Hadronic structure on the light front. IX. Orbital-spin-isospin wave functions of baryons

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This paper, which is part of a series, is devoted to several technical issues. In the first part of the paper, we discuss the usual wave functions in the center of mass frame for baryons, by clarifying the representations of the three-quark permutation group S_3. We extend the analysis for up to five "spinors" with ρ, λ-symmetry, and derive explicitly the totally symmetric wave functions modulo color. They are explicitly used to describe the excited nucleons N^* states, in the P- and D-shell. We also show how to use symbolic operations in *Mathematica*, in spin-tensor notations to make explicit these states. For the S- and P-shells, the totally antisymmetric wave functions are given, and the pertinent matrix elements for the spin-dependent operators calculated, including the mixing between states with different total spin S. In the second part of the paper we turn to the light front wave functions, with an emphasis on the longitudinal wave functions, with a novel basis set. We also discuss their symmetries under permutations, and select the proper combinations for the transverse and longitudinal excitations for N^* on the light front.

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I. INTRODUCTION

The nonrelativistic quark model, which originated in the 1960s, was first based on spin-flavor $SU(6)$ symmetry. Using its multiplets it was possible to define the baryon wave functions for the lowest S-shell, and calculate their quantum numbers and important properties, e.g. magnetic moments. With the advent of QCD and quarkonia in the 1970s, perturbative and confining forces between quarks were added, including spin-dependent interactions, leading to a qualitatively successful spectroscopy. One important simplification was the use of quadratic (oscillatory) confinement, see, e.g., [1,2].

The extension of this theory to excited baryons, especially to the P-shell with negative parity [3], revealed a number of issues. To simplify some of the symmetry issues for three-light-quark states qqq, they focused on the mixed qqs states (Σ, Λ) ones, and then discussed the light quark limit $m_s \to m_{u,d}$.

They have shown that the perturbative predictions from the one-gluon exchange worked qualitatively well only for spin-spin and tensor forces, while the spin-orbit calculated in the same approximation was absent. A key shortcoming at the time, was the missing D-shell states predicted but not observed ("missing resonances problem"). Further refinements, including in particular the inclusion of relativized quarks, were made by Capstick and Isgur [4]. With time, the issue of "missing resonances" faded away, as the states in the second and the third shells were nearly all observed.

More recent quarks models for mesons and baryons were defined using light front variables, with light front wave functions (LFWFs). The advantage of the light front, is the potential to relate the LFWFs to partonic observables, DAs, PDFs, GPDs etc. This formulation, treats democratically heavy and light quarks, with no need for a nonrelativitic approximation. The LFWFs for hadrons made of light and heavy quarks are developed in the same setting, which is very convenient for the discussion of the multiquark hadrons discovered in the last decade.

The first step for the description of the LFWFs, has been made by Ji *et al.* [5], who have classified the components of the spin-up proton in terms of spin/orbital helicities. In this formulation the wave functions of the 3-quark sector are multicomponent, e.g., the (ground shell) nucleon is ascribed to possess six unrelated functions. For excited baryon states the issues related with required quark permutation symmetry were not resolved. Also, the LFWFs for baryonic excitations with nontrivial orbital-spin-isospin wave functions, has not been addressed to our knowledge.

The aims of this technical paper are twofold. First, we wish to clarify how to build symmetric representations of

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quark permutation symmetry S_3, as it follows from quark Fermi statistics. There is no need to take the indirect root via strange baryons again and again, we rederive the explicit wave functions for the ground S-shell, negative parity P-shell and D-shell nucleons. We propose to use standard spin-tensor forms with many components, conveniently manipulated symbolically in *Mathematica*. We then turn to the derivation of the LFWFs, clarifying their orbital-spin-isospin structure.

Our other aim is to use the available phenomenological information on nucleon excitations, and translate it into matrix elements of the central and spin-dependent interquark forces. Since the 1970's, spectroscopists still use the one-gluon exchange forces plus linear potential due to the electric flux tubes. But even for heavy quarkonia, we found in [6], that its central potential can be derived from instanton-based model of the vacuum fields, with the benefit of also predicting spin-dependent nonperturbative forces, at least comparable in magnitude to the perturbative ones. In [7] we calculated the instanton-induced quark-quark forces in various baryons, and compared them to the empirical and lattice data.

In our next publication we will continue along this line, focusing on interquark forces in baryons. While we start with generic three-Wilson-lines, the outcome are binary interactions only. We will also generalize our approach to spin-dependent forces. All of that can also be further generalized to the recently discovered multiquark hadrons.

II. LIGHT QUARK BARYONS IN THE c.m. FRAME

The physics of baryon resonances is a very broad field. It is also rather old. It started in the 1950s with discoveries of the first N^* and Δ resonances. The QCD-based quark models are also about 50 years old. The systematic analysis of the data are in the Particle Data Tables, for recent review see, e.g., [8]. Basically, it appears that the "missing resonance" problem is solved, and the states belonging to the ground, second and third shells are all observed, with quantum numbers for nearly all of them determined.

So, the reader may think that all of the main questions in this field are answered, and doubt that anything useful (rather than new experimental data) can be added to it. In fact, the situation is far from that. The structure and especially the physical origin of the quark-quark forces are not yet understood, especially of the spin-dependent part. Lattice studies of the static spin-dependent potentials are sketchy, and in so far reduced to either $\bar{q}q$ channel or diquarks. These issues are now discussed again, in view of the recent renaissance in the hadronic spectroscopy, due to the discovery of many new multiquark hadrons.

The question whether spin-dependent splittings of baryons are due to perturbative one-gluon exchange, or instanton-induced 't Hooft Lagrangian has been discussed in [9]. Another proposed mechanism, via pion exchanges between quarks [10], appears as a higher order effect of the 't Hooft Lagrangian, as the pions themselves are mostly bound by it. (We will not include those, trying to avoid a double counting.) In this vain, it is worth mentioning also, the solitonic construction based on the large number of colors limit [11], which has been also extended to the exotics via the holographic principle [12,13].

In this work we would like to start with another reanalysis of the situation, with a phenomenological analysis of the data, mostly of positive parity (ground shell) and negative parity (second shell) light baryons and discuss whether the existing data can accommodate certain matrix elements of all spin-depending operators, namely spin-spin, spin-orbit, tensor as well as instanton-induced 't Hooft operator.

In order to see the importance of various spin-dependent forces in action, it is *not* sufficient to look at the lowest shell baryons, N and Δ, as there is basically just one mass difference between them. To reach a quantitative understanding of the magnitude of the spin-dependent forces, we need to look at more states. An important step in this direction was made by Isgur and Karl (IK) [1] who focused on the P-shell negative parity baryons. The Jacobi coordinates $\vec{\rho}$, $\vec{\lambda}$ are antisymmetric and symmetric under permutation of the 1–2 quarks, which leads naturally to the use of blocks with the same ρ, λ structure also for the orbital, spin and isospin parts of the wave functions [14]. This observation was used extensively in the work of IK and their followers. IK focused on hyperons with a strange quark, for which the wave function with 12-symmetry was sufficient. The light quark baryons were then constructed as certain limits from the combination of hyperons, with the strange quark mass reduced to the light ones.

Unlike Isgur-Karl, we decided not to deal with extra parameters related with strangeness, and focus entirely on light quark states. As it is well known, negative parity u, d baryons belonging to the second $L = 1$ (or P-shell) have seven states, five N^* and two Δ^*. Five masses of the N^* and two matrix elements of mixing ($J = 3/2$ and $J = 1/2$ pairs with different spins S) are seven inputs. We will use the inverse logic adopted by most spectroscopists: instead of formulating a model and then comparing its prediction with various data, we decided to start with estimates of the *phenomenological values* of the matrix elements of several contributing operators. With the structure of the wave functions at hand, and spin-orbit-isospin structure of the operators known, one can write masses as linear combinations of matrix elements. There is enough information to fix those uniquely.

A. Excited nucleons, negative parity

We recall that the spin states of three quarks have 2^3 states: four of those belong to the $S = 3/2$ case, and the remaining two pairs of states have spin $S = 1/2$ but different symmetries under permutations. Adding orbital momentum $L = 1$ to the former set leads to $J = 5/2, 3/2, 1/2$ N^* states. In the latter case two $S = 1/2$

TABLE I. The second shell baryons made of light quarks. The original Isgur-Karl predictions [1] are compared to the experimental masses from RPP. The last column shows the masses after "unmixing" (see text).

States J^P	Isgur-Karl	Experiment	Unmixed
$N^*_{1/2^-}$	1490	1535	1567.3, $S=1/2$
$N^*_{1/2^-}$	1655	1650	1617.7, $S=3/2$
$N^*_{3/2^-}$	1535	1520	1521.97, $S=1/2$
$N^*_{3/2^-}$	1745	1700	1698.0, $S=3/2$
$N^*_{5/2^-}$	1670	1675	$S=3/2$

structures need to be combined with two isospin $I=1/2$ structures, to get a single combination with the correct permutation symmetry: this leads to another pair of $J=3/2,1/2$ of N^*. Two pairs of N^* from those two families, with the same J, are intermixed by the tensor forces.

We start our phenomenological discussion with Table I, where the current experimental masses for the P-shell states are recorded. We also compare these empirical masses with the original predictions by Isgur and Karl [1].

There are also mixing angles of the $S=3/2$ and $1/2$ states determined from the decays: those are listed in the RPP reviews as being $\theta_{S1/2}=-32°$, $\theta_{S3/2}=6°$. The relation of the energies of the mixed states to the unmixed ones is

$$M_\pm=\frac{1}{2}\left(M_1+M_2\pm\sqrt{4H^2_{\rm mix}+(M_1-M_2)^2}\right) \tag{1}$$

and the mixing angle to the nondiagonal mixing matrix element is

$$\tan\theta=\frac{2H_{\rm mix}}{M_1-M_2-\sqrt{4H^2_{\rm mix}+(M_1-M_2)^2}}. \tag{2}$$

We may use the values of the observed masses $M_\pm$, and the observed mixing angles to derive the "unmixed masses" $M_{1,2}$ with fixed total spin S, which we give in the last column in the Table I. Similarly, the numerical values of two mixing matrix elements are

$$\begin{aligned} H_{\rm mix}(J=1/2)&=51.7\ {\rm MeV}\\ H_{\rm mix}(J=3/2)&=-18.7\ {\rm MeV}. \end{aligned} \tag{3}$$

Below we will use five masses to get five matrix elements, and compare the value of the tensor matrix element to these two mixing matrix elements.

B. Permutation symmetry S_3 and three quark wave function

Our first task is to clarify the use of S_3 for the construction of baryon orbital-spin-isospin wave functions, which can be made uniquely. After which, we will use phenomenology to evaluate the matrix elements of all pertinent operators.

For completeness, we start with well known basic facts. Fermi statistics require antisymmetry over all permutations of quarks. With antisymmetric color wave function $\sim\epsilon_{ijk}$, the remainder of the wave function needs to be totally symmetric

$$\text{color}_A\times(\text{space}\times\text{flavor}\times\text{spin})_S. \tag{4}$$

For orbital momenta $L_i=0$ (S-shell) states, the traditional classification follows the representations of the spin-flavor $SU(6)$ group of the nonrelativistic quark model

$$\begin{aligned} 6\otimes(6\otimes 6)&=6\otimes(21_S\oplus 15_A)\\ &=(56_S\oplus 70_M)\oplus(20_A\oplus 70_M). \end{aligned} \tag{5}$$

The nucleon belongs to the mixed symmetry 8_M octet, which is part of the symmetric 56-plet. As we do not include strange hyperons, flavor $SU(3)$ is reduced to the isospin $SU(2)$ group. This helps to make spin and isospin notations more similar.

The three quarks coordinate vectors are compressed into two Jacobi coordinates

$$\vec\rho=\frac{1}{\sqrt{2}}\vec r_{12}\qquad \lambda=\frac{1}{\sqrt{6}}(\vec r_{13}+\vec r_{23}) \tag{6}$$

with $\vec r_{ij}=\vec r_i-\vec r_j$ and 3D vectors assumed. As noted by Isgur and Karl [3], simple symmetry properties of $\varphi^{\rho,\lambda}_{LM}$ under the permutation group, suggest a convenient set of basis functions, both in spin and isospin, out of which the baryon wave functions can be constructed. Most of the details are given in Appendix A, with only the main ideas presented here.

To compose the states with proper symmetry under the permutation group S_3, we recall that this group is composed of 6 elements

$$P_{i=1,\ldots,6}=I,(12),(13),(23),(123),(132). \tag{7}$$

It will be enough to enforce the symmetry under (12) and (23) permutations. Let us add that there are three Young tableau's for this group, symmetric, antisymmetric and mixed (two boxes in the top line and one below).

The Jacobi coordinates (6) transform under permutations as follows

$$[P_2=(12)]\begin{pmatrix}\rho\\ \lambda\end{pmatrix}=\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}\rho\\ \lambda\end{pmatrix}$$
$$[P_4=(23)]\begin{pmatrix}\rho\\ \lambda\end{pmatrix}=\begin{pmatrix}\frac{1}{2} & \frac{\sqrt{3}}{2}\\ \frac{\sqrt{3}}{2} & -\frac{1}{2}\end{pmatrix}\begin{pmatrix}\rho\\ \lambda\end{pmatrix}. \tag{8}$$

Thus ρ, λ doublet transformation under (12) is simply antisymmetric and symmetric, but it is of mixed symmetry under the (23) permutations. Therefore, the construction of the states out of the ρ or λ-type blocks (as done e.g. by Isgur and Karl [1]) allows for the easy removal of the contributions that are not (12) symmetric. However, it is not enough: one needs to enforce full S_3 symmetry. The corresponding matrices of (23) permutation for two and three quark is required (see the Appendix A), ultimately fixing the wave functions uniquely.

Let us start with the (well known) proton state. In the literature one can find it in several forms, using either products of blocks with different symmetries, or a sets of "monoms" (basic states):

$$\begin{aligned}&\left|J=\frac{1}{2},J_z=\frac{1}{2},I_z=\frac{1}{2}S_z=I_z=\frac{1}{2}\right\rangle_{p_M^+}\\ &=\frac{1}{\sqrt{18}}[(\uparrow\downarrow\uparrow-\downarrow\uparrow\uparrow)(udu-duu)+(\uparrow\uparrow\downarrow-\uparrow\downarrow\uparrow)(uud-udu)+(\uparrow\uparrow\downarrow-\downarrow\uparrow\uparrow)(uud-duu)]\\ &=\frac{1}{\sqrt{18}}[2(u^\uparrow d^\downarrow u^\uparrow)+2(d^\downarrow u^\uparrow u^\uparrow)+2(u^\uparrow u^\uparrow d^\downarrow)-(d^\uparrow u^\downarrow u^\uparrow)-(u^\downarrow d^\uparrow u^\uparrow)\\ &\quad-(u^\uparrow d^\uparrow u^\downarrow)-(d^\uparrow u^\uparrow u^\downarrow)-(u^\downarrow u^\uparrow d^\uparrow)-(u^\uparrow u^\downarrow d^\uparrow)].\end{aligned} \tag{9}$$

Its spin and isospin components have mixed symmetry, but taken together, they are symmetric under all permutations. (Among its classic applications are the neutron-to-proton ratio of magnetic moments $-2/3$, etc.) But in order to derive the wave functions for the baryon excitations, we need a more systematic approach.

There are $2^3=8$ spin (or isospin) states of three quarks. Four of those correspond to total spin $S=3/2$, and four other to $S=1/2$. It is convenient to split them according to their symmetry under (12). Examples of such doublets are the spin $-\frac{1}{2}$ states

$$S^\rho_{\frac{11}{22}}=\frac{1}{\sqrt{2}}(\uparrow\downarrow-\downarrow\uparrow)\uparrow$$
$$S^\lambda_{\frac{11}{22}}=-\frac{1}{\sqrt{6}}(\uparrow\downarrow\uparrow+\downarrow\uparrow\uparrow-2\uparrow\uparrow\downarrow). \tag{10}$$

Similar isospin-$\frac{1}{2}$ states are defined as

$$F^\rho_{\frac{11}{22}}=\frac{1}{\sqrt{2}}(ud-du)u$$
$$F^\lambda_{\frac{11}{22}}=-\frac{1}{\sqrt{6}}(udu+duu-2uud). \tag{11}$$

The $S=\frac{3}{2}$ (or $I=\frac{3}{2}$) are fully symmetric

$$S^S_{\frac{3}{2}m}=\left(\uparrow\uparrow\uparrow,\frac{1}{\sqrt{3}}(\uparrow\uparrow\downarrow+\text{perm}),\frac{1}{\sqrt{3}}(\uparrow\downarrow\downarrow+\text{perm}),\downarrow\downarrow\downarrow\right). \tag{12}$$

To streamline the states with S_3 symmetry $R=S,A,M$ (symmetric, antisymmetric, mixed), we introduce the spectroscopic notation

$$|LSJm\rangle_{X^P_R}$$

with the hadronic label X^P_R for $X=p,\Delta,\ldots$ with parity $P=\pm$, and $L,S,J,J_z=m$ for orbital, spin, and total angular momentum with projection $J_z=m$, respectively.

The simplest baryon Δ have $S=I=\frac{3}{2}$ and both spin and isospin wave functions are symmetric S^S constructions (12)

$$\left|0\frac{3}{2}\frac{3}{2}m\right\rangle_{\Delta_S^+}=C_A\varphi_{00}S^S_{\frac{3}{2}m}F^S_{\frac{3}{2}} \tag{13}$$

The proton ground state WF should have spin-isospin $S=I=\frac{1}{2}$, so it should be constructed out of (II B). There are four combinations. Two of them $S^\rho F^\lambda$, $S^\lambda F^\rho$ are asymmetric under (12) and should be rejected. Two others are symmetric: any combinations of those is symmetric under (12). To fix the wave function uniquely, we need to calculate their transformation under (23) permutations. As shown in the Appendix, the only combination symmetric under both (12) and (23) is

$$\left|0\frac{1}{2}\frac{1}{2}m\right\rangle_{p_M^+}\sim\varphi_{00}\frac{1}{\sqrt{2}}(S^\rho_{\frac{1}{2}m}F^\rho_{\frac{1}{2}}+S^\lambda_{\frac{1}{2}m}F^\lambda_{\frac{1}{2}}) \tag{14}$$

where the radial wave function $(\rho^2+\lambda^2)$ depends on 6d "hyperdistance" $\sim(\rho^2+\lambda^2)$

Let us now proceed to the more general case of excited baryons, with the orbital part of the wave function included. In general, there could be angular functions depending on the angles of both vectors $\vec{\rho}$, $\vec{\lambda}$, but for the P-shell we are interested in, it is either $L_\rho = 1, L_\lambda = 0$ or $L_\lambda = 1, L_\rho = 0$. So, for the spatial WFs $\varphi^{\rho,\lambda}_{LM}$ with orbital content of mixed symmetry under S_3, we define

$$\varphi^\rho_{1m} = (\rho_-, \sqrt{2}\rho_z, -\rho_+)\varphi_{00} \equiv z^\rho_m \varphi_{00}$$
$$\varphi^\lambda_{1m} = (\lambda_-, \sqrt{2}\lambda_z, -\lambda_+)\varphi_{00} \equiv z^\lambda_m \varphi_{00} \tag{15}$$

with $\rho_\pm = (\rho_1 \pm i\rho_2)$ and $\lambda_\pm = (\lambda_1 \pm i\lambda_2)$. Up to module of the two vectors, they are the standard angular functions Y^m_1 with $m = -1, 0, +1$.

They combine with the possible spins to give 5 nucleons negative parity proton excited states,

$$J^P = L^P \oplus S^P = 1^- \otimes \left(\frac{1^+}{2}, \frac{3^+}{2}\right) \to 2 \times \left(\frac{1^-}{2}, \frac{3^-}{2}\right), \frac{5^-}{2} \tag{16}$$

i.e., two $\frac{1}{2}^-$, two $\frac{3}{2}^-$, and one $\frac{5}{2}^-$. Yet to construct the wave functions, we need to include also the isospin in a proper way. Modulo color, the baryon wave function has three components, orbital, spin, and isospin. (Not to be confused with the number of quarks: those will be there for any quark number.) As shown in Appendix A, three objects are combined together into a unique combination (A10).

1. 3-quarks with $S = \frac{3}{2}$ with $J = \frac{5}{2}, \frac{3}{2}, \frac{1}{2}$

The totally antisymmetrized wave function for 3 quarks with the maximally stretched spin are

$$\left|1\frac{3}{2}\frac{5}{2}\frac{5}{2}\right\rangle_{p^-_M} = C_A S^S_{\frac{33}{22}} \frac{1}{\sqrt{2}}(F^\rho_{\frac{1}{2}}\varphi^\rho_{11} + F^\lambda_{\frac{1}{2}}\varphi^\lambda_{11}) \tag{17}$$

$$\left|1\frac{3}{2}\frac{3}{2}\frac{3}{2}\right\rangle_{p^-_M} = C_A\left(\sqrt{\frac{3}{5}}S^S_{\frac{33}{22}}\frac{1}{\sqrt{2}}(F^\rho_{\frac{1}{2}}\varphi^\rho_{10} + F^\lambda_{\frac{1}{2}}\varphi^\lambda_{10}) - \sqrt{\frac{2}{5}}S^S_{\frac{31}{22}}\frac{1}{\sqrt{2}}(F^\rho_{\frac{1}{2}}\varphi^\rho_{11} + F^\lambda_{\frac{1}{2}}\varphi^\lambda_{11})\right)$$
$$\left|1\frac{3}{2}\frac{1}{2}\frac{1}{2}\right\rangle_{p^-_M} = C_A\left(-\frac{1}{\sqrt{2}}S^S_{\frac{33}{22}}\frac{1}{\sqrt{2}}(F^\rho_{\frac{1}{2}}\varphi^\rho_{1-1} + F^\lambda_{\frac{1}{2}}\varphi^\lambda_{1-1}) + \frac{1}{\sqrt{3}}S^S_{\frac{31}{22}}\frac{1}{\sqrt{2}}(F^\rho_{\frac{1}{2}}\varphi^\rho_{10} + F^\lambda_{\frac{1}{2}}\varphi^\lambda_{10}) - \frac{1}{\sqrt{6}}S^S_{\frac{3}{2}-\frac{1}{2}}\frac{1}{\sqrt{2}}(F^\rho_{\frac{1}{2}}\varphi^\rho_{11} + F^\lambda_{\frac{1}{2}}\varphi^\lambda_{11})\right) \tag{18}$$

with $S^S_{\frac{33}{22}} = \uparrow\uparrow\uparrow$ the totally symmetric 3-quark spin $\frac{3}{2}$ state.

In general, the addition of spin S states with orbital functions to a particular J, requires a sum of kinematically possible states, $m = J_z = m_S + m_L$, using the standard Clebsch-Gordon coefficients

$$\left|1\frac{3}{2}Jm\right\rangle_{p^-_M} = \sum_{m_S} \mathbf{C}^{Jm}_{1m_L\frac{3}{2}m_S}\left[C_A S^S_{\frac{3}{2}m_s}\frac{1}{\sqrt{2}}(F^\rho_{\frac{1}{2}}\varphi^\rho_{1m_L} + F^\lambda_{\frac{1}{2}}\varphi^\lambda_{1m_L})\right]. \tag{19}$$

Note that we use convention for the Clebsch-Gordon coefficients in terms of Wigner 3-j symbol

$$\mathbf{C}^{Jm}_{Lm_LSm_s} = (-1)^{S-L-m}\sqrt{2J+1}\begin{pmatrix} L & S & J \\ m_L & m_s & -m \end{pmatrix}. \tag{20}$$

2. 3-quarks in spin $\frac{1}{2}$ with $J = \frac{3}{2}, \frac{1}{2}$

The maximally stretched J-states for fixed L, S are

$$\left|1\frac{1}{2}\frac{3}{2}\frac{3}{2}\right\rangle_{p^-_M} = C_A\frac{1}{\sqrt{2}}\left(F^\rho_{\frac{1}{2}}\frac{1}{\sqrt{2}}(\varphi^\rho_{11}S^\lambda_{\frac{11}{22}} + \varphi^\lambda_{11}S^\rho_{\frac{11}{22}}) + F^\lambda_{\frac{1}{2}}\frac{1}{\sqrt{2}}(\varphi^\rho_{11}S^\rho_{\frac{11}{22}} - \varphi^\lambda_{11}S^\lambda_{\frac{11}{22}})\right)$$
$$\left|1\frac{1}{2}\frac{1}{2}\frac{1}{2}\right\rangle_{p^-_M} = C_A\left(\sqrt{\frac{2}{3}}\frac{1}{\sqrt{2}}\left(F^\rho_{\frac{1}{2}}\frac{1}{\sqrt{2}}(\varphi^\rho_{11}S^\lambda_{\frac{1}{2}-\frac{1}{2}} + \varphi^\lambda_{11}S^\rho_{\frac{1}{2}-\frac{1}{2}}) + F^\lambda_{\frac{1}{2}}\frac{1}{\sqrt{2}}(\varphi^\rho_{11}S^\rho_{\frac{1}{2}-\frac{1}{2}} - \varphi^\lambda_{11}S^\lambda_{\frac{1}{2}-\frac{1}{2}})\right)\right.$$
$$\left. - \sqrt{\frac{1}{3}}\frac{1}{\sqrt{2}}\left(F^\rho_{\frac{1}{2}}\frac{1}{\sqrt{2}}(\varphi^\rho_{10}S^\lambda_{\frac{1}{2}-\frac{1}{2}} + \varphi^\lambda_{10}S^\rho_{\frac{1}{2}-\frac{1}{2}}) + F^\lambda_{\frac{1}{2}}\frac{1}{\sqrt{2}}(\varphi^\rho_{10}S^\rho_{\frac{1}{2}-\frac{1}{2}} - \varphi^\lambda_{10}S^\lambda_{\frac{1}{2}-\frac{1}{2}})\right)\right) \tag{21}$$

$$\left|1\frac{1}{2}\frac{3}{2}\frac{3}{2}\right\rangle_{\Delta_M^-} = C_A F^S_{\frac{3}{2}} \frac{1}{\sqrt{2}}(S^\rho_{\frac{11}{22}}\varphi^\rho_{11} + S^\lambda_{\frac{11}{22}}\varphi^\lambda_{11}). \tag{22}$$

The lower J-states given L, S follow by Clebsch-Gordoning,

$$\left|1\frac{1}{2}Jm\right\rangle_{p_M^-} = \sum_{m_S} \mathbf{C}^{Jm}_{1m_L\frac{1}{2}m_s}\left[C_A\frac{1}{\sqrt{2}}\left(F^\rho_{\frac{1}{2}}\frac{1}{\sqrt{2}}(\varphi^\rho_{1m_L}S^\lambda_{\frac{1}{2}m_s} + \varphi^\lambda_{1m_L}S^\rho_{\frac{1}{2}m_s}) + F^\lambda_{\frac{1}{2}}\frac{1}{\sqrt{2}}(\varphi^\rho_{1m_L}S^\rho_{\frac{1}{2}m_S} - \varphi^\lambda_{1m_L}S^\lambda_{\frac{1}{2}m_s})\right)\right] \tag{23}$$

and for the odd parity $J = \frac{1}{2}, \frac{3}{2}$ shells in the isobar,

$$\left|1\frac{1}{2}Jm\right\rangle_{\Delta_M^-} = \sum_{m_S} \mathbf{C}^{Jm}_{1m_L\frac{1}{2}m_s}\left[C_A F^S_{\frac{3}{2}}\frac{1}{\sqrt{2}}(S^\rho_{\frac{1}{2}m_s}\varphi^\rho_{1m_L} + S^\lambda_{\frac{1}{2}m_s}\varphi^\lambda_{1m_L})\right]. \tag{24}$$

The method for deriving the orbital-spin-isospin wave functions out of the ρ, λ blocks of mixed permutation symmetry, is basically known. Their use using Jacobi-like combination was carried by Isgur and Karl. However, the explicit wave functions in the current literature, are not in a standard form useful for applications to multiquark states. Since while working on this paper we have developed them, we will explain the simple rules on how it was done in *Mathematica*, and present a full set of explicit wave functions in Appendix B.

III. $J^P = 2^+$ EXCITED STATES

Representation theory of angular momentum tell us that one can construct positive parity excited nucleon states with J^P assignments

$$L^\pi \otimes S = 2^+ \otimes \frac{1}{2},\frac{3}{2} = \left(\frac{3}{2}^+,\frac{5}{2}^+\right),\left(\frac{1}{2}^+,\frac{3}{2}^+,\frac{5}{2}^+,\frac{7}{2}^+\right)$$

However, the explicit construction of the wave functions symmetric under quark permutations needs further attention. These wave functions were discussed, e.g., in [15]. However the full treatment is better achieved using the representations of the S_3 permutation group, to be discussed below in Appendix A.

Since orbital part of the wave function for $J^P = 2^+$ are symmetric tensors constructed out of coordinate vectors ρ^i, λ^i, one has three options for this

$$\rho^i\rho^j, \qquad \lambda^i\lambda^j, \qquad (\rho^i\lambda^j + \lambda^i\rho^j)$$

The first two are symmetric under [12], the last is antisymmetric. Under [23] permutation their transformation is involved under (A3). To get totally symmetric wave functions, they need to be supplemented by spin and isospin wave functions with appropriate symmetries.

For the total spin $S = \frac{3}{2}$, the corresponding spin wave functions are symmetric (e.g., $\uparrow\uparrow\uparrow$), and we need to apply the S_3 representation with three objects (two coordinates and isospin) $X^{A1}X^{A2}X^{A3}$ with binary indices $A = \rho$, λ. We have shown that it is a single combination (A10).

For the total spin $S = \frac{1}{2}$, there are two options S^ρ, S^λ. The most straightforward way to construct the wave functions is via building symmetric representation of *four* spinorlike objects

$$X^{A1}X^{A2}X^{A3}X^{A4}$$

with binary indices $A = \rho$, λ. Their total number is $2^4 = 16$, half of them symmetric and half antisymmetric under [12] permutations. The 16×16 matrix (not shown) of [23] permutations have 8 symmetric and 8 antisymmetric eigenvectors. In the Appendix, we show that there are *two* symmetric under *all* permutations, which can be rewritten as

$$\lambda^i\lambda^j S_\lambda I_\lambda + \rho^i\lambda^j S_\lambda I_\rho + \lambda^i\rho^j S_\rho I_\lambda + \rho^i\rho^j S_\rho I_\rho \tag{25}$$

$$\lambda^i\lambda^j S_\rho I_\rho - \rho^i\lambda^j S_\lambda I_\rho - \lambda^i\rho^j S_\rho I_\lambda + \rho^i\rho^j S_\lambda I_\lambda. \tag{26}$$

Their linear combinations generate tensor excited states with spin $S = \frac{1}{2}$. For those with fixed J, J_z one needs, as usual, to calculate the Clebsch-Gordan coefficients.

In the Appendix we have pushed the method one step further, to 5 ρ, λ blocks. Out of 32 states, we found 3 symmetric eigenstates.

IV. SPLITTINGS OF NEGATIVE PARITY NUCLEONS

The details of the operators and calculation methods are presented below and in the Appendices, but we start by summarizing the results.

All the matrix elements contain 6-dimensional integrals over $\vec{\rho}$, $\vec{\lambda}$ coordinates which are split into two sets, 4D angular and two remaining radial

$$\int\left(\frac{d\Omega_\rho}{4\pi}\frac{d\Omega_\lambda}{4\pi}\right)(\rho^2 d\rho\lambda^2 d\lambda)\ldots \tag{27}$$

times pertinent matrix elements. The wave functions consist of a common 6D spherically symmetric function $\phi_{00}(R), R^2 \equiv \rho^2 + \lambda^2$, times various orbital parts. For the P-wave baryons under consideration, the orbital parts are linear in $\vec{\rho}$, $\vec{\lambda}$. The matrix elements receive multicontributions from the different parts of the symmetrized wave functions, as detailed in the Appendix. Four angular integrals will be performed in matrix elements to follow, while integrals over the moduli ρ, λ remains undone, as no assumptions about $\phi_{00}(R)$ is made. We introduce a shorthand notations for such integrals

$$\langle \hat{V}(\rho,\lambda)\rangle \equiv \int\int d\rho d\lambda \cdot \rho^2\lambda^2 |\phi_{00}|^2 V(\rho,\lambda). \quad (28)$$

For example, in such notations the normalization integral ($\hat{V} = \hat{1}$) "averages" the orbital wave function squared, namely

$$\langle(\rho^2+\lambda^2)\rangle = \int\int d\rho d\lambda \cdot \rho^2\lambda^2 |\phi_{00}|^2 (\rho^2+\lambda^2). \quad (29)$$

This combination will appear in denominators of all terms.

The masses of the N_J^* excited nucleons have some overall constant (not written below) plus contributions of four spin operators, e.g., spin-spin, spin-orbit, tensor and 't Hooft terms. The resulting mixing matrices in the P-subshells are

$$\mathbb{M}'_{\frac{5}{2}} = \frac{3}{4}\frac{\langle(\rho^2+\lambda^2)V_S(\sqrt{2}\rho)\rangle}{\langle(\rho^2+\lambda^2)\rangle} + 3\frac{\langle\rho^2 V_{LS}(\sqrt{2}\rho)\rangle}{\langle(\rho^2+\lambda^2)\rangle} - \frac{1}{5}\frac{\langle\rho^4 V_T(\sqrt{2}\rho)\rangle}{\langle(\rho^2+\lambda^2)\rangle} \quad (30)$$

$$\mathbb{M}'_{\frac{3}{2}} = \begin{pmatrix} 3/4 & 0 \\ 0 & -3/4 \end{pmatrix}\frac{\langle(\rho^2+\lambda^2)V_S(\sqrt{2}\rho)\rangle}{\langle(\rho^2+\lambda^2)\rangle} + \begin{pmatrix} -2 & -\sqrt{5/2} \\ -\sqrt{5/2} & 1 \end{pmatrix}\frac{\langle\rho^2 V_{SL}(\sqrt{2}\rho)\rangle}{\langle(\rho^2+\lambda^2)\rangle}$$
$$+ \begin{pmatrix} \frac{4}{5} & \frac{1}{\sqrt{10}} \\ \frac{1}{\sqrt{10}} & 0 \end{pmatrix}\frac{\langle\rho^4 V_T(\sqrt{2}\rho)\rangle}{\langle(\rho^2+\lambda^2)\rangle} + \begin{pmatrix} 0 & 0 \\ 0 & (1+3a) \end{pmatrix}\frac{G_{tH}\langle\lambda^2\rangle}{8\sqrt{2}\pi\langle(\rho^2+\lambda^2)\rangle} \quad (31)$$

$$\mathbb{M}'_{\frac{1}{2}} = \begin{pmatrix} 3/4 & 0 \\ 0 & -3/4 \end{pmatrix}\frac{\langle(\rho^2+\lambda^2)V_S(\sqrt{2}\rho)\rangle}{\langle(\rho^2+\lambda^2)\rangle} - \begin{pmatrix} 5 & 1 \\ 1 & -2 \end{pmatrix}\frac{\langle\rho^2 V_{SL}(\sqrt{2}\rho)\rangle}{\langle(\rho^2+\lambda^2)\rangle}$$
$$+ \begin{pmatrix} -1 & -1 \\ -1 & 0 \end{pmatrix}\frac{\langle\rho^4 V_T(\sqrt{2}\rho)\rangle}{\langle(\rho^2+\lambda^2)\rangle} + \begin{pmatrix} 0 & 0 \\ 0 & 1+3a \end{pmatrix}\frac{G_{tH}\langle\lambda^2\rangle}{8\sqrt{2}\pi\langle(\rho^2+\lambda^2)\rangle}. \quad (32)$$

The rows and columns of the 2×2 matrices, correspond to states with $S = \frac{3}{2}, \frac{1}{2}$, respectively,

$$\mathbb{M}_J = \begin{pmatrix} \langle S=\frac{3}{2}|\mathbb{V}_{S+T+SL+V_{TH}}|S=\frac{3}{2}\rangle & \langle S=\frac{3}{2}|\mathbb{V}_{S+T+SL+V_{TH}}|S=\frac{1}{2}\rangle \\ \langle S=\frac{1}{2}|\mathbb{V}_{S+T+SL+V_{TH}}|S=\frac{3}{2}\rangle & \langle S=\frac{1}{2}|\mathbb{V}_{S+T+SL+V_{TH}}|S=\frac{1}{2}\rangle \end{pmatrix}_J. \quad (33)$$

Some of these matrices have nondiagonal (or mixing) matrix elements, which means that the total spin S is not a good quantum number of the observed states.

We now explain how the matrix elements are calculated. All operators used should be symmetric under quark permutations, as are all the orbital-spin-isospin wave functions. In most cases we calculated the pair 12-interactions, and then multiplied the result by 3 (accounting for three pairs, 12, 13, 23). In Jacobi coordinates, the 12-distance is $d_{12} = \sqrt{2}\rho$, so most potentials are fixed at that distance.

The operators are defined as follows

$$H^{12}_{SS} = V_S(d_{12})(\vec{S}_1\vec{S}_2)$$
$$H^{12}_{LS} = V_{LS}(d_{12})\frac{(S_1+S_2)^i}{2}\rho^j\left(\frac{\partial}{i\partial\rho^k}\right)\epsilon^{ijk}$$
$$H^{12}_{T} = V_T(d_{12})[(\vec{S}_1\vec{d}_{12})(\vec{S}_2 d_{12}) - (\vec{S}_1\vec{S}_2)d^2_{12}] \quad (34)$$

Since, the corresponding terms are proportional to $\langle\rho^2 V_{LS}(\sqrt{2}\rho)\rangle$. the numerical factors in front each contribution, stem from the convolution over all indices of operators, wave functions and 4D angular integrals.

The exception is the instanton-induced interaction taken in the local approximation,

$$H^{12}_{\rm tHooft} = G_{tH}\delta^3(\vec{d}_{12})(1-\vec{\tau}_1\vec{\tau}_2)(1-a\vec{\sigma}_1\vec{\sigma}_2) \quad (35)$$

where $\vec{\tau}_i$, $\vec{\sigma}_i$ are Pauli matrices for isospin and spin, respectively. The 't Hooft interaction, is quasi-local in nature, and for the 12-quark pair is proportional to $\sim G_{tH}\delta^3(\vec{d}_{12})$. As a result, the integral over $\vec{\rho}$ drops out, and only the integral over λ remains, so we define for it a lambda-only averaging

$$I_\lambda \equiv \int d\lambda\lambda^4|\phi_{00}(\lambda)|^2. \quad (36)$$

A. Extracting matrix elements from N^* data

The knowledge of the coefficients of the relevant operators for the five (unmixed) masses of negative parity N^*, allows to fix the pertinent matrix elements. This immediately reveals one striking feature, already noted by Isgur and Karl [2]: the role of spin-orbit is an order of magnitude (or more) suppressed relative to the spin-spin and tensor force. The same observation follows for the 't Hooft operator which we tried to include: it is not improving the fit and its expectation value is within the error bars. (For P-shell baryons, not so for the ground state nucleon).

In view of this we resorted to what we call "an optimized IK model," which ignores LS and 't Hooft terms and keep only the spin-spin and the tensor forces. In this case, the fitted matrix elements can be considered reliable.

$$\begin{aligned}\langle H_0\rangle &= 1607.\ \text{MeV}\\ \frac{\langle\rho^2 V_{SS}\rangle}{\langle\rho^2+\lambda^2\rangle} &= 83.2\ \text{MeV},\\ \frac{\langle\rho^4 V_{\rm tensor}\rangle}{\langle\rho^2+\lambda^2\rangle} &= 43.7\ \text{MeV}\end{aligned} \quad (37)$$

where H_0 is the spin-independent part of the Hamiltonian, common to all five resonances.

Our only assumptions so far are using spin-dependent forces to the first order, and ignoring near-threshold effects. (Isgur and Karl made additional assumptions, such as a Gaussian S-wave function, used to evaluate them.) The quality of the overall description of masses can be seen in Fig. 1. One can see that the agreement of this model with the data is very good. In all cases, it is significantly better than the half-widths of these resonances (which provides a scale for the ignored threshold effects).

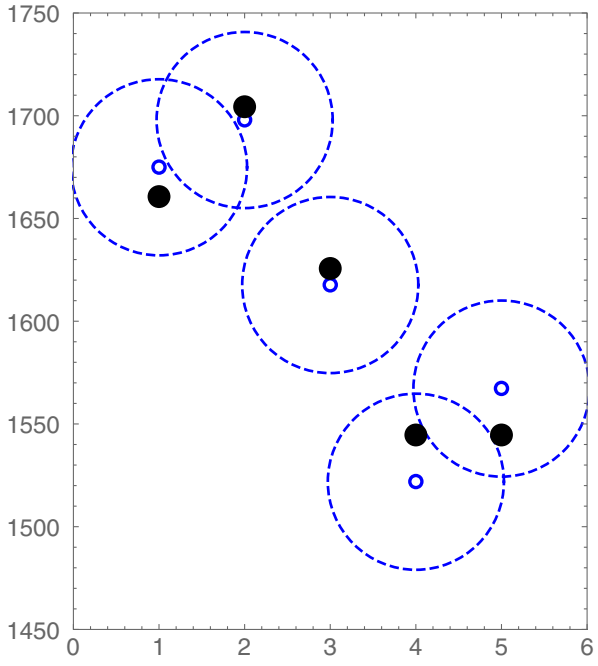

FIG. 1. The black dots are masses of negative parity P-shell baryons, from the "optimized IK model" with fitted radial matrix elements (37), in the order $|J=\frac{5}{2}\rangle, |J=\frac{3}{2}, S=\frac{3}{2}\rangle, |J=\frac{1}{2}, S=\frac{3}{2}\rangle, |J=\frac{3}{2}, S=\frac{1}{2}\rangle, |J=\frac{1}{2}, S=\frac{1}{2}\rangle$. The small blue circles show the experimental pole positions. They are surrounded by large dashed blue circles, with radius of the order of typical half width $\frac{1}{2}\Gamma$.

As "external tests" of the accuracy of such a model, we use the empirical mixing matrix elements (3). The observed ratio for the magnitude of the mixing matrix elements in the tensor case is

$$\frac{H_{\rm mix}(J=1/2)}{H_{\rm mix}(J=3/2)} \approx -2.76. \quad (38)$$

If only the tensor operator is left, the ratio of its coefficients $-\sqrt{10} = -3.16$ is sufficiently close to it. The value from the fitted tensor matrix element to the splittings in Fig. 1, we have the value (37) or 43.7 MeV. This is to be compared to 51.6 MeV from the mixing matrix element in the $J=\frac{1}{2}$- shell pair as discussed above. Hence, some other effects, not yet accounted for here, contribute at the level of ~8 MeV. A similar conclusion can also be inferred from the deviations between the observed and fitted masses seen in Fig. 1. These deviations should be due to higher order corrections not yet accounted for. They are reasonably small compared to the splitting themselves ~100 MeV as well as to $\Gamma/2$–50 MeV used as the maximal possible shifts due to threshold locations.

V. SPIN-DEPENDENT INTERACTIONS IN BARYONS

Having summarized so to say the "applied aspects" of the calculations, let us return to the theory. Before discussing the spin-dependent effects, few words about the spin-*independent* potentials. Those are expressed in terms of Wilson-line correlators, or complicated nonlocal correlations of (Euclidean) vacuum gauge fields. They can be computed numerically on the lattice, or evaluated in some models of gauge fields in the vacuum. An instanton-based model we used for quarkonia in [6], and claimed that it can explain most of the confining potential, at least till the relevant distances < 1 fm. For baryons, there are correlators of three Wilson lines, see their calculation in the instanton model [7] and detailed comparison with lattice results. In a recent paper [16] we extended such calculations for the three Wilson lines (for baryons) and four ones (for tetraquarks). In the latters, the hyperditance potential was shown to describe the distances between the three $cc\bar{c}\,\bar{c}$ states recently discovered, and interpreted as $1S$, $2S$, $3S$ states. Application to $uuu = \mathrm{Delta}^{++}$ S-shell states were also successful. No attempt so far were made along these lines to describe baryons or multiquark states made of different flavor quarks. In such cases we need to go beyond the hyperdistance approximation.

The generic spin-dependent five potentials were defined in [17], in terms of Wilson-lines dressed by pertinent gauge field strengths. Specific relations between them and the central potential, follow for one-gluon exchange. For example, the spin-spin potential is famously a Laplacian of the central potential, a delta function for the Coulomb force. Similar but distinct relations follow for the instanton-induced potentials, mostly constrained by self-duality of the instanton fields. However, if a significant fraction of the vacuum fields are due to the overlapping instantons and antiinstantons, there could be significant corrections. The spin-dependent interactions in the baryons and tetraquarks due to instantons can be evaluated, and we hope to report on them in our subsequent publications.

In this section we only include some general comments related to the definition of the operators. The specifics of the calculation for fixed states, are summarized in the Appendix.

A. Spin-spin and tensor interactions

We use the same strategy in the evaluation of all the spin interactions. More specifically, the 12-interactions are folded over the pertinent wave functions, and the overall results are multiplied by 3 by symmetry. The results for spin-spin forces agree with the much simpler "quantum-mechanical" method.

For the "unmixed" states with fixed total spin of three quarks, $S = \frac{3}{2}$ or $S = \frac{1}{2}$, we have $\vec{S} = \vec{S}_1 + \vec{S}_2 + \vec{S}_3$, so that

$$\langle \vec{S}_1\vec{S}_2 + \vec{S}_1\vec{S}_3 + \vec{S}_3\vec{S}_2\rangle = \frac{1}{2}(S(S+1) - 9/4) \quad (39)$$

hence $\pm\frac{3}{4}$ for the cases of interest. The left-hand side (lhs) is symmetric under quark permutations, and is the only form for the spin-spin forces. Of course, each term is convoluted with potentials, that depend on the relative distance, e.g.,

$$(\vec{S}_1 \cdot \vec{S}_2)V_{SS}(d_{12}), \qquad \vec{d}_{12} = \sqrt{2}\vec{\rho}$$

the average of which multiplied by 3.

The tensor operator we used is given in (34) in a standard way, and also evaluated it using the 12-pair. Note that in the tensor, we are using the actual distance vector $\vec{d}_{12}$ rather than its unit vector version. Hence the extra $2\rho^2$ in the matrix element (to be accounted for in the potential V_T).

B. Spin-orbit operator

Two-body systems (mesons) have one orbital momentum L and one total spin S, so the natural spin-orbit force is proportional to $(\vec{S}\,\vec{L})$. For three quarks there are $\vec{L}_\rho$ and $\vec{L}_\lambda$ orbital and three different quark spin structures. In general, it is still not all, since there can be products of one coordinate vector times momentum of another one. The requirement of permutation symmetry helps. Suppose the spin-dependent potentials are binary (two-body). With this in mind, the spin-orbit contribution is a sum of binaries

$$\begin{aligned}\mathbb{V}_{SL}(1,2,3) = &\sum_{i<j=1,2,3}^{3} V_{SL}(d_{ij})[(r_{ij}\times p_i)\\ &\cdot\sigma_i - (r_{ij}\times p_j)\cdot\sigma_i)]\\ \equiv\ &\mathbb{V}_{SL}(1,2) + V_{SL}(1,3) + V_{SL}(2,3). \end{aligned} \quad (40)$$

Recall that the spin-orbit relation to the central potential is suppressed $V_{SL} \sim \frac{1}{m_Q^2}$. This is the case of all relativistic corrections.

For the instanton-induced spin-orbit potential we have

$$V_{SL} = \frac{1}{m_Q^2}\frac{1}{r_{ij}}\frac{dV_C(r_{ij})}{dr_{ij}}$$

in terms of the instanton-induced central potential. To recast it in terms of the Jacobi coordinates, we recall that

$$r_{12} = \sqrt{2}\rho \qquad r_{13,23} = \frac{1}{\sqrt{2}}(\rho^2 + 3\lambda^2 \pm \sqrt{3}\rho\cdot\lambda)^{\frac{1}{2}} \quad (41)$$

and the corresponding momenta

$$\begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & 0 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & 0 \\ 0 & -\frac{2}{\sqrt{6}} & 0 \end{pmatrix} = \begin{pmatrix} p_\rho \\ p_\lambda \\ 0 \end{pmatrix}. \quad (42)$$

In particular, for the 12-spin-orbit contribution we have

$$\mathbb{H}_{SL}(1,2) = V_{SL}(d_{12})\left(\left(\frac{\sigma_1+\sigma_2}{2}\right)\cdot(\rho\times p_\rho) + \frac{1}{\sqrt{3}}(\sigma_1-\sigma_2)\cdot(\rho\times p_\lambda)\right) \quad (43)$$

Here the first contribution is the standard spin-orbit contribution, proportional to L_ρ, rotation of a 12-pair. The second term is also symmetric under 12-permutation, but has a vector product of the coordinate ρ and momentum p_λ, so it is neither two-body nor even an angular momentum. Fortunately, we found that all the second term contributions vanish after angular integration, basically because $\vec{\rho}$ and $\vec{\lambda}$ have independent directions. Therefore we do not need to include this complicated part of the operator.

C. 't Hooft induced interactions

The instanton-induced forces follow from the fermionic zero modes discovered by 't Hooft. Those are antisymmetric in flavors, and form generically 6-fermion operators of the $\bar{u}u\bar{d}d\bar{s}s$ flavor structure. If we ignore strangeness (technically, by substituting $\bar{s}s$ part by its vacuum average value), the interaction reduces to a 4-fermion ud-interaction. It can be shown to be of the form

$$\mathbb{V}_{TH}(1,2,3) = \sum_{i<j=1,2,3} V_{TH}(r_{ij})(1-\tau_i\cdot\tau_j)(1-a\sigma_i\cdot\sigma_j) \quad (44)$$

in the ultralocal approximation, with

$$V_{TH}(r_{ij}) \to -\frac{1}{4}|\kappa_2|A_{2N} = -\frac{1}{4}|\kappa_2|\frac{2N_c-1}{2N_c(N_c^2-1)}$$

and

$$a = 4B_{2N} = \frac{1}{(2N_c-1)}. \quad (45)$$

The coefficient a comes from terms with color matrices, and assuming that we discuss baryons and carry the average over all instanton color orientations, we get $a=\frac{1}{5}$ for $N_c=3$ and $a\to 0$ for $N_c\to\infty$.

The first factor in (44) with isospin matrices $\vec{\tau}$, is a projector on the flavor singlet states (zero in Δ). So it contributes to the $N-\Delta$ splitting in the S-shell. There is also no problem to evaluate this operator over the five N^* states in the P-shell. In the quasi-local approximation $V_{\rm tHooft}\sim\delta^3(d_{12})$, of the $\sim\rho^2,\lambda^2$ contributions arising from the orbital wave functions, only the λ^2 remains.

VI. BARYON WAVE FUNCTIONS ON THE LIGHT FRONT

There are many differences between the center of mass (c.m.) and the LF wave functions induced simply by kinematics. For example, the former uses nonrelativistic (or semirelativistic [4]) descriptions, not well justified for light quarks. The latter (in the form developed in our previous papers [6,7,18–21]) does not need this assumption, it takes the same form, from light u, d, s to heavy c, b quarks. Furthermore, it naturally provides an oscillatorlike behavior for transverse momenta, which translate into a *linear* dependence of the baryon *mass squared*, on the number of excitation quanta.

However, on the LF manifest rotational symmetry is lost, as transverse and longitudinal motions are treated differently. The states are no longer classified by their total angular momentum J, orbital L or total spin S. Only the helicity projection of those, J_z, L_z and S_z, can be used. As we will see in this case, the importance of permutation symmetry (as it was illustrated in previous sections and Appendix A) is even broader.

A simple Fock state representation of the spin-up proton wave functions on the LF was originally given in [22]

$$|p^\uparrow\rangle_{L_z=0} = \int d[1,2,3](\psi_1(1,2,3)+i\epsilon^{\alpha\beta}k_{1\alpha\perp}k_{2\beta\perp}\psi_2(1,2,3))\frac{\epsilon^{ABC}}{\sqrt{6}}b^{A\dagger}_{u\uparrow}(1)\left(b^{B\dagger}_{u\downarrow}(2)b^{C\dagger}_{d\uparrow}(3)-b^{B\dagger}_{d\downarrow}(2)b^{C\dagger}_{u\uparrow}(3)\right)|0\rangle$$
$$|p^\uparrow\rangle_{L_z=+1} = \int d[1,2,3](k^+_{1\perp}\psi_3(1,2,3)+k^+_{2\perp}\psi_4(1,2,3))\frac{\epsilon^{ABC}}{\sqrt{6}}\left(b^{A\dagger}_{u\uparrow}(1)b^{B\dagger}_{u\downarrow}(2)b^{C\dagger}_{d\downarrow}(3)-b^{A\dagger}_{d\uparrow}(1)b^{B\dagger}_{u\downarrow}(2)b^{C\dagger}_{u\downarrow}(3)\right)|0\rangle$$
$$|p^\uparrow\rangle_{L_z=-1} = \int d[1,2,3]k^-_{2\perp}\psi_5(1,2,3)\frac{\epsilon^{ABC}}{\sqrt{6}}b^{A\dagger}_{u\uparrow}(1)\left(b^{B\dagger}_{u\uparrow}(2)b^{C\dagger}_{d\uparrow}(3)-b^{B\dagger}_{d\uparrow}(2)b^{C\dagger}_{u\uparrow}(3)\right)|0\rangle$$
$$|p^\uparrow\rangle_{L_z=+2} = \int d[1,2,3]k^+_{1\perp}k^+_{3\perp}\psi_6(1,2,3)\frac{\epsilon^{ABC}}{\sqrt{6}}b^{A\dagger}_{u\downarrow}(1)\left(b^{B\dagger}_{d\downarrow}(2)b^{C\dagger}_{u\downarrow}(3)-b^{B\dagger}_{u\downarrow}(2)b^{C\dagger}_{d\downarrow}(3)\right)|0\rangle \quad (46)$$

where generically $b^{C\dagger}_{fs}(1)=b^{C\dagger}_{fs}(x_i,k_{i\perp})$, $k^\pm_\perp=k_x\pm k_y$, and

$$d[1,2,3]=(2\pi)^3\delta^3\left(\sum_{i=1}^{3}k_{i\perp}\right)\delta\left(1-\sum_{i=1}^{3}x_i\right)\prod_{i=1}^{3}\frac{dx_i dk_{i\perp}}{\sqrt{2x_i}(2\pi)^3} \tag{47}$$

The fermionic creation operators anticommute, and once multiplied by the color indices ϵ^{ABC}, they generate 12 terms.

The first term ψ_1 in the upper raw (times the second raw) above reproduces the correct permutation-symmetric S-wave function of the proton, in the form we already used in the upper raw of (9). The expressions for the proton with $L_z=\pm1$ are linear in (transverse) momentum, and obviously ψ_1 refer to a spin-up proton with *negative parity*. The 3 $\psi_{3,4,5}$ LF wave functions refer to part of the nucleon P-shell, we discussed above. They do not cover all the 5 P-nucleon states we derived, as they have different wave functions, with different permutation content. Also, their longitudinal momentum content under permutation symmetry is not specified. Finally, the second term in the first raw and the last one are quadratic in momenta. So, the 2 $\psi_{2,6}$ wave functions correspond to D-shell excitation or their admixtures, with positive parity. Their permutation structure is incomplete. This issue has been addressed in the c.m. above in Sec. III, and will be extended below to the LF.

To clarify the argument, let us first recall again the situation in the c.m. frame, returning to our main example of the P-shell nucleons. Physical states in the c.m. formulation have spherical symmetry, and therefore fixed total angular momentum J. They are specific combinations of all possible values of the orbital and spin helicities L_z, S_z. For example,

$$\begin{aligned}|N^*J_z=1/2,P=-1\rangle&\\ =+f_1|L_z=1,S_z=-1/2\rangle&\\ +f_0|L_z=0,S_z=1/2\rangle&\\ +f_{-1}|L_z=-1,S_z=3/2\rangle&\end{aligned}$$

are superpositions of states with all three L_z values, with some coefficients f_{L_z}. For states with fixed J those are defined by standard Clebsch-Gordon rules, which for this particular example prescribe the coefficients f_1, f_0, f_{-1} to be

$$\begin{aligned}&\sqrt{6}\left(\frac{Y_{11}(\theta,\phi)}{2\sqrt{15}},\frac{Y_{10}(\theta,\phi)}{\sqrt{10}},\frac{Y_{1-1}(\theta,\phi)}{2\sqrt{5}}\right),\\ &2\left(\frac{Y_{11}(\theta,\phi)}{\sqrt{10}},\frac{Y_{10}(\theta,\phi)}{2\sqrt{15}},-\frac{2Y_{1-1}(\theta,\phi)}{\sqrt{15}}\right),\\ &\sqrt{2}\left(\frac{Y_{11}(\theta,\phi)}{2},-\frac{Y_{10}(\theta,\phi)}{\sqrt{6}},\frac{Y_{1-1}(\theta,\phi)}{2\sqrt{3}}\right)\end{aligned} \tag{48}$$

for $J=5/2,3/2,1/2$, respectively, and $J_z=1/2$. Note that these three vectors are indeed mutually orthogonal and normalized. Under the zeroth order Hamiltonian, they all have the same energy, and only the spin-dependent forces create the observed splittings.These states are defined in the c.m. frame as eigenstates of the total angular momentum $\vec{J}$, but this option is not available in the LF formulation.

Furthermore, as emphasized above, Clebsching is not enough for baryons. Their wave functions are *linear* in coordinates, which bring in their negative parity. There are 6 $\vec{\rho}$, $\vec{\lambda}$, and one expects 6 basic orbital wave functions. The same number of basis states needs to be defined in the LF formulation. The transverse polarization component $L_z=0$, proportional to the *longitudinal* coordinates/momenta ρ_z, λ_z in the c.m. frame, needs to be redefined. As we will show below, they should be substituted by solutions on the triangle, of ρ and λ-type respectively

$$Y_{10}(\theta_\rho)\to D^\rho,\qquad Y_{10}(\theta_\lambda)\to D^\lambda. \tag{49}$$

Also, the spin $S_z=\pm\frac{1}{2}$ structure in fact exists in three forms, the fully symmetric one corresponding to $S=\frac{3}{2}$, and two more mixed with ρ-like and λ-like permutation properties. The same statement applies to isospin wave function. Constructing the correct combinations is not a trivial task, and it was the subject of the preceding sections.

A. Light front wave functions for negative parity baryons

Now we switch to the main subject of this work, the construction of the corresponding wave functions on the light front (LF), with a focus on the P-shell baryons. Four out of the six coordinates corresponding to the transverse 12-plane, remains unchanged. Since we proceed on LF in momentum representation, the angular functions with $L_z=\pm1$ are proportional to either $k_1^\rho\pm ik_2^\rho$ or $k_1^\lambda\pm ik_2^\lambda$. The requirements of 12-permutation symmetry helps to eliminate many impossible combinations, and to reduce the number of functions necessary. In particular, the spin-isospin-transverse functions should be superpositions of *permutation-symmetric* blocks, such as

$$(e^{\pm i\phi_\rho}I_\rho+e^{\pm i\phi_\lambda}I_\lambda)S_{\rm sym}$$

for p-shell nucleons with spin $\frac{3}{2}$ and

$$(e^{\pm i\phi_\rho}S_\rho+e^{\pm i\phi_\lambda}S_\lambda)I_{\rm sym}$$

for Δ baryons. Other combinations are formed as in the c.m. frame.

The question remains what are the other two other basis states, with longitudinal momenta. Using Bjorken-Feynman longitudinal momenta *fractions* $x_i\in[0,1]$, $i=1,2,3$ as variables and enforcing the kinematical constraint $x_1+x_2+x_3=1$, we will again use Jacobi variables to characterize them. We will refer to them by ρ, λ without any

indices. (They should not be confused with the lengths of the coordinate vectors $\vec{\rho}$, $\vec{\lambda}$ used in previous sections.)

As discussed in our paper [7], the physical domain in this case is the equilateral triangle, between three points at which one of the x_i reaches unity and two others vanish,

$$(\rho,\lambda)\in\text{Triangle}\left(\left\{0,-\sqrt{\frac{2}{3}}\right\},\left\{\frac{1}{\sqrt{2}},\frac{1}{\sqrt{6}}\right\},\left\{-\frac{1}{\sqrt{2}},\frac{1}{\sqrt{6}}\right\}\right).$$

The role of the permutation group S_3 is now clearly seen,. In these notations, it is a set of reflections and rotations of this triangle.

Quantization on this equilateral triangle has been done in [7], which was achieved both analytically and numerically. We recall that the confining potential, after an "einbine trick," can be made quadratic in coordinates. In the momentum representation we use the sum of the squared coordinates, to turn it Laplacian in momenta ρ, λ. The analytic solutions follow in the form of six waves, interfering at the triangle boundaries enforcing Dirichlet (zero) boundary condition, see (54) of [7]. The spectrum of the Laplacian is given by two integers m_L, n_L (L refers to the longitudinal directions).

$$\epsilon_{m_L,n_L}=\frac{8\pi^2}{3^2}(m_L^2+n_L^2-m_Ln_L). \tag{50}$$

The single-degenerate solutions correspond to $m_L=2n_L$, with the two lowest already shown and discussed in [7].

Now we focus on the solutions with $m_L>2n_L$, which are in fact *double* degenerate

$$\begin{aligned}D^{\lambda}_{m,n}(\lambda,\rho)=\frac{4}{L3^{\frac{3}{4}}}\Bigg[&\cos\left(\frac{2\pi(2m_L-n_L)\rho}{3L}\right)\sin\left(\frac{2\pi n_L\tilde{\lambda}}{\sqrt{3}L}\right)\\&-\cos\left(\frac{2\pi(2n_L-m_L)\rho}{3L}\right)\sin\left(\frac{2\pi m_L\tilde{\lambda}}{\sqrt{3}L}\right)\\&+\cos\left(\frac{2\pi(m_L+n_L)\rho}{3L}\right)\sin\left(\frac{2\pi(m_L-n_L)\tilde{\lambda}}{\sqrt{3}L}\right)\Bigg]\\D^{\rho}_{m,n}(\lambda,\rho)=\frac{4}{L3^{\frac{3}{4}}}\Bigg[&\sin\left(\frac{2\pi(2m_L-n_L)\rho}{3L}\right)\sin\left(\frac{2\pi n_L\tilde{\lambda}}{\sqrt{3}L}\right)\\&-\sin\left(\frac{2\pi(2n_L-m_L)\rho}{3L}\right)\sin\left(\frac{2\pi m_L\tilde{\lambda}}{\sqrt{3}L}\right)\\&-\sin\left(\frac{2\pi(m_L+n_L)\rho}{3L}\right)\sin\left(\frac{2\pi(m_L-n_L)\tilde{\lambda}}{\sqrt{3}L}\right)\Bigg]\end{aligned}$$

with $\tilde{\lambda}=\lambda+L/\sqrt{3}$. Their symmetry properties include (1–2) (or $\rho\to-\rho$ symmetry)

$$D^{\rho,\lambda}_{m,n}(\lambda,-\rho)=\pm D^{\rho,\lambda}_{m,n}(\lambda,\rho).$$

(Yes, the triangle has obvious triple symmetry by 120° rotations, but those produce linear combinations of these two solutions.) Those were called D^c, D^s before, because the former includes combinations only with $\cos(C_i\rho)$, and the latter similar set of $\sin(C_i\rho)$. In the present paper, where permutation symmetry is central, we would like to rename them into D^{λ}, D^{ρ}, respectively. Indeed, the former is even under 12-permutation, and the latter is odd for $\rho\to-\rho$, see Fig. 2. These are the solutions of ρ and λ types, which on the LF are substitutes for zero orbital momentum components in the c.m., $L^{\lambda}_z=0$ and $L^{\rho}_z=0$, essentially simple linear coordinates λ_z, ρ_z.

Since these are new solutions, let us discuss how the spin operators act on them. The spin-orbit interactions include orbital momenta such as

$$L^i_{\rho}=\epsilon^{ijk}\rho^j(i\partial/\partial\rho^k).$$

In the coordinate representation the momentum is a derivative, and in the momentum representation (we are using on the LF) the coordinates are derivatives over

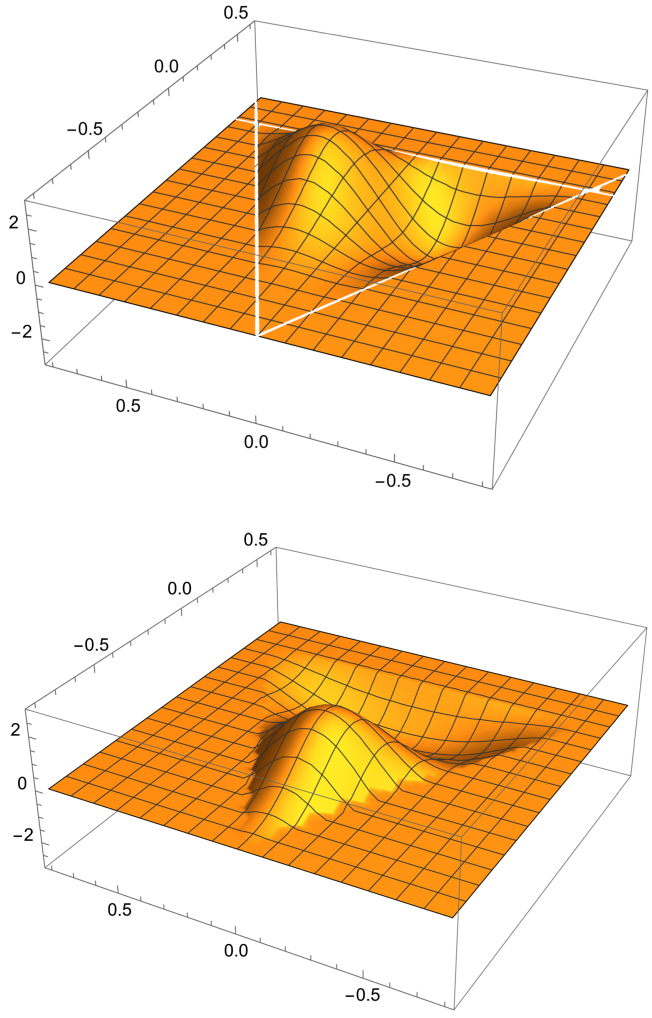

FIG. 2. The D^{ρ}, D^{λ} solutions of the Laplacian on the equilateral triangle, for $m_L=3$, $n_L=1$.

momenta. Acting by L on the longitudinal function of longitudinal momentum $\rho \sim p_z^\rho$, the vector product can only get other—transverse—components $p_{x,y}^\rho$ due to the first factor. Since those appear linearly, it means the L operation converts longitudinal functions into transverse ones. As a result, on the LF the angular momentum operator mixes two transverse functions with one longitudinal, as it does in the c.m. description for the three angular functions in the P-shell.

Another thing to notice is that the derivative in L interchanges cos and sin. With the appearance of ρ- type transverse coordinates, we see that this operator does not change parity under 12-permutation. Yet, the differentiation changes the function, so it is not just $D^\rho \leftrightarrow D^\lambda$ as there are additional admixtures of higher functions, with other m_L, n_L.

Similar thing happens with the tensor force. In the c.m. formulation, the interaction generates nonzero nondiagonal matrix elements for $L=0 \leftrightarrow L=2$, $L=1 \leftrightarrow L=3$ shells etc. The spin part in the tensor causes mixing of the $S=\frac{3}{2}$ and $S=\frac{1}{2}$ states, as we discussed above. Yet, since the coordinate tensor times the spin tensor is rotationally a scalar, J remains unchanged. In the LF formulation, there is no rotational symmetry or J, and the tensor force mixes all D^ρ, D^λ functions.

Let us now address the issue of parity. The standard P mirror reflection, sign change of all coordinates, makes odd-L states with negative parity, which cannot be mixed with positive parity states. On the LF the notion of parity is remedied by an additional rotation by 180°, so that longitudinal momenta do not flip. In particular, our equilateral triangle in the ρ, λ plane maps into itself. So, formally, the D^ρ, D^λ functions do not have parity as such.

Yet these LF functions have certain parities under particle permutations. As one can see from Fig. 2, D^ρ is odd under flipping of ρ and D^λ under flipping of $\tilde{\lambda} = \lambda + \sqrt{2/3}$ (since they contain the corresponding sin functions). Those are sufficient to define odd and even shells, which cannot be mixed by any forces/operators preserving the permutation symmetry.

In sum, the six wave functions out of which the P-shell baryons are made on the LF, consist of four transverse ones, with $\exp(\pm i\phi_\rho), \exp(\pm i\phi_\lambda)$, complemented by two new D^ρ, D^λ functions on the triangular domain for longitudinal momenta fraction. The P-shell baryons can be constructed out of these mixed-symmetry blocks by the same expressions as used above.

More explicitly, the $S=\frac{3}{2}$ with fixed J_z the spin WF is symmetric, and the orbital and isospin ones are mixed symmetries of ρ, λ types. We know that the full S_3 symmetric combination out of the two mixed blocks, has the structure $X_1^\rho X_2^\rho + X_1^\lambda X_2^\lambda$. Therefore the *permutations-symmetric* wave functions are

$$|N^{*\uparrow}_{S=\frac{3}{2},J_z}\rangle = \int d[1,2,3] C_A \sum_m S^S_{\frac{3}{2}(J_z-m)} \times \left(\left(A_m^{\frac{3}{2}J_z} e^{im\phi_\rho} + B_m^{\frac{3}{2}J_z} D^\rho_{3,1}\right) F^\rho_{\frac{1}{2}} + \left(\rho \to \lambda\right)\right) \quad (51)$$

and similarly for the $S=\frac{1}{2}$ and fixed J_z we have

$$|N^{*\uparrow}_{S=\frac{1}{2},J_z}\rangle = \int d[1,2,3] C_A \sum_m \times \left(F^\rho_{\frac{1}{2}}\left(\left(A_m^{\frac{1}{2}J_z} e^{im\phi_\rho} + B_m^{\frac{1}{2}J_z} D^\rho_{3,1}\right) S^\lambda_{\frac{1}{2}(J_z-m)} + \left(\lambda \leftrightarrow \rho\right)\right) + F^\lambda_{\frac{1}{2}}\left(\left(A_m^{\frac{1}{2}J_z} e^{im\phi_\rho} + B_m^{\frac{1}{2}J_z} D^\rho_{3,1}\right) S^\rho_{\frac{1}{2}(J_z-m)} - \left(\rho \to \lambda\right)\right)\right). \quad (52)$$

We note that the two $L=\pm 1$ lines in (46) should be substituted by the 5 states (51) and (52) for the P-shell N^*. The same applies to the 2 Δ^*. The coefficients A_m, B_m are not fixed kinematically in the absence of rotational symmetry. They are determined dynamically through the diagonalization of the LF Hamiltonian. Clearly, a fine tuning of the parameters is likely needed to achieve the same mass spectrum for different J_z. This is in sharp contrast with the c.m. where rotational symmetry fixes these coefficients as Clebsch-Gordon coefficients as in (48).

Finally, note that since the transverse part of the LF Hamiltonian is an oscillator, we have $M_n^2 \sim n_\perp$. The longitudinal eigenvalues (50) are quadratic in quantum numbers, yet after minimization over the einbine parameter, they enter via $(m_L^2 + n_L^2 - m_L n_L)^{\frac{1}{2}}$. They are not linear, but relatively close to linear. More specifically, the first three double-degenerate excitations, counted from from the ground state (n = 2m = 2), have energies

$$\sqrt{(m_L^2 + n_L^2 - m_L n_L)} - \sqrt{3} = 0.913, 1.87, 2.85 \quad (53)$$

to be compared to 1, 2, 3 for an oscillator. So, with a pertinent fit to the parameters, one should be able to get agreement between the masses for different J_z components in the same J-multiplet, with few percent accuracy. Similar accuracy is expected for Regge phenomenology. Furthermore, if one would only compare states with the same J_z and different J [as we did for negative parity baryons in (48)] such deviations are further reduced, as they only appear in the longitudinal component, a part of the wave function. So far, our discussion refers to the bare Hamiltonian H_0. The spin-dependent potentials cause larger shifts, ∼10% for the P-shell baryons, and we expect that this imperfection in the longitudinal part of the spectrum on the LF, will not seriously affect the spectroscopy.

B. Wave functions on LF for positive parity excitations

In the c.m. frame the D-shell baryons have wave functions *quadratic* in coordinates, as we already discussed above. Each of the coordinates have 2 transverse and one longitudinal components, of ρ or λ types. So, in general, one may approach the problem asking for representations with *four* structures $X_1 \otimes X_2 \otimes X_3 \otimes X_4$, of ρ or λ types, and seek their totally symmetric combinations. It can be done along the same reasoning as for the $X_1 \otimes X_2 \otimes X_3$ case we addressed in Appendix A. This case would correspond to the case of the $S = \frac{3}{2}$ symmetric in spin.

VII. LEADING TWIST PROTON DAS

The simple classification of the lowest Fock states for a spin-up proton state made of three constituent quarks, consists of the 6 LF wave functions in (46). As we noted earlier, this Fock state representation mixes the ground and excited states of the nucleon with fixed spin $\frac{1}{2}$. Boosting the nucleon in its ground S-state, should leave the nucleon in its ground S-state. Also the simple Fock state representation forgoes the issue of the center of mass and symmetrization, which are important for the description of the nucleon S, P, D states as we have discussed.

This notwithstanding, the Fock state representation is useful for the characterization of generic spin-$\frac{1}{2}$ nucleon distribution amplitudes (DAs), in the leading twist approximation. More specifically, three quarks with spin-$\frac{1}{2}$ can combine in four different ways to form a proton with spin up $p^\uparrow$, i.e.,

$$p^\uparrow = p_0^\uparrow + p_{+1}^\uparrow + p_{-1}^\uparrow + p_{+2}^\uparrow \tag{54}$$

with the spin-orbital arrangements

$$\begin{aligned} p_0^\uparrow &= \left(\uparrow\downarrow\uparrow = +\frac{1}{2}\right) + (L_z = 0) \\ p_{+1}^\uparrow &= \left(\downarrow\downarrow\uparrow = -\frac{1}{2}\right) + (L_z = +1) \\ p_{-1}^\uparrow &= \left(\uparrow\uparrow\uparrow = +\frac{3}{2}\right) + (L_z = -1) \\ p_{+2}^\uparrow &= \left(\downarrow\downarrow\downarrow = -\frac{3}{2}\right) + (L_z = +2). \end{aligned} \tag{55}$$

On the LF, the leading twist operators for the proton *uud* with positive parity, are typically of the form

$$(u_\uparrow C\gamma^+ u_\uparrow)d_\downarrow \qquad (u_\uparrow C i\sigma^{+i} u_\uparrow)d_\downarrow$$

with twist $\tau = \frac{7}{2} - \frac{3}{2} = 2$, and $C = i\gamma^2\gamma^0$ the charge conjugation matrix. Note that the two independent Ioffe′s currents

$$(u_\uparrow C\gamma_\mu u_\uparrow)\gamma^5\gamma^\mu d_\downarrow \qquad (u_\uparrow C i\sigma_{\mu\nu} u_\uparrow)\gamma^5\sigma^{\mu\nu} d_\downarrow$$

are twist $\tau = \frac{9}{2} - \frac{3}{2} = 3$ on the LF, hence subleading. In terms of the good component quark fields q_{+S}^C, the leading twist nucleon DAs for a spin-up proton are tied to the Fock states (46) and read [23]

$\mathbf{L_z = 0}$:

$$\begin{aligned} &\frac{\epsilon^{ABC}}{\sqrt{6}}\frac{1}{p^+}\langle 0|(u_{+\uparrow}^A(z_1)C\gamma^+ u_{+\downarrow}^B(z_2))d_{+\uparrow}^C(z_3)|p^\uparrow\rangle = \psi_1(z_1,z_2,z_3)N_+^\uparrow(p) \\ &\frac{\epsilon^{ABC}}{\sqrt{6}}\frac{1}{p^+}\langle 0|(u_{+\uparrow}^A(z_1)Ci\sigma^{+i} u_{+\uparrow}^B(z_2))d_{+\downarrow}^C(z_3)|p^\uparrow\rangle = (\psi_2(z_1,z_3,z_2) + \psi_2(z_2,z_3,z_1))\gamma^i N_+^\uparrow(p) \end{aligned} \tag{56}$$

$\mathbf{L_z = +1}$:

$$\begin{aligned} &\frac{\epsilon^{ABC}}{\sqrt{6}}\frac{1}{p^+}\langle 0|(u_{+\uparrow}^A(z_1)C\gamma^+ u_{+\downarrow}^B(z_2))d_{+\downarrow}^C(z_3)|p^\uparrow\rangle = (-i\not{\nabla}_{1\perp}\psi_3(z_1,z_2,z_3) - i\not{\nabla}_{2\perp}\psi_4(z_1,z_2,z_3))N_+^\uparrow(p) \\ &\frac{\epsilon^{ABC}}{\sqrt{6}}\frac{1}{p^+}\langle 0|(u_{+\downarrow}^A(z_1)Ci\sigma^{+i} u_{+\downarrow}^B(z_2))d_{+\uparrow}^C(z_3)|p^\uparrow\rangle = (i\nabla_{+1}^i(\psi_4(z_3,z_1,z_2) - \psi_3(z_3,z_1,z_2) - \psi_3(z_3,z_2,z_1)) \\ &+ (i\nabla_{+2}^i(\psi_4(z_3,z_2,z_1) - \psi_3(z_3,z_2,z_1) - \psi_3(z_3,z_1,z_2))))\gamma^i N_+^\uparrow(p) \end{aligned} \tag{57}$$

$\mathbf{L_z = -1}$:

$$\begin{aligned} &\frac{\epsilon^{ABC}}{\sqrt{6}}\frac{1}{p^+}\langle 0|(u_{+\uparrow}^A(z_1)iC\sigma^{+i} u_{+\uparrow}^B(z_2))d_{+\uparrow}^C(z_3)|p^\uparrow\rangle = (i\nabla_{-1}^i(\psi_5(z_1,z_3,z_2) - \psi_5(z_1,z_2,z_3)) \\ &+ i\nabla_{-2}^i(\psi_5(z_2,z_3,z_1) - \psi_5(z_2,z_1,z_3)))N_+^\uparrow(p) \end{aligned} \tag{58}$$

$$\mathbf{L_z = +2}:$$
$$\frac{\epsilon^{ABC}}{\sqrt{6}}\frac{1}{p^+}\langle 0|(u^A_{+\downarrow}(z_1)Ci\sigma^{+i}u^B_{+\downarrow}(z_2))d^C_{+\downarrow}(z_3)|p^\uparrow\rangle = (i\nabla^{\{i}_{+1}i\nabla^{j\}}_{\perp 2}(\psi_6(z_1,z_3,z_2)+\psi_6(z_2,z_1,z_3))$$
$$-\psi_6(z_1,z_3,z_2))-\psi_6(z_2,z_3,z_1))+i\nabla^{\{i}_{+1}i\nabla^{j\}}_{\perp 1}(\psi_6(z_1,z_3,z_2)+i\nabla^{\{i}_{+2}i\nabla^{j\}}_{\perp 2}(\psi_6(z_2,z_1,z_3))N^\uparrow_+(p). \quad (59)$$

Here $N^\uparrow_+(p)$ is the good nucleon on-shell component with $p^2 = m_N^2$, and the shorthand notation for the traceless symmetrization

$$\nabla^{\{i}\nabla^{j\}} = \nabla^i\nabla^j + \nabla^j\nabla^i - \delta^{ij}\nabla^k\nabla^k. \quad (60)$$

VIII. SUMMARY AND OUTLOOK

This paper is in many respects a methodical paper, devoted to novel technical tools of few-body quantum mechanics. We now provide a brief summary, and put forth few outlooks.

Summary: We started this paper by clarifying the use of the permutation group S_3, in particular its pertinent representations, essential for defining the symmetry properties of the wave functions for three quarks. As it is well known since the early 1960s, the baryon spin and flavor (isospin) parts of the wave function for the S-shell, do not factorize, as only their permutation symmetric and non-factorizable combination is allowed.

For the P-shell states (negative parity baryons) we have in addition the orbital $L = 1$ part of the wave functions to symmetrize. This is best achieved using the analogs of the ρ-type or λ-type Jacobi coordinates. In this case we need to construct triple tensor products of permutation matrices, and find their *totally symmetric* states. How to do that, without guessing and a loss of generality, is shown in Appendix A. A nontrivial result (not new but explicitly derived) is that there is a *unique* permutation symmetric wave function. In the next D-shell, the coordinates appear as products of two, so one has to find the totally symmetric wave function of four objects, and so on. All of those are found by the proposed method.

Our other methodical suggestion is to use the natural spin-tensor notations for the wave functions, and apply symbolic manipulation capabilities of programs like Maple or *Mathematica*. The use of symbolic "monoms" is traditional, yet it is better done in generic basis. Specifically, all possible spin-isospin monoms of three quarks are $4^3 = 64$. While for the nucleon only 9 monoms are needed, for the P-shell and D-shell baryons, the wave functions are much more involved. The universal spin-tensor notations allows for any type of symbolic operation when coded, e.g., spin-orbit with differentiation etc. In *Mathematica*, whether an operator is acting on the wave function with 1 or 64 components, makes no practical difference.

As a demonstration of this technique, we repeated the well known Isgur-Karl calculations of the splitting of the five negative parity N^* resonances. Unlike them, we explicitly construct the wave functions with proper permutation symmetry, and not as certain limits of less complicated Σ, Λ wave functions when $m_s \to m_u, m_d$. Like them, we also show that using only the spin-spin and tensor forces (but without spin-orbit) we can get a very good description of the mass spectrum for these states.

In this paper we stated that, apart from the orbital part, the basic wave function of all N^* should be spherically symmetric in 6 dimensions, $\phi(\sqrt{\vec{\rho}^2+\vec{\lambda}^2})$, yet we have not evaluated it. However, we note that in our recent study [16], we have used the appropriate "hyperdistance approximation" and the reduced radial Schrodinger equation, to obtain such wave functions, for (flavor-symmetric) baryons and (all-charmed) tetraquarks. The reason we focused on all-charm hadrons first, is that for heavy quarks the spin-dependent interactions—coming from relativistic corrections—are small and can be neglected in the zeroth approximation.

Construction of LF wave functions for negative parity baryons needs explicit definition of the longitudinal wave functions (depending on momentum fractions x_1, x_2, x_3) with symmetries of Jacobi coordinates ρ, λ. We have shown what these functions are in Sec. VI A.

Outlook: All of the present study is a preparation leading to the proper definitions of the light front wave functions (LFWFs) for multiquark hadrons, or their multiquark components. What we mean can be explained by examples: to a heavy $\bar{Q}Q$ mesons, we can add a light $\bar{q}q$ pair to form a *tetraquark*, while to a baryon we can add a pair to turn it to a *pentaquark*. With pertinent quantum numbers, these are "exotic hadrons," minimizing the Hamiltonian. Also, they can be considered as "virtual clouds" (as is the 5-quark component of the nucleon, seeding its antiquark sea, e.g., discussed in [19]) or separate "pentaquark states" orthogonal to the nucleon with all its cloud (as the recently discovered pentaquark resonances with hidden charm $uud + c\bar{c}$ states). Their discovery was helped by their narrow widths, as they happen to be just above the thresholds of baryon-meson states to which they can decay. The further discoveries of stable pentaquarks below such thresholds (e.g., with hidden $\bar{b}b$) are still ahead of us. Those with $\bar{u}u$, $\bar{d}d$, $\bar{s}s$ are perhaps not likely to be seen as separate resonances. However, the theory still needs to answer: where are they? what contribution to the known states (e.g., nucleon) they actually make?

When there are several identical quarks, the issues of appropriate symmetries of the wave functions need to be resolved. We have discussed in details the S-shell and P-shell baryons N^*. Our important point was that the LFWFs should have *the same structure under the S_3 permutation group* for their spin-isospin wave functions, as they have in the c.m. approach. The orbital part should not only contain the transverse $k_\perp^\pm$ momenta, as was proposed, but also it should include the longitudinal doubly degenerate excited states. We showed that those are $D^{\rho,\Lambda}_{n_L,m_L}$ consisting of six waves on an equilateral triangle for longitudinal momenta fractions. We have further shown, that while spherical symmetry is strictly speaking absent on the LF, the numerical deviations from the energies (and fixed-J wave functions in the c.m. frame) for P-shell and D-shell states constitute only several percents. Hopefully these differences are smaller than the splitting generated by the spin-spin and tensor forces, In this way, the squared mass splittings will not be affected by these "nonsphericity" corrections.

As parting comments, we note that one would not be able to calculate the $\bar{u}$, $\bar{d}$, $\bar{s}$, $\bar{c}$ contributions to the PDFs and other density matrices, *without* solving the many-body Hamiltonian, and obtain the full wave functions. And, e.g., for the five-quark components of the LFWFs with flavors content $uudu\bar{u}$, $uudd\bar{d}$, one would need also to combine the isospin representations with the color, spin, and orbital wave functions, to get the correct Fermi statistics. We hope to address those issues in subsequent publications.

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APPENDIX A: PERMUTATIONS AND THE WAVE FUNCTIONS

1. Permutation group S_3

The (12) and (23) permutations are realized as improper O(3) rotations on the Jacobi 2-vector, with determinant equal to -1. This is rather natural if we think of the initial three 1, 2, 3 particles set on an equilateral triangle. The six permutations (7) are then 3 in-plane rotations by $\frac{\pi}{3}$ of which P_4 is an example, and three out-of plane rotations by π along each of the 3 bisectors of which P_2 is an example.

With this in mind, the Jacobi 2-vector under the 3-particle permutation, transforms analogously to a spin-$\frac{1}{2}$ under SU(2) rotations. In retrospect this is expected, since the mixed representation Young tableau for 3 spin-$\frac{1}{2}$ carries the same dimension 2_M as the primitive spin-$\frac{1}{2}$ Young tableau. This observation will be repeatedly used below to construct the excited states of baryons with 3 quarks, with totally symmetric space-spin-flavor wave functions under S_3.

In general, the representations of S_3 fall into: 1/totally symmetric states (S), totally antisymmetric states (A), and states with mixed symmetry (M), under P_i. Three quark spin states can be split into $S=\frac{3}{2}$ symmetric states, $\uparrow\uparrow\uparrow$ and so on, and then two mutually orthogonal $S=\frac{1}{2}$ states which, following Jacobi coordinates, we call S^ρ, S^λ states. The following development consists of two steps. The first one, going back at least to [3], is related to symmetry under (12) reflection (8). With this in mind, any characterization of a 3-particle state (space, spin, flavor) can be composed by paralleling the Jacobi coordinates, with manifest symmetry under S_3. The idea, is to construct mixed symmetry wave functions $M^{\rho,\lambda}$, out of ρ-like and λ-like blocks which have pure S or A permutation properties,

$$[P_2=(12)]\begin{pmatrix}M^\rho\\M^\lambda\end{pmatrix}=\begin{pmatrix}-1&0\\0&1\end{pmatrix}\begin{pmatrix}M^\rho\\M^\lambda\end{pmatrix}$$
$$[P_4=(23)]\begin{pmatrix}M^\rho\\M^\lambda\end{pmatrix}=\begin{pmatrix}\frac{1}{2}&\frac{\sqrt{3}}{2}\\\frac{\sqrt{3}}{2}&-\frac{1}{2}\end{pmatrix}\begin{pmatrix}M^\rho\\M^\lambda\end{pmatrix} \tag{A1}$$

The construction of the representations of S_3 is carried as for any other groups, e.g., through the familiar generalization from the spinor representations of $O(3)$ to spin-1, spin-$\frac{3}{2}$ etc. The tensor product of two generic representations X_a and X_b with different symmetries under S_3, is a sum of representations X_{ab}, each with symmetries S, A, $M^{\rho,\lambda}$. While it is clear that the symmetries of the S, A products are

$$\begin{aligned}S_a\otimes S_b&=S_{ab}\\A_a\otimes A_b&=S_{ab}\\S_a\otimes A_b&=A_{ab}\end{aligned} \tag{A2}$$

the product of the mixed representations $M^{\rho,\lambda}$ viewed as a primitive 2_M doublets, is more subtle and requires more detailed studies.

For two ρ, λ-type blocks there are $2^2=4$ combinations, that can be constructed using say a basis $(X_1^\rho X_2^\rho, X_1^\lambda X_2^\rho, X_1^\rho X_2^\lambda, X_1^\lambda X_2^\lambda)$. Half of the states are symmetric and half are antisymmetric under 12-interchange. Yet what needs to be done, is to enforce say 23-permutation on these combinations. Using (8) two times, we obtain matrices corresponding to this permutation for two objects. The corresponding M(23) matrix in this basis takes the form

$$\begin{bmatrix}1/4&\sqrt{3}/4&\sqrt{3}/4&3/4\\\sqrt{3}/4&3/4&-1/4&-\sqrt{3}/4\\\sqrt{3}/4&-1/4&3/4&-\sqrt{3}/4\\3/4&-\sqrt{3}/4&-\sqrt{3}/4&1/4\end{bmatrix}. \tag{A3}$$

The commutator of M(12) with M(23) matrices, yields two imaginary and two zero eigenvalues. The latter correspond to the *totally* symmetric and antisymmetric combinations, respectively. The symmetric in this notations is (1, 0, 0, 1) or $X_1^\rho X_2^\rho + X_1^\lambda X_2^\lambda$, the only combination of two objects *simultaneously* symmetric under (12) and (23) (and in fact all) permutations. This is the one which we are seeking for the wave functions of the baryons in the ground S-shell, as well as, e.g., the three spin-$\frac{3}{2}$ nucleons in the P-shell. Although it may indeed be easy to simply guess this form, the method we follow eliminates any guessing, and can be used for any number of objects.

In analogy of what we do for the rotational (spin) group $O(3)$, we derived the tensor product of two representations into irreducible representations of S_3

$$2_M \otimes 2_M = 1_A \oplus 3_S$$

with 1_A a singlet antisymmetric, and 3_S a triplet of symmetric representations. The singlet 1_A antisymmetric representation is

$$A_{ab} = \frac{1}{\sqrt{2}} M_a^T \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} M_b = \frac{1}{\sqrt{2}} (M_a^\rho M_b^\lambda - M_a^\lambda M_b^\rho) \quad \text{(A4)}$$

while one of the triplet 3_S is, e.g.,

$$M_{ab}^\rho = \frac{1}{\sqrt{2}} M_a^T \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} M_b = \frac{1}{\sqrt{2}} (M_a^\rho M_b^\lambda + M_a^\lambda M_b^\rho). \quad \text{(A5)}$$

The remaining part of the triplet 3_S with projection-$\pm$, are regrouped in the manifestly orthogonal combinations

$$S_{ab} = \frac{1}{\sqrt{2}} M_a^T \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} M_b = \frac{1}{\sqrt{2}} (M_a^\rho M_b^\rho + M_a^\lambda M_b^\lambda) \quad \text{(A6)}$$

$$M_{ab}^\lambda = \frac{1}{\sqrt{2}} M_a^T \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} M_b = \frac{1}{\sqrt{2}} (M_a^\rho M_b^\rho - M_a^\lambda M_b^\lambda). \quad \text{(A7)}$$

S_{ab} is invariant under all the six rotations, hence all the six permutations. It is manifestly symmetric. The combinations $M_{ab}^{\rho,\lambda}$ can be checked to transform as a doublet under all permutations, e.g.,

$$[P_4 = (23)] \begin{pmatrix} M_{ab}^\rho \\ M_{ab}^\lambda \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} M_{ab}^\rho \\ M_{ab}^\lambda \end{pmatrix}. \quad \text{(A8)}$$

With this in mind, the spin-flavor combination

$$S_{ab} = \frac{1}{\sqrt{2}} (S_a^\rho F_b^\rho + S_a^\lambda F_b^\lambda) \quad \text{(A9)}$$

uniquely defines the symmetric part of the proton wave function.

For three objects of ρ, λ types, there are 2^3 combinations possible, written, e.g., in the basis of the following 8 monoms,

$$\begin{array}{llll} X_1^\lambda X_2^\lambda X_3^\lambda, & X_1^\rho X_2^\lambda X_3^\lambda, & X_1^\lambda X_2^\rho X_3^\lambda, & X_1^\rho X_2^\rho X_3^\lambda, \\ X_1^\lambda X_2^\lambda X_3^\rho, & X_1^\rho X_2^\lambda X_3^\rho, & X_1^\lambda X_2^\rho X_3^\rho, & X_1^\rho X_2^\rho X_3^\rho. \end{array}$$

One should proceed as we did before, calculating their transformation under (23)

$$M_{23}^{3\,\text{objects}} = \begin{bmatrix} -1/8 & \sqrt{3}/8 & \sqrt{3}/8 & -3/8 & \sqrt{3}/8 & -3/8 & ,-3/8, & 3\sqrt{3}/8 \\ \sqrt{3}/8, & 1/8, & -3/8, & -\sqrt{3}/8, & -3/8, & -\sqrt{3}/8, & 3\sqrt{3})/8, & 3/8 \\ \sqrt{3}/8, & -3/8, & 1/8, & -\sqrt{3}/8, & -3/8, & 3\sqrt{3}/8, & -\sqrt{3}/8, & 3/8 \\ -3/8, & -\sqrt{3}/8, & -\sqrt{3}/8, & -1/8, & 3\sqrt{3}/8, & 3/8, & 3/8, & \sqrt{3}/8 \\ \sqrt{3}/8, & -3/8, & -3/8, & 3\sqrt{3}/8, & 1/8, & -\sqrt{3}/8, & -\sqrt{3}/8, & 3/8 \\ -3/8, & -\sqrt{3}/8, & 3\sqrt{3}/8, & 3/8, & -\sqrt{3}/8, & -1/8, & 3/8, & \sqrt{3}/8 \\ -3/8, & 3\sqrt{3}/8, & -\sqrt{3}/8, & 3/8, & -\sqrt{3}/8, & 3/8, & -1/8, & \sqrt{3}/8 \\ 3\sqrt{3}/8, & 3/8, & 3/8, & \sqrt{3}/8, & 3/8, & \sqrt{3}/8, & \sqrt{3}/8, & 1/8 \end{bmatrix}.$$

Although this matrix may appear involved, its determinant is 1, with four eigenvalues (-1) and four $(+1)$. Note that it is the same set of eigenvalues as for the (diagonal) matrix of the (12) permutation. So, there is a 4-dimensional subspace which is symmetric under (23). The other 4-dimensional subspace (half of our basis) corresponds to the symmetric combinations under (12). The eigensystem of the commutator of these matrices, yields two zero eigenvalues, of which *only one* is the symmetric combination, $(-1, 0, 0, 1, 0, 1, 1, 0)$ in our basis, corresponding to the sought after symmetric wave function under S_3,

$$-X^\lambda X^\lambda X^\lambda + X^\rho X^\rho X^\lambda + X^\rho X^\lambda X^\rho + X^\lambda X^\rho X^\rho \quad \text{(A10)}$$

Again, no guessing is needed, the *unique* totally symmetric wave functions is shown by construction. We recall that

while our derivation is generic, we used it for three blocks here being the orbital, spin and isospin parts of the wave function.

This method can be used for any number of blocks, and we now proceed to four, as, e.g., those needed for the D-shell resonances. The matrix for the 23-permutation in this case is 16×16, with 8 antisymmetric and 8 symmetric eigenvectors. In fact it can be generated in *Mathematica* by using command *KroneckerProduct[P23,P23,P23]* (in which case the basis set is automatically selected by Kroneker product as well). The commutator of (12) and (23) matrices has 6 zero eigenvalues, of which 3 are symmetric. Their linear combinations can be used as wave functions in the D-shell. They are

$$\begin{aligned}&\{1,0,0,0,0,0,0,1,0,0,1,0,0,0,0,0,0,1\}\\&\{0,0,0,1,0,0,-1,0,0,-1,0,0,1,0,0,0\}\\&\{0,0,0,0,0,1,-1,0,0,-1,1,0,0,0,0,0\}.\end{aligned}\tag{A11}$$

Note that all of them are combination of four monoms weighted by simple ± 1 coefficients.

Although we do not have immediate applications for it, we repeated the procedure for 5 blocks, in the representation of dimension $2^5=32$. The corresponding (12) and (23) matrices were generated. Their commutator has 10 zero eigenvalues, with half corresponding to symmetric and half antisymmetric wave functions. The corresponding combinations are more complicated and have more than 4 nonzero terms.

APPENDIX B: DETAILS OF MATRIX ELEMENT CALCULATIONS

We have calculated the matrix elements of all the operators in two ways: by direct calculations using the explicit wave functions quoted earlier, and using symbolic manipulations of the wave functions represented as spin-tensors via *Mathematica*. We will detail here some parts of the explicit calculations for all the spin interactions, starting from the unmixed states with $J=\frac{5}{2}$, then proceed to the mixing matrix in the $J=\frac{3}{2}$ shell, and only quote the final results for the mixing matrix in the $J=\frac{1}{2}$ shell.

1. Spin-spin coupling in $J=\frac{3}{2}$

Using the explicit states (19)–(21) and short-hand notations, the spin-spin coupling contributions gives

$$\left\langle\frac{3}{2}\middle|\mathbb{V}_S\middle|\frac{3}{2}\right\rangle_{J=\frac{3}{2}}=\frac{3}{2}\left((\varphi^\rho\mathbb{V}_S(\sqrt{2}\rho)\varphi^\rho+\varphi^\lambda\mathbb{V}_S(\sqrt{2}\rho)\varphi^\lambda)\times\left(S^S_{\frac{3}{2}}\frac{1}{4}\sigma_1\cdot\sigma_2 S^S_{\frac{3}{2}}\right)\right)\tag{B1}$$

with the Clebsches and some indices omitted to avoid clutter. The overall factor of 3 follows from the permutation symmetry as we noted. The spin-spin contribution is reduced by the identity

$$\sigma_1\cdot\sigma_2 S^S_{\frac{3}{2},\frac{1}{2}}=\frac{1}{6}(4S^2_{\rm tot}-9)S^S_{\frac{3}{2},\frac{1}{2}}=S^S_{\frac{3}{2},\frac{1}{2}}\tag{B2}$$

in the permutation symmetric S-state, and gives

$$\left\langle\frac{3}{2}\middle|\mathbb{V}_S\middle|\frac{3}{2}\right\rangle_{J=\frac{3}{2}}=\frac{1}{4}\int d\vec{\rho}d\vec{\lambda}(\vec{\rho}^2+\vec{\lambda}^2)|\varphi_{00}|^2\mathbb{V}_S(\sqrt{2}\rho).\tag{B3}$$

The same arguments yield

$$\begin{aligned}\left\langle\frac{1}{2}\middle|\mathbb{V}_S\middle|\frac{1}{2}\right\rangle_{J=\frac{3}{2}}&=-\frac{1}{4}\int d\vec{\rho}d\vec{\lambda}(\vec{\rho}^2+\vec{\lambda}^2)|\varphi_{00}|^2\mathbb{V}_S(\sqrt{2}\rho)\\\left\langle\frac{3}{2}\middle|\mathbb{V}_S\middle|\frac{1}{2}\right\rangle_{J=\frac{3}{2}}&=0\end{aligned}\tag{B4}$$

where we used the spin identities

$$\begin{aligned}\sigma_1\cdot\sigma_2 S^\lambda_{\frac{11}{22}}&=+S^\lambda_{\frac{11}{22}}\\\sigma_1\cdot\sigma_2 S^\rho_{\frac{11}{22}}&=-3S^\rho_{\frac{11}{22}}.\end{aligned}\tag{B5}$$

a. Tensor coupling in $J=\frac{3}{2}$

The tensor contribution mixes configurations with different spin content, and is more involved. To evaluate it, we reestablish the Clebsches and azimuthal labelings

$$\left\langle\frac{3}{2}\middle|\mathbb{V}_T\middle|\frac{3}{2}\right\rangle_{J=\frac{3}{2}}=\frac{3}{2}\left(\left(\sqrt{\frac{3}{5}}\varphi^\rho_{10}S^S_{\frac{33}{22}}-\sqrt{\frac{2}{5}}\varphi^\rho_{11}S^S_{\frac{31}{22}}\right)\times\mathbb{V}_T\left(\sqrt{\frac{3}{5}}\varphi^\rho_{10}S^S_{\frac{33}{22}}-\sqrt{\frac{2}{5}}\varphi^\rho_{11}S^S_{\frac{31}{22}}\right)\right)\tag{B6}$$

where only the manifestly nonzero contributions are retained. Using the spin-space dependence of the tensor interaction, we find that the spin valued parts in (B13) can be reduced, using the matrix elements

$$\begin{aligned}S^S_{\frac{33}{22}}\mathbb{V}^\rho_T S^S_{\frac{33}{22}}&=+\frac{2}{3}\sqrt{\frac{\pi}{5}}Y^0_2(\hat{\rho})\\S^S_{\frac{31}{22}}\mathbb{V}^\rho_T S^S_{\frac{31}{22}}&=-\frac{2}{3}\sqrt{\frac{\pi}{5}}Y^0_2(\hat{\rho})\\S^S_{\frac{31}{22}}\mathbb{V}^\rho_T S^S_{\frac{33}{22}}&=-\frac{2}{3}\sqrt{\frac{2\pi}{5}}Y^{+1}_2(\hat{\rho})\\S^S_{\frac{33}{22}}\mathbb{V}^\rho_T S^S_{\frac{31}{22}}&=+\frac{2}{3}\sqrt{\frac{2\pi}{5}}Y^{-1}_2(\hat{\rho})\end{aligned}\tag{B7}$$

with

$$\mathbb{V}_T^\rho = \frac{1}{2}\left(\sigma_1\cdot\hat{\rho}\sigma_2\cdot\hat{\rho} - \frac{1}{3}\sigma_1\cdot\sigma_2\right) \tag{B8}$$

to give in shorthand notations

$$\left\langle\frac{3}{2}\middle|\mathbb{V}_T\middle|\frac{3}{2}\right\rangle_{J=\frac{3}{2}} = \frac{\sqrt{\pi}}{\sqrt{5}}\left(\frac{3}{5}\varphi_{10}Y_2^0\varphi_{10} - \frac{2}{5}\varphi_{11}Y_2^0\varphi_{11} - \frac{2\sqrt{3}}{5}\varphi_{10}Y_2^{-1}\varphi_{11} + \frac{2\sqrt{3}}{5}\varphi_{11}Y_2^{+1}\varphi_{10}\right). \tag{B9}$$

If we recall that

$$\varphi_{1m}^\rho = \sqrt{\frac{8\pi}{3}}Y_1^m(\hat{\rho})\rho\varphi_{00}$$
$$\varphi_{1m}^\lambda = \sqrt{\frac{8\pi}{3}}Y_1^m(\hat{\lambda})\lambda\varphi_{00} \tag{B10}$$

the integration over the three spherical harmonics in (B9) can be undone using the identity

$$\int d\hat{\rho}Y_{l_1}^{m_1}(\hat{\rho})Y_{l_2}^{m_2}(\hat{\rho})Y_{l_3}^{m_3}(\hat{\rho}) = \left(\frac{(2l_1+1)(2l_2+1)(2l_3+1)}{4\pi}\right)^{\frac{1}{2}}\begin{pmatrix} l_1 & l_2 & l_3\\ 0 & 0 & 0\end{pmatrix} \times \begin{pmatrix} l_1 & l_2 & l_3\\ m_1 & m_2 & m_3\end{pmatrix} \tag{B11}$$

with the result

$$\left\langle\frac{3}{2}\middle|\mathbb{V}_T\middle|\frac{3}{2}\right\rangle_{J=\frac{3}{2}} = \frac{4}{15}\int d\vec{\rho}d\vec{\lambda}\vec{\rho}^2|\varphi_{00}|^2\mathbb{V}_T(\sqrt{2}\rho). \tag{B12}$$

The remaining spin-spin and tensor matrix elements can be done similarly, with the results

$$\left\langle\frac{3}{2}\middle|\mathbb{V}_T\middle|\frac{1}{2}\right\rangle_{J=\frac{3}{2}} = -\frac{1}{3\sqrt{10}}\int d\vec{\rho}d\vec{\lambda}\vec{\rho}^2|\varphi_{00}|^2\mathbb{V}_T(\sqrt{2}\rho)$$
$$\left\langle\frac{1}{2}\middle|\mathbb{V}_T\middle|\frac{1}{2}\right\rangle_{J=\frac{3}{2}} = 0 \tag{B13}$$

for the remaining mixing matrix entries in the $J=\frac{3}{2}$ P-shell. We used the fact that the off-diagonal and nonvanishing tensor matrix elements are

$$S_{\frac{33}{22}}^S\mathbb{V}_T^\rho S_{\frac{11}{22}}^\lambda = +\frac{2}{3}\sqrt{\frac{\pi}{5}}Y_2^{-1}(\hat{\rho})$$
$$S_{\frac{31}{22}}^S\mathbb{V}_T^\rho S_{\frac{11}{22}}^\lambda = -\frac{2}{3}\sqrt{\frac{2\pi}{5}}Y_2^0(\hat{\rho}) \tag{B14}$$

with the vanishing diagonal ones

$$S_{\frac{11}{22}}^{\rho,\lambda}\mathbb{V}_T^\rho S_{\frac{11}{22}}^{\rho,\lambda} = 0. \tag{B15}$$

b. Spin-orbit coupling in $J=\frac{3}{2}$

When we calculate the LS interaction of one pair of quarks, 1–2, all orbital terms $\sim\lambda$ are eliminated by derivative over $\vec{\rho}$ and only ρ-dependent ones remain.

The spin-orbit contributions simplify considerably, if we note that φ^ρ spins along the λ-direction, and φ^λ spins along the ρ-direction. This means that the standard spin-orbit contribution in (43) has only a nonvanishing matrix element for the combination

$$\varphi^\rho(\rho\times p_\rho)\varphi^\rho = l_\rho \tag{B16}$$

as the two other entries vanish

$$\varphi^\lambda(\rho\times p_\rho)\varphi^\rho = \varphi^\lambda(\rho\times p_\rho)\varphi^\lambda = 0. \tag{B17}$$

The mixed spin-orbit contribution in (43) vanishes identically, as all entries

$$\varphi^{\rho,\lambda}(\rho\times p_\lambda)\varphi^{\rho,\lambda} = 0 \tag{B18}$$

are seen to angle average to zero. As a result, the 12-spin-orbit contribution (43) simplifies to

$$\mathbb{V}_{SL}(1,2) \to V_{SL}(\sqrt{2}\rho)\frac{1}{2}(\sigma_1+\sigma_2)\cdot l^\rho. \tag{B19}$$

The spin-orbit contribution mixes configurations with different spin content as well. Its evaluation follows that of the tensor coupling detailed above. More specifically, we have

$$\left\langle\frac{3}{2}\middle|\mathbb{V}_{SL}\middle|\frac{3}{2}\right\rangle_{J=\frac{3}{2}} = \frac{3}{2}\left(\left(\sqrt{\frac{3}{5}}\varphi_{10}^\rho S_{\frac{33}{22}}^S - \sqrt{\frac{2}{5}}\varphi_{11}^\rho S_{\frac{31}{22}}^S\right) \times \mathbb{V}_{SL}\left(\sqrt{\frac{3}{5}}\varphi_{10}^\rho S_{\frac{33}{22}}^S - \sqrt{\frac{2}{5}}\varphi_{11}^\rho S_{\frac{31}{22}}^S\right)\right). \tag{B20}$$

We can simplify the orbital contributions in (B20) if we recall that

$$l_\rho^z\varphi_{1m}^\rho = m\varphi_{1m}^\rho \qquad l_\rho^\pm\varphi_{1m}^\rho = \sqrt{2-m(m\pm1)}\varphi_{1m\pm1}^\rho \tag{B21}$$

with the result

$$\left\langle \frac{3}{2}\middle|\mathbb{V}_{SL}\middle|\frac{3}{2}\right\rangle_{J=\frac{3}{2}} = \frac{3}{10}\left(2|\varphi^{\rho}_{11}|^2 S^S_{\frac{31}{22}}\frac{1}{2}(\sigma_1+\sigma_2)^z S^S_{\frac{31}{22}} - \sqrt{3}|\varphi^{\rho}_{10}|^2\left(S^S_{\frac{33}{22}}\frac{1}{2}(\sigma_1+\sigma_2)^+ S^S_{\frac{31}{22}}+\text{c.c.}\right)\right). \tag{B22}$$

Using the permutation symmetry of the in-out spin symmetric S-states, we can make use of the identities

$$S^S_{\frac{31}{22}}\frac{1}{2}(\sigma_1+\sigma_2)^z S^S_{\frac{31}{22}} = \frac{2}{3}S^S_{\frac{31}{22}}S^z_{\text{tot}}S^S_{\frac{31}{22}} = \frac{1}{3}$$
$$S^S_{\frac{33}{22}}\frac{1}{2}(\sigma_1+\sigma_2)^+ S^S_{\frac{31}{22}} = \frac{2}{3}S^S_{\frac{31}{22}}S^+_{\text{tot}}S^S_{\frac{31}{22}} = \frac{2}{\sqrt{3}} \tag{B23}$$

which allow for the simplification of (B22) into

$$\left\langle \frac{3}{2}\middle|\mathbb{V}_{SL}\middle|\frac{3}{2}\right\rangle_{J=\frac{3}{2}} = \frac{1}{5}\int d\vec{\rho}d\vec{\lambda}(|\rho_-|^2 - 12\rho_z^2)|\varphi_{00}|^2\mathbb{V}_{SL}(\sqrt{2}\rho) = -\frac{2}{3}\int d\vec{\rho}d\vec{\lambda}\vec{\rho}^2|\varphi_{00}|^2\mathbb{V}_{SL}(\sqrt{2}\rho). \tag{B24}$$

The remaining spin-orbit contributions to the mixing matrix for $J=\frac{3}{2}$, can be found similarly with the results

$$\left\langle \frac{3}{2}\middle|\mathbb{V}_{SL}\middle|\frac{1}{2}\right\rangle_{J=\frac{3}{2}} = -\frac{\sqrt{10}}{6}\int d\vec{\rho}d\vec{\lambda}\vec{\rho}^2|\varphi_{00}|^2\mathbb{V}_{SL}(\sqrt{2}\rho)$$
$$\left\langle \frac{1}{2}\middle|\mathbb{V}_{SL}\middle|\frac{1}{2}\right\rangle_{J=\frac{3}{2}} = +\frac{1}{3}\int d\vec{\rho}d\vec{\lambda}\vec{\rho}^2|\varphi_{00}|^2\mathbb{V}_{SL}(\sqrt{2}\rho). \tag{B25}$$

c. 't Hooft coupling in $J=\frac{3}{2}$

The 't Hooft coupling contribution to the mixing matrix, involves flavor matrix elements. The calculations simplifies considerably if we note that the flavor states $F^{S,\rho,\lambda}$ are eigenstates of the flavor singlet projector,

$$(1-\tau_1\cdot\tau_2)F^S = 0F^S$$
$$(1-\tau_1\cdot\tau_2)F^\rho = 4F^\rho$$
$$(1-\tau_1\cdot\tau_2)F^\lambda = 0F^\lambda. \tag{B26}$$

With this in mind, a rerun of the preceding arguments gives

$$\left\langle \frac{3}{2}\middle|\mathbb{V}_{TH}\middle|\frac{3}{2}\right\rangle_{J=\frac{3}{2}} = \frac{8}{5}(1-a)\int d\vec{\rho}d\vec{\lambda}\vec{\rho}^2|\varphi_{00}|^2\mathbb{V}_{TH}(\sqrt{2}\rho)$$
$$\left\langle \frac{1}{2}|\mathbb{V}_{TH}\middle|\frac{1}{2}\right\rangle_{J=\frac{3}{2}} = 2\int d\vec{\rho}d\vec{\lambda}((1-a)\vec{\rho}^2 + (1+3a)\vec{\lambda}^2)|\varphi_{00}|^2\mathbb{V}_{TH}(\sqrt{2}\rho). \tag{B27}$$

d. Spin splittings in the $J=\frac{5}{2}$ shell

The spin $J=\frac{5}{2}$ shell is unmixed. The spin interactions in this shell are the simplest to evaluate. Using their explicit wave function, and some of the spin and flavor identities we derived earlier, we obtain

$$\left\langle \frac{3}{2}\middle|\mathbb{V}_S\middle|\frac{3}{2}\right\rangle_{\frac{5}{2}} = \frac{1}{4}\int d\vec{\rho}d\vec{\lambda}(\vec{\rho}^2+\vec{\lambda}^2)|\varphi_{00}|^2\mathbb{V}_S(\sqrt{2}\rho)$$
$$\left\langle \frac{3}{2}\middle|\mathbb{V}_T\middle|\frac{3}{2}\right\rangle_{\frac{5}{2}} = -\frac{1}{15}\int d\vec{\rho}d\vec{\lambda}\vec{\rho}^2|\varphi_{00}|^2\mathbb{V}_T(\sqrt{2}\rho)$$
$$\left\langle \frac{3}{2}\middle|\mathbb{V}_{SL}\middle|\frac{3}{2}\right\rangle_{\frac{5}{2}} = \int d\vec{\rho}d\vec{\lambda}\vec{\rho}^2|\varphi_{00}|^2\mathbb{V}_{SL}(\sqrt{2}\rho)$$
$$\left\langle \frac{3}{2}\middle|\mathbb{V}_{TH}\middle|\frac{3}{2}\right\rangle_{\frac{5}{2}} = 4(1-a)\int d\vec{\rho}d\vec{\lambda}\vec{\rho}^2|\varphi_{00}|^2\mathbb{V}_{TH}(\sqrt{2}\rho) \tag{B28}$$

with the short-hand notation $|\frac{3}{2}\rangle_{\frac{5}{2}} \equiv |1\,\frac{3}{2}\,\frac{5}{2}\,\frac{5}{2}\rangle_{p^-}$.

2. Mixing matrix

The hyperfine interactions in the degenerate P-sub-shells of fixed $J=\frac{1}{2},\frac{3}{2}$ are fixed by

$$\mathbb{M}_J = \frac{1}{\langle p^-|p^-\rangle}\begin{pmatrix} \langle\frac{3}{2}|\mathbb{V}_{S+T+SL+V_{TH}}|\frac{3}{2}\rangle & \langle\frac{3}{2}|\mathbb{V}_{S+T+SL+V_{TH}}|\frac{1}{2}\rangle \\ \langle\frac{1}{2}|\mathbb{V}_{S+T+SL+V_{TH}}|\frac{3}{2}\rangle & \langle\frac{1}{2}|\mathbb{V}_{S+T+SL+V_{TH}}|\frac{1}{2}\rangle \end{pmatrix}_J \tag{B29}$$

where we used the S-labeling as a short-hand for the degenerate nucleon P-states, which are normalized by

$$\langle p^-|p^-\rangle = \frac{1}{3}\mathbb{N}_P = \frac{1}{3}\int d\vec{\rho}d\vec{\lambda}(\vec{\rho}^2+\vec{\lambda}^2)|\varphi_{00}|^2. \tag{B30}$$

The diagonalization of (B29) yields the two mixing angles in the P-shell with $J=\frac{1}{2},\frac{3}{2}$. For $J=\frac{3}{2}$, their explicit forms we already given in results (30)

APPENDIX C: EXPLICIT WAVE FUNCTIONS OF BARYONS IN SPIN-TENSOR NOTATIONS IN *MATHEMATICA*

Mathematica is a platform for symbolic calculations widely used in many branches of physics. In some (e.g., general relativity) its ability to handle multicomponent tensors and perform, e.g., hundreds of differentiations, is crucial for progress in the field, as recognized long ago. (A side remark about Maple: of course one can do anything in it as well. Yet its elaborate structures—sets, arrays, vectors, matrices seem a bit cumbersome, at least for beginners. A single notion of *Table* in *Mathematica* is the only one needed.)

In quantum mechanics, with applications like the ones discussed in this paper, *Mathematica*'s ability to do *analytic* (rather than numeric) computations are not yet sufficiently utilized. A wave function may have hundreds or thousands of components, which can be naturally added, combined by tensor products, acting upon by spin-dependent or differential operators, squared and integrated. The rules of operation with Tables of any number of dimensions are simple or even elementary, yet some explanation may be helpful to some readers. That is why in this paper we focused on the 3-quark systems, as their wave functions have $2^6 = 64$ "monoms." It is already complicated enough, so that issues of, e.g., symmetries of identical quarks are nontrivial. This way of solving the few-quark problems, should prove useful for the other (newly discovered) tetraquarks, and pentaquarks states as well.

The usual representation of quantum states is done with certain *monoms* times coordinate functions. The important case of the nucleon is a well known example. Three light quarks, with 2 spin times 2 isospin states, lives in a basis of $2^6 = 64$ possible monoms. Only nine are in (9) for this example. A different number with other functional coefficients appear for the excited (e.g., P-shell) states. The natural standardized way to represent all wave functions is provided by *spin-tensors*. The suggested order of the indices in their definition is arbitrary, so we propose to keep the spin and then isospin s_1, s_2, s_3, i_1, i_2, i_3 sequentially, each with binary values 1 or 2. Up and down notations can be held either symbolically (in formulas) or explicitly, for which we define the elementary states

$$\text{up} = \{1, 0\}; \qquad \text{down} = \{0, 1\}$$

and then use the *substitution* function while forming actual spin-tensor. Let us take as an example the $S = \frac{1}{2}$, $S_z = \frac{1}{2}$ expressions for the ρ, λ blocks we used above $S^\rho = (\uparrow\downarrow\uparrow - \downarrow\uparrow\uparrow)/\sqrt{2}$ and show the *Mathematica* command converting it to explicit spin-tensor form

$$\begin{aligned} S\rho &:= (u[[s1]]d[[s2]]u[[s3]] - d[[s1]]u[[s2]]u[[s3]])/\sqrt{2}; \\ S\rho^{\text{numeric}} &= Table[(S\rho/.\{u- > up, d- > down\}, \{s1, 1, 2\}, \{s2, 1, 2\}, \{s3, 1, 2\}] \\ &= \{\{\{0, 0\}, \{-1/\sqrt{2}, 0\}\}, \{\{1/\sqrt{2}, 0\}, \{0, 0\}\}\} \end{aligned} \tag{C1}$$

Note that we are using the u, d notations in any symbolic expression. The explicit numerical form follows through the substitution to the *up*, *down* monoms. While keeping unnecessary components represented by 6 zeros, it is still quite compact and convenient to use. The isospin expressions with $I = \frac{1}{2}$ are the same, with only the indices redefined as $i1$, $i2$, $i3$.

The spin-isospin wave function of the proton (spin up) is then

$$\begin{aligned} p\uparrow &= Table[(S\rho[[s1, s2, s3]]S\rho[[i1, i2, i3]] + S\lambda[[s1, s2, s3]]S\lambda[[i1, i2, i3]])/\sqrt{2}, \\ &\quad \{s1, 1, 2\}, \{s2, 1, 2\}, \{s3, 1, 2\}, \{i1, 1, 2\}, \{i2, 1, 2\}, \{i3, 1, 2\}] \\ &= \{\{\{\{\{0, 0\}, \{0, 0\}\}, \{\{0, 0\}, \{0, 0\}\}\}, \{\{\{0, \sqrt{2}/3\}, \{-1/(3\sqrt{2}), 0\}\}, \{\{-1/(3\sqrt{2}), 0\}, \\ &\quad \{0, 0\}\}\}\}, \{\{\{\{0, -1/(3\sqrt{2})\}, \{\sqrt{2}/3, 0\}\}, \{\{-1/(3\sqrt{2}), 0\}, \{0, 0\}\}\}, \{\{\{0, 0\}, \{0, 0\}\}, \\ &\quad \{\{0, 0\}, \{0, 0\}\}\}\}, \{\{\{\{0, -1/(3\sqrt{2})\}, \{-1/(3\sqrt{2}), 0\}\}, \{\{\sqrt{2}/3, 0\}, \{0, 0\}\}\}, \{\{\{0, 0\}, \{0, 0\}\}, \{\{0, 0\}, \\ &\quad \{0, 0\}\}\}\}, \{\{\{\{0, 0\}, \{0, 0\}\}, \{\{0, 0\}, \{0, 0\}\}\}, \{\{\{0, 0\}, \{0, 0\}\}, \{\{0, 0\}, \{0, 0\}\}\}\}\}. \end{aligned} \tag{C2}$$

In this example only 9 elements (out of 64) are nonzero. The matrix elements are evaluated as usual, e.g., the normalization is the sum over all indices

$$\langle p|p\rangle = Sum[p[[s1, s2, s3, i1, i2, i3]]^2, \{s1, 1, 2\}, \{s2, 1, 2\}, \{s3, 1, 2\}, \{i1, 1, 2\}, \{i2, 1, 2\}, \{i3, 1, 2\}].$$

If complex, the outgoing wave function needs to be conjugated, as usual.

For nonzero orbital momentum (e.g. $L = 1$ to be discussed), the explicit orbital functions are linear in coordinates, and will be defined below. With 6 coordinates $\vec{\rho}$, $\vec{\lambda}$ in any matrix elements, we will perform explicitly the integration over the 4 angles in both solid angles, $d\Omega_\rho d\Omega_\lambda$. All the wave functions to be shown below have the same normalization factor

$$N_{\text{norm}} = \frac{32\pi^2}{3}\langle\lambda^2 + \rho^2\rangle \tag{C3}$$

by which the matrix elements of operators should be divided. (The meaning of angular brackets here was defined in (29), it contains a double integral over the ρ, λ moduli, with the remaining radial wave functions.).

The matrix elements of any operators including spin, isospin and coordinate variables (in matrix or differential form) can easily be calculated from these wave functions, with the summation over all indices. For P-shell nucleons, we constructed nine of them, with $|J, J_z, S\rangle$ selections

$$\left|\frac{5}{2},\frac{5}{2},\frac{3}{2}\right\rangle\left|\frac{5}{2},\frac{3}{2},\frac{3}{2}\right\rangle,\quad \left|\frac{5}{2},\frac{1}{2},\frac{3}{2}\right\rangle,$$
$$\left|\frac{3}{2},\frac{3}{2},\frac{3}{2}\right\rangle,\quad \left|\frac{3}{2},\frac{1}{2},\frac{3}{2}\right\rangle,$$
$$\left|\frac{3}{2},\frac{3}{2},\frac{1}{2}\right\rangle,\quad \left|\frac{3}{2},\frac{1}{2},\frac{1}{2}\right\rangle,|$$
$$\left|\frac{1}{2},\frac{1}{2},\frac{3}{2}\right\rangle,\quad \left|\frac{1}{2},\frac{1}{2},\frac{1}{2}\right\rangle.$$

After checking normalization and mutual orthogonality, the matrix elements of all spin-dependent interactions have been calculated. Since operators are rotationally scalars, all states which differ in J_z (orientation) only, must yield the same matrix elements.

We explicitly show in Figs. 3–7 five wave functions, all with $J_z = 1/2$, for which pertinent Clebsch-Gordon coefficients and coordinates are included. (Let us remind the reader that those are shown *before* mixing, when they still have definite total spins, $S = \frac{3}{2}$ and $S = \frac{1}{2}$.) We present those spin-tensors in full, with 6 indices and 6 curly brackets, and also keep all indices in operators to avoid confusion.

(While it is not necessary, one can further use even more compressed notations. By *Flatten* command, the wave functions are reduced from spin-tensors to 64-d vectors in the "monom space." All operators can also be redefined in it, as 64×64 matrices. The benefit of this, is that in this form they can be multiplied as ordinary matrices. Let us demonstrate how it works for the spin-spin interaction

```
s := Table[1/2*PauliMatrix[i], {i, 1, 3}]
Slist[i_, m_] := Insert[Table[IdentityMatrix[2], 5], s[[m]], i]
Si[i_, m_] := KroneckerProduct @@ Slist[i, m]
SS[i_, j_] := Sum[Si[i, m].Si[j, m], {m, 1, 3}]
```

Note that Slist is a list of 6 matrices, with Pauli spin/isospin matrices at the position i. Next line (Si) promotes it to an operator in the monom space. In such notations complex operators can be written as the usual sum of their products, e.g., the spin-spin interaction is $(\vec{S}_1\vec{S}_2) \to SS[1,2]$.)

Before we present the wave functions for these states, we show their "coordinate density" $|\psi|^2$ summed over all indices,

$$|N^*_{J=\frac{5}{2},J_z=1/2,S=3/2}|^2 \sim (2/5)(\lambda_1^2+\lambda_2^2+3\lambda_3^2+\rho_1^2+\rho_2^2+3\rho_3^2)$$
$$|N^*_{J=3/2,J_z=1/2,S=3/2}|^2 \sim (2/15)(7\lambda_1^2+7\lambda_2^2+\lambda_3^2+7\rho_1^2+7\rho_2^2+\rho_3^2)$$
$$|N^*_{J=1/2,J_z=1/2,S=3/2}|^2 \sim (2/3)(\lambda_1^2+\lambda_2^2+\lambda_3^2+\rho_1^2+\rho_2^2+\rho_3^2)$$
$$|N^*_{J=3/2,J_z=1/2,S=1/2}|^2 \sim (1/3)(\lambda_1^2+\lambda_2^2+4\lambda_3^2+\rho_1^2+\rho_2^2+4\rho_3^2)$$
$$|N^*_{J=1/2,J_z=1/2,S=1/2}|^2 \sim (2/3)(\lambda_1^2+\lambda_2^2+\lambda_3^2+\rho_1^2+\rho_2^2+\rho_3^2).$$

Note that all the $J = \frac{1}{2}$ states have 6d spherical shape, as they should, while the $J = \frac{5}{2}$ have deformations of opposite signs to both the $J = \frac{3}{2}$ states.

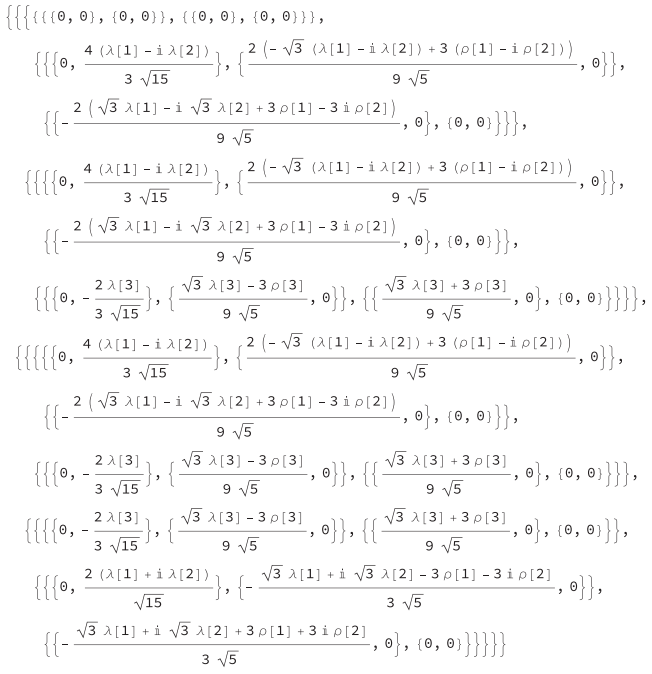

FIG. 3. Orbital-Orbital-spin-isospin wave function for $|N^*, J = \frac{5}{2}, Jz = 1/2, S = 3/2\rangle$.

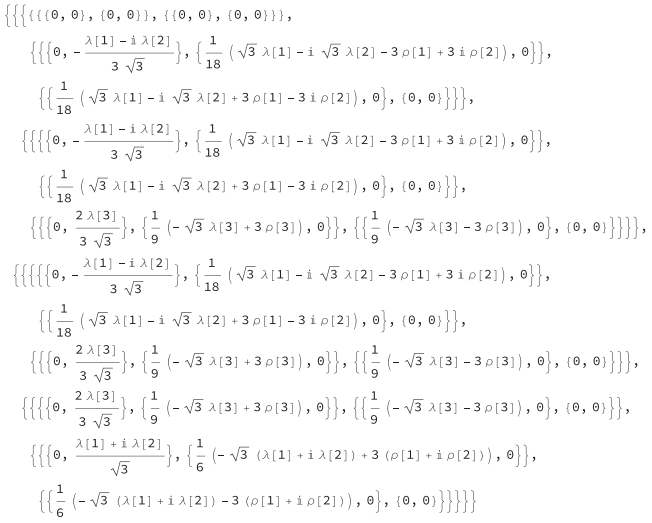

FIG. 4. Orbital-spin-isospin wave function for $|N^*, J = 3/2, Jz = 1/2, S = 3/2\rangle$.

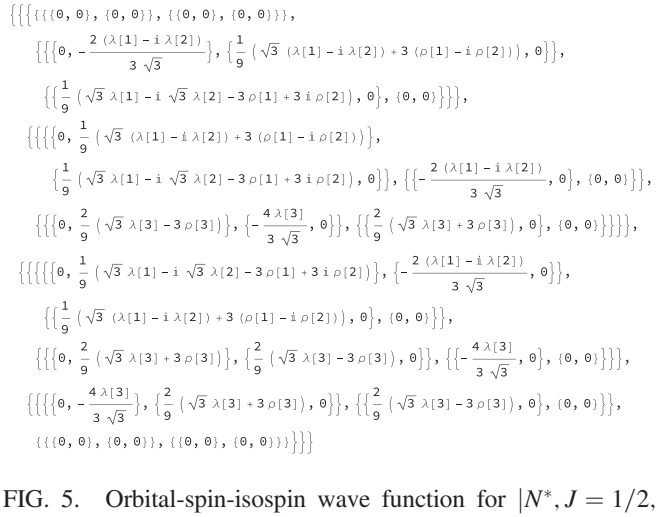

FIG. 5. Orbital-spin-isospin wave function for $|N^*, J = 1/2, Jz = 1/2, S = 3/2\rangle$.

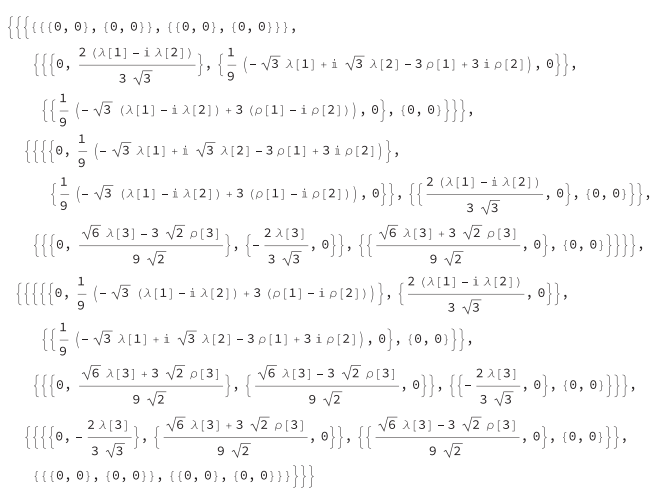

FIG. 6. Orbital-spin-isospin wave function for $|N^*, J = 3/2, Jz = 1/2, S = 1/2\rangle$.

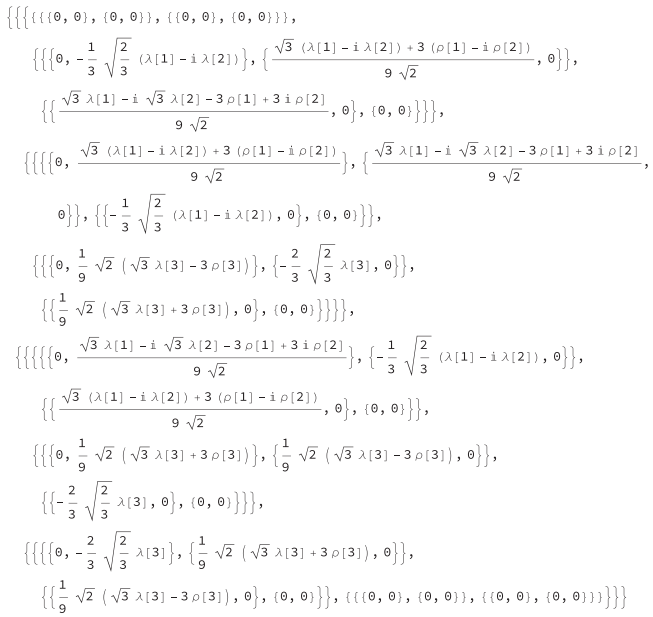

FIG. 7. Orbital-spin-isospin wave function for $|N^*, J = 1/2, Jz = 1/2, S = 1/2\rangle$.

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