac{d}{dz}\right)F_{1LT}^{(1)}(z) = -2\int_z^\infty \frac{dz_1}{(z_1)^2}\frac{\mathrm{Re}\left[\tilde{F}_{LT}(z,z_1)\right]}{\left(\frac{1}{z}-\frac{1}{z_1}\right)^2}, \tag{35}$$

and this is a new LIR for tensor-polarized FFs. If we integrate Eq. (34) over the momentum fraction z, one can have

$$\begin{aligned}F_{LT}(z) = -\frac{3z}{2}\int_z^1 dz_1\frac{F_{1LL}(z_1)}{(z_1)^2} + z\int_z^1\frac{dz_1}{z_1}\int_{z_1}^\infty\frac{dz_2}{(z_2)^2}\Bigg\{&\frac{\left[1+\frac{1}{z_1}\delta\left(\frac{1}{z_1}-\frac{1}{z}\right)\right]\mathrm{Re}\left[\hat{G}_{LT}(z_1,z_2)\right]}{\frac{1}{z_1}-\frac{1}{z_2}}\\ &-\frac{\left[\frac{3}{z_1}-\frac{1}{z_2}+\frac{1}{z_1}\left(\frac{1}{z_1}-\frac{1}{z_2}\right)\delta\left(\frac{1}{z_1}-\frac{1}{z}\right)\right]\mathrm{Re}\left[\hat{F}_{LT}(z_1,z_2)\right]}{\left(\frac{1}{z_1}-\frac{1}{z_2}\right)^2}\Bigg\},\end{aligned} \tag{36}$$

where it should be understood that z_1 falls within the range of integration $(z,1)$, namely, $\int_z^1 dz_1 F(z_1)\delta(1/z_1 - 1/z) = z^2F(z)$. The intrinsic twist-3 FF $F_{LT}(z)$ is decomposed into the contributions of a twist-2 FF F_{1LL} and the dynamical FFs. We can obtain a similar expression for the kinematical twist-3 FF $F_{1LT}^{(1)}(z)$ by inserting Eq. (36) into the e.o.m. relation of Eq. (23),

$$F_{1LT}^{(1)}(z) = \frac{3}{2}\int_z^1 dz_1\frac{F_{1LL}(z_1)}{(z_1)^2} + \int_z^1\frac{dz_1}{z_1}\int_{z_1}^\infty\frac{dz_2}{(z_2)^2}\left\{\frac{\left(\frac{3}{z_1}-\frac{1}{z_2}\right)\mathrm{Re}\left[\hat{F}_{LT}(z_1,z_2)\right]}{\left(\frac{1}{z_1}-\frac{1}{z_2}\right)^2} - \frac{\mathrm{Re}\left[\hat{G}_{LT}(z_1,z_2)\right]}{\frac{1}{z_1}-\frac{1}{z_2}}\right\}. \tag{37}$$

By dropping the contributions of dynamical FFs into Eqs. (36) and (37), they become the WW-type relations. Then, the twist-3 intrinsic and kinematical FFs can be estimated by using the twist-2 FF $F_{1LL}(z)$, and the latter should be much easier to be extracted from experimental measurements compared with the former.

If we choose $\Gamma = \sigma^{\mu\alpha}$ in Eq. (28) and $\Gamma_1 = 1$ in Eq. (29), the following identity can be derived [9]:

$$\begin{aligned}\frac{\partial}{\partial\xi_\alpha}\langle 0|q(-\xi)|P_h,T;X\rangle\langle P_h,T;X|\bar{q}(\xi)\sigma^{\xi\alpha}|0\rangle = &\int_{-1}^\infty dtt\langle 0|igF_{\alpha\xi}(t\xi)q(-\xi)|P_h,T;X\rangle\langle P_h,T;X|\bar{q}(\xi)\sigma^{\xi\alpha}|0\rangle\\ &+\int_\infty^1 dtt\langle 0|q(-\xi)|P_h,T;X\rangle\langle P_h,T;X|\bar{q}(\xi)\sigma^{\xi\alpha}igF_{\alpha\xi}(t\xi)|0\rangle.\end{aligned} \tag{38}$$

Similarly, the matrix element of the nonlocal operator $\bar{q}(\xi)\sigma^{\xi\sigma}q(-\xi)$ is expressed in terms of the FFs $\tilde{H}_{1LT}(z)$ and $\tilde{H}_{LT}(z)$ at twist 3,

$$\langle 0|q(-\xi)|P_h,T;X\rangle\langle P_h,T;X|\bar{q}(\xi)\sigma^{\xi\sigma}|0\rangle = -4N_cM\int d\left(\frac{1}{z}\right)e^{\frac{2iP\xi}{z}}\left[2(W^\sigma+V^\sigma)\tilde{H}_{1LT}(z)+\frac{3}{2}V^\sigma\tilde{H}_{LL}(z)\right], \tag{39}$$

where W^σ and V^σ are defined as

$$W^\sigma = T^{\sigma\mu}\xi_\mu - \frac{\xi\cdot T\cdot\xi}{P_h\cdot\xi}P_h^\sigma, \tag{40}$$

$$V^\sigma = M^2\frac{\xi\cdot T\cdot\xi}{(P_h\cdot\xi)^2}\left[\xi^\sigma - \frac{\xi^2}{P_h\cdot\xi}P_h^\sigma\right], \tag{41}$$

and they satisfy the relations of $W\cdot\xi=0$ and $V\cdot\xi=0$. In the light-cone limit $\xi^2\to 0$, Eq. (39) goes back to Eq. (5). We obtain the following identity by calculating the matrix element in Eq. (38):

$$4\tilde{H}_{1LT}(z) - \frac{1}{z^2}\frac{dH_{LL}(z)}{d(1/z)} = -2\int d\left(\frac{1}{z_1}\right)\mathcal{P}\left(\frac{1}{\frac{1}{z}-\frac{1}{z_1}}\right)\left(\frac{\partial}{\partial(1/z)}-\frac{\partial}{\partial(1/z_1)}\right)\mathrm{Im}\left[\tilde{H}_{LL}^{\perp}(z,z_1)\right]. \tag{42}$$

Moreover, we obtain $d(\tilde{H}_{LL}(z)/z)/d(1/z)$ by using the expression in Eq. (20), and the sum of $d(\tilde{H}_{LL}(z)/z)/d(1/z)$ and Eq. (42) leads to

$$\tilde{H}_{LL}(z) + 2\tilde{H}_{1LT}(z) + \left(1 - z\frac{d}{dz}\right)H_{1LL}^{(1)}(z) = -2\int_z^\infty \frac{dz_1}{(z_1)^2}\frac{\mathrm{Im}\left[\tilde{H}_{LL}^{\perp}(z,z_1)\right]}{\left(\frac{1}{z}-\frac{1}{z_1}\right)^2}, \tag{43}$$

which is also a LIR for tensor-polarized FFs. The integration of Eq. (42) gives

$$H_{LL}(z) = 4\int_z^1 dz_1\frac{H_{1LT}(z_1)}{z_1} + 4\int_z^1 dz_1\int_{z_1}^\infty \frac{dz_2}{(z_2)^2}\frac{\frac{2}{z_1}-\frac{1}{z_2}+\frac{1}{2z_1}\left(\frac{1}{z_1}-\frac{1}{z_2}\right)\delta\left(\frac{1}{z_1}-\frac{1}{z}\right)}{\left(\frac{1}{z_1}-\frac{1}{z_2}\right)^2}\mathrm{Im}\left[\hat{H}_{LL}^{\perp}(z_1,z_2)\right], \tag{44}$$

and the intrinsic twist-3 FF $H_{LL}(z)$ is expressed in terms of the twist-2 FF $H_{1LT}(z)$ and the dynamical FF $\hat{H}_{LL}^{\perp}(z_1,z_2)$. If we combine Eq. (44) with the e.o.m. relation of Eq. (18),

$$H_{1LL}^{(1)}(z) = -\frac{2}{z}\int_z^1 dz_1\frac{H_{1LT}(z_1)}{z_1} - \frac{2}{z}\int_z^1 dz_1\int_{z_1}^\infty \frac{dz_2}{(z_2)^2}\frac{\frac{2}{z_1}-\frac{1}{z_2}}{\left(\frac{1}{z_1}-\frac{1}{z_2}\right)^2}\mathrm{Im}\left[\hat{H}_{LL}^{\perp}(z_1,z_2)\right], \tag{45}$$

which also decomposes the kinematical twist-3 FF $H_{1LL}^{(1)}(z)$ into the contributions of $H_{1LT}(z)$ and $\hat{H}_{LL}^{\perp}(z_1,z_2)$. We can obtain the WW-type relations for $H_{LL}(z)$ and $H_{1LL}^{(1)}(z)$ by dropping the terms of the dynamical FF in Eqs. (44) and (45).

If we consider the matrix elements of Eqs. (28) and (29) with $\Gamma=\epsilon^{\alpha\mu\rho S_{LT}}\gamma_\mu\gamma_5$ and $\Gamma_1=\frac{i}{2}(\gamma^\rho\sigma^{S_{LT}\xi}-\sigma^{S_{LT}\xi}\gamma^\rho)$, respectively, one can derive

$$\begin{aligned}
&\epsilon^{\alpha\mu\rho S_{LT}}\xi_\rho\frac{\partial}{\partial\xi^\alpha}\langle 0|q(-\xi)|P_h,T;X\rangle\langle P_h,T;X|\bar{q}(\xi)\gamma_\mu\gamma_5|0\rangle\\
&= \int_{-1}^\infty dt\langle 0|gF_{\xi S_{LT}}(t\xi)q(-\xi)|P_h,T;X\rangle\langle P_h,T;X|\bar{q}(\xi)\not{\xi}|0\rangle + \int_\infty^1 dt\langle 0|q(-\xi)|P_h,T;X\rangle\langle P_h,T;X|\bar{q}(\xi)\not{\xi}gF_{\xi S_{LT}}(t\xi)|0\rangle\\
&\quad + i\epsilon^{\alpha\mu\xi S_{LT}}\left[\int_{-1}^\infty dtt\langle 0|gF_{\alpha\xi}(t\xi)q(-\xi)|P_h,T;X\rangle\langle P_h,T;X|\bar{q}(\xi)\gamma_\mu\gamma_5|0\rangle + \int_\infty^1 dtt\langle 0|q(-\xi)|P_h,T;X\rangle\right.\\
&\quad\left.\times\langle P_h,T;X|\bar{q}(\xi)\gamma_\mu\gamma_5 gF_{\alpha\xi}(t\xi)|0\rangle\right] - 2im_q\langle 0|q(-\xi)|P_h,T;X\rangle\langle P_h,T;X|\bar{q}(\xi)\sigma^{\xi S_{LT}}|0\rangle,
\end{aligned} \tag{46}$$

and the left-hand side is related to the matric element of $\bar{q}(\xi)\gamma^\mu\gamma_5 q(-\xi)$, which is given by

$$\langle 0|q(-\xi)|P_h,T;X\rangle\langle P_h,T;X|\bar{q}(\xi)\gamma^\mu\gamma_5|0\rangle = 4N_c M\frac{\epsilon^{\mu\xi\alpha P_h}}{P_h\cdot\xi}Y_\alpha\int d\left(\frac{1}{z}\right)e^{\frac{2iP\xi}{z}}\tilde{G}_{LT}(z), \tag{47}$$

and the vector Y is defined as

$$Y^\alpha = \frac{2M}{P_h\cdot\xi}\left[T^{\alpha\mu}\xi_\mu - \frac{\xi\cdot T\cdot\xi}{P_h\cdot\xi}P_h^\alpha + M^2\frac{\xi\cdot T\cdot\xi}{(P_h\cdot\xi)^2}\left(\xi^\alpha - \frac{\xi^2}{P_h\cdot\xi}P_h^\alpha\right)\right]. \tag{48}$$

If we take the light-cone limit $\xi^2\to 0$, one can obtain $Y^\alpha\to S_{LT}^\alpha$. Thus, Eq. (46) leads to the following identity:

$$\begin{aligned}\frac{1}{z}\frac{d\tilde{G}_{LT}(z)}{d(1/z)} + \frac{m_q}{M}\frac{d\tilde{H}_{1LT}(z)}{d(1/z)} = \int d\left(\frac{1}{z_1}\right)\mathcal{P}\left(\frac{1}{\frac{1}{z}-\frac{1}{z_1}}\right)\Bigg\{&\left(\frac{\partial}{\partial(1/z)}-\frac{\partial}{\partial(1/z_1)}\right)\mathrm{Im}\left[\tilde{G}_{LT}(z,z_1)\right] \\ &-\left(\frac{\partial}{\partial(1/z)}+\frac{\partial}{\partial(1/z_1)}\right)\mathrm{Im}\left[\tilde{F}_{LT}(z,z_1)\right]\Bigg\}.\end{aligned} \tag{49}$$

Combining Eq. (49) with the e.o.m. relation of Eq. (24), another LIR can be derived for tensor-polarized FFs,

$$\tilde{G}_{LT}(z) + \left(1 - z\frac{d}{dz}\right)G_{1LT}^{(1)}(z) = -2\int_z^\infty \frac{dz_1}{(z_1)^2}\frac{\mathrm{Im}\left[\tilde{G}_{LT}(z,z_1)\right]}{\left(\frac{1}{z}-\frac{1}{z_1}\right)^2}, \tag{50}$$

and the quark mass term in Eq. (49) is canceled in this LIR. From Eqs. (24) and (49), one can also express the twist-3 FFs $G_{LT}(z)$ and $G_{1LT}^{(1)}(z)$ in terms of $H_{1LT}(z)$, $\hat{F}_{LT}(z_1,z_2)$, and $\hat{G}_{LT}(z_1,z_2)$,

$$\begin{aligned}G_{LT}(z) = &-\frac{m_q}{M}\left[zH_{1LT}(z) + z\int_z^1 dz_1\frac{H_{1LT}(z_1)}{z_1}\right] - z\int_z^1\frac{dz_1}{z_1}\int_{z_1}^\infty\frac{dz_2}{(z_2)^2}\Bigg\{\frac{\left[1+\frac{1}{z_1}\delta\left(\frac{1}{z_1}-\frac{1}{z}\right)\right]\mathrm{Im}\left[\hat{F}_{LT}(z_1,z_2)\right]}{\frac{1}{z_1}-\frac{1}{z_2}} \\ &-\frac{\left[\frac{3}{z_1}-\frac{1}{z_2}+\frac{1}{z_1}\left(\frac{1}{z_1}-\frac{1}{z_2}\right)\delta\left(\frac{1}{z_1}-\frac{1}{z}\right)\right]\mathrm{Im}\left[\hat{G}_{LT}(z_1,z_2)\right]}{\left(\frac{1}{z_1}-\frac{1}{z_2}\right)^2}\Bigg\},\end{aligned} \tag{51}$$

$$G_{1LT}^{(1)}(z) = \frac{m_q}{M}\int_z^1 dz_1\frac{H_{1LT}(z_1)}{z_1} + \int_z^1\frac{dz_1}{z_1}\int_{z_1}^\infty\frac{dz_2}{(z_2)^2}\left\{\frac{\mathrm{Im}\left[\hat{F}_{LT}(z_1,z_2)\right]}{\frac{1}{z_1}-\frac{1}{z_2}} - \frac{\left(\frac{3}{z_1}-\frac{1}{z_2}\right)\mathrm{Im}\left[\hat{G}_{LT}(z_1,z_2)\right]}{\left(\frac{1}{z_1}-\frac{1}{z_2}\right)^2}\right\}. \tag{52}$$

If we consider the production of a tensor-polarized hadron h in the lepton-nucleon collision, namely $l+N\to h+X$, the twist-3 cross sections are dependent on the chosen frame, which is induced by the arbitrariness in the choice of light-cone vectors for distribution and fragmentation correlators. The LIRs we derive can be used to remove the frame dependence of the twist-3 cross sections for this process, such as twist-3 SSAs and double-spin asymmetries.

V. SUMMARY

The tensor-polarized FFs of a spin-1 hadron (h) can be measured in the various hard semi-inclusive processes such as $e^+e^-\to hX$ and $ep\to ehX$ (semi-inclusive deep inelastic scattering), and the former process is accessible at BESIII and Belle II, while the latter is possible at JLab and the Electron-Ion Colliders in the U.S. and China. Inspired by the ongoing measurement of the tensor-polarized FFs for ϕ at BESIII, we investigate the theoretical relations among the tensor-polarized intrinsic, kinematical, and dynamical FFs for a spin-1 hadron in this work. First, the QCD e.o.m. relations are obtained for the tensor-polarized FFs. Second, we derive the operator identities where the nonlocal quark-quark operators are expressed in terms of quark-gluon-quark operators. Three new Lorentz invariance relations (LIRs) are presented for the tensor-polarized FFs, and they can be used to remove the frame dependence of the twist-3 spin observables in the hard semi-inclusive reactions so that Lorentz invariance properties are satisfied. Finally, we also show that the intrinsic and kinematical twist-3 FFs are expressed in terms of the twist-2 FFs and the dynamical twist-3 FFs, and the Wandzura-Wilczek-type relations are obtained by neglecting the dynamical FFs. Since the twist-2 FFs are much easier to be accessed in experiment than the twist-3 ones, one can give a rough estimate for the twist-3 FFs by such relations. Our results will be valuable for the future experimental measurements and theoretical studies of tensor-polarized FFs.

ACKNOWLEDGMENTS

We acknowledge useful discussions with Shunzo Kumano, Bernard Pire, Ji Xu, and Ya-Teng Zhang. Q.-T. S. was supported by the National Natural Science Foundation of China under Grant No. 12005191.

[1] S. Wandzura and F. Wilczek, Phys. Lett. **72B**, 195 (1977).
[2] A. Accardi, A. Bacchetta, W. Melnitchouk, and M. Schlegel, J. High Energy Phys. 11 (2009) 093.
[3] P. J. Mulders and R. D. Tangerman, Nucl. Phys. **B461**, 197 (1996); **B484**, 538(E) (1997).
[4] A. V. Belitsky, Int. J. Mod. Phys. A **32**, 1730018 (2017).
[5] R. D. Tangerman and P. J. Mulders, arXiv:hep-ph/9408305.
[6] R. Kundu and A. Metz, Phys. Rev. D **65**, 014009 (2002).
[7] K. Goeke, A. Metz, P. V. Pobylitsa, and M. V. Polyakov, Phys. Lett. B **567**, 27 (2003).
[8] A. Metz, P. Schweitzer, and T. Teckentrup, Phys. Lett. B **680**, 141 (2009).
[9] K. Kanazawa, Y. Koike, A. Metz, D. Pitonyak, and M. Schlegel, Phys. Rev. D **93**, 054024 (2016).
[10] Y. Koike, K. Yabe, and S. Yoshida, Phys. Rev. D **101**, 054017 (2020).
[11] Y. Koike, A. Metz, D. Pitonyak, K. Yabe, and S. Yoshida, Phys. Rev. D **95**, 114013 (2017).
[12] Y. Koike, K. Yabe, and S. Yoshida, Phys. Rev. D **104**, 054023 (2021).
[13] P. Hoodbhoy, R. L. Jaffe, and A. Manohar, Nucl. Phys. **B312**, 571 (1989).
[14] L. L. Frankfurt and M. I. Strikman, Nucl. Phys. **A405**, 557 (1983).
[15] R. L. Jaffe and A. Manohar, Phys. Lett. B **223**, 218 (1989).
[16] M. Nzar and P. Hoodbhoy, Phys. Rev. D **45**, 2264 (1992).
[17] F. E. Close and S. Kumano, Phys. Rev. D **42**, 2377 (1990).
[18] A. Airapetian *et al.* (HERMES Collaboration), Phys. Rev. Lett. **95**, 242001 (2005).
[19] W. Cosyn, Y. B. Dong, S. Kumano, and M. Sargsian, Phys. Rev. D **95**, 074036 (2017).
[20] G. A. Miller, Phys. Rev. C **89**, 045203 (2014).
[21] J. P. Ma, C. Wang, and G. P. Zhang, arXiv:1306.6693.
[22] S. Kumano and Q. T. Song, Phys. Rev. D **101**, 054011 (2020).
[23] S. Kumano and Q. T. Song, Phys. Rev. D **101**, 094013 (2020).
[24] J.-P. Chen *et al.*, Proposal to Jefferson Lab PAC-38, PR12-11-110 (2011), https://www.jlab.org/exp_prog/proposals/11/PR12-11-110.pdf.
[25] M. Jones *et al.*, A letter of intent to Jefferson Lab PAC 44, LOI12-16-006 (2016), https://www.jlab.org/exp_prog/proposals/16/LOI12-16-006.pdf.
[26] D. Keller, D. Crabb, and D. Day, Nucl. Instrum. Methods Phys. Res., Sect. A **981**, 164504 (2020).
[27] D. Keller (SpinQuest Collaboration), arXiv:2205.01249.
[28] J. Clement and D. Keller, Nucl. Instrum. Methods Phys. Res., Sect. A **1050**, 168177 (2023).
[29] A. Arbuzov, A. Bacchetta, M. Butenschoen, F. G. Celiberto, U. D'Alesio, M. Deka, I. Denisenko, M. G. Echevarria, A. Efremov, N. Y. Ivanov *et al.*, Prog. Part. Nucl. Phys. **119**, 103858 (2021).
[30] S. Kumano and Q. T. Song, J. High Energy Phys. 09 (2021) 141.
[31] S. Kumano and Q. T. Song, Phys. Lett. B **826**, 136908 (2022).
[32] W. Cosyn and B. Pire, Phys. Rev. D **98**, 074020 (2018).
[33] A. Bacchetta and P. J. Mulders, Phys. Rev. D **62**, 114004 (2000).
[34] D. Boer, S. Cotogno, T. van Daal, P. J. Mulders, A. Signori, and Y. J. Zhou, J. High Energy Phys. 10 (2016) 013.
[35] Y. Ninomiya, W. Bentz, and I. C. Cloët, Phys. Rev. C **96**, 045206 (2017).
[36] S. Kumano and Qin-Tao Song, Phys. Rev. D **103**, 014025 (2021).
[37] X. D. Ji, Phys. Rev. D **49**, 114 (1994).
[38] K. B. Chen, W. H. Yang, S. Y. Wei, and Z. T. Liang, Phys. Rev. D **94**, 034003 (2016).
[39] Dr. Ya-Teng Zhang on the measreument of tensor-polarized FFs for ϕ in $e^+e^- \to \phi X$ (private communication).
[40] J. C. Collins and D. E. Soper, Nucl. Phys. **B194**, 445 (1982).
[41] J. C. Collins, Nucl. Phys. **B396**, 161 (1993).
[42] D. Boer, P. J. Mulders, and F. Pijlman, Nucl. Phys. **B667**, 201 (2003).

[43] A. Bacchetta, M. Diehl, K. Goeke, A. Metz, P. J. Mulders, and M. Schlegel, J. High Energy Phys. 02 (2007) 093.

[44] A. Metz and A. Vossen, Prog. Part. Nucl. Phys. **91**, 136 (2016).

[45] K. B. Chen, T. Liu, Y. K. Song, and S. Y. Wei, Particles **6**, 515 (2023).

[46] S. Meissner and A. Metz, Phys. Rev. Lett. **102**, 172003 (2009).

[47] A. Metz and D. Pitonyak, Phys. Lett. B **723**, 365 (2013); **762**, 549(E) (2016).

[48] I. I. Balitsky and V. M. Braun, Nucl. Phys. **B311**, 541 (1989).

[49] I. I. Balitsky, V. M. Braun, and A. V. Kolesnichenko, Nucl. Phys. **B312**, 509 (1989).

[50] I. I. Balitsky and V. M. Braun, Nucl. Phys. **B361**, 93 (1991).

[51] P. Ball and M. Lazar, Phys. Lett. B **515**, 131 (2001).

[52] P. Ball and V. M. Braun, Nucl. Phys. **B543**, 201 (1999).

[53] P. Ball and V. M. Braun, Phys. Rev. D **54**, 2182 (1996).

[54] V. M. Braun and I. E. Filyanov, Z. Phys. C **48**, 239 (1990).

[55] J. Kodaira and K. Tanaka, Prog. Theor. Phys. **101**, 191 (1999).

[56] H. Eguchi, Y. Koike, and K. Tanaka, Nucl. Phys. **B752**, 1 (2006).