Causality and stability analysis for the minimal causal spin hydrodynamics

Xin-Qing Xie,[1] Dong-Lin Wang,[1,*] Chen Yang,[1] and Shi Pu[1,2,†]

[1]*Department of Modern Physics, University of Science and Technology of China, Anhui 230026, China*
[2]*Southern Center for Nuclear-Science Theory (SCNT), Institute of Modern Physics, Chinese Academy of Sciences, Huizhou 516000, Guangdong Province, China*

(Received 11 August 2023; accepted 25 October 2023; published 21 November 2023)

We perform the linear analysis of causality and stability for a minimal extended spin hydrodynamics up to second order of the gradient expansion. The first order spin hydrodynamics, with a rank-3 spin tensor being antisymmetric for only the last two indices, are proved to be acausal and unstable. We then consider the minimal causal spin hydrodynamics up to second order of the gradient expansion. We derive the necessary causality and stability conditions for this minimal causal spin hydrodynamics. Interestingly, the satisfaction of the stability conditions relies on the equations of state for the spin density and chemical potentials. Moreover, different with the conventional relativistic dissipative hydrodynamics, the stability of the theory seems to be broken at the finite wave vector when the stability conditions are fulfilled at small and large wave vector limits. It implies that the behavior in small and large wave vector limits may be insufficient to determine the stability conditions for spin hydrodynamics in linear mode analysis.

DOI: 10.1103/PhysRevD.108.094031

I. INTRODUCTION

Relativistic heavy ion collisions provide a novel platform to study the spin physics. In noncentral relativistic heavy-ion collisions, the quark-gluon plasma (QGP) with large angular momentum perpendicular to the reaction plane is created. Because of the total angular momentum conservation, the averaged spin of final particles produced from QGP is polarized along the direction of the initial orbital angular momentum [1–3], as known as the global polarization. The measurements of the global polarization for Λ, $\bar{\Lambda}$, and other hyperons [4–10] can be understood well by various phenomenological models [11–26]. The experimental data also indicates that the QGP generated in noncentral relativistic heavy-ion collisions is the most vortical fluid ever observed [4]. STAR [6,27] and ALICE [28] collaborations also measured the local polarization of Λ and $\bar{\Lambda}$ along the beam and out-of-plane directions. Interestingly, the sign of local polarization in theoretical calculations is opposite to that of experimental data [15,16,23,29–31]. To resolve the disagreement, a great deal of effort has been taken in feed-down effects [32,33], hadronic interactions [34,35], relativistic spin hydrodynamics [30,36–74], statistical models [29,75,76], quantum kinetic theory [77–114], effective theories [115–117], and other phenomenological models [16,20,21,23,31–33,118–122]. Although there is much important progress [26,117,121–129], the local polarization has not been fully understood. Another important phenomenon related to spin, called the spin alignment of vector mesons proposed by Refs. [1–3], has drawn a lot of attention. The spin alignment is characterized by the deviation of ρ_{00} from $1/3$, where ρ_{00} is the 00 component of the spin density matrix of vector mesons [130]. A nonvanishing $\rho_{00} - 1/3$ indicates a net spin alignment of vector mesons. The experimental results [131–136] show that the magnitude of the spin alignment of the vector meson is much larger than that caused by vorticity and other conventional effects [2,137–141]. Such unexpectedly large spin alignment may arise from a fluctuating strong force field of ϕ [142–146].

The above novel phenomena related to spin triggered the rapid developments of spin hydrodynamics [30,36–73]. The spin hydrodynamics is a natural extension of the conventional hydrodynamics coupled with the dynamic evolution of spin through the total angular momentum conservation. The spin hydrodynamics incorporating the quantum property may serve as a powerful tool to understand the novel phenomena about spin in noncentral relativistic heavy-ion collisions. Over the past few years, various approaches have been proposed to construct spin hydrodynamics, such as entropy current analysis [45,48,50,57–59,67,73], quantum kinetic theory

*donglinwang@mail.ustc.edu.cn
†shipu@ustc.edu.cn

[38–40,49,51,55,56,63,65,66,69,86,94,101,111,147], holographic duality [52,53], and the effective Lagrangian method [36,37].

In spite of the substantial efforts, the arbitrariness due to pseudogauge transformations in spin hydrodynamics is not fully understood. Through the pseudogauge transformations [148,149], one can obtain new forms of energy momentum tensor and spin tensor without affecting the conservation law. Although such transformations have no impact on the total conserved charges, they indeed change the values of locally defined quantities, e.g., energy momentum tensor and spin tensor [41,147–149]. Thus, different pseudogauge transformations give rise to different frameworks of the spin hydrodynamics, e.g., canonical [45,59,150], Belinfante [48], Hilgevoord-Wouthuysen [65,151], de Groot-van Leeuwen-van Weert [43,152] forms. Which framework is suitable for understanding the experimental data leads to intense discussions [41,48,66,147,153–155].

So far, the spin hydrodynamics in the first order of the gradient expansion, with a rank-3 spin tensor that exhibits antisymmetry solely in its last two indices, has been established [45,48]. Before simulating the spin hydrodynamics, it is necessary to investigate the theory's causality and stability, as is done in conventional hydrodynamics. In fact, the first order conventional relativistic hydrodynamics at the Landau frame in the gradient expansion are always acausal and unstable, e.g., see the discussions in Refs. [156–159]. Therefore, the question whether the first order spin hydrodynamics can be casual or stable arises. Several studies conclude that the spin hydrodynamics up to the first order in the gradient expansion may be acausal and unstable in the linear modes analysis [70,71]. In the early study [45], the authors have modified the constitutive relations for the antisymmetric part of energy momentum tensor through the equations of motion for the fluid and the stability conditions of this first order theory in the rest frame of fluid seem to be satisfied in the linear modes analysis. Later on, Ref. [71] shows that this first order theory may be acausal, while Ref. [70] finds the stability conditions [which corresponds to Eq. (46) in this work] may not be satisfied.

In this work, we systematically investigate the linear causality and stability for the spin hydrodynamics proposed in Refs. [45,48]. Our findings indicate that the spin hydrodynamics up to the first order in the gradient expansion is acausal and unstable even when using the replacement mentioned by Ref. [45]. The acausal and unstable modes can usually be removed when extending the theory up to the second order in the gradient expansion. Therefore, we follow the method outlined in the conventional hydrodynamics [158–162] to consider the minimal causal spin hydrodynamics. It is sufficient to see whether the causality and stability can be recovered up to the second order in the gradient expansion [158–162]. We then analyze the causality and stability for this minimal extended theory.

The paper is organized as follows. We first review the first order spin hydrodynamics introduced in Refs. [45,48] in Sec. II and show it is acausal and unstable in Sec. III. In Sec. IV, we consider the minimal causal spin hydrodynamics following the method outlined in the conventional hydrodynamics. In Sec. V, we analyze the causality and stability for the minimal causal spin hydrodynamics in the rest frame and comment on the results in moving frames. We summarize this work in Sec. VI.

Throughout this work, we work with the metric $g_{\mu\nu} = \mathrm{diag}\{+,-,-,-\}$ and $\Delta_{\mu\nu} = g_{\mu\nu} - u_\mu u_\nu$. For a rank-2 tensor $A^{\mu\nu}$, we introduce the short-hand notations $A^{(\mu\nu)} \equiv (A^{\mu\nu} + A^{\nu\mu})/2$, $A^{[\mu\nu]} \equiv (A^{\mu\nu} - A^{\nu\mu})/2$, and $A^{\langle\mu\nu\rangle} \equiv \frac{1}{2}[\Delta^{\mu\alpha}\Delta^{\nu\beta} + \Delta^{\mu\beta}\Delta^{\nu\alpha}]A_{\alpha\beta} - \frac{1}{3}\Delta^{\mu\nu}(\Delta^{\alpha\beta}A_{\alpha\beta})$.

II. FIRST ORDER SPIN HYDRODYNAMICS

In this section, let us briefly review the first order relativistic spin hydrodynamics. In spin hydrodynamics, we have the conservation equations for energy, momentum, total angular momentum, and particle number, i.e., [45,48,50,58,59,67,163]

$$\partial_\mu \Theta^{\mu\nu} = 0, \qquad \partial_\lambda J^{\lambda\mu\nu} = 0, \qquad \partial_\mu j^\mu = 0, \tag{1}$$

where $\Theta^{\mu\nu}$ is the energy momentum tensor, $J^{\lambda\mu\nu}$ is the total angular momentum current, and j^μ is the current for particle number. Different from conventional relativistic hydrodynamics, the total angular momentum conservation equation in Eq. (1) plays a crucial role to describe the evolution of spin. The total angular momentum current can be written as [45,48]

$$J^{\lambda\mu\nu} = x^\mu \Theta^{\lambda\nu} - x^\nu \Theta^{\lambda\mu} + \Sigma^{\lambda\mu\nu}, \tag{2}$$

where the first two terms correspond to the conventional orbital angular momentum, and $\Sigma^{\lambda\mu\nu}$ is the rank-3 spin tensor. Using Eq. (2), the conservation equation $\partial_\lambda J^{\lambda\mu\nu} = 0$ can be rewritten as the spin evolution equation,

$$\partial_\lambda \Sigma^{\lambda\mu\nu} = -2\Theta^{[\mu\nu]}. \tag{3}$$

Equation (3) implies that the antisymmetric part of energy momentum tensor $\Theta^{[\mu\nu]}$ is the source for spin, and the spin can be viewed as a conserved quantity if and only if $\Theta^{[\mu\nu]} = 0$.

After introducing the spin degrees of freedom, the thermodynamic relations in spin hydrodynamics are modified as [45,48,50,58,59,67,163]

$$e + p = Ts + \mu n + \omega_{\mu\nu} S^{\mu\nu}, \tag{4}$$

$$de = Tds + \mu dn + \omega_{\mu\nu} dS^{\mu\nu}, \tag{5}$$

where $e, p, T, s, n, \mu, \omega_{\mu\nu}$, and $S^{\mu\nu}$ denote energy density, pressure, temperature, entropy density, particle number density, chemical potential, spin chemical potential, and spin density. The spin density is defined as

$$S^{\mu\nu} \equiv u_\lambda \Sigma^{\lambda\mu\nu} \tag{6}$$

with the fluid velocity u^μ. Analogous to the relationship between μ and n, here we introduce the antisymmetric spin chemical potential $\omega_{\mu\nu}$ as the conjugate of $S^{\mu\nu}$.

Before decomposing the $\Theta^{\mu\nu}$ and $\Sigma^{\lambda\mu\nu}$, we emphasize that there exist different choices for them. For example, by applying the Nöther theorem to two equivalent Lagrangian density for Dirac field,

$$\mathcal{L}_1 = \bar{\psi}(i\gamma\cdot\partial - m)\psi, \tag{7}$$

$$\mathcal{L}_2 = \frac{1}{2}\bar{\psi}i\gamma\cdot\overleftrightarrow{\partial}_\mu\psi - m\bar{\psi}\psi, \tag{8}$$

where $\overleftrightarrow{\partial}_\mu \equiv \overrightarrow{\partial}_\mu - \overleftarrow{\partial}_\mu$, two distinct sets of energy momentum tensors and spin tensors emerge,

$$\Theta_1^{\mu\nu} = \bar{\psi}i\gamma^\mu\partial^\nu\psi, \qquad \Sigma_1^{\lambda\mu\nu} = \frac{1}{4}\bar{\psi}i\gamma^\lambda[\gamma^\mu,\gamma^\nu]\psi, \tag{9}$$

$$\Theta_2^{\mu\nu} = \frac{i}{2}\bar{\psi}\gamma^\mu\overleftrightarrow{\partial^\nu}\psi, \qquad \Sigma_2^{\lambda\mu\nu} = \frac{1}{8}\bar{\psi}i\{\gamma^\lambda,[\gamma^\mu,\gamma^\nu]\}\psi. \tag{10}$$

Here, $\Sigma_1^{\lambda\mu\nu}$ is antisymmetric only with respect to μ and ν indices, while $\Sigma_2^{\lambda\mu\nu}$ is totally antisymmetric. In principle, one can derive the spin hydrodynamics from the microscopic theories, as demonstrated in Refs. [43,65,66,69,164] for kinetic theories and Refs. [41,165,166] for statistical methods. An alternative method to derive the spin hydrodynamics is to map the tensor structure of hydrodynamic variables to operators mentioned above, e.g. see Refs. [45,48,59]. In this work, we follow Refs. [45,48] and adopt the energy momentum tensor and spin tensor sharing the similar tensor structure with $\Theta_1^{\mu\nu}$ and $\Sigma_1^{\lambda\mu\nu}$, respectively. For other choices, one can refer to Refs. [59,147,155] and references therein.

Following Refs. [45,48], the energy momentum tensor and particle current can be decomposed as

$$\Theta^{\mu\nu} = eu^\mu u^\nu - (p+\Pi)\Delta^{\mu\nu} + 2h^{(\mu}u^{\nu)} + \pi^{\mu\nu} + 2q^{[\mu}u^{\nu]} + \phi^{\mu\nu}, \tag{11}$$

$$j^\mu = nu^\mu + \nu^\mu, \tag{12}$$

where h^μ, ν^μ, Π, and $\pi^{\mu\nu}$ stand for heat current, particle diffusion, bulk viscous pressure, and shear stress tensor, respectively, and the antisymmetric parts $2q^{[\mu}u^{\nu]}$ and $\phi^{\mu\nu}$ are related to the spin effects. As for the rank-3 spin tensor $\Sigma^{\lambda\mu\nu}$, we have [45,48]

$$\Sigma^{\lambda\mu\nu} = u^\lambda S^{\mu\nu} + \Sigma_{(1)}^{\lambda\mu\nu}, \tag{13}$$

where the spin density $S^{\mu\nu}$ defined in Eq. (6) has six independent degrees of freedom.

In this work, we follow the power counting scheme in Refs. [48,62,64],

$$S^{\mu\nu} \sim O(1), \qquad \omega_{\mu\nu} \sim O(\partial), \qquad \Sigma_{(1)}^{\lambda\mu\nu} \sim O(\partial). \tag{14}$$

The spin density $S^{\mu\nu}$ is chosen as the leading order in the gradient expansion. It corresponds to the case in which most of the particles in the system are polarized, i.e. the order of $S^{\mu\nu}$ is considered as the same as the one for the number density n. In Refs. [45,59], the authors have chosen a different power counting scheme, $S^{\mu\nu} \sim O(\partial)$, $\omega_{\mu\nu} \sim O(\partial)$, $\Sigma_{(1)}^{\lambda\mu\nu} \sim O(\partial^2)$.

Following [45,48], it is straightforward to get the entropy production rate,

$$\begin{aligned}\partial_\mu \mathcal{S}^\mu_{\mathrm{can}} = {}& \left(h^\mu - \frac{e+p}{n}\nu^\mu\right)\left[\partial_\mu \frac{1}{T} + \frac{1}{T}(u\cdot\partial)u_\mu\right] \\ &+ \frac{1}{T}\pi^{\mu\nu}\partial_\mu u_\nu - \frac{1}{T}\Pi(\partial\cdot u) + \frac{1}{T}\phi^{\mu\nu}(\partial_\mu u_\nu + 2\omega_{\mu\nu}) \\ &+ \frac{q^\mu}{T}\left[T\partial_\mu\frac{1}{T} - (u\cdot\partial)u_\mu + 4\omega_{\mu\nu}u^\nu\right] + O(\partial^3),\end{aligned} \tag{15}$$

where $\mathcal{S}^\mu_{\mathrm{can}}$ is the entropy density current. The second law of thermodynamics $\partial_\mu \mathcal{S}^\mu_{\mathrm{can}} \geq 0$ can give us the first order constitutive relations [45,48],

$$h^\mu - \frac{e+p}{n}\nu^\mu = \kappa\Delta^{\mu\nu}\left[\frac{1}{T}\partial_\nu T - (u\cdot\partial)u_\nu\right], \tag{16}$$

$$\pi^{\mu\nu} = 2\eta\partial^{\langle\mu}u^{\nu\rangle}, \tag{17}$$

$$\Pi = -\zeta\partial_\mu u^\mu, \tag{18}$$

$$q^\mu = \lambda\Delta^{\mu\nu}\left[\frac{1}{T}\partial_\nu T + (u\cdot\partial)u_\nu - 4\omega_{\nu\alpha}u^\alpha\right], \tag{19}$$

$$\phi^{\mu\nu} = 2\gamma_s\Delta^{\mu\rho}\Delta^{\nu\sigma}(\partial_{[\rho}u_{\sigma]} + 2\omega_{\rho\sigma}), \tag{20}$$

where the heat conductivity coefficient κ, shear viscosity coefficient η, and bulk viscosity ζ also exist in conventional hydrodynamics, while λ and γ_s are new coefficients corresponding to the interchange of spin and orbital angular momentum. The entropy principle also requires that the transport coefficients,

$$\kappa, \eta, \zeta, \lambda, \gamma_s > 0, \tag{21}$$

are positive. As the system approaches global equilibrium, the entropy production rate in Eq. (15) tends to zero. It yields the well-known killing condition [41,165], which causes the right-hand sides of Eqs. (16)–(20) to vanish. Especially, we have $q^\mu, \phi^{\mu\nu} = 0$ such that the energy momentum tensor $\Theta^{\mu\nu}$ is symmetric in the global equilibrium state. Note that, pointed out by Refs. [67,167], some cross terms between the different dissipative currents may also exist due to the Onsager relation, but here we neglect them for simplicity.

Before ending this section, we would like to comment on the heat flow h^μ. Interestingly, when we set $\nu^\mu = 0$ and $n = 0$, we find that one cannot fix the expression for heat current h^μ in the first order of gradient expansion. By using $\Delta_{\nu\alpha}\partial_\mu\Theta^{\mu\nu} = 0$ and Eqs. (4) and (5), we find that $(\partial_\mu \frac{1}{T} + \frac{1}{T} Du_\mu) \sim O(\partial^2)$ when $\nu^\mu = 0$ and $n = 0$. In that case, the term $h^\mu(\partial_\mu \frac{1}{T} + \frac{1}{T} Du_\mu) \sim O(\partial^3)$ will be neglected in the entropy production rate (15), i.e., we cannot determine the expression of h^μ by the entropy principle there. A similar behavior was also observed in conventional hydrodynamics [168,169].

III. UNSTABLE AND ACAUSAL MODES IN THE FIRST ORDER SPIN HYDRODYNAMICS

In this section, we analyze the causality and stability for the first order spin hydrodynamics. It is well known that the conventional relativistic hydrodynamics in the Landau frame up to the first order in gradient expansion are always acausal, e.g., see Refs. [156,157] as the early pioneer works.

In linear modes analysis, one can consider the perturbations δX to the hydrodynamical quantities X in the equilibrium. By assuming the $\delta X \sim \delta\tilde{X} e^{i\omega t - ikx}$ with $\delta\tilde{X}$ being constant in space-time, one can solve the dispersion relation $\omega = \omega(k)$ from the conservation equations. In the conventional hydrodynamics, the causality condition is usually given by [159,168,170–172]

$$\lim_{k\to\infty} \left| \mathrm{Re} \frac{\omega}{k} \right| \leq 1, \tag{22}$$

where the condition (22) can also be written as $\lim_{k\to\infty} |\mathrm{Re} \frac{\partial\omega}{\partial k}| \leq 1$ in some literature [159,171,172]. However, the above condition is insufficient to guarantee the causality. We need an extra condition that [173]

$$\lim_{k\to\infty} \left| \frac{\omega}{k} \right| \text{ is bounded.} \tag{23}$$

As pointed out by the early pioneer work [173], the unbounded $\lim_{k\to\infty} |\frac{\omega}{k}|$ gives the infinite propagating speed of the perturbation, even if the ω is pure imaginary. One simple example is the nonrelativistic diffusion equation, $\partial_t n - D_n \partial_x^2 n = 0$ with D_n being the diffusion constant. It is easy to check that its dispersion relation gives $\omega = iD_n k^2$, which satisfies condition (22) but does not obey condition (23). Therefore, the perturbation in the nonrelativistic diffusion equations has the unlimited propagating speed, i.e. with any compact initial value for $n(t_0, x)$, the $n(t_0 + \Delta t, x)$ at $x \to \infty$ can still get the influence [174]. We emphasize that the conditions (22) and (23) are necessary but not sufficient to guarantee that the theory is casual [175–177]. One example is the transverse perturbations of an Eckart fluid with shear viscous tensor, whose dispersion relation satisfies the conditions (22) and (23), but the velocity can exceed the speed of light [see Eqs. (47) and (48) in Ref. [157] for the perturbation equations and the propagating velocity].

The stability means that the imaginary part of $\omega = \omega(k)$ must be positive for $k \neq 0$, i.e.

$$\mathrm{Im}\omega(k) > 0. \tag{24}$$

Note that the case of $\mathrm{Im}\omega = 0$ corresponds to the neutral equilibrium, which means the equilibrium state is not unique. In this work, we will not consider such special cases, and we only consider condition (24) to study the stability of spin hydrodynamics as in Ref. [70].

It is necessary to study the causality and stability for the relativistic spin hydrodynamics in the first order. To see whether the first order spin hydrodynamics can be casual or not, we consider the linear modes analysis to the system, i.e. we take the small perturbations on top of static equilibrium. Following Refs. [156,157], the static equilibrium background is assumed to be an irrotational global equilibrium state. We label the quantities with subscript (0) as those at the global equilibrium state, while we use "δX" to denote the small perturbations of the quantity X, e.g., $e_{(0)}$ and δe stand for the energy density at the global equilibrium and the small perturbations of energy density, respectively.

From now on, unless specified otherwise, we adopt the Landau frame, and neglect the conserved charge current j^μ.

We now consider the small perturbations on top of static equilibrium. Not all of the perturbations are independent of each other, and we can choose

$$\delta e, \quad \delta u^i, \quad \delta S^{\mu\nu}, \tag{25}$$

as independent variables.

The variation of pressure δp and spin chemical potential $\delta\omega^{\mu\nu}$ can be expressed as functions of δe and $\delta S^{\mu\nu}$ through

$$\delta p = c_s^2 \delta e, \qquad \delta\omega^{0i} = \chi_b \delta S^{0i} + \chi_e^{0i} \delta e,$$
$$\delta\omega^{ij} = \chi_s \delta S^{ij} + \chi_e^{ij} \delta e, \tag{26}$$

where the speed of sound c_s, and χ_b, χ_s,$\chi_e^{\mu\nu}$ are in general the functions of thermodynamic variables. For simplicity, we take c_s, χ_b, χ_s,$\chi_e^{\mu\nu}$ as constants in the linear modes analysis. Note that $\chi_e^{\mu\nu}$ comes from the anisotropy of the system. Under the assumption of an irrotational global equilibrium, from Eq. (19) the spin chemical potential vanishes $\omega_{(0)}^{\mu\nu} = 0$. For simplicity, we further choose

$S^{\mu\nu}_{(0)} = 0$. The variation of the temperature δT can be obtained by the thermodynamics relations, with the help of Eqs. (4) and (5),

$$\delta T = \frac{T_{(0)}}{e_{(0)} + p_{(0)}}\left[\delta p - T_{(0)} S^{\mu\nu}_{(0)} \delta\left(\frac{\omega_{\mu\nu}}{T}\right)\right] = \frac{T_{(0)} c_s^2 \delta e}{e_{(0)} + p_{(0)}}. \tag{27}$$

Next, we consider the variation of the conservation equations $\partial_\mu \delta\Theta^{\mu\nu} = 0$ and $\partial_\lambda \delta J^{\lambda\mu\nu} = 0$, where the perturbations $\delta\Theta^{\mu\nu}$ and $\delta J^{\lambda\mu\nu}$ can be derived from the constitutive relations in Eqs. (2), (11), and (16)–(20). It is straightforward to obtain the linearized equations for the independent perturbations $\delta e, \delta\vartheta^i, \delta S^{\mu\nu}$,

$$0 = \left(\partial_0 + \frac{1}{2}\lambda' c_s^2 \partial_i \partial^i + 4\lambda \chi_e^{0i} \partial_i\right)\delta e + \left(\partial_i + \frac{1}{2}\lambda' \partial_i \partial_0\right)\delta\vartheta^i + D_b \partial_i \delta S^{0i}, \tag{28}$$

$$0 = \left(4\gamma_s \chi_e^{ij} \partial_i - c_s^2 \partial^j - \frac{1}{2} c_s^2 \lambda' \partial_0 \partial^j - 4\lambda \chi_e^{0j} \partial_0\right)\delta e + (\gamma_\parallel - \gamma_\perp - \gamma')\partial^j \partial_i \delta\vartheta^i + \left[\partial_0 - \frac{1}{2}\lambda' \partial_0 \partial_0 + (\gamma_\perp + \gamma')\partial^i \partial_i\right]\delta\vartheta^j - D_b \partial_0 \delta S^{0j} + D_s \partial_i \delta S^{ij}, \tag{29}$$

$$0 = (\lambda' c_s^2 \partial^i + 8\lambda \chi_e^{0i})\delta e + \lambda' \partial_0 \delta\vartheta^i + (2D_b - \partial_0)\delta S^{0i}, \tag{30}$$

$$0 = 8\gamma_s \chi_e^{ij} \delta e + 2\gamma' \partial^i \delta\vartheta^j - 2\gamma' \partial^j \delta\vartheta^i + (2D_s + \partial_0)\delta S^{ij}. \tag{31}$$

Here we introduce the following shorthand notations:

$$D_s \equiv 4\gamma_s \chi_s, \qquad D_b \equiv 4\lambda \chi_b,$$
$$\delta\vartheta^i \equiv (e_{(0)} + p_{(0)})\delta u^i, \qquad \lambda' \equiv \frac{2\lambda}{e_{(0)} + p_{(0)}},$$
$$\gamma' \equiv \frac{\gamma_s}{e_{(0)} + p_{(0)}}, \qquad \gamma_\perp \equiv \frac{\eta}{e_{(0)} + p_{(0)}}, \qquad \gamma_\parallel \equiv \frac{\frac{4}{3}\eta + \zeta}{e_{(0)} + p_{(0)}}. \tag{32}$$

In linear modes analysis, the perturbations are assumed along the x direction only,

$$\delta e = \delta\tilde{e} e^{i\omega t - ikx}, \qquad \delta\vartheta^i = \delta\tilde{\vartheta}^i e^{i\omega t - ikx}, \qquad \delta S^{\mu\nu} = \delta\tilde{S}^{\mu\nu} e^{i\omega t - ikx}, \tag{33}$$

where $\delta\tilde{e}$, $\delta\tilde{\vartheta}^i$, and $\delta\tilde{S}^{\mu\nu}$ are independent of space and time.

Inserting the perturbations in Eq. (33) into Eqs. (28)–(31) yields

$$\mathcal{M}_1 \delta\tilde{X}_1 = 0, \tag{34}$$

where

$$\delta\tilde{X}_1 \equiv (\delta\tilde{e}, \delta\tilde{\vartheta}^x, \delta\tilde{S}^{0x}, \delta\tilde{\vartheta}^y, \delta\tilde{S}^{0y}, \delta\tilde{S}^{xy}, \delta\tilde{\vartheta}^z, \delta\tilde{S}^{0z}, \delta\tilde{S}^{xz}, \delta\tilde{S}^{yz})^{\mathrm{T}}, \tag{35}$$

and

$$\mathcal{M}_1 \equiv \begin{pmatrix} M_1 & 0 & 0 & 0 \\ A_1 & M_2 & 0 & 0 \\ A_2 & 0 & M_2 & 0 \\ A_3 & 0 & 0 & M_3 \end{pmatrix}, \tag{36}$$

with

$$M_1 \equiv \begin{pmatrix} i\omega + \frac{1}{2}\lambda' c_s^2 k^2 - 4ik\lambda \chi_e^{0x} & \frac{1}{2}\lambda' k\omega - ik & -ikD_b \\ \frac{1}{2}\lambda' c_s^2 k\omega - ikc_s^2 - 4i\omega\lambda \chi_e^{0x} & \gamma_\parallel k^2 + i\omega + \frac{1}{2}\lambda' \omega^2 & -i\omega D_b \\ ik\lambda' c_s^2 + 8\lambda \chi_e^{0x} & i\omega\lambda' & 2D_b - i\omega \end{pmatrix}, \tag{37}$$

$$M_2 \equiv \begin{pmatrix} k^2(\gamma_\perp + \gamma') + i\omega + \frac{1}{2}\lambda' \omega^2 & -i\omega D_b & -ikD_s \\ i\omega\lambda' & 2D_b - i\omega & 0 \\ 2ik\gamma' & 0 & 2D_s + i\omega \end{pmatrix}, \tag{38}$$

$$M_3 \equiv 2D_s + i\omega. \tag{39}$$

The off-diagonal blocks A_1, A_2, A_3 in the matrix $\mathcal{M}_1$, whose expressions are shown in Appendix A, and are irrelevant to the following discussions. The nontrivial solutions in Eq. (34) requires

$$0 = \det \mathcal{M}_1 = \det M_1 \cdot (\det M_2)^2 \cdot \det M_3. \tag{40}$$

From Eqs. (37)–(39), we find that Eq. (40) is a polynomial equation for two variables ω and k. Solving this equation gives the dispersion relations $\omega = \omega(k)$.

The $\det M_3 = 0$ gives a nonhydrodynamic mode,

$$\omega = 2iD_s, \tag{41}$$

which corresponds to the spin relaxation [45,59]. The stability condition (24) requires that $D_s > 0$.

The dispersion relations solved from $\det M_1 = 0$ and $\det M_2 = 0$ are lengthy and complicated, so here we only discuss the relations in small k and large k limits to analyze stability and causality. In the $k \to 0$ limit, the dispersion relations are

$$\omega = \pm c_s k + \frac{i}{2}(\gamma_\parallel \mp 4c_s\lambda\chi_e^{0x}D_b^{-1})k^2 + O(k^3), \tag{42}$$

$$\omega = (-i \pm \sqrt{4D_b\lambda' - 1})\lambda'^{-1} + O(k), \tag{43}$$

$$\omega = i\gamma_\perp k^2 + O(k^3), \tag{44}$$

$$\omega = 2iD_s + O(k^2), \tag{45}$$

where the dispersion relations (42) and (43) and (43)–(45) are solved from $\det M_1 = 0$ and $\det M_2 = 0$, respectively. The modes in Eqs. (42) and (44) correspond to the sound and shear modes in the conventional hydrodynamics [156,158,159,171], respectively. The stability condition (24) for the dispersion relation in Eqs. (42)–(45) gives

$$D_s > 0, \qquad \lambda' < 0, \qquad D_b < -4c_s\lambda\gamma_\parallel^{-1}|\chi_e^{0x}| \leq 0. \tag{46}$$

However, conditions (46) contradict the entropy principle in Eq. (21), i.e. $\lambda' = 2\lambda/(e_{(0)} + p_{(0)}) > 0$ defined in Eq. (32) with $\lambda > 0$ and $e_{(0)} + p_{(0)} > 0$.

In the $k \to \infty$ limit, the dispersion relations become

$$\omega = -4iD_b\gamma_\parallel^{-1}\lambda'^{-1}k^{-2} + O(k^{-3}), \tag{47}$$

$$\omega = -ic_s^{2/3}\gamma_\parallel^{1/3}k^{4/3} + O(k), \tag{48}$$

$$\omega = (-1)^{1/6}c_s^{2/3}\gamma_\parallel^{1/3}k^{4/3} + O(k), \tag{49}$$

$$\omega = (-1)^{5/6}c_s^{2/3}\gamma_\parallel^{1/3}k^{4/3} + O(k), \tag{50}$$

$$\omega = -2iD_b + O(k^{-1}), \tag{51}$$

$$\omega = 2iD_s\gamma_\perp(\gamma' + \gamma_\perp)^{-1} + O(k^{-1}), \tag{52}$$

$$\omega = \pm ik\sqrt{2\lambda'^{-1}(\gamma' + \gamma_\perp)} + O(k^0), \tag{53}$$

where the first four modes come from $\det M_1 = 0$, and others can be derived by $\det M_2 = 0$. Obviously, Eq. (53) contains an unstable mode.

On the other hand, we also find that in Eqs. (48)–(50) $|\omega/k|$ is unbounded, which violates the causality condition (23). We also notice that Ref. [71] has also analyzed the causality for the first order spin hydrodynamics in the small k limit.

We find that the first order spin hydrodynamics is acausal and unstable similar to the conventional relativistic hydrodynamics in the Landau frame.

Before ending this section, we comment on condition (46). We notice that the dispersion relations in Refs. [45,70,71] are different from ours in Eqs. (41)–(53). Let us explain what happens here. The energy momentum conservation equation $\Delta_{\mu\alpha}\partial_\nu\Theta^{\mu\nu} = 0$ gives the acceleration equations for the fluid velocity,

$$(u\cdot\partial)u^\mu = \frac{1}{T}\Delta^{\mu\nu}\partial_\nu T + O(\partial^2). \tag{54}$$

In Refs. [45,70,71], the authors have replaced $(u\cdot\partial)u^\mu$ in q^μ in Eq. (19) by Eq. (54) and gotten another expression for q^μ:

$$q^\mu = \lambda\left(\frac{2\Delta^{\mu\nu}\partial_\nu p}{e+p} - 4\omega^{\mu\nu}u_\nu\right) + O(\partial^2). \tag{55}$$

Although q^μ in Eq. (55) (also in Refs. [45,70,71]) is equivalent to our q^μ in Eq. (19) up to the first order in gradient expansion, we emphasize that these two q^μ correspond to different hydrodynamic frames and will lead to different hydrodynamic equations (also see Refs. [168,178] for the general discussion for these kinds of replacement in relativistic hydrodynamics). Different from our Eqs. (47)–(53), the dispersion relations computed with the q^μ in Eq. (55) are stable and satisfy causality condition (22) in the rest frame under certain conditions. However, they do not obey the causality condition (23) and the whole theory becomes acausal, e.g., one mode in Refs. [45,70,71] is

$$\omega = i(\gamma' + \gamma_\perp)k^2 \quad \text{as } k \to \infty, \tag{56}$$

and breaks the causality condition (23).

We now conclude that the first order spin hydrodynamics at the static equilibrium state are unstable and acausal in the rest frame. We do not need to discuss the stability and causality of the first order spin hydrodynamics in moving frames again.

IV. MINIMAL CAUSAL SPIN HYDRODYNAMICS

In the previous section, we have shown that the first order spin hydrodynamics in Landau frame are acausal and unstable. The acausal and unstable theory is not physical, we therefore need to consider the second order spin hydrodynamics in gradient expansion. In this section we follow the idea of minimal causal extension in conventional hydrodynamics and implement it to the spin hydrodynamics.

Up to now, there are two ways to establish causal hydrodynamics. The first way is to add the second order corrections to the dissipative terms, such as the Müller-Israel-Stewart (MIS) theory [160,161] or other related second order hydrodynamics. The MIS theory is a famous causal conventional hydrodynamic theory up to $O(\partial^2)$ in gradient expansion. Here, we consider a relativistic dissipative hydrodynamics with the bulk viscous pressure Π only as an example to explain why the MIS theory can be casual. The entropy current in MIS theory is assumed to be [161,179,180]

$$\mathcal{S}^{\mu} = su^{\mu} - \frac{\mu}{T}\nu^{\mu} + \frac{1}{T}h^{\mu} - \frac{1}{2T}\beta_0 u^{\mu}\Pi^2 + \cdots, \quad (57)$$

where the coefficient $\beta_0 > 0$ and the ellipsis stands for other possible $O(\partial^2)$ terms. Then the second law of thermodynamics $\partial_{\mu}\mathcal{S}^{\mu} \geq 0$ leads to

$$\tau_{\Pi}\frac{d}{d\tau}\Pi + \Pi = -\zeta\partial_{\mu}u^{\mu} + \cdots, \quad (58)$$

where $d/d\tau \equiv u^{\mu}\partial_{\mu}$, and $\tau_{\Pi} = \zeta\beta_0 > 0$ is defined as the relaxation time for the bulk viscous pressure. If $\tau_{\Pi} \to 0$, the hydrodynamic equations reduce to parabolic equations and become acausal. With a finite τ_{Π}, the hydrodynamic equations are hyperbolic and can be causal under certain conditions [158,159,171,181,182]. In linear modes analysis, the dispersion relations from Eq. (58) satisfy causal conditions (22) and (23) when the relaxation time τ_{Π} is sufficiently large. The second order constitutive equations for shear viscous tensor $\pi^{\mu\nu}$, heat flow h^{μ} and heat current ν^{μ} can be obtained in a similar way. These equations represent evolution equations that incorporate the respective relaxation time [161,179,180]. Apart from the MIS theory, many other second order causal conventional hydrodynamic theories, e.g., Baier-Romatschke-Son-Starinets-Stephanov (BRSSS) theory [183] and the Denicol-Niemi-Molnar-Rischke (DNMR) theory [184], have been established. All of them contain the terms proportional to the relaxation times and can be causal and stable under certain conditions [183,185,186]. Following this discussion, we can say that the key to recover the causality of the theory is to introduce the terms proportional to relaxation time.

Different with the above second order theories, the Bemfica-Disconzi-Noronha-Kovtun (BDNK) [168–170,187–189] is a first order hydrodynamic theory in general (fluid) frames. It roughly says that one can choose some preferred frames to satisfy the causality and stability conditions. Unfortunately, the commonly used Landau or Eckart frame are not the preferred fluid frames in the BDNK theory. Therefore, we will not discuss the spin hydrodynamics in the BDNK theory in this work. We also notice that recent studies in Ref. [190] discuss the casual spin hydrodynamics in the first order similar to BDNK theory.

In this work, we follow the basic idea in MIS, BRSSS, and DNMR theories to construct a simplified causal spin hydrodynamics. Instead of considering the complete second order spin hydrodynamics, we only analyze the called "minimal" extended second order spin hydrodynamics. Here, the word minimal means that we concentrate on the essential terms in the second order of gradient expansion to get a causal theory and neglect the other terms which do not contribute to the dispersion relations in the linear modes analysis. As mentioned below Eq. (58), the key to get the causal theory is to add the terms proportional to the relaxation times similar to $\tau_{\Pi}d\Pi/d\tau$, in the left-hand side of Eq. (58). Following this idea, the constitutive equations (16)–(20) in the minimal extended causal spin hydrodynamics can be rewritten as

$$\tau_q\Delta^{\mu\nu}\frac{d}{d\tau}q_{\nu} + q^{\mu} = \lambda(T^{-1}\Delta^{\mu\alpha}\partial_{\alpha}T + Du^{\mu} - 4\omega^{\mu\nu}u_{\nu}), \quad (59)$$

$$\tau_{\phi}\Delta^{\mu\alpha}\Delta^{\nu\beta}\frac{d}{d\tau}\phi_{\alpha\beta} + \phi^{\mu\nu} = 2\gamma_s\Delta^{\mu\alpha}\Delta^{\nu\beta}(\partial_{[\alpha}u_{\beta]} + 2\omega_{\alpha\beta}), \quad (60)$$

$$\tau_{\pi}\Delta^{\alpha<\mu}\Delta^{\nu>\beta}\frac{d}{d\tau}\pi_{\alpha\beta} + \pi^{\mu\nu} = 2\eta\partial^{<\mu}u^{\nu>}, \quad (61)$$

$$\tau_{\Pi}\frac{d}{d\tau}\Pi + \Pi = -\zeta\partial_{\mu}u^{\mu}, \quad (62)$$

where $\tau_q, \tau_{\phi}, \tau_{\pi}$ and τ_{Π} are positive relaxation times for $q^{\mu}, \phi^{\mu\nu}, \pi^{\mu\nu}, \Pi$, respectively. Equations (61) and (62) are the same as those in the conventional hydrodynamics[1] [158,159,171]. Recently, the second order spin hydrodynamics similar to MIS theory has been introduced in Ref. [73] by using the entropy principle. Our minimal causal spin hydrodynamics can be regarded as a simplified version of it. We also notice that in Ref. [60] the authors have proposed the same expressions for q^{μ} and $\phi^{\mu\nu}$ as presented in Eqs. (59) and (60) for minimal causal spin hydrodynamics.

Let us give some physical interpretation for Eqs. (59)–(62). The nonzero relaxation times imply that the system requires time to transition from a nonequilibrium state to an equilibrium state. In other words, the dissipative fluxes Π, $\pi^{\mu\nu}$, q^{μ}, and $\phi^{\mu\nu}$ do not undergo sudden transitions from nonzero to zero [161,179]. As an example, we consider the general solution for Π [162]

$$\Pi = \Pi_0 e^{-(\tau-\tau_0)/\tau_{\Pi}} - \int_{\tau_0}^{\tau} d\tau' G(\tau,\tau')\zeta\partial_{\mu}u^{\mu}, \quad (63)$$

where Π_0 is constant, and the Green's function is defined as follows: $G(\tau,\tau') = 0$ for $\tau < \tau'$, $G(\tau,\tau') = 1/(2\tau_{\Pi})$ for $\tau = \tau'$, and $G(\tau,\tau') = \frac{1}{\tau_{\Pi}}e^{-(\tau-\tau')/\tau_{\Pi}}$ for $\tau > \tau'$. The general solutions for $\pi^{\mu\nu}$, q^{μ}, and $\phi^{\mu\nu}$ in Eqs. (59)–(61) share a structure similar to that of Eq. (63). Now, we assume that $\zeta\partial_{\mu}u^{\mu}$ jumps from nonzero to zero at time τ_0. Because of the

[1] Another kind of minimal causal theory is discussed in Refs. [162,191], in which the extended dissipative terms cannot be determined from the entropy principle $\partial_{\mu}\mathcal{S}^{\mu} \geq 0$.

nonzero relaxation time τ_Π, the solution (63) indicates that Π cannot instantaneously switch from a nonzero (nonequilibrium) value to zero (equilibrium). However, if $\tau_\Pi = 0$, the solution (63) reduces to $\Pi = -\zeta\partial_\mu u^\mu$, and then Π undergoes sudden change from nonzero to zero and it thus causes acausality. Therefore, to obtain a physical theory, we introduce the nonzero relaxation times and treat Π, $\pi^{\mu\nu}$, q^μ, and $\phi^{\mu\nu}$ as dynamical variables in Eqs. (59)–(62). In principle, we can also consider the nonzero $\Sigma_{(1)}^{\lambda\mu\nu}$ in Eq. (13), which might involve corrections similar to the relaxation terms for q^μ and $\phi^{\mu\nu}$. In this work, we concentrate on the simplest extension of the second-order terms and leave the more general discussion for future research.

V. CAUSALITY AND STABILITY ANALYSIS FOR MINIMAL CAUSAL SPIN HYDRODYNAMICS

In this section we analyze the causality and stability of the minimal causal spin hydrodynamics. We use similar notations in Sec. III, i.e., for a physical quantity X, we use $X_{(0)}$ and δX to denote the X at the global equilibrium state and the small perturbations of the quantity X, respectively. We adopt the independent perturbations as

$$\delta e, \quad \delta u^i, \quad \delta S^{\mu\nu}, \quad \delta\Pi, \quad \delta\pi^{ij}, \tag{64}$$

where $\delta\pi^i{}_i = 0$ and $\delta\pi^{ij} = \delta\pi^{ji}$.

We first start from the spin hydrodynamics in the rest frame, i.e., $u^\mu_{(0)} = (1, 0)$. The conservation equations $\partial_\mu\delta\Theta^{\mu\nu} = 0$ and $\partial_\lambda\delta J^{\lambda\mu\nu} = 0$ with the constitutive equations (59)–(62) read

$$0 = (\lambda' c_s^2\partial^i + 8\lambda\chi_e^{0i})\delta e + \lambda'\partial_0\delta\vartheta^i + (2D_b - \tau_q\partial_0\partial_0 - \partial_0)\delta S^{0i}, \tag{65}$$

$$0 = 8\gamma_s\chi_e^{ij}\delta e + 2\gamma'(\partial^i\delta\vartheta^j - \partial^j\delta\vartheta^i) + (\tau_\phi\partial_0\partial_0 + \partial_0 + 2D_s)\delta S^{ij}, \tag{66}$$

$$0 = \tau_\pi\partial_0\delta\pi^{ij} + \delta\pi^{ij} - \gamma_\perp\left(\partial^i\delta\vartheta^j + \partial^j\delta\vartheta^i - \frac{2}{3}g^{ij}\partial_k\delta\vartheta^k\right), \tag{67}$$

$$0 = \tau_\Pi\partial_0\delta\Pi + \delta\Pi + \left(\gamma_\parallel - \frac{4}{3}\gamma_\perp\right)\partial_i\delta\vartheta^i, \tag{68}$$

$$0 = \partial_0\delta e + \partial_i\delta\vartheta^i + \frac{1}{2}\partial_0\partial_i\delta S^{0i}, \tag{69}$$

$$0 = -c_s^2\partial^j\delta e + \partial_0\delta\vartheta^j - \partial^j\delta\Pi + \partial_i\delta\pi^{ij} - \frac{1}{2}\partial_0\partial_0\delta S^{0j} - \frac{1}{2}\partial_0\partial_i\delta S^{ij}, \tag{70}$$

where $\chi_b, \chi_e^{\mu\nu}, \chi_s, D_s, D_b, \delta\vartheta^i, \lambda', \gamma', \gamma_\perp, \gamma_\parallel$ are defined in Eqs. (26) and (32) and we have used the spin evolution equation (3) to replace δq^i and $\delta\phi^{ij}$ by $\delta S^{\mu\nu}$,

$$\delta q^i = \frac{1}{2}\partial_0\delta S^{0i}, \qquad \delta\phi^{ij} = -\frac{1}{2}\partial_0\delta S^{ij}. \tag{71}$$

A. Zero modes for the spin hydrodynamics with zero viscous effects

Following the conventional hydrodynamics, we consider a fluid with the dissipative terms q^μ and $\phi^{\mu\nu}$ only for simplicity, i.e., we remove Eqs. (67) and (68) and take $\delta\Pi = 0$ and $\delta\pi^{ij} = 0$ in Eqs. (65), (66), (69), and (70). The detail of the calculation is shown in Appendix C 1. The causality condition requires

$$0 \le \frac{c_s^2(3\lambda' + 2\tau_q)}{2\tau_q - \lambda'} \le 1, \qquad 0 \le \frac{2\gamma'\tau_q}{(2\tau_q - \lambda')\tau_\phi} \le 1. \tag{72}$$

The stability conditions give

$$\tau_q > \lambda'/2, \qquad D_s > 0, \qquad D_b < 0, \qquad \chi_e^{0x} = 0. \tag{73}$$

The above conditions are derived from the small k and large k limits only. We can implement the Routh-Hurwitz criterion [168–170,188,192,193] to prove that the condition (73) is sufficient and necessary for stability. More discussion can be found in Appendix C 2.

Interestingly, there exist zero modes, i.e., $\omega = 0$ for all k, coming from Eq. (70) with vanishing $\delta\Pi, \delta\pi^{ij}$. Generally, the zero modes in the linear mode analysis do not mean the perturbations are not decaying with time. It indicates that the nonlinear modes should be included in Eq. (70) if $\delta\Pi = \delta\pi^{ij} = 0$. To continue our linear mode analysis, we need to set nonvanishing $\delta\Pi, \delta\pi^{ij}$.

B. Causality analysis in the rest frame

Next, we substitute the plane wave solutions Eq. (33) and

$$\delta\Pi = \delta\tilde{\Pi}e^{i\omega t - ikx}, \qquad \delta\pi^{ij} = \delta\tilde{\pi}^{ij}e^{i\omega t - ikx}, \tag{74}$$

with $\delta\tilde{\Pi}, \delta\tilde{\pi}^{ij}$, being constants, into Eqs. (65)–(70), and obtain the matrix equation

$$\mathcal{M}_2\delta\tilde{X}_2 = 0, \tag{75}$$

where $\delta\tilde{X}_2$ and $\mathcal{M}_2$ are given by

$$\delta\tilde{X}_2 \equiv (\delta\tilde{e}, \delta\tilde{\vartheta}^x, \delta\tilde{S}^{0x}, \delta\tilde{\Pi}, \delta\tilde{\pi}^{xx}, \delta\tilde{\vartheta}^y, \delta\tilde{S}^{0y}, \delta\tilde{S}^{xy}, \delta\tilde{\pi}^{xy}, \delta\tilde{\vartheta}^z, \delta\tilde{S}^{0z}, \delta\tilde{S}^{xz}, \delta\tilde{\pi}^{xz}, \delta\tilde{S}^{yz}, \delta\tilde{\pi}^{yy}, \delta\tilde{\pi}^{yz})^{\mathrm{T}}, \tag{76}$$

and

$$\mathcal{M}_2 = \begin{pmatrix} M_4 & 0 & 0 & 0 \\ A_4 & M_5 & 0 & 0 \\ A_5 & 0 & M_5 & 0 \\ A_6 & 0 & 0 & M_6 \end{pmatrix}, \tag{77}$$

with

$$M_4 = \begin{pmatrix} i\omega & -ik & \frac{1}{2}\omega k & 0 & 0 \\ -ikc_s^2 & i\omega & \frac{1}{2}\omega^2 & -ik & -ik \\ ik\lambda' c_s^2 + 8\lambda\chi_e^{0x} & i\omega\lambda' & 2D_b + \tau_q\omega^2 - i\omega & 0 & 0 \\ 0 & -ik\left(\gamma_\parallel - \frac{4}{3}\gamma_\perp\right) & 0 & i\omega\tau_\Pi + 1 & 0 \\ 0 & -\frac{4}{3}ik\gamma_\perp & 0 & 0 & i\omega\tau_\Pi + 1 \end{pmatrix}, \tag{78}$$

$$M_5 = \begin{pmatrix} 2ik\gamma' & 0 & -\tau_\phi\omega^2 + i\omega + 2D_s & 0 \\ i\omega & \frac{1}{2}\omega^2 & -\frac{1}{2}\omega k & -ik \\ i\omega\lambda' & 2D_b + \tau_q\omega^2 - i\omega & 0 & 0 \\ -ik\gamma_\perp & 0 & 0 & i\omega\tau_\Pi + 1 \end{pmatrix}, \tag{79}$$

$$M_6 = \begin{pmatrix} -\tau_\phi\omega^2 + i\omega + 2D_s & 0 & 0 \\ 0 & i\omega\tau_\Pi + 1 & 0 \\ 0 & 0 & i\omega\tau_\Pi + 1 \end{pmatrix}. \tag{80}$$

The submatrices $A_{4,5,6}$ in Eq. (77) are shown in Appendix A. If there exist nonzero plane wave solutions, we have

$$0 = \det\mathcal{M}_2 = \det M_4 \cdot (\det M_5)^2 \cdot \det M_6. \tag{81}$$

We observe the zero modes in Eq. (70) disappear. It indicates that the current analysis is consistent with the assumption of linear response. The dispersion relations $\omega = \omega(k)$ are the solutions to the polynomial equation (81).

The $\det M_6 = 0$ gives

$$\omega = \frac{i}{\tau_\pi}, \tag{82}$$

$$\omega = \frac{1}{2\tau_\phi}(i \pm \sqrt{8D_s\tau_\phi - 1}), \tag{83}$$

which are nonpropagating modes or nonhydrodynamic modes.

In the $k \to 0$ limit, the $\det M_4 = 0$ and $\det M_5 = 0$ give

$$\omega = \frac{i}{\tau_\pi} + O(k), \tag{84}$$

$$\omega = \frac{i}{\tau_\Pi} + O(k), \tag{85}$$

$$\omega = \pm c_s k + \frac{i}{2}(\gamma_\parallel \mp 4c_s\lambda\chi_e^{0x}D_b^{-1})k^2 + O(k^3), \tag{86}$$

$$\omega = \left[i \pm \sqrt{-4D_b(2\tau_q - \lambda') - 1}\right](2\tau_q - \lambda')^{-1} + O(k), \tag{87}$$

$$\omega = i\gamma_\perp k^2 + O(k^3), \tag{88}$$

$$\omega = \frac{1}{2\tau_\phi}(i \pm \sqrt{8D_s\tau_\phi - 1}) + O(k), \tag{89}$$

where Eqs. (84) and (87) are doubly degenerate. In the large k limit, we have

$$\omega = -4iD_b\gamma_\parallel^{-1}\lambda'^{-1}k^{-2} + O(k^{-3}), \tag{90}$$

$$\omega = \frac{3i\gamma_\parallel}{\tau_\pi(3\gamma_\parallel - 4\gamma_\perp) + 4\gamma_\perp\tau_\Pi} + O(k^{-1}), \tag{91}$$

$$\omega = c_1 k + i\frac{c_2}{c_3} + O(k^{-1}), \tag{92}$$

$$\omega = \pm\sqrt{\frac{2\tau_q(\gamma'\tau_\pi + \gamma_\perp\tau_\phi)}{(2\tau_q - \lambda')\tau_\pi\tau_\phi}}k + ic_4 + O(k^{-1}), \tag{93}$$

$$\omega = \frac{i \pm \sqrt{-1 - 8D_b\tau_q}}{2\tau_q} + O(k^{-1}), \tag{94}$$

$$\omega = \frac{i(\gamma' + \gamma_\perp) \pm c_5}{2(\gamma'\tau_\pi + \gamma_\perp\tau_\phi)} + O(k^{-1}), \tag{95}$$

where the expressions of these k-independent coefficients $c_{1,2,3,4,5}$ are shown in Appendix B. The $\det M_4 = 0$ gives Eqs. (84)–(87) and (90)–(92), while $\det M_5 = 0$ gives Eqs. (84), (87)–(89), and (93)–(95).

Now, let us analyze the causality conditions. From Eqs. (90)–(95), we find that all modes in minimal causal spin hydrodynamics correspond to finite propagation speed since $|\omega/k|$ is bounded as $k \to +\infty$. Imposing Eq. (22) on the propagating modes in Eqs. (92) and (93), the causality requires

$$0 \le \frac{b_1^{1/2} \pm (b_1 - b_2)^{1/2}}{6(2\tau_q - \lambda')\tau_\pi\tau_\Pi} \le 1 \quad \text{and} \quad 0 \le \frac{2\tau_q(\gamma'\tau_\pi + \gamma_\perp\tau_\phi)}{(2\tau_q - \lambda')\tau_\pi\tau_\phi} \le 1, \tag{96}$$

where $b_{1,2}$ are defined in Appendix B. The causality conditions imply that the relaxation times $\tau_q, \tau_\pi, \tau_\Pi, \tau_\phi$ cannot be arbitrarily small, which is consistent with the discussion in Sec. IV. We also notice that Eq. (96) reduces to Eq. (C17) when we take a smooth limit $\tau_\pi, \tau_\Pi, \gamma_\perp, \gamma_\parallel \to 0$.

C. Nontrivial stability conditions in rest frame

The requirement of stability is nontrivial. Inserting Eq. (24) into Eqs. (82)–(95) yields

$$\tau_q > \lambda'/2, \tag{97}$$

$$D_s > 0, \qquad D_b < -4c_s\lambda\gamma_\parallel^{-1}|\chi_e^{0x}| \le 0, \tag{98}$$

$$b_1 > b_2 > 0, \qquad \frac{c_2}{c_3} > 0. \tag{99}$$

The stability condition $\lambda' < 0$ in Eq. (46) for the first order spin hydrodynamics becomes $\lambda' < 2\tau_q$ in Eq. (97). When the relaxation time τ_q is sufficiently large, the inequality $\lambda' < 2\tau_q$ is satisfied, and then the previous unstable modes are removed. We also notice that the conditions (97) and (98) agree with Eq. (C18) except that $\chi_e^{0x} = 0$. The strong constraint $\chi_e^{0x} = 0$ is released in this case.

The satisfaction of the stability condition (98) relies on the specific equation of state governing $S^{\mu\nu}$ and $\omega^{\mu\nu}$. In Ref. [70], it was found that the stability condition (98) cannot be satisfied if $\delta S^{\mu\nu} \sim T^2\delta\omega^{\mu\nu}$ [62,64]. In more general cases, we can have

$$u_\mu\delta\omega^{\mu\nu} = \chi_1 u_\mu \delta S^{\mu\nu}, \tag{100}$$

$$\Delta^{\mu\alpha}\Delta^{\nu\beta}\delta\omega_{\alpha\beta} = (\chi_1 + \chi_2)\Delta^{\mu\alpha}\Delta^{\nu\beta}\delta S_{\alpha\beta}, \tag{101}$$

where $\chi_{1,2}$ are susceptibility corresponding to the S^{0i} and S^{ij} in the rest frame. In this case, according to the definitions in Eqs. (26) and (32), the stability condition (98) is satisfied if $\chi_2 > -\chi_1 > 0$. Details can be found in Appendix D. Note that the parameters χ_1 and χ_2 strongly depend on the equation of state for $S^{\mu\nu}$ and $\omega^{\mu\nu}$. To determine the equation of state, we need the microscopic theories, and we will leave it for the future studies.

Another remarkable observation for the stability conditions is that there exist unstable modes at finite k. Equations (97)–(99) are the stability conditions in small k and large k limits only. We still need to study the $\mathrm{Im}\omega$ in the finite k region. One analytic method, named the Routh-Hurwitz criterion [168–170,188,192,193], is usually implemented to study the sign of $\mathrm{Im}\omega$ in the finite k region. Unfortunately, $\det \mathcal{M}_2$ cannot be reduced to the form that Routh-Hurwitz criterion applies, thus, we analyze the behavior of $\mathrm{Im}\omega$ numerically instead of the Routh-Hurwitz criterion. For a finite k, we find that $\mathrm{Im}\omega$ can be negative, even if all the conditions (97)–(99) are satisfied. In Fig. 1, we present an example to show that $\mathrm{Im}\omega$ can be negative for finite k. We choose the parameters as

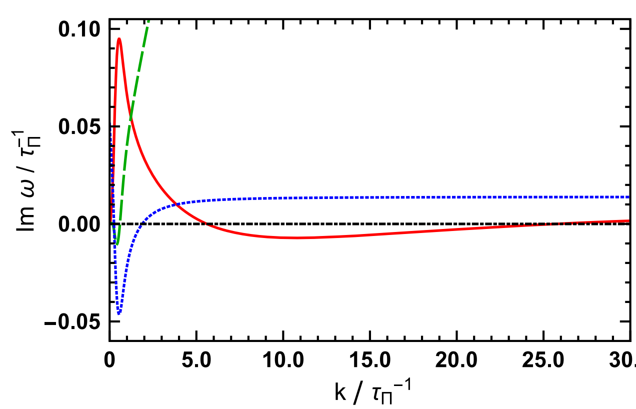

FIG. 1. We plot the imaginary parts of $\omega\tau_\Pi$ as a function of $k\tau_\Pi$ in three modes derived from $\det M_4 = 0$. The parameters are chosen as in Eq. (102), which satisfy the causality and stability conditions equations (22)–(24). The solid, dashed and dotted lines stand for three unstable modes.

$$c_s = \frac{1}{\sqrt{3}}, \qquad \lambda\chi_e^{0x} = \frac{1}{8}, \qquad \tau_\pi = 4\tau_\Pi,$$
$$\tau_\phi = 2\tau_\Pi, \qquad \tau_q = 10\tau_\Pi,$$
$$\lambda' = \frac{1}{2}\tau_\Pi, \qquad \gamma_\parallel = \frac{7}{10}\tau_\Pi, \qquad \gamma_\perp = \frac{1}{2}\tau_\Pi,$$
$$\gamma' = \tau_\Pi, \qquad D_s = \frac{1}{2\tau_\Pi}, \qquad D_b = -\frac{1}{2\tau_\Pi}. \tag{102}$$

It is straightforward to verify that the parameters in Eq. (102) satisfy the stability and causality constraints (22)–(24). We pick up three modes derived from $\det M_4 = 0$. We observe that the $\mathrm{Im}\omega$ at both small and large k limits are positive, while it becomes negative when $k\tau_\Pi \sim 0.5$ and $k\tau_\Pi \sim 10.0$, i.e., the modes are unstable in the finite k region.

We comment on the unstable modes at finite k. The unstable modes in the minimal causal spin hydrodynamics are significantly different from those in the conventional hydrodynamics. As discussed in Refs. [158,159,168,170,171,188], the stability conditions obtained in $k \to 0$ and $k \to +\infty$ limits are sufficient to ensure the stability at any real k. However, it looks failed in minimal causal spin hydrodynamics. It implies that the conditions (97)–(99) are necessary but may not be sufficient. At last, it is still unclear whether the unstable modes at finite k indicate the fluid becomes unstable or not.

D. Causality and stability analysis for extended q^μ and $\phi^{\mu\nu}$

In principle, we can introduce the coupling terms for Π, $\pi^{\mu\nu}$, q^μ, and $\phi^{\mu\nu}$ on the right-hand side of Eqs. (59) and (60). These terms will alter the linearized hydrodynamic equations and then the causality and stability conditions can be changed. In the current work, as the first step, we focus on the simplest coupling between q^μ and $\phi^{\mu\nu}$,

$$\tau_q \Delta^{\mu\nu}\frac{d}{d\tau}q_\nu + q^\mu = \lambda(T^{-1}\Delta^{\mu\nu}\partial_\nu T + u^\nu\partial_\nu u^\mu - 4u_\nu\omega^{\mu\nu}) + g_1\Delta^{\mu\nu}\partial^\rho\phi_{\nu\rho}, \tag{103}$$

$$\tau_\phi \Delta^{\mu\alpha}\Delta^{\nu\beta}\frac{d}{d\tau}\phi_{\alpha\beta} + \phi^{\mu\nu} = 2\gamma_s\Delta^{\mu\alpha}\Delta^{\nu\beta}(\partial_{[\alpha}u_{\beta]} + 2\omega_{\alpha\beta}) + g_2\Delta^{\mu\alpha}\Delta^{\nu\beta}\partial_{[\alpha}q_{\beta]}, \tag{104}$$

where $g_{1,2}$ are new transport coefficients describing the coupling between q^μ and $\phi^{\mu\nu}$. For more general coupling terms, one can refer to Ref. [73].

Following the same method, Eqs. (65) and (66) become

$$0 = (\lambda' c_s^2\partial^i + 8\lambda\chi_e^{0i})\delta e + \lambda'\partial_0\delta\vartheta^i + (2D_b - \tau_q\partial_0\partial_0 - \partial_0)\delta S^{0i} - g_1\partial_j\partial_0\delta S^{ij}, \tag{105}$$

$$0 = 8\gamma_s\chi_e^{ij}\delta e + 2\gamma'(\partial^i\delta\vartheta^j - \partial^j\delta\vartheta^i) + (\tau_\phi\partial_0\partial_0 + \partial_0 + 2D_s)\delta S^{ij} + \frac{1}{2}g_2\partial^i\partial_0\delta S^{0j} - \frac{1}{2}g_2\partial^j\partial_0\delta S^{0i}. \tag{106}$$

We first consider the cases without viscous effects. The causality condition (72) becomes

$$0 \le \frac{c_s^2(3\lambda' + 2\tau_q)}{2\tau_q - \lambda'} \le 1, \qquad 0 \le \frac{m}{4(2\tau_q - \lambda')\tau_\phi} \le 1, \tag{107}$$

where m is defined in Eq. (B12). While the stability condition (73) is changed to

$$\tau_q > \lambda'/2, \qquad D_s > 0, \qquad D_b < 0,$$
$$\chi_e^{0x} = 0, \qquad m > 8\gamma'\left(\frac{2}{2\tau_q - \lambda'} + \frac{1}{\tau_\phi}\right)^{-1}. \tag{108}$$

Details can be found in Appendix C 3. We implement the Routh-Hurwitz criterion [168–170,188,192,193] again to prove that these conditions (108) are sufficient and necessary for stability. Details for the proof can be found in Appendix C 4. Similar to Sec. V A, we still find the zero modes coming from Eq. (70).

Therefore, we need to consider the nonvanishing viscous effects. Now, the submatrix M_5 shown in Eq. (79) is replaced with

$$M_5 = \begin{pmatrix} 2ik\gamma' & -\frac{1}{4}g_2\omega k & -\tau_\phi\omega^2 + i\omega + 2D_s & 0 \\ i\omega & \frac{1}{2}\omega^2 & -\frac{1}{2}\omega k & -ik \\ i\omega\lambda' & 2D_b + \tau_q\omega^2 - i\omega & g_1\omega k & 0 \\ -ik\gamma_\perp & 0 & 0 & i\omega\tau_\Pi + 1 \end{pmatrix}, \tag{109}$$

while other submatrices are unaffected by g_1 and g_2. Equations (93)–(95) become

$$\omega = \pm\sqrt{\frac{f+f'}{8(2\tau_q-\lambda')\tau_\pi\tau_\phi}}k + i\frac{f+f'}{4(2\tau_q-\lambda')\tau_\pi\tau_\phi}c_6 + \mathcal{O}(k^{-1}), \tag{110}$$

$$\omega = \pm\sqrt{\frac{f-f'}{8(2\tau_q-\lambda')\tau_\pi\tau_\phi}}k + i\frac{f-f'}{4(2\tau_q-\lambda')\tau_\pi\tau_\phi}c_7 + \mathcal{O}(k^{-1}), \tag{111}$$

$$\omega = \pm 4\sqrt{\frac{-D_bD_s}{g_1g_2}}k^{-1} + 4i\frac{[D_s\gamma_\perp - D_b(\gamma_\perp+\gamma')]}{g_1g_2\gamma_\perp}k^{-2} + \mathcal{O}(k^{-3}), \tag{112}$$

where the definitions of f, f', c_6, and c_7 are defined in Appendix B.

From these new dispersion relations, we obtain causality conditions,

$$0 \le \frac{b_1^{1/2} \pm (b_1-b_2)^{1/2}}{6(2\tau_q-\lambda')\tau_\pi\tau_\Pi} \le 1 \quad \text{and} \quad 0 \le \frac{f \pm f'}{8(2\tau_q-\lambda')\tau_\pi\tau_\phi} \le 1, \tag{113}$$

which reproduce Eq. (96) when $g_1, g_2 \to 0$.

Similarly, the stability conditions are given by

$$\tau_q - \frac{\lambda'}{2} > 0, \tag{114}$$

$$D_s > 0, \qquad -4c_s\lambda\gamma_\parallel^{-1}|\chi_e^{0x}| - D_b > 0, \tag{115}$$

$$b_1 > b_2 > 0, \qquad \frac{c_2}{c_3} > 0, \tag{116}$$

$$g_1g_2 > 0, \qquad f > 0, \qquad f' > 0, \tag{117}$$

$$\mathrm{Re}\,c_6 > 0, \qquad \mathrm{Re}\,c_7 > 0. \tag{118}$$

Unfortunately, we find that the extended q^μ and $\phi^{\mu\nu}$ cannot remove the unstable modes at finite k coming from $\det M_4 = 0$. We choose the parameters satisfying the causality conditions (113) and stability conditions (114)–(118), and consider the influence on dispersion relations of g_1, g_2. For simplicity, we choose the parameters as the same as in Eq. (102) with $(g_1/\tau_\Pi, g_2/\tau_\Pi) = (0.0, 0.0), (2.0, 0.1), (6.0, 0.1), (6.0, 0.05)$. We find that one modes from $\det M_5 = 0$ becomes unstable at finite k with $(g_1/\tau_\Pi, g_2/\tau_\Pi) = (6.0, 0.1), (6.0, 0.05)$ as shown in Fig. 2.

As a brief summary, the extended q^μ and $\phi^{\mu\nu}$ can modify the causality and stability conditions, but cannot remove the zero modes when we turn off other dissipative effects. The unstable modes at finite k cannot be cured by the extended q^μ and $\phi^{\mu\nu}$.

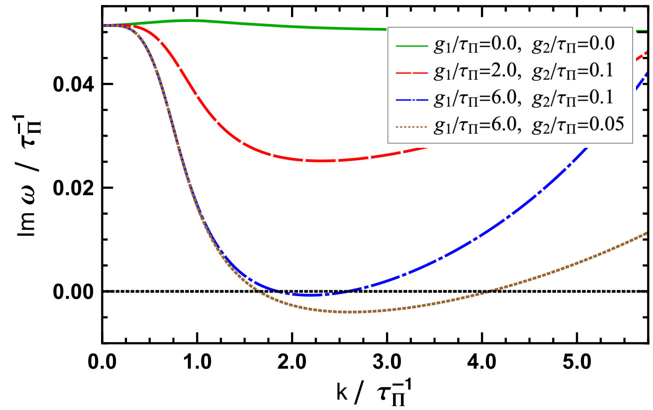

FIG. 2. Imaginary parts of $\omega\tau_\Pi$ as a function of $k\tau_\Pi$ in one mode derived from $\det M_5 = 0$. The green solid, red dashed, blue dash-dotted and brown dotted lines stand for the results with $(g_1/\tau_\Pi, g_2/\tau_\Pi) = (0.0, 0.0), (2.0, 0.1), (6.0, 0.1), (6.0, 0.05)$. Other parameters are also chosen as in Eq. (102).

E. Causality and stability in moving frames

Let us briefly discuss the causality and stability of the minimal causal spin hydrodynamics in moving frames.

For the causality in a moving frame, we refer to the studies in Refs. [168,176,177]. The authors in Refs. [168,176,177] have studied the dispersion relations at the large k limit in moving frames and demonstrate that the system is causal in moving frames if it is causal in the rest frame. Thus, the minimal causal spin hydrodynamics is causal in moving frames when the causality condition (96) in the rest frame is satisfied.

For the stability, it has also been proved that if a causal theory is unstable in the rest frame, then it is also unstable in moving frames (see Theorem 2 of Ref. [194]). We now apply this theorem to the minimal causal spin hydrodynamics. If the equation of state gives $\delta\omega^{\mu\nu} = \chi_1\delta S^{\mu\nu}$ with constant χ_1, the minimal causal spin hydrodynamics will be unstable in moving frames since it has unstable modes in the rest frame. For more general cases, the stability of the theory in both moving frames and the rest frame depends on the equation of state for $S^{\mu\nu}$ and $\omega^{\mu\nu}$.

In summary, the minimal causal spin hydrodynamics is causal in any reference frame when Eq. (96) is fulfilled. Hence, we have solved the problem of acausality by introducing the minimal causal spin hydrodynamics. However, the stability of minimal causal spin hydrodynamics remains unclear. Our findings indicate that the validity of the stability condition (98) is highly contingent upon the equation of state governing spin density and spin chemical potential. Moreover, we also find that the stability conditions (97)–(99) obtained at $k \to 0$ and $k \to +\infty$ are necessary but not sufficient.

VI. CONCLUSION

In this work, we investigate the linear causality and stability of the spin hydrodynamics proposed in Refs. [45,48].

In linear modes analysis, we consider perturbations to the spin hydrodynamics near the static equilibrium. We obtain the dispersion relations $\omega = \omega(k)$ and analyze all of the possible modes. The results show the stability condition (46) cannot be fulfilled. Moreover, the value of $|\omega/k|$ in Eqs. (48)–(50) is unbounded, which violates the causality condition (23). In Refs. [45,70,71], the expression of q^μ is modified by using the equation of motion for the fluid. We emphasize that the first order spin hydrodynamics in Refs. [45,70,71] are still acausal since one mode shown in Eq. (56) breaks the causality condition (23). We conclude that the spin hydrodynamics in the first order of gradient expansion are acausal and unstable.

We then follow the basic idea in MIS, BRSSS, and DNMR theories and consider the minimal causal spin hydrodynamics. The constitutive equations (16)–(20) in a minimal extended causal spin hydrodynamics are replaced by Eqs. (59)–(62). One can view it as a natural extension of the first order spin hydrodynamics or a simplified version of the complete second order spin hydrodynamics [73]. We investigate the causality and stability for this minimal causal spin hydrodynamics. We analyze the causality and stability for dissipative fluids with q^μ and $\phi^{\mu\nu}$ only and find the zero modes in the linear modes analysis. This suggests that linear mode analysis is inadequate in this case. Therefore, we consider dissipative spin fluids with shear viscous tensor and bulk viscous pressure.

For causality, we find that the modes with infinite speed disappear and all modes are causal in the rest frame if the conditions in Eq. (96) are fulfilled. Following the statement in Refs. [168,176,177], we comment that the minimal causal spin hydrodynamics are causal in any reference frame when the conditions (96) are fulfilled.

For the stability, although we obtain the stability conditions in Eqs. (97)–(99) from the constraints in the $k \to 0$ and $k \to +\infty$ limits, the stability of the theory in both moving frames and the rest frame remains unclear. Two kinds of problems can lead to instabilities. The first one is related to stability condition (98). Interestingly, we prove that the coefficients D_s, D_b do not obey the stability condition (98) if the equation of state $S^{\mu\nu} \sim T^2\omega^{\mu\nu}$ is adopted. In more general cases, the fulfillment of the stability condition (98) hinges on the specific equations of state. One has to assess the condition (98) on a case-by-case basis. Surprisingly, different with the conventional hydrodynamics, we find that the stability condition (24) breaks at finite k as shown in Fig. 1. It implies that the conditions (97)–(99) are necessary but may not be sufficient.

We also considered the extended q^μ and $\phi^{\mu\nu}$, in which the q^μ and $\phi^{\mu\nu}$ are coupled in the second order constitutive equations. The causality and stability conditions are modified in this case. However, in dissipative fluids with q^μ and $\phi^{\mu\nu}$ only the zero modes cannot be removed. The unstable modes at finite wavelength are still there.

We conclude that the spin hydrodynamics in the first order of gradient expansion, proposed in Refs. [45,48], are always acausal and unstable. The minimal causal extension of it makes the theory be causal in the sense of Eqs. (22) and (23). However, the linear stability of the minimal causal spin hydrodynamics remains unclear. The studies beyond the linear modes analysis may provide us a better and clear answer to the problem of stability.

ACKNOWLEDGMENTS

We thank Francesco Becattini, Matteo Buzzegoli, Asaad Daher, Xu-Guang Huang, Jin Hu, Masoud Shokri, and David Wagner for helpful discussion during the 7th International Conference on Chirality, Vorticity, and Magnetic Field in Heavy Ion Collisions. This work is supported in part by the National Key Research and Development Program of China under Contract No. 2022YFA1605500, by the Chinese Academy of Sciences (CAS) under Grant No. YSBR-088, and by National Nature Science Foundation of China (NSFC) under Grants No. 12075235 and No. 12135011.

APPENDIX A: OFF-DIAGONAL SUBMATRICES IN EQS. (36) AND (77)

In this appendix, we list all of the off-diagonal submatrices introduced in Eqs. (36) and (77):

$$A_1 \equiv \begin{pmatrix} -4i(\omega\lambda\chi_e^{0y} + k\gamma_s\chi_e^{xy}) & 0 & 0 \\ 8\lambda\chi_e^{0y} & 0 & 0 \\ 8\gamma_s\chi_e^{xy} & 0 & 0 \end{pmatrix},$$

$$A_2 \equiv \begin{pmatrix} -4i(\omega\lambda\chi_e^{0z} + k\gamma_s\chi_e^{xz}) & 0 & 0 \\ 8\lambda\chi_e^{0z} & 0 & 0 \\ 8\gamma_s\chi_e^{xz} & 0 & 0 \end{pmatrix}, \tag{A1}$$

$$A_3 = \begin{pmatrix} 8\gamma_s\chi_e^{yz}, & 0, & 0 \end{pmatrix}, \qquad A_4 = \begin{pmatrix} 8\gamma_s\chi_e^{xy} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 8\lambda\chi_e^{0y} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \qquad A_5 = \begin{pmatrix} 8\gamma_s\chi_e^{xz} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 8\lambda\chi_e^{0z} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \tag{A2}$$

$$A_6 = \begin{pmatrix} 2\gamma_s\chi_e^{yz} & 0 & 0 & 0 & 0 \\ 0 & \frac{2}{3}ik\gamma_\perp & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{A3}$$

APPENDIX B: DEFINITIONS OF COEFFICIENTS

The coefficients introduced in Eqs. (90)–(95) are defined as follows:

$$c_1 = \sqrt{\frac{b_1^{1/2} \pm (b_1 - b_2)^{1/2}}{6(2\tau_q - \lambda')\tau_\pi\tau_\Pi}}, \quad \text{or} \quad -\sqrt{\frac{b_1^{1/2} \pm (b_1 - b_2)^{1/2}}{6(2\tau_q - \lambda')\tau_\pi\tau_\Pi}}, \tag{B1}$$

$$c_2 = -3c_1^4[2\tau_\pi\tau_\Pi + (2\tau_q - \lambda')(\tau_\pi + \tau_\Pi)] + 48c_1^3\lambda\chi_e^{0x}\tau_\pi\tau_\Pi - 3c_s^2\gamma_\parallel\lambda' + c_1^2\{6\gamma_\parallel\tau_q + (6\gamma_\parallel - 8\gamma_\perp)\tau_\pi + 8\gamma_\perp\tau_\Pi + 3c_s^2[2\tau_\pi\tau_\Pi + (3\lambda' + 2\tau_q)(\tau_\pi + \tau_\Pi)]\} - 8c_1\lambda\chi_e^{0x}[(3\gamma_\parallel - 4\gamma_\perp)\tau_\pi + 4\gamma_\perp\tau_\Pi], \tag{B2}$$

$$c_3 = -2c_s^2\lambda'[(3\gamma_\parallel - 4\gamma_\perp)\tau_\pi + 4\gamma_\perp\tau_\Pi] - 18c_1^4(2\tau_q - \lambda')\tau_\pi\tau_\Pi + 4c_1^2[3c_s^2(3\lambda' + 2\tau_q)\tau_\pi\tau_\Pi + 2(3\gamma_\parallel - 4\gamma_\perp)\tau_q\tau_\pi + 8\gamma_\perp\tau_q\tau_\Pi], \tag{B3}$$

$$c_4 = \frac{\gamma_\perp[\tau_q(2\tau_q - \lambda') + \lambda'\tau_\pi]\tau_\phi^2 + \gamma'\tau_\pi^2[\tau_q(2\tau_q - \lambda') + \lambda'\tau_\phi]}{2(2\tau_q - \lambda')\tau_q\tau_\pi\tau_\phi(\gamma'\tau_\pi + \gamma_\perp\tau_\phi)}, \tag{B4}$$

$$c_5 = \sqrt{8D_s\gamma_\perp(\gamma'\tau_\pi + \gamma_\perp\tau_\phi) - (\gamma' + \gamma_\perp)^2}, \tag{B5}$$

where

$$b_1 = \{8\gamma_\perp\tau_q\tau_\Pi + \tau_\pi[2\tau_q(3\gamma_\parallel - 4\gamma_\perp) + 3\tau_\Pi c_s^2(3\lambda' + 2\tau_q)]\}^2, \tag{B6}$$

$$b_2 = 12c_s^2\lambda'(2\tau_q - \lambda')\tau_\pi\tau_\Pi[\tau_\pi(3\gamma_\parallel - 4\gamma_\perp) + 4\gamma_\perp\tau_\Pi]. \tag{B7}$$

The coefficients used in Eqs. (110)–(112) are given by

$$f = m\tau_\pi + 8\gamma_\perp\tau_q\tau_\phi, \tag{B8}$$

$$f' = \{-32g_1g_2\gamma_\perp(2\tau_q - \lambda')\tau_\pi\tau_\phi + (m\tau_\pi + 8\gamma_\perp\tau_q\tau_\phi)^2\}^{1/2}, \tag{B9}$$

$$c_6 = -\frac{1}{(f'^2 + fd^{1/2})}[m\tau_\pi(2\tau_q - \lambda')(\tau_\phi - \tau_\pi) + 8\gamma_\perp\tau_q\tau_\phi(\tau_q - \tau_\pi)(\lambda' - 2\tau_\phi) + 16\gamma'\tau_\pi^2\tau_\phi(2\tau_q - \lambda') - f'(2\tau_q - \lambda')(\tau_\pi + \tau_\phi) + 2\tau_\pi\tau_\phi(-m\tau_\pi - 8\gamma_\perp\lambda'\tau_\phi + 8\gamma'\tau_q^2 - f')], \tag{B10}$$

$$c_7 = -\frac{c_{72}}{c_{71}}, \tag{B11}$$

where

$$m = 2g_1g_2 + 8g_1\gamma' + g_2\lambda' + 8\gamma'\tau_q, \tag{B12}$$

$$c_{71} = -4g_1^2(g_2 + 4\gamma')^2\tau_\pi^2 + [g_2\lambda'\tau_\pi + 8\tau_q(\gamma'\tau_\pi + \gamma_\perp\tau_\phi)](-g_2\lambda'\tau_\pi - 8\gamma'\tau_q\tau_\pi - 8\gamma_\perp\tau_q\tau_\phi + d^{1/2}) - 2g_1\tau_\pi\{2g_2^2\lambda'\tau_\pi + g_2[8\gamma'(\lambda' + 2\tau_q)\tau_\pi + 16\gamma_\perp\lambda'\tau_\phi - 16\gamma_\perp\tau_q\tau_\phi - d^{1/2}] - 4\gamma'(16\gamma'\tau_q\tau_\pi + 16\gamma_\perp\tau_q\tau_\phi - d^{1/2})\}, \tag{B13}$$

$$c_{72} = -f'(2\tau_q - \lambda')(\tau_\pi + \tau_\phi) + m\tau_\pi(2\tau_q - \lambda')(\tau_\pi - \tau_\phi) - 16\gamma'\tau_\pi^2\tau_\phi(2\tau_q - \lambda') - 8\gamma_\perp\tau_q\tau_\phi(2\tau_q - \lambda')(\tau_\pi - \tau_\phi) + \tau_\pi\tau_\phi(-2f' + 2m\tau_\pi - 16\gamma_\perp\tau_q\tau_\phi + 16\gamma_\perp\lambda'\tau_\phi), \tag{B14}$$

$$d = 4g_1^2(g_2 + 4\gamma')^2\tau_\pi^2 + [g_2\lambda'\tau_\pi + 8\tau_q(\gamma'\tau_\pi + \gamma_\perp\tau_\phi)]^2 + 4g_1\tau_\pi[g_2^2\lambda'\tau_\pi + 4g_2\gamma'(\lambda' + 2\tau_q)\tau_\pi + 8g_2\gamma_\perp(\lambda' - \tau_q)\tau_\phi + 32\gamma'\tau_q(\gamma'\tau_\pi + \gamma_\perp\tau_\phi)]. \tag{B15}$$

APPENDIX C: CAUSALITY AND STABILITY OF THE MINIMAL CAUSAL SPIN HYDRODYNAMICS WITH q^μ AND $\phi^{\mu\nu}$ ONLY

In this appendix, we study the causality and stability of minimal causal spin hydrodynamics, considering only q^μ and $\phi^{\mu\nu}$. We will discuss two different cases. We name the system in which q^μ and $\phi^{\mu\nu}$ are not coupled, as depicted in Eqs. (59) and (60), as case I. Conversely, we name the system in which q^μ and $\phi^{\mu\nu}$ are coupled, as described by Eqs. (103) and (104), as case II.

1. Analysis for the case I

We let $\delta\Pi = \delta\pi^{ij} = 0$ in Eqs. (65), (66), (69), and (70) and remove Eqs. (67) and (68). Then we substitute the plane wave solutions Eq. (33) into Eqs. (65), (66), (69), and (70) and derive

$$\mathcal{M}_2'\delta\tilde{X}_2' = 0, \tag{C1}$$

where $\delta\tilde{X}_2'$ and $\mathcal{M}_2'$ are given by

$$\delta\tilde{X}_2' \equiv (\delta\tilde{e}, \delta\tilde{\vartheta}^x, \delta\tilde{S}^{0x}, \delta\tilde{\vartheta}^y, \delta\tilde{S}^{0y}, \delta\tilde{S}^{xy}, \delta\tilde{\vartheta}^z, \delta\tilde{S}^{0z}, \delta\tilde{S}^{xz}, \delta\tilde{S}^{yz})^{\mathrm{T}}, \tag{C2}$$

and

$$\mathcal{M}_2' \equiv \begin{pmatrix} M_4' & 0 & 0 & 0 \\ A_4' & M_5' & 0 & 0 \\ A_5' & 0 & M_5' & 0 \\ A_6' & 0 & 0 & M_6' \end{pmatrix}, \tag{C3}$$

with

$$M_4' = \begin{pmatrix} i\omega & -ik & \frac{1}{2}\omega k \\ -ikc_s^2 & i\omega & \frac{1}{2}\omega^2 \\ \lambda' c_s^2 ik + 8\lambda\chi_e^{0x} & \lambda' i\omega & 2D_b + \tau_q\omega^2 - i\omega \end{pmatrix}, \tag{C4}$$

$$M_5' = \begin{pmatrix} i\omega & \frac{1}{2}\omega^2 & -\frac{1}{2}\omega k \\ \lambda' i\omega & 2D_b + \tau_q\omega^2 - i\omega & 0 \\ 2\gamma' ik & 0 & -\tau_\phi\omega^2 + i\omega + 2D_s \end{pmatrix}, \tag{C5}$$

$$M_6' = -\tau_\phi\omega^2 + i\omega + 2D_s. \tag{C6}$$

The off-diagonal matrices $A_{4,5,6}'$ are given by

$$A_4' \equiv \begin{pmatrix} 0 & 0 & 0 \\ 8\lambda\chi_e^{0y} & 0 & 0 \\ 8\gamma_s\chi_e^{xy} & 0 & 0 \end{pmatrix}, \qquad A_5' \equiv \begin{pmatrix} 0 & 0 & 0 \\ 8\lambda\chi_e^{0z} & 0 & 0 \\ 8\gamma_s\chi_e^{xz} & 0 & 0 \end{pmatrix}, \qquad A_6' = \begin{pmatrix} 8\gamma_s\chi_e^{yz}, & 0, & 0 \end{pmatrix}. \tag{C7}$$

The dispersion relations $\omega = \omega(k)$ are derived from

$$\det\mathcal{M}_2' = \det M_4' \cdot (\det M_5')^2 \cdot \det M_6' = 0. \tag{C8}$$

We find that there exist two zero modes coming from the equation $\det M_5' = 0$. Now, let us focus on the nonzero modes. The $\det M_6' = 0$ gives two nonhydrodynamic modes

$$\omega = \frac{1}{2\tau_\phi}(i \pm \sqrt{8D_s\tau_\phi - 1}). \tag{C9}$$

From $\det M_4' = 0$ and $\det M_5' = 0$, we obtain the dispersion relation in the small k limit,

$$\omega = \pm c_s k \mp 2ic_s\lambda\chi_e^{0x}D_b^{-1}k^2 + O(k^3), \tag{C10}$$

$$\omega = \left[i \pm \sqrt{-4D_b(2\tau_q - \lambda') - 1}\right](2\tau_q - \lambda')^{-1} + O(k), \tag{C11}$$

$$\omega = \frac{1}{2\tau_\phi}(i \pm \sqrt{8D_s\tau_\phi - 1}) + O(k), \tag{C12}$$

and, in the large k limit,

$$\omega = \pm k\sqrt{\frac{c_s^2(3\lambda' + 2\tau_q)}{2\tau_q - \lambda'}} + \frac{4i\lambda'}{(2\tau_q - \lambda')(2\tau_q + 3\lambda')} \mp \frac{8\lambda\chi_e^{0x}}{c_s\sqrt{(\lambda' - 2\tau_q)(3\lambda' + 2\tau_q)}} + O(k^{-1}), \tag{C13}$$

$$\omega = \frac{i \pm \sqrt{-1 - 4D_b(2\tau_q + 3\lambda')}}{2\tau_q + 3\lambda'} + O(k^{-1}), \tag{C14}$$

$$\omega = \pm\sqrt{\frac{2\gamma'\tau_q}{(2\tau_q-\lambda')\tau_\phi}}k + i\frac{[\tau_q(2\tau_q-\lambda')+\lambda'\tau_\phi]}{2\tau_q\tau_\phi(2\tau_q-\lambda')} + O(k^{-1}), \tag{C15}$$

$$\omega = \frac{i\pm\sqrt{-1-8D_b\tau_q}}{2\tau_q} + O(k^{-1}). \tag{C16}$$

The causality conditions (22) and (23) require

$$0 \leq \frac{c_s^2(3\lambda'+2\tau_q)}{2\tau_q-\lambda'} \leq 1, \qquad 0 \leq \frac{2\gamma'\tau_q}{(2\tau_q-\lambda')\tau_\phi} \leq 1, \tag{C17}$$

which implies that the relaxation times τ_q, τ_ϕ cannot be arbitrarily small. It is consistent with the discussion in Sec. IV.

The stability condition (24) leads to

$$\tau_q > \lambda'/2, \qquad D_s > 0, \qquad D_b < 0, \qquad \chi_e^{0x} = 0, \tag{C18}$$

where $\chi_e^{0x} = 0$ comes from the stability of the sound mode (C10). Although the conditions in Eq. (C18) are derived from the small k and large k limits only, we can implement the Routh-Hurwitz criterion [168–170,188,192,193] to prove that the conditions (C18) are sufficient and necessary for stability, i.e., if (C18) are satisfied, then $\mathrm{Im}\omega > 0$ for all $k \neq 0$. Details for the proof are given below.

2. Condition (C18) is sufficient and necessary for the stability

As mentioned, we have derived the stability condition (C18) from the linear modes analysis in small and large k limits only. Now, we implement the Routh-Hurwitz criterion [168–170,188,192,193] to prove that the condition (C18) guarantees stability for all real nonzero k.

We only need to prove that the nonzero modes derived from $\det M_4' = 0$ and $\det M_5' = 0$ satisfy $\mathrm{Im}\omega > 0$ for all k. First, we discuss the modes coming from the $\det M_4' = 0$. The $\det M_4' = 0$ gives

$$a_0\omega^4 - ia_1\omega^3 - a_2\omega^2 + ia_3\omega + a_4 = 0, \tag{C19}$$

with

$$\begin{aligned} a_0 &= \frac{1}{2}(2\tau_q-\lambda'),\\ a_1 &= 1,\\ a_2 &= \frac{1}{2}c_s^2k^2(3\lambda'+2\tau_q) - 2D_b,\\ a_3 &= c_s^2k^2,\\ a_4 &= -2c_s^2D_bk^2. \end{aligned} \tag{C20}$$

We redefine $\omega = -i\Delta$ and rewrite Eq. (C19) as

$$a_0\Delta^4 + a_1\Delta^3 + a_2\Delta^2 + a_3\Delta + a_4 = 0. \tag{C21}$$

Notice that the coefficients $a_{0,1,2,3,4}$ are pure real. According to the Routh-Hurwitz criterion [168–170,188,192,193], the stability condition (24), i.e., $\mathrm{Im}\omega > 0$ or $\mathrm{Re}\Delta < 0$, is fulfilled for all nonzero k if and only if

$$\begin{aligned} &a_i > 0,\\ &a_1a_2a_3 - a_1^2a_4 - a_0a_3^2 > 0. \end{aligned} \tag{C22}$$

When the conditions in Eq. (C18) are fulfilled, the first inequality $a_i > 0$ is automatically satisfied. The second inequality can be expressed as $\lambda' = 2\lambda/[e_{(0)} + p_{(0)}] > 0$, which has already been guaranteed by entropy principle (21). Thus, the modes derived from $\det M_4' = 0$ are stable for all k if condition (C18) is satisfied.

Second, we consider the nonzero modes derived from $\det M_5' = 0$. The $\det M_5' = 0$ gives $\omega = 0$ or

$$a_0'\omega^4 - ia_1'\omega^3 - a_2'\omega^2 + ia_3'\omega + a_4' = 0, \tag{C23}$$

where

$$\begin{aligned} a_0' &= \frac{1}{2}\tau_\phi(2\tau_q-\lambda'),\\ a_1' &= \tau_\phi + \frac{1}{2}(2\tau_q-\lambda'),\\ a_2' &= 1 + D_s(2\tau_q-\lambda') + k^2\gamma'\tau_q - 2D_b\tau_\phi,\\ a_3' &= \gamma'k^2 + 2D_s - 2D_b,\\ a_4' &= -4D_bD_s - 2D_b\gamma'k^2. \end{aligned} \tag{C24}$$

Similarly, the Routh-Hurwitz criterion provides the necessary and sufficient conditions for $\mathrm{Im}\omega > 0$ in Eq. (C23),

$$a_i' > 0, \tag{C25}$$

$$a_1'a_2'a_3' - a_1'^2a_4' - a_0'a_3'^2 > 0. \tag{C26}$$

Each $a_i' > 0$ does not give new constraints for stability. We now show that the second inequality holds for all k if the conditions in Eq. (C18) are fulfilled. Define a new function $F(D_b, D_s, k)$,

$$\begin{aligned} F(D_b, D_s, k) &\equiv a_1'a_2'a_3' - a_1'^2a_4' - a_0'a_3'^2\\ &= 4\tau_\phi^2D_b^2 + \frac{1}{2}[8D_s(2\tau_q-\lambda')\tau_\phi + G(k)]D_b\\ &\quad + H(D_s, k), \end{aligned} \tag{C27}$$

with

$$G(k) \equiv -(2+k^2\gamma'\lambda')(2\tau_q-\lambda') - 2[2+k^2\gamma'(3\lambda'-4\tau_q)]\tau_\phi, \tag{C28}$$

$$H(D_s,k)\equiv\frac{1}{2}(2D_s+k^2\gamma')(2\tau_q-\lambda')\times[1+D_s(2\tau_q-\lambda')+k^2\gamma'\tau_q]+\frac{1}{2}(2D_s+k^2\gamma')(2+k^2\gamma'\lambda')\tau_\phi. \quad (C29)$$

Since $\tau_q>\lambda'/2$ in Eq. (C18), we have $H(D_s,k)>0$ for any k and any $D_s>0$.

Then, we discuss two cases. When

$$8D_s(2\tau_q-\lambda')\tau_\phi+G(k)\leq 0, \quad (C30)$$

we find $F(D_b,D_s,k)>0$ for any $D_b<0$. In another case, $8D_s(2\tau_q-\lambda')\tau_\phi+G(k)>0$, i.e.,

$$D_s>\frac{-G(k)}{8(2\tau_q-\lambda')\tau_\phi}, \quad (C31)$$

for each fixed $D_s>0$ and k, the function $F(D_b,D_s,k)$ gets its minimal value

$$F(D_b,D_s,k)\geq F(D_b,D_s,k)|_{D_b=-[8D_s(2\tau_q-\lambda')\tau_\phi+G(k)]/(16\tau_\phi^2)}=\frac{1}{64\tau_\phi^2}(2+k^2\gamma'\lambda')(\lambda'-2\tau_q-2\tau_\phi)^2\times[16\tau_\phi D_s-2-k^2\gamma'(\lambda'-8\tau_\phi)], \quad (C32)$$

at

$$D_b=-[8D_s(2\tau_q-\lambda')\tau_\phi+G(k)]/(16\tau_\phi^2). \quad (C33)$$

Substituting Eq. (C31) into Eq. (C32) leads to

$$F(D_b,D_s,k)\geq\frac{(2+k^2\gamma'\lambda')^2(\lambda'-2\tau_q-2\tau_\phi)^2(2\tau_q-\lambda'+4\tau_\phi)}{64(2\tau_q-\lambda')\tau_\phi^2}>0, \quad (C34)$$

where we have used $\tau_q>\lambda'/2$ in Eq. (C18). Thus, the nonzero modes derived from $\det M_5'=0$ are stable for all k if the conditions in Eq. (C18) are fulfilled.

Therefore, the conditions in Eq. (C18) are sufficient and necessary for the stability of fluids with q^μ and $\phi^{\mu\nu}$ only.

3. Analysis for case II

We now consider a more general case where q^μ and $\phi^{\mu\nu}$ are coupled as shown in Eqs. (103) and (104). Here we consider the q^μ and $\phi^{\mu\nu}$ only and neglect other dissipative terms for simplicity. In this case, M_5' in Eq. (C5) should be replaced with

$$M_5'=\begin{pmatrix} i\omega & \frac{1}{2}\omega^2 & -\frac{1}{2}\omega k\\ \lambda' i\omega & 2D_b+\tau_q\omega^2-i\omega & g_1\omega k\\ 2\gamma' ik & -\frac{1}{4}g_2\omega k & -\tau_\phi\omega^2+i\omega+2D_s\end{pmatrix}, \quad (C35)$$

while the matrix M_4' is the same as before. The dispersion relations in Eqs. (C15) and (C16) become

$$\omega=\pm\sqrt{\frac{m}{4(2\tau_q-\lambda')\tau_\phi}}k+\frac{1}{2}i\left(\frac{2}{2\tau_q-\lambda'}+\frac{1}{\tau_\phi}-\frac{8\gamma'}{m}\right)+\mathcal{O}(k^{-1}), \quad (C36)$$

$$\omega=\frac{4\gamma'(i\pm\sqrt{-1-D_bm\gamma'^{-1}})}{m}+\mathcal{O}(k^{-1}), \quad (C37)$$

where $m=2g_1g_2+8g_1\gamma'+g_2\lambda'+8\gamma'\tau_q$. We also notice that the zero modes mentioned before cannot be solved by introducing the coupling between q^μ and $\phi^{\mu\nu}$.

Imposing Eq. (22) to the propagating modes in Eqs. (92) and (93), the causality conditions (C17) are replaced with

$$0\leq\frac{c_s^2(3\lambda'+2\tau_q)}{2\tau_q-\lambda'}\leq 1,\qquad 0\leq\frac{m}{4(2\tau_q-\lambda')\tau_\phi}\leq 1. \quad (C38)$$

Inserting Eq. (24) into the new dispersion relations, we obtain the new stability conditions

$$\tau_q>\lambda'/2,\qquad D_s>0,\qquad D_b<0,\qquad \chi_e^{0x}=0,\qquad m>8\gamma'\left(\frac{2}{2\tau_q-\lambda'}+\frac{1}{\tau_\phi}\right)^{-1}. \quad (C39)$$

Similarly, we can still implement the Routh-Hurwitz criterion to verify that the conditions in Eq. (C39) are sufficient and necessary for stability.

4. Condition (C39) is sufficient and necessary for the stability

Let us now prove that the condition (C39) ensures $\mathrm{Im}\omega>0$ for all nonzero real k. Consider the nonzero modes derived from $\det M_5'=0$. The $\det M_5'=0$ gives

$$a_0'\omega^4-ia_1'\omega^3-a_2'\omega^2+ia_3'\omega+a_4'=0, \quad (C40)$$

where

$$
\begin{aligned}
a'_0 &= \frac{1}{2}\tau_\phi(2\tau_q - \lambda'),\\
a'_1 &= \tau_\phi + \frac{1}{2}(2\tau_q - \lambda'),\\
a'_2 &= 1 + D_s(2\tau_q - \lambda') + \frac{1}{8}k^2 m - 2D_b\tau_\phi,\\
a'_3 &= \gamma' k^2 + 2D_s - 2D_b,\\
a'_4 &= -4D_bD_s - 2D_b\gamma' k^2. \qquad \text{(C41)}
\end{aligned}
$$

The necessary and sufficient conditions for $\mathrm{Im}\omega > 0$ in Eq. (C40) are

$$a'_i > 0, \qquad \text{(C42)}$$

$$a'_1 a'_2 a'_3 - a'^2_1 a'_4 - a'_0 a'^2_3 > 0. \qquad \text{(C43)}$$

The first conditions are automatically satisfied when we have the constraints for stability. Then we need to analyze whether Eq. (C43) is satisfied under the existing constraints.

Define a function $F(D_b, D_s, k)$,

$$
\begin{aligned}
F(D_b, D_s, k) &\equiv a'_1 a'_2 a'_3 - a'^2_1 a'_4 - a'_0 a'^2_3\\
&= F_a D_b^2 + F_b D_b + F_c, \qquad \text{(C44)}
\end{aligned}
$$

where

$$
\begin{aligned}
F_a &\equiv 4\tau_\phi^2,\\
F_b &\equiv \left[\frac{1}{2}k^2\gamma'(2\tau_q - \lambda') + (4D_s + 3k^2\gamma')\tau_\phi\right](2\tau_q - \lambda') - \frac{1}{8}(mk^2 + 8)(2\tau_\phi + 2\tau_q - \lambda'),\\
F_c &\equiv \frac{1}{16}(2D_s + k^2\gamma')\{8D_s(2\tau_q - \lambda')^2 + (2\tau_q - \lambda')[8 + k^2(m - 8\gamma'\tau_\phi)] + 2(8 + k^2 m)\tau_\phi\}\\
&> \frac{1}{2}(2D_s + k^2\gamma')\{2\tau_\phi + (2\tau_q - \lambda')[1 + D_s(2\tau_q - \lambda')]\} > 0. \qquad \text{(C45)}
\end{aligned}
$$

When $F_b < 0$, i.e.,

$$D_s < \frac{(mk^2 + 8)(2\tau_\phi + 2\tau_q - \lambda')}{32(2\tau_q - \lambda')\tau_\phi} - \frac{k^2\gamma'(2\tau_q - \lambda')}{8\tau_\phi} - \frac{3}{4}k^2\gamma', \qquad \text{(C46)}$$

we get

$$F(D_b, D_s, k) > F(0, D_s, k) = F_c > 0. \qquad \text{(C47)}$$

In another case, $F_b \geq 0$, i.e.,

$$D_s \geq \frac{(mk^2 + 8)(2\tau_\phi + 2\tau_q - \lambda')}{32(2\tau_q - \lambda')\tau_\phi} - \frac{k^2\gamma'(2\tau_q - \lambda')}{8\tau_\phi} - \frac{3}{4}k^2\gamma', \qquad \text{(C48)}$$

the function has its minimal value

$$
\begin{aligned}
F(D_b, D_s, k)_{\min} &= F(D_b, D_s, k)|_{D_b = -F_b/(2F_a)}\\
&= -\frac{(2\tau_\phi + 2\tau_q - \lambda')^2}{1024\tau_\phi^2}\{8 + k^2[m - 4\gamma'(2\tau_q - \lambda')]\}\{8 + k^2[m - 4\gamma'(2\tau_q - \lambda')] - 32k^2\tau_\phi(\gamma' + 2D_s)\}\\
&\geq \frac{\{8 + k^2[m - 4\gamma'(2\tau_q - \lambda')]\}^2(2\tau_\phi + 2\tau_q - \lambda')^3}{1024\tau_\phi^2(2\tau_q - \lambda')} > 0, \qquad \text{(C49)}
\end{aligned}
$$

at

$$D_b = -\frac{F_b}{2F_a}, \qquad \text{(C50)}$$

$$D_s = \frac{(mk^2 + 8)(2\tau_\phi + 2\tau_q - \lambda')}{32(2\tau_q - \lambda')\tau_\phi} - \frac{k^2\gamma'(2\tau_q - \lambda')}{8\tau_\phi} - \frac{3}{4}k^2\gamma'. \qquad \text{(C51)}$$

Therefore, the nonzero modes are stable for all k if the stability condition (C39) is satisfied.

APPENDIX D: DISCUSSIONS ON THE STABILITY CONDITIONS (98)

Here, we discuss the stability conditions (98), i.e., $D_s > 0$, $D_b < 0$.

Let us consider an isotropic fluid at equilibrium, i.e., we assume that there are not preferred directions induced by spin and external fields. In this case, the variation of spin chemical potential is

$$\delta\omega^{\mu\nu} = \chi^{\mu\nu\alpha\beta}\delta S_{\alpha\beta} + \chi_e^{\mu\nu}\delta e, \tag{D1}$$

with a rank-4 tensor $\chi^{\mu\nu\alpha\beta}$ and rank-2 tensor $\chi_e^{\mu\nu}$. We find that $\chi^{\mu\nu\alpha\beta}$ satisfies $\chi^{\mu\nu\alpha\beta} = -\chi^{\nu\mu\alpha\beta} = -\chi^{\mu\nu\beta\alpha}$.

In an irrotational isotropic background fluid without any external fields, any rank-n tensor can only be constructed by $u^\mu, g^{\mu\nu}, \partial^\mu, \epsilon^{\mu\nu\alpha\beta}$. Back to rank-4 tensor $\chi^{\mu\nu\alpha\beta}$, in the linear modes analysis, we do not need to consider the part in $\chi^{\mu\nu\alpha\beta}$ proportional to space-time derivatives ∂^μ since those terms in $\chi^{\mu\nu\alpha\beta}\delta S_{\alpha\beta}$ becomes nonlinear and will be dropped. The tensor $\epsilon^{\mu\nu\alpha\beta}$ violates the reflection symmetry and cannot be used there. According to the antisymmetric properties of $\chi^{\mu\nu\alpha\beta}$, the only possible expression is

$$\chi^{\mu\nu\alpha\beta} = \frac{\chi_1}{2}(g^{\mu\alpha}g^{\nu\beta} - g^{\mu\beta}g^{\nu\alpha}) + \frac{\chi_2}{2}(\Delta^{\mu\alpha}\Delta^{\nu\beta} - \Delta^{\mu\beta}\Delta^{\nu\alpha}), \tag{D2}$$

where χ_1 and χ_2 are scalars.

Substituting Eq. (D2) into Eq. (D1), we obtain

$$\delta\omega^{\mu\nu} = \chi_1\delta S^{\mu\nu} + \chi_2\Delta^{\mu\alpha}\Delta^{\nu\beta}\delta S_{\alpha\beta}. \tag{D3}$$

One can also write it as

$$u_\mu\delta\omega^{\mu\nu} = \chi_1 u_\mu\delta S^{\mu\nu}, \tag{D4}$$

$$\Delta^{\mu\alpha}\Delta^{\nu\beta}\delta\omega_{\alpha\beta} = (\chi_1 + \chi_2)\Delta^{\mu\alpha}\Delta^{\nu\beta}\delta S_{\alpha\beta}. \tag{D5}$$

From the definitions in Eqs. (26) and (32), we then have

$$D_s = 4\gamma_s(\chi_1 + \chi_2), \qquad D_b = 4\lambda\chi_1. \tag{D6}$$

Since $\gamma_s > 0, \lambda > 0$, the stability condition (98), $D_s > 0$, $D_b < 0$, is equivalent to

$$\chi_2 > -\chi_1 > 0. \tag{D7}$$

The equation of state used in our previous works [62,64] corresponds to $\chi_2 = 0$ [see Eq. (17) of Ref. [62] and Eq. (38) of Ref. [64]]. In that case, Eq. (D7) cannot be satisfied and there exist unstable modes, although the analytic solutions in Refs. [62,64] do not rely on it. For general cases where $\chi_2 \neq 0$, whether the stability condition (98) $D_s > 0$, $D_b < 0$ is satisfied depends on χ_1, χ_2, which relates with the equation of state for $S^{\mu\nu}$ and $\omega^{\mu\nu}$. To determine the value of χ_1, χ_2, further investigations should be done from the microscopic theory.

[1] Z.-T. Liang and X.-N. Wang, Phys. Rev. Lett. **94**, 102301 (2005); **96**, 039901(E) (2006).
[2] Z.-T. Liang and X.-N. Wang, Phys. Lett. B **629**, 20 (2005).
[3] J.-H. Gao, S.-W. Chen, W.-t. Deng, Z.-T. Liang, Q. Wang, and X.-N. Wang, Phys. Rev. C **77**, 044902 (2008).
[4] STAR Collaboration, Nature (London) **548**, 62 (2017).
[5] J. Adam *et al.* (STAR Collaboration), Phys. Rev. C **98**, 014910 (2018).
[6] J. Adam *et al.* (STAR Collaboration), Phys. Rev. Lett. **123**, 132301 (2019).
[7] S. Acharya *et al.* (ALICE Collaboration), Phys. Rev. C **101**, 044611 (2020); **105**, 029902(E) (2022).
[8] J. Adam *et al.* (STAR Collaboration), Phys. Rev. Lett. **126**, 162301 (2021).
[9] F. J. Kornas (HADES Collaboration), Springer Proc. Phys. **250**, 435 (2020).
[10] M. S. Abdallah *et al.* (STAR Collaboration), Phys. Rev. C **104**, L061901 (2021).
[11] F. Becattini and F. Piccinini, Ann. Phys. (Amsterdam) **323**, 2452 (2008).
[12] F. Becattini, F. Piccinini, and J. Rizzo, Phys. Rev. C **77**, 024906 (2008).
[13] F. Becattini, V. Chandra, L. Del Zanna, and E. Grossi, Ann. Phys. (Amsterdam) **338**, 32 (2013).
[14] R.-h. Fang, L.-g. Pang, Q. Wang, and X.-n. Wang, Phys. Rev. C **94**, 024904 (2016).
[15] I. Karpenko and F. Becattini, Eur. Phys. J. C **77**, 213 (2017).
[16] Y. Xie, D. Wang, and L. P. Csernai, Phys. Rev. C **95**, 031901 (2017).
[17] H. Li, L.-G. Pang, Q. Wang, and X.-L. Xia, Phys. Rev. C **96**, 054908 (2017).
[18] Y. Sun and C. M. Ko, Phys. Rev. C **96**, 024906 (2017).
[19] S. Shi, K. Li, and J. Liao, Phys. Lett. B **788**, 409 (2019).
[20] D.-X. Wei, W.-T. Deng, and X.-G. Huang, Phys. Rev. C **99**, 014905 (2019).
[21] X.-L. Xia, H. Li, Z.-B. Tang, and Q. Wang, Phys. Rev. C **98**, 024905 (2018).
[22] O. Vitiuk, L. V. Bravina, and E. E. Zabrodin, Phys. Lett. B **803**, 135298 (2020).
[23] B. Fu, K. Xu, X.-G. Huang, and H. Song, Phys. Rev. C **103**, 024903 (2021).
[24] S. Ryu, V. Jupic, and C. Shen, Phys. Rev. C **104**, 054908 (2021).
[25] A. Lei, D. Wang, D.-M. Zhou, B.-H. Sa, and L. P. Csernai, Phys. Rev. C **104**, 054903 (2021).
[26] X.-Y. Wu, C. Yi, G.-Y. Qin, and S. Pu, Phys. Rev. C **105**, 064909 (2022).

[27] T. Niida (STAR Collaboration), Nucl. Phys. **A982**, 511 (2019).
[28] S. Acharya *et al.* (ALICE Collaboration), Phys. Rev. Lett. **128**, 172005 (2022).
[29] F. Becattini and I. Karpenko, Phys. Rev. Lett. **120**, 012302 (2018).
[30] W. Florkowski, A. Kumar, R. Ryblewski, and A. Mazeliauskas, Phys. Rev. C **100**, 054907 (2019).
[31] H.-Z. Wu, L.-G. Pang, X.-G. Huang, and Q. Wang, Phys. Rev. Res. **1**, 033058 (2019).
[32] X.-L. Xia, H. Li, X.-G. Huang, and H. Z. Huang, Phys. Rev. C **100**, 014913 (2019).
[33] F. Becattini, G. Cao, and E. Speranza, Eur. Phys. J. C **79**, 741 (2019).
[34] L. P. Csernai, J. I. Kapusta, and T. Welle, Phys. Rev. C **99**, 021901 (2019).
[35] M. G. L. Nogueira-Santos and C. C. Barros, Int. J. Mod. Phys. E **29**, 2050013 (2020).
[36] D. Montenegro, L. Tinti, and G. Torrieri, Phys. Rev. D **96**, 056012 (2017); **96**, 079901(A) (2017).
[37] D. Montenegro, L. Tinti, and G. Torrieri, Phys. Rev. D **96**, 076016 (2017).
[38] W. Florkowski, B. Friman, A. Jaiswal, R. Ryblewski, and E. Speranza, Phys. Rev. D **97**, 116017 (2018).
[39] W. Florkowski, B. Friman, A. Jaiswal, and E. Speranza, Phys. Rev. C **97**, 041901 (2018).
[40] W. Florkowski, E. Speranza, and F. Becattini, Acta Phys. Pol. B **49**, 1409 (2018).
[41] F. Becattini, W. Florkowski, and E. Speranza, Phys. Lett. B **789**, 419 (2019).
[42] D.-L. Yang, Phys. Rev. D **98**, 076019 (2018).
[43] W. Florkowski, A. Kumar, and R. Ryblewski, Prog. Part. Nucl. Phys. **108**, 103709 (2019).
[44] W. Florkowski, A. Kumar, and R. Ryblewski, Phys. Rev. C **98**, 044906 (2018).
[45] K. Hattori, M. Hongo, X.-G. Huang, M. Matsuo, and H. Taya, Phys. Lett. B **795**, 100 (2019).
[46] W. Florkowski, A. Kumar, R. Ryblewski, and R. Singh, Phys. Rev. C **99**, 044910 (2019).
[47] K. Fukushima and S. Pu, Lect. Notes Phys. **987**, 381 (2021).
[48] K. Fukushima and S. Pu, Phys. Lett. B **817**, 136346 (2021).
[49] S. Shi, C. Gale, and S. Jeon, Phys. Rev. C **103**, 044906 (2021).
[50] S. Li, M. A. Stephanov, and H.-U. Yee, Phys. Rev. Lett. **127**, 082302 (2021).
[51] S. Bhadury, W. Florkowski, A. Jaiswal, A. Kumar, and R. Ryblewski, Phys. Lett. B **814**, 136096 (2021).
[52] A. D. Gallegos and U. Gürsoy, J. High Energy Phys. 11 (2020) 151.
[53] M. Garbiso and M. Kaminski, J. High Energy Phys. 12 (2020) 112.
[54] S. Shi, C. Gale, and S. Jeon, Nucl. Phys. **A1005**, 121949 (2021).
[55] S. Bhadury, W. Florkowski, A. Jaiswal, A. Kumar, and R. Ryblewski, Phys. Rev. D **103**, 014030 (2021).
[56] R. Singh, G. Sophys, and R. Ryblewski, Phys. Rev. D **103**, 074024 (2021).
[57] A. D. Gallegos, U. Gürsoy, and A. Yarom, SciPost Phys. **11**, 041 (2021).
[58] D. She, A. Huang, D. Hou, and J. Liao, Sci. Bull. **67**, 2265 (2022).
[59] M. Hongo, X.-G. Huang, M. Kaminski, M. Stephanov, and H.-U. Yee, J. High Energy Phys. 11 (2021) 150.
[60] Y.-C. Liu and X.-G. Huang, Nucl. Sci. Tech. **31**, 56 (2020).
[61] W. Florkowski, R. Ryblewski, R. Singh, and G. Sophys, Phys. Rev. D **105**, 054007 (2022).
[62] D.-L. Wang, S. Fang, and S. Pu, Phys. Rev. D **104**, 114043 (2021).
[63] H.-H. Peng, J.-J. Zhang, X.-L. Sheng, and Q. Wang, Chin. Phys. Lett. **38**, 116701 (2021).
[64] D.-L. Wang, X.-Q. Xie, S. Fang, and S. Pu, Phys. Rev. D **105**, 114050 (2022).
[65] N. Weickgenannt, D. Wagner, E. Speranza, and D. H. Rischke, Phys. Rev. D **106**, 096014 (2022).
[66] N. Weickgenannt, D. Wagner, and E. Speranza, Phys. Rev. D **105**, 116026 (2022).
[67] Z. Cao, K. Hattori, M. Hongo, X.-G. Huang, and H. Taya, Prog. Theor. Exp. Phys. **2022**, 071D01 (2022).
[68] A. D. Gallegos, U. Gursoy, and A. Yarom, J. High Energy Phys. 05 (2023) 139.
[69] N. Weickgenannt, D. Wagner, E. Speranza, and D. H. Rischke, Phys. Rev. D **106**, L091901 (2022).
[70] A. Daher, A. Das, and R. Ryblewski, Phys. Rev. D **107**, 054043 (2023).
[71] G. Sarwar, M. Hasanujjaman, J. R. Bhatt, H. Mishra, and J.-e. Alam, Phys. Rev. D **107**, 054031 (2023).
[72] R. Biswas, A. Daher, A. Das, W. Florkowski, and R. Ryblewski, Phys. Rev. D **107**, 094022 (2023).
[73] R. Biswas, A. Daher, A. Das, W. Florkowski, and R. Ryblewski, Phys. Rev. D **108**, 014024 (2023).
[74] P. Shi and H. Xu-Guang, Acta Phys. Sin. **72**, 071202 (2023).
[75] F. Becattini, Rep. Prog. Phys. **85**, 122301 (2022).
[76] A. Palermo and F. Becattini, Eur. Phys. J. Plus **138**, 547 (2023).
[77] M. A. Stephanov and Y. Yin, Phys. Rev. Lett. **109**, 162001 (2012).
[78] D. T. Son and N. Yamamoto, Phys. Rev. D **87**, 085016 (2013).
[79] J.-H. Gao, Z.-T. Liang, S. Pu, Q. Wang, and X.-N. Wang, Phys. Rev. Lett. **109**, 232301 (2012).
[80] J.-W. Chen, S. Pu, Q. Wang, and X.-N. Wang, Phys. Rev. Lett. **110**, 262301 (2013).
[81] C. Manuel and J. M. Torres-Rincon, Phys. Rev. D **89**, 096002 (2014).
[82] C. Manuel and J. M. Torres-Rincon, Phys. Rev. D **90**, 076007 (2014).
[83] J.-Y. Chen, D. T. Son, M. A. Stephanov, H.-U. Yee, and Y. Yin, Phys. Rev. Lett. **113**, 182302 (2014).
[84] J.-Y. Chen, D. T. Son, and M. A. Stephanov, Phys. Rev. Lett. **115**, 021601 (2015).
[85] Y. Hidaka, S. Pu, and D.-L. Yang, Phys. Rev. D **95**, 091901 (2017).
[86] Y. Hidaka, S. Pu, and D.-L. Yang, Phys. Rev. D **97**, 016004 (2018).

[87] N. Mueller and R. Venugopalan, Phys. Rev. D **97**, 051901 (2018).
[88] Y. Hidaka and D.-L. Yang, Phys. Rev. D **98**, 016012 (2018).
[89] Y. Hidaka, S. Pu, and D.-L. Yang, Nucl. Phys. **A982**, 547 (2019).
[90] A. Huang, S. Shi, Y. Jiang, J. Liao, and P. Zhuang, Phys. Rev. D **98**, 036010 (2018).
[91] Y.-C. Liu, L.-L. Gao, K. Mameda, and X.-G. Huang, Phys. Rev. D **99**, 085014 (2019).
[92] J.-H. Gao, Z.-T. Liang, Q. Wang, and X.-N. Wang, Phys. Rev. D **98**, 036019 (2018).
[93] J.-H. Gao and Z.-T. Liang, Phys. Rev. D **100**, 056021 (2019).
[94] N. Weickgenannt, X.-L. Sheng, E. Speranza, Q. Wang, and D. H. Rischke, Phys. Rev. D **100**, 056018 (2019).
[95] K. Hattori, Y. Hidaka, and D.-L. Yang, Phys. Rev. D **100**, 096011 (2019).
[96] Z. Wang, X. Guo, S. Shi, and P. Zhuang, Phys. Rev. D **100**, 014015 (2019).
[97] S. Li and H.-U. Yee, Phys. Rev. D **100**, 056022 (2019).
[98] S. Lin and A. Shukla, J. High Energy Phys. 06 (2019) 060.
[99] S. Lin and L. Yang, Phys. Rev. D **101**, 034006 (2020).
[100] N. Weickgenannt, X.-L. Sheng, E. Speranza, Q. Wang, and D. H. Rischke, Nucl. Phys. **A1005**, 121963 (2021).
[101] N. Weickgenannt, E. Speranza, X.-l. Sheng, Q. Wang, and D. H. Rischke, Phys. Rev. Lett. **127**, 052301 (2021).
[102] D.-L. Yang, K. Hattori, and Y. Hidaka, J. High Energy Phys. 07 (2020) 070.
[103] Y.-C. Liu, K. Mameda, and X.-G. Huang, Chin. Phys. C **44**, 094101 (2020); **45**, 089001(E) (2021).
[104] N. Yamamoto and D.-L. Yang, Astrophys. J. **895**, 56 (2020).
[105] J.-H. Gao, G.-L. Ma, S. Pu, and Q. Wang, Nucl. Sci. Tech. **31**, 90 (2020).
[106] A. Huang, S. Shi, X. Zhu, L. He, J. Liao, and P. Zhuang, Phys. Rev. D **103**, 056025 (2021).
[107] J.-H. Gao, Z.-T. Liang, and Q. Wang, Int. J. Mod. Phys. A **36**, 2130001 (2021).
[108] Z. Wang, X. Guo, S. Shi, and P. Zhuang, Nucl. Phys. **A1005**, 121976 (2021).
[109] N. Weickgenannt, E. Speranza, X.-l. Sheng, Q. Wang, and D. H. Rischke, Phys. Rev. D **104**, 016022 (2021).
[110] Z. Wang and P. Zhuang, arXiv:2105.00915.
[111] X.-L. Sheng, N. Weickgenannt, E. Speranza, D. H. Rischke, and Q. Wang, Phys. Rev. D **104**, 016029 (2021).
[112] Y.-C. Liu and X.-G. Huang, Sci. China Phys. Mech. Astron. **65**, 272011 (2022).
[113] Y. Hidaka, S. Pu, Q. Wang, and D.-L. Yang, Prog. Part. Nucl. Phys. **127**, 103989 (2022).
[114] S. Fang, S. Pu, and D.-L. Yang, Phys. Rev. D **106**, 016002 (2022).
[115] J.-j. Zhang, R.-h. Fang, Q. Wang, and X.-N. Wang, Phys. Rev. C **100**, 064904 (2019).
[116] S. Y. F. Liu and Y. Yin, Phys. Rev. D **104**, 054043 (2021).
[117] S. Y. F. Liu and Y. Yin, J. High Energy Phys. 07 (2021) 188.
[118] S. A. Voloshin, EPJ Web Conf. **171**, 07002 (2018).
[119] S. Y. F. Liu, Y. Sun, and C. M. Ko, Phys. Rev. Lett. **125**, 062301 (2020).
[120] H.-Z. Wu, L.-G. Pang, X.-G. Huang, and Q. Wang, Nucl. Phys. **A1005**, 121831 (2021).
[121] W. Florkowski, A. Kumar, A. Mazeliauskas, and R. Ryblewski, Phys. Rev. C **105**, 064901 (2022).
[122] Y. Sun, Z. Zhang, C. M. Ko, and W. Zhao, Phys. Rev. C **105**, 034911 (2022).
[123] B. Fu, S. Y. F. Liu, L. Pang, H. Song, and Y. Yin, Phys. Rev. Lett. **127**, 142301 (2021).
[124] F. Becattini, M. Buzzegoli, and A. Palermo, Phys. Lett. B **820**, 136519 (2021).
[125] F. Becattini, M. Buzzegoli, G. Inghirami, I. Karpenko, and A. Palermo, Phys. Rev. Lett. **127**, 272302 (2021).
[126] C. Yi, S. Pu, and D.-L. Yang, Phys. Rev. C **104**, 064901 (2021).
[127] C. Yi, S. Pu, J.-H. Gao, and D.-L. Yang, Phys. Rev. C **105**, 044911 (2022).
[128] S. Alzhrani, S. Ryu, and C. Shen, Phys. Rev. C **106**, 014905 (2022).
[129] C. Yi *et al.*, arXiv:2304.08777.
[130] K. Schilling, P. Seyboth, and G. E. Wolf, Nucl. Phys. **B15**, 397 (1970); **B18**, 332(E) (1970).
[131] S. Acharya *et al.* (ALICE Collaboration), Phys. Rev. Lett. **125**, 012301 (2020).
[132] B. Mohanty, S. Kundu, S. Singha, and R. Singh, Mod. Phys. Lett. A **36**, 2130026 (2021).
[133] ALICE Collaboration, Phys. Rev. Lett. **131**, 042303 (2023).
[134] M. S. Abdallah *et al.* (STAR Collaboration), Nature (London) **614**, 244 (2023).
[135] S. Singha (STAR Collaboration), Acta Phys. Pol. B Proc. Suppl. **16**, 34 (2023).
[136] J. Chen, Z.-T. Liang, Y.-G. Ma, and Q. Wang, Sci. Bull. **68**, 874 (2023).
[137] Y.-G. Yang, R.-H. Fang, Q. Wang, and X.-N. Wang, Phys. Rev. C **97**, 034917 (2018).
[138] X.-L. Xia, H. Li, X.-G. Huang, and H. Zhong Huang, Phys. Lett. B **817**, 136325 (2021).
[139] J.-H. Gao, Phys. Rev. D **104**, 076016 (2021).
[140] B. Müller and D.-L. Yang, Phys. Rev. D **105**, L011901 (2022); **106**, 039904(E) (2022).
[141] A. Kumar, B. Müller, and D.-L. Yang, Phys. Rev. D **108**, 016020 (2023).
[142] X.-L. Sheng, L. Oliva, and Q. Wang, Phys. Rev. D **101**, 096005 (2020); **105**, 099903(E) (2022).
[143] X.-L. Sheng, Q. Wang, and X.-N. Wang, Phys. Rev. D **102**, 056013 (2020).
[144] X.-L. Sheng, L. Oliva, Z.-T. Liang, Q. Wang, and X.-N. Wang, Phys. Rev. Lett. **131**, 042304 (2023).
[145] X.-L. Sheng, L. Oliva, Z.-T. Liang, Q. Wang, and X.-N. Wang, arXiv:2206.05868.
[146] X.-L. Sheng, S. Pu, and Q. Wang, arXiv:2308.14038.
[147] E. Speranza and N. Weickgenannt, Eur. Phys. J. A **57**, 155 (2021).
[148] F. W. Hehl, Rep. Math. Phys. **9**, 55 (1976).
[149] E. Leader and C. Lorcé, Phys. Rep. **541**, 163 (2014).
[150] S. Dey, W. Florkowski, A. Jaiswal, and R. Ryblewski, Phys. Lett. B **843**, 137994 (2023).

[151] J. Hilgevoord and E. De Kerf, Physica (Amsterdam) **31**, 1002 (1965).
[152] S. R. De Groot, *Relativistic Kinetic Theory. Principles and Applications* (North-Holland Publishing Company, Amsterdam, 1980).
[153] A. Das, W. Florkowski, R. Ryblewski, and R. Singh, Phys. Rev. D **103**, L091502 (2021).
[154] M. Buzzegoli, Phys. Rev. C **105**, 044907 (2022).
[155] A. Daher, A. Das, W. Florkowski, and R. Ryblewski, Phys. Rev. C **108**, 024902 (2023).
[156] W. A. Hiscock and L. Lindblom, Phys. Rev. D **31**, 725 (1985).
[157] W. A. Hiscock and L. Lindblom, Phys. Rev. D **35**, 3723 (1987).
[158] G. S. Denicol, T. Kodama, T. Koide, and P. Mota, J. Phys. G **35**, 115102 (2008).
[159] S. Pu, T. Koide, and D. H. Rischke, Phys. Rev. D **81**, 114039 (2010).
[160] W. Israel and J. M. Stewart, Proc. R. Soc. A **365**, 43 (1979).
[161] W. Israel and J. M. Stewart, Ann. Phys. (N.Y.) **118**, 341 (1979).
[162] T. Koide, G. S. Denicol, P. Mota, and T. Kodama, Phys. Rev. C **75**, 034909 (2007).
[163] J. Hu, Phys. Rev. D **103**, 116015 (2021).
[164] S. Bhadury, W. Florkowski, A. Jaiswal, A. Kumar, and R. Ryblewski, Phys. Rev. Lett. **129**, 192301 (2022).
[165] F. Becattini, Phys. Rev. Lett. **108**, 244502 (2012).
[166] F. Becattini, Lect. Notes Phys. **987**, 15 (2021).
[167] J. Hu, Phys. Rev. C **107**, 024915 (2023).
[168] P. Kovtun, J. High Energy Phys. 10 (2019) 034.
[169] F. S. Bemfica, M. M. Disconzi, and J. Noronha, Phys. Rev. X **12**, 021044 (2022).
[170] R. E. Hoult and P. Kovtun, J. High Energy Phys. 06 (2020) 067.
[171] C. V. Brito and G. S. Denicol, Phys. Rev. D **102**, 116009 (2020).
[172] C. V. Brito and G. S. Denicol, Phys. Rev. D **105**, 096026 (2022).
[173] E. Krotscheck and W. Kundt, Commun. Math. Phys. **60**, 171 (1978).
[174] L. C. Evans, *Partial Differential Equations*, 2nd ed. (American Mathematical Society, Providence, 2010).
[175] L. Gavassino, M. M. Disconzi, and J. Noronha, arXiv:2307.05987.
[176] D.-L. Wang and S. Pu, arXiv:2309.11708.
[177] R. E. Hoult and P. Kovtun, arXiv:2309.11703.
[178] P. Kovtun, J. Phys. A **45**, 473001 (2012).
[179] A. Muronga, Phys. Rev. C **69**, 034903 (2004).
[180] L. Rezzolla and O. Zanotti, *Relativistic Hydrodynamics* (Oxford University Press, New York, 2013).
[181] W. A. Hiscock and L. Lindblom, Ann. Phys. (N.Y.) **151**, 466 (1983).
[182] T. S. Olson, Ann. Phys. (N.Y.) **199**, 18 (1990).
[183] R. Baier, P. Romatschke, D. T. Son, A. O. Starinets, and M. A. Stephanov, J. High Energy Phys. 04 (2008) 100.
[184] G. S. Denicol, H. Niemi, E. Molnar, and D. H. Rischke, Phys. Rev. D **85**, 114047 (2012); **91**, 039902(E) (2015).
[185] S. Floerchinger and E. Grossi, J. High Energy Phys. 08 (2018) 186.
[186] F. S. Bemfica, M. M. Disconzi, V. Hoang, J. Noronha, and M. Radosz, Phys. Rev. Lett. **126**, 222301 (2021).
[187] F. S. Bemfica, M. M. Disconzi, and J. Noronha, Phys. Rev. D **98**, 104064 (2018).
[188] F. S. Bemfica, M. M. Disconzi, and J. Noronha, Phys. Rev. D **100**, 104020 (2019); **105**, 069902(E) (2022).
[189] R. E. Hoult and P. Kovtun, Phys. Rev. D **106**, 066023 (2022).
[190] N. Weickgenannt, Phys. Rev. D **108**, 076011 (2023).
[191] G. S. Denicol, T. Kodama, T. Koide, and P. Mota, J. Phys. G **36**, 035103 (2009).
[192] M. Gopal, *Control Systems: Principles and Design* (McGraw-Hill, New York, 2006).
[193] I. Gradshteuin, I. Ryzhik, A. Jeffrey, and D. Zwillinger, *Table of Integrals, Series, and Products* (Elsevier, Amsterdam, 2007).
[194] L. Gavassino, Phys. Rev. X **12**, 041001 (2022).