Entanglement renormalization of the class of continuous matrix product states

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Continuous tensor networks give a variational ansatz for the ground state of the quantum field theories (QFTs). Notable examples are the continuous matrix product state (cMPS) and the continuous multiscale entanglement renormalization ansatz (cMERA). While cMPS is just adapted to the nonrelativistic QFTs, only the Gaussian cMERA is well understood, which we cannot use to approximate the ground state of the interacting relativistic QFTs. But, instead, cMERA also corresponds to a real-space renormalization group flow in the context of the wave functions. In this paper, we investigate the backward Gaussian cMERA renormalization group flow of the class of cMPS by putting the standard cMPS at the IR scale. At the UV scale, for the bosonic systems in the thermodynamic limit, we achieve the variational class of states that has been proposed recently, as the relativistic cMPS (RCMPS) is adapted to the relativistic QFTs without requiring one to introduce of any additional IR or UV cutoff. We also extend the RCMPS to fermionic systems and theories on a finite circle.

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I. INTRODUCTION

Tensor network states are the entanglement-based ansatz that has arisen in recent years based on the renormalization group (RG) ideas and later on developed using tools and concepts from quantum information theory. The main examples include *matrix product states* (MPS) [1], *projected entangled-pair states* (PEPS) [2], and *multiscale entanglement renormalization ansatz* (MERA) [3]. By construction, they obey the entropy or area law [4–7] and are able to encode both global and local symmetries [8–12]. Therefore, they provide an efficient class of symmetric variational ansatz to approximate the ground state of the local Hamiltonian. In general, the understanding of the low-energy behavior of many-body quantum systems is one of the major challenges of modern physics, in both high-energy and condensed matter physics. There are plenty of methods based on RG introduced to tackle this problem. To study the weakly coupled system, one can use the momentum space RG [13–17]. But instead, in the case of the strongly interacting systems where the perturbation theory fails, this question is usually addressed by real-space RG methods.

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In the case of the many-body system on the lattice, Kadanoff's spin-blocking idea [18] was replaced by Wilson's real-space RG [17], which was improved later by White's *density matrix renormalization group* (DMRG) [19,20]. This technique is extraordinarily powerful in the study of quantum systems on the 1D lattice. It has been generalized as *tensor renormalization group* (TRG) by Levin and Nave [21] to study the Euclidean path integral of 1D quantum systems or the 2D classical lattice models. Although both the DMRG and TRG are very successful, they provide a coarse-grained system that still contains irrelevant microscopic information which implies the breakdown of both methods at criticality [21], and the resulting RG flow has the wrong structure of noncritical fixed points [22]. In the context of wave functions, this problem was resolved with the introduction of *entanglement renormalization* (ER) by Vidal [3]. A key aspect of ER is the ability to remove the short-range entanglement at each coarse-graining step by introducing a disentangler operator. This leads to the restoration of scale invariance at criticality and results in a proper RG flow with the correct structure of fixed points both at criticality and off criticality. More recently, this technique has been adapted to tackle the same problem in TRG in the context of the Euclidean path integral of quantum many-body systems and the partition function of a classical statistical system by removing short-range correlations this time from the partition function, known as *tensor network renormalization* (TNR) [23]. ER and TNR represent a powerful alternative to Wilsonian real-space RG methods in the context of the wave function and partition function, respectively.

Beginning with the DMRG, it has been shown that this technique can be understood as a variational method within the class of MPS [24]. In addition, it justifies the point that DMRG is powerful just in one spatial dimension because of the area law. More generally, any variational class corresponds to an RG scheme. As another important example, the ER is naturally associated with the class of MERA [3].

Tensor network formalism can be also applied to study the low-energy limit of quantum field theories (QFTs) after an appropriate discretization of the theory on the lattice [25–30]. However, the symmetries of spacetime in this way will be destroyed. Thus, it would be desirable to work directly in the continuum which can provide a powerful nonperturbative approach for studying the strongly interacting QFTs. In the past decade, the generalization from lattice to continuum has been done for some classes of tensor network states [31–35], in particular, the continuous version of MPS and MERA, known as cMPS [31] and cMERA [32], respectively. To date, only the Gaussian cMERA is well understood, which limits the interest of cMERA to use as a variational ansatz to study the strongly coupled QFTs. Instead, cMERA has already attracted considerable attention in the context of holography [36–49]. On the other hand, the cMPS provides a variational class of non-Gaussian wave functional which is just adapted to the nonrelativistic interacting QFTs in $1+1$ dimensions. In the case of relativistic QFTs, the cMPS construction suffers from regularization ambiguity. One can still use cMPS to study the low-energy limit of the theory in practice by introducing a UV cutoff [50,51]. But still by construction, using the cMPS approach, one cannot capture the short-distance behavior of the system. Moreover, defining a UV cutoff by itself is in contrast with the purpose of working directly in the continuum.

In this paper, motivated by [52], we study the one-parameter family of cMPS generated by ER which maps a free nonrelativistic theory at the IR scale to a free relativistic theory at UV. We will find that, at the UV scale, the resulting wave functional is exactly the variational ansatz known as *relativistic* cMPS (RCMPS) introduced in [52] which is adapted to relativistic QFTs in $1+1$ dimensions. In the following, one can find the brief review of the ER and the class of cMPS that we need in the main discussion of the paper and introduce the notation there.

II. ENTANGLEMENT RENORMALIZATION IN CONTINUUM

cMERA [32] was originally introduced as an ansatz wave functional for the ground states of QFT Hamiltonians. The same as the ER that corresponds to MERA tensor network, the continuous version of it implements a real-space RG in the continuum. MERA on a lattice can also be visualized as a quantum circuit [53]. In this representation, the physical state can be obtained by evolving a simple product state with no entanglement that factories with respect to the lattice sites—usually considered as “all sites 0”—by a unitary operator to create entanglement at different scales. The generalization to the continuum is conceptually straightforward. To describe cMERA, first assume a QFT and impose a UV cutoff Λ. It is required to start with a finite Λ in order to define the process, but, in the end, it can be sent to infinity. One parameter family of scale-dependent states is produced through continuous unitary evolution in scale u as

$$|\Psi(u)\rangle = U(u, u_{\mathrm{IR}})|\Omega\rangle = \mathcal{P}e^{-i\int_{u_{\mathrm{IR}}}^{u}(K(s)+L)ds}|\Omega\rangle, \quad (1)$$

where the symbol $\mathcal{P}$ is path ordering. $|\Omega\rangle$ is the IR state that is the continuum limit of a product state on the lattice that contains no entanglement between spatial regions, and the UV state is what describes the system we are studying, usually, the ground state of the system. Moreover, it has been shown that any spacetime symmetry of the ground state is also a symmetry of the cMERA representation of it [54]. Only the difference between UV and IR limits is fixed as $u_{\mathrm{UV}} - u_{\mathrm{IR}} = O(\log \xi\Lambda)$ when ξ is the correlation length of the theory. It is convenient to set $u_{\mathrm{UV}} = 0$ and $u_{\mathrm{IR}} = -O(\log \xi\Lambda)$. For critical systems $u_{\mathrm{IR}} \to -\infty$.

On the other hand, L is the generator of the scale transformation in spacial directions and $K(u)$ is the so-called entangler (or disentangler, depending on the direction of the RG flow) which contains the variational parameters of the cMERA. The IR state is scale invariant; thus,

$$L|\Omega\rangle = 0. \quad (2)$$

Consider a set of field operators of the theory $\psi(x)$, $\psi^{\dagger}(x)$ satisfying $[\psi(x), \psi^{\dagger}(y)]_{\pm} = \delta(x-y)$ with $+(-)$ for fermions (bosons). If the IR state is the vacuum of this set of annihilation and creation operators, i.e.,

$$\psi(x)|\Omega\rangle = 0 \quad \forall \; x, \quad (3)$$

the generator of scale transformation can be read as

$$L = -\frac{i}{2}\int \psi^{\dagger}(x)x\frac{d\psi(x)}{dx} - x\frac{d\psi^{\dagger}(x)}{dx}\psi(x)dx. \quad (4)$$

Although some steps have been taken toward finding the form of the entangler operator for interacting theories, both at the perturbative level [55–57] and nonperturbatively [58], it has been explicitly studied only for free theories [32,59]. The entangler operator for quadratic interactions is the generator of Bogoliubov transformation given by

$$K(u) = \frac{i}{2}\int dk\big(g(k,u)\psi_k^\dagger\psi_{-k}^\dagger - g^*(k,u)\psi_{-k}\psi_k\big), \quad (5)$$

where $\psi_k = \frac{1}{\sqrt{2\pi}}\int dx e^{-ikx}\psi(x)$ and $g(k/\Lambda, u)$ is even and odd in its first argument for bosons and fermions, respectively. Finally, we mention that the cMERA unitary process provides a RG flow for the operators as

$$\frac{dO(u)}{du} = -i\big[K(u) + L, O(u)\big] \quad (6)$$

when the physical or bare operators of the theory are defined at the UV scale.

III. CONTINUOUS MPS

The cMPS was originally proposed in [31] by Verstraete and Cirac as a variational ansatz for the ground state of nonrelativistic QFT Hamiltonians in $1+1$ dimensions. It can be obtained as the continuum limit of a certain family of MPS which is selected in such a way to have a valid continuum limit.

The most generic form of a MPS for a lattice with N sites is given by

$$|\psi\rangle = \sum_{i_1,\ldots,i_N}\mathrm{Tr}\big[A_1^{i_1}A_2^{i_2}\ldots A_N^{i_N}\big]|i_1, i_2, \ldots, i_N\rangle, \quad (7)$$

where $A_n^{i_n}$ are $D\times D$ complex matrices containing the variational parameters of this ansatz. Therefore, the MPS representation of the many-body wave function is specified by just $O(D^2)$ variational parameters instead of exponential growth with N, which makes it a powerful variational ansatz for interacting theories. To find a generalization of MPS in the continuum, we can approximate the QFT on a line of length L by a lattice with lattice spacing ϵ and $N = L/\epsilon$ sites. At each site of the lattice, there is a bosonic (or fermionic) mode a_i obeys $[a_i, a_j^\dagger]_\pm = \delta_{ij}$ and, thus, the Hilbert space spanned by $\{|n_i\rangle\}$, where $|n_i\rangle$ corresponds to having n_i particles on that site. The many-body state $|i_1, i_2, \ldots, i_N\rangle$ can be rewritten as $a_1^{\dagger i_1}a_2^{\dagger i_2}\ldots a_N^{\dagger i_N}|\mathbf{0}\rangle$, where $|\mathbf{0}\rangle = \otimes_{n=1}^N |0\rangle_n$. On this lattice, one can define a family of MPS as

$$A_i^0 = I + \epsilon Q(i\epsilon),$$
$$A_i^n = \frac{1}{n!}\big(\sqrt{\epsilon}R(i\epsilon)\big)^n \quad n \geq 1. \quad (8)$$

By taking the $\epsilon \to 0$ limit of it, one can find the class of cMPS as

$$|\psi[Q,R]\rangle = \mathrm{Tr}_{\mathrm{aux}}\bigg\{\mathcal{P}\exp\int_{-L/2}^{L/2}dx \times\big(Q(x)\otimes I + R(x)\otimes\psi^\dagger(x)\big)\bigg\}|\Omega\rangle, \quad (9)$$

where $\mathrm{Tr}_{\mathrm{aux}}$ denotes a partial trace over the auxiliary system where the matrices Q and R act. For the translational invariant cMPS, the matrices Q and R are position independent. The field $\psi(x)$ is the continuum limit of the rescale modes satisfying $[\psi(x), \psi^\dagger(y)]_\pm = \delta(x-y)$, and $|\Omega\rangle$, the empty vacuum defined as

$$\psi(x)|\Omega\rangle = 0 \quad \forall\ x, \quad (10)$$

the same as the IR state of the cMERA. The expectation value of local operators and, in particular, the Hamiltonian on the cMPS representation of the ground state can be easily expressed in terms of the matrices Q and R. In particular, all normal ordered correlation functions of local field operators can be deduced from a generating functional as

$$\langle : F[\psi^\dagger(x), \psi(y)] : \rangle = F\left[\frac{\delta}{\delta\bar{j}(x)}, \frac{\delta}{\delta j(y)}\right]\mathcal{Z}_{\bar{j},j}|_{\bar{j},j=0}, \quad (11)$$

when its explicit form can be given in terms of the cMPS matrices

$$\mathcal{Z}_{\bar{j},j} = \mathrm{Tr}\bigg\{\mathcal{P}\exp\bigg[\int dxT + j(x)R\otimes I + \bar{j}(x)I\otimes\bar{R}\bigg]\bigg\}, \quad (12)$$

where $T = Q\otimes I + I\otimes\bar{Q} + R\otimes\bar{R}$ is the cMPS transfer matrix [60]. To find the cMPS approximation of the ground state, it just needed to minimize the expectation value of the Hamiltonian over the cMPS matrices. After that, correlation functions can be straightforwardly computed. The same as MPS, the cMPS representation has gauge freedom that one can use to impose certain conditions on the cMPS matrices, including symmetry conditions. Moreover, for the continuum version, the left orthogonality condition of MPS can be read as $Q(x) + Q^\dagger(x) + R^\dagger(x)R(x) = 0$ for all x. By increasing D, one can find a better approximation of the ground state. In the past decade, several optimization algorithms have been developed to study a number of theories, both bosonic and fermionic [50,61–72]. As mentioned, the cMPS provides an efficient variational ansatz for nonrelativistic QFTs. It is not adapted to relativistic theories because of a lack of sensitivity to short-distance behavior. One can look at [52,73] for a complete explanation of the difficulties of applying the cMPS to the relativistic QFTs.

IV. cMERA RG FLOW OF THE CLASS OF cMPS

The cMPS representation is a mathematical framework used in the context of QFT. It is a way of describing the ground state of a nonrelativistic QFT as a special kind of state generated by transforming the ground state of the free part of the nonrelativistic QFT's Hamiltonian, i.e., $|\Omega\rangle$ in (10).

Now, when we are dealing with relativistic QFT, things get more complicated due to the relativistic nature of the theory. However, this interpretation of cMPS suggests a way to adapt it to represent the ground state of a relativistic QFT. This adaptation involves transforming the ground state of the free relativistic QFT in a certain manner. Interestingly, there exists a concept known as cMERA RG flow. This concept establishes a connection or flow between the ground states of the nonrelativistic and relativistic free theories. In simpler terms, it provides a way to relate the ground states of these two different types of quantum field theories.

In the context of this discussion, the focus is on studying a particular family of cMPS that evolves using the corresponding cMERA evolution. This means we are investigating how a specific set of cMPS changes or transforms as we apply the principles of cMERA, particularly in the context of the ground states of both nonrelativistic and relativistic free quantum field theories. In summary, this discussion revolves around using the cMPS framework to represent ground states in QFT, extending it to relativistic cases, and exploring the connections and transformations between these states through the concept of cMERA RG flow (Fig. 1).

There is a cMERA RG flow that relates the ground states of the nonrelativistic and relativistic free theories to each other. To do this, we start by placing the vacuum of the cMPS representation at the "IR level" in the cMERA framework. In simpler terms, we are setting up our system with the nonrelativistic vacuum state as a starting point. Then, we follow this process as we move up to the "UV level" within the cMERA framework. At this UV level, the goal is to reach the vacuum state of the free relativistic quantum field theory.

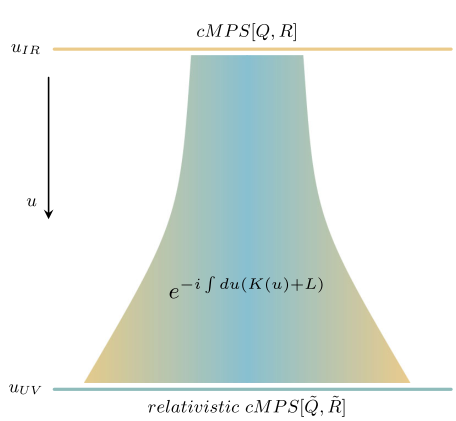

FIG. 1. Entanglement renormalization group flow of the class of cMPS.

In the following, we study the one-parameter family of the cMPS evolves with the corresponding cMERA evolution. The cMERA formalism was originally formulated for infinite systems [32]. However, its generalization to systems with open boundary conditions [74] and on a finite circle [75] has been introduced more recently. First, we work in the thermodynamic limit, i.e., $L\to\infty$, and after that, we will discuss the extension for the theories defined on a finite circle. To proceed, we should find the generator of the RG flow in the case that we are interested in, i.e., the cMERA generator of mapping the IR state $|\Omega\rangle$ to the ground state of the relativistic field theory.

Consider the free scalar field in the $1+1$ dimension. The Hamiltonian is given by

$$H_{\text{fb}}=\frac{1}{2}\int dx\left[\pi^2(x)+(\partial_x\phi(x))^2+m^2\phi(x)^2\right],\qquad(13)$$

where the field operator and its conjugate momentum satisfy $[\phi(x),\pi(y)]=i\delta(x-y)$. One can expand them in terms of creation and annihilation operators a_k and $a_k^\dagger$ satisfying $[a_k,a_{k'}^\dagger]=2\pi\delta(k-k')$. The ground state of the theory is known to be the Fock space vacuum denoted by $|\mathbf{0}\rangle_a$, while $a_k|\mathbf{0}\rangle_a=0$ for all k.

In order to specify the cMERA representation of the ground state, we need to first define the unentangled reference state $|\Omega\rangle$ which is the same as the vacuum of the cMPS state in terms of the fundamental fields of the given theory. In general, one can define a Gaussian factorized state with width Δ^{-1} as $\psi(x)|\Omega\rangle=0$ for all x, while [32]

$$\psi(x)=\sqrt{\frac{\Delta}{2}}\phi(x)+i\sqrt{\frac{1}{2\Delta}}\pi(x).\qquad(14)$$

Notice that the operator (14) here is equivalent to the cMPS operators $\psi(x)$. By substituting (14) into (4) and (5), we have the form of the cMERA Hamiltonian in terms of ϕ and π. The function $g(k,u)$ in (5) is assumed to be real valued in the form

$$g(k,u)=\chi(u)\Theta(1-|k|/\Lambda),\qquad(15)$$

where $\Theta(x)$ is the step function. By considering

$$|\psi(u=0)\rangle=|\mathbf{0}\rangle_a\qquad(16)$$

in (1), we find an ansatz to represent the exact ground state of the theory as a circuit cMERA. As the last step, one should apply the variational principle and minimize the energy

$$E=\langle\psi(u=0)|H_{\text{fb}}|\psi(u=0)\rangle\qquad(17)$$

to exactly find Δ and $\chi(u)$.

In order to do the calculation, it is useful to go to the interaction picture where L can be understood as the free part of the cMERA Hamiltonian while $K(u)$ is the interacting part. One can rewrite the unitary evolution in scale in the interaction picture as

$$U(u_1,u_2) = e^{-iu_1L}\hat{U}(u_1,u_2)e^{iu_2L} = e^{-iu_1L}\mathcal{P}e^{-i\int_{u_2}^{u_1}\hat{K}(u)du}e^{iu_2L}, \tag{18}$$

where $\hat{K}(u) = e^{iuL}K(u)e^{-iuL}$ can be read off as

$$\hat{K}(u) = \frac{i}{2}\int dkg(ke^{-u},u)\left(a_k^\dagger a_{-k}^\dagger - a_{-k}a_k\right). \tag{19}$$

Finally, by requiring $\delta E/\delta\chi(u) = 0$ for every u, we find that

$$\Delta = \sqrt{\Lambda^2 + m^2} \tag{20}$$

and

$$\chi(u) = \Lambda^2 e^{2u}/2\left(\Lambda^2 e^{2u} + m^2\right). \tag{21}$$

Before going ahead to find the RG flow of the class of cMPS, in order to find the renormalized operators via the evolution in scale, it is good to know that

$$e^{-iuL}\psi(k)e^{iuL} = e^{-u/2}\psi(ke^{-u}) \tag{22}$$

and under the action of $\hat{U}(u,u_{\mathrm{IR}})$

$$\begin{pmatrix} a_k \\ a_{-k}^\dagger \end{pmatrix} \to \begin{pmatrix} \cosh\theta(u) & -\sinh\theta(u) \\ -\sinh\theta(u) & \cosh\theta(u) \end{pmatrix}\begin{pmatrix} a_k \\ a_{-k}^\dagger \end{pmatrix}, \tag{23}$$

where $\theta(u) = \int_{u_{\mathrm{IR}}}^{u} dsg(ke^{-s},s)$ and $\theta(u=0) = \ln\sqrt{\frac{\Delta}{\omega_k}}$, while $\omega_k = \sqrt{k^2+m^2}$ [41].

Now, we are ready to define a one-parameter family of states by relating them to the IR state through the entangling evolution in scale as

$$|\Psi(u)\rangle = U(u,u_{\mathrm{IR}})|\psi[Q,R]\rangle. \tag{24}$$

Here, $|\Psi(u_{\mathrm{IR}})\rangle = |\psi[Q,R]\rangle$ is the standard class of cMPS which is suitable for the ground state of the nonrelativistic QFT, and $U(u,u_{\mathrm{IR}})$ is the cMERA RG flow that maps the state $|\Omega\rangle$ to the ground state of the free relativistic theory, i.e., $|0\rangle_a = U(0,u_{\mathrm{IR}})|\Omega\rangle$.

In the standard cMPS, one can explicitly expand the path order in (9) and obtain

$$|\psi[Q,R]\rangle = \sum_{n=0}^{\infty}\frac{1}{n!}\int_{-\infty}^{\infty}dx_1dx_2\ldots dx_n \times \Phi_n(x_1,x_2,\ldots,x_n)\psi^\dagger(x_1)\psi^\dagger(x_2)\ldots\psi^\dagger(x_n)|\Omega\rangle \tag{25}$$

while

$$\Phi_n(x_1,\ldots,x_n) = \mathrm{Tr}\left[\mathcal{P}\left\{e^{\int_{-\infty}^{\infty}Q(y)dy}R(x_1)\ldots R(x_n)\right\}\right]. \tag{26}$$

Therefore, the one-parameter family of states in (24) can be read as

$$|\Psi(u)\rangle = \sum_{n=0}^{\infty}\frac{1}{n!}\int_{-\infty}^{\infty}dx_1dx_2\ldots dx_n \times \Phi_n(x_1,x_2,\ldots,x_n)U(u,u_{\mathrm{IR}})\psi^\dagger(x_1) \times \psi^\dagger(x_2)\ldots\psi^\dagger(x_n)|\Omega\rangle. \tag{27}$$

Therefore, to find the explicit form of $|\Psi(u)\rangle$, it is enough to determine the transformation of the $\psi^\dagger(x_1)\ldots\psi^\dagger(x_n)|\Omega\rangle$ under the action of unitary evolution, which is

$$U(u_{\mathrm{IR}},u)\psi^\dagger(x_1)\ldots\psi^\dagger(x_n)|\Omega\rangle = \psi^\dagger(x_1,u)\ldots\psi^\dagger(x_n,u)|\psi(u)\rangle, \tag{28}$$

where we define

$$\psi^\dagger(x,u) = U(u,u_{\mathrm{IR}})\psi^\dagger(x)U^{-1}(u,u_{\mathrm{IR}}) \tag{29}$$

and

$$|\psi(u)\rangle = U(u,u_{\mathrm{IR}})|\Omega\rangle. \tag{30}$$

In particular, by using (22) and (23), one can obtain that at the UV scale

$$\psi(x,u=0) = e^{u_{\mathrm{IR}}/2}a(xe^{u_{\mathrm{IR}}}), \tag{31}$$

where

$$a(x) = 1/\sqrt{2\pi}\int dke^{ikx}a_k \tag{32}$$

is defined to be the Fourier transform of the annihilation operator a_k. By construction, we also have

$$|\psi(0)\rangle = |\mathbf{0}\rangle_a. \tag{33}$$

In the end, we obtain the UV state as

$$|\Psi(u=0)\rangle = |\Psi[\tilde{Q},\tilde{R}]\rangle = \mathrm{Tr}_{\mathrm{aux}}\left\{\mathcal{P}\exp\int_{-\infty}^{\infty}dx\left(\tilde{Q}(x)\otimes I + \tilde{R}(x)\otimes a^\dagger(x)\right)\right\}|\mathbf{0}\rangle_a, \tag{34}$$

while $\tilde{Q}(x)$ and $\tilde{R}(x)$ in terms of the $Q(x)$ and $R(x)$ can be given as

$$\tilde{Q}(x) = e^{-u_{\rm IR}} Q(x e^{-u_{\rm IR}}), \qquad \tilde{R}(x) = e^{-u_{\rm IR}/2} R(x e^{-u_{\rm IR}}). \tag{35}$$

It is nothing but the class of RCMPS introduced in [52] as an ansatz to approximate the ground state of a relativistic QFT without requiring any additional UV cutoff, and, thus, the result is valid even at high momenta. As the operator $a(x)$ has the same algebra as $\psi(x)$, RCMPS inherits the properties of the class of cMPS by replacing $\psi(x)$ with $a(x)$. Specifically, the correlation function of the $a(x), a^\dagger(x)$ can be obtained via the same generation functional (12). The only important point is that, since $a(x)$ is not local in terms of ϕ and π, the computation of the expectation value of the Hamiltonian is more difficult than in the nonrelativistic cases. Moreover, the naive optimization, which works well for the standard cMPS, fails for RCMPS, and one should use some more precise methods like the tangent space approach [76]. In [52], RCMPS was used to study the self-interacting ϕ^4 theory and provided some remarkable results.

Finally, one can also check the cMERA RG flow of the Hamiltonian. We define the Hamiltonian as $H(u=0) = H_{\rm fb}$. Here, $H_{\rm fb}$ represents the Hamiltonian of the free boson system in a relativistic context given in (13). At the IR scale, we will get

$$H(u_{\rm IR}) = U^\dagger(u=0, u_{\rm IR}) H(u=0) U(u=0, u_{\rm IR}). \tag{36}$$

One can explicitly find that, at the IR scale, we reach exactly the Hamiltonian of the nonrelativistic free boson as

$$H(u_{\rm IR}) = \frac{1}{2\tilde{m}} \int dx \partial_x \psi^\dagger(x) \partial_x \psi(x) + \mu \int dx \psi^\dagger(x) \psi(x), \tag{37}$$

while $\tilde{m} = m e^{2u_{\rm IR}}$ and $\mu = m$ is the so-called chemical potential. Thus, $|\Omega\rangle$ really represents the ground state of the free nonrelativistic field theory.

V. RCMPS FOR FERMIONIC THEORIES

The free relativistic fermions in the $1+1$ dimensions are given by Dirac Hamiltonian

$$H_{\rm Dirac} = \int dx [\bar{\psi}(x) \sigma_2 \partial_x \psi(x) + m \bar{\psi}\psi], \tag{38}$$

where $\psi = (\psi_1, \psi_2)^T$ is the two-component complex fermions and $\bar{\psi} = \psi^\dagger \sigma_3$. Here, one can choose the unentangled state as

$$\psi_1(x)|\Omega\rangle = 0 = \psi_2^\dagger(x)|\Omega\rangle. \tag{39}$$

The standard class of cMPS is defined as

$$|\psi[Q, R_1, R_2]\rangle = {\rm Tr}_{\rm aux}\left\{\mathcal{P}\exp\int dx (Q(x)\otimes I + R_1(x) \otimes \psi_1^\dagger(x) + R_2(x)\otimes \psi_2(x))\right\}|\Omega\rangle. \tag{40}$$

To find the related class of states appropriate for relativistic theories, we need to find the exact form of the RG flow such that $|\mathbf{0}\rangle = U(u=0, u_{\rm IR})|\Omega\rangle$, where $|\mathbf{0}\rangle$ is the exact ground state of the Dirac Hamiltonian. The entangler is given as

$$K(u) = i \int dk g(k,u) (\psi_1^\dagger \psi_2(k) + \psi_1(k)\psi_2(k)^\dagger). \tag{41}$$

In this case, the Bogoliubov angle is antisymmetric, and we can suppose its form as

$$g(k,u) = k\chi(u)\theta(1 - |k|/\Lambda). \tag{42}$$

The same as free bosons, one can find $\chi(u)$ by minimizing the expectation value of the Hamiltonian [32]. Moreover, one can derive that

$$e^{-iuL}\psi_{1,2}(k) e^{iuL} = e^{-u/2}\psi_{1,2}(k e^{-u}) \tag{43}$$

while $\psi_i(k)$ is the Fourier transform of $\psi_i(x)$, and under the action of the unitary evolution in the interaction picture

$$\begin{pmatrix} \psi_1(k) \\ \psi_2(k) \end{pmatrix} \to \begin{pmatrix} \cos\theta_f(u) & -\sin\theta_f(u) \\ \sin\theta_f(u) & \cos\theta_f(u) \end{pmatrix} \begin{pmatrix} \psi_1(k) \\ \psi_2(k) \end{pmatrix}, \tag{44}$$

where

$$\theta_f(u) = \int_{u_{\rm IR}}^{u} ds g(k e^{-s}, s) \tag{45}$$

and $\theta_f(u=0) = \frac{1}{2}\arcsin(-k/\omega_k)$. By considering (40) as the IR state, we can find the fermionic RCMPS at UV scale, i.e., $u=0$, as

$$|\Psi[\tilde{Q}, \tilde{R}_1, \tilde{R}_2]\rangle = {\rm Tr}_{\rm aux}\left\{\mathcal{P}\exp\int dx \big(\tilde{Q}(x)\otimes I + \tilde{R}_1(x) \otimes b_1^\dagger(x) + \tilde{R}_2(x)\otimes b_2(x)\big)\right\}|\mathbf{0}\rangle, \tag{46}$$

while $\tilde{Q}$ and $\tilde{R}$ defined by (35), and $b_{1,2}(x)$ are the Fourier transform of the $b_{1,2}(k)$ which can be found in terms of $\psi_{1,2}(k)$ as

$$\begin{aligned} b_1(k) &= \alpha_k \psi_1(k) + \beta_k \psi_2(k), \\ b_2(k) &= -\beta_k \psi_1(k) + \alpha_k \psi_2(k), \end{aligned} \tag{47}$$

while

$$\alpha_k = -k/\sqrt{k^2 + (\omega_k - m)^2},$$
$$\beta_k = (m - \omega_k)/\sqrt{k^2 + (\omega_k - m)^2}. \tag{48}$$

One can check that $[H, b_1^\dagger(k)] = \omega_k b_1^\dagger(k)$ and $[H, b_2(k)] = \omega_k b_2(k)$ or, in other words, the set of operators $b_{1,2}(k)$ are the modes diagonalizing the Dirac Hamiltonian.

VI. RCMPS ON A CIRCLE

Finding the RCMPS on a circle requires having the cMERA RG flow for relativistic free fields on the circle. In [75], it has been shown that if a Gaussian cMERA describes the ground state of a theory on a line, the ground state of the same theory on a circle has a cMERA representation as well. Furthermore, the cMERA entangler can be obtained using the method of images. The unentangled reference state is defined as

$$\psi(x)|\Omega^c\rangle = 0 \tag{49}$$

for $x \in [0, l_c)$ when $\psi(x)$ is again given by (14). The entangler has the form of

$$K^c(u) = \frac{i}{2}\sum_{n\in\mathbb{N}} \tilde{g}^c(n,u)\left[\psi_n^\dagger \psi_{-n}^\dagger - \psi_n \psi_{-n}\right], \tag{50}$$

where

$$\psi_n = 1/\sqrt{l_c}\int_0^{l_c} dx e^{-ik_n x}\psi(x) \tag{51}$$

for $n \in \mathbb{Z}$ and $k_n = 2\pi n/l_c$. The entangling profile on the circle defined as

$$\tilde{g}^c(x,u) = 1/\sqrt{l_c}\sum_n e^{ik_n x}\tilde{g}^c(n,u) \tag{52}$$

can be obtained from the one on the line $g(x,u)$ through the method of images as

$$\tilde{g}^c(x,u) = \sum_{n\in\mathbb{Z}} g(x + n l_c, u). \tag{53}$$

It implies that $\tilde{g}^c(n,u) = g(k,u)|_{k=k_n}$. Following the procedure described above, one can generalize RCMPS to an ansatz as a variational class to approximate the ground state of the relativistic theory on a finite circle as

$$|\Psi[Q,R]\rangle^c = \mathrm{Tr}_{\mathrm{aux}}\left\{\mathcal{P}e^{\int_0^{l_c} dx(\tilde{Q}(x)\otimes I + \tilde{R}(x)\otimes a^{c\dagger}(x))}\right\}|\mathbf{0^c}\rangle_a, \tag{54}$$

where $a^c(x)|\mathbf{0^c}\rangle_a = 0$ for all $x \in [0, l_c)$ and $a^c(x)$ is defined as the Fourier transform of the modes which diagonalize the free theory on the circle [75].

VII. DISCUSSION

In this paper, we could obtain the class of RCMPS via an RG flow generated by an appropriate cMERA circuit. They can be used to approximate the ground states of the relativistic QFTs in $1+1$ dimensions containing both bosonic theories like the sine-Gordon model and fermionic ones such as the Gross-Neveu and Thirring models. Moreover, since the Gaussian cMERA is known in higher dimensions for all bosonic, fermionic, and gauge fields [32,59], the procedure above can provide a way to find appropriate wave functionals for relativistic theories in higher dimensions, especially the relativistic version of the continuous PEPS in $2+1$ dimensions. Furthermore, an alternative approach to RCMPS for relativistic theories is the interacting cMERA (icMERA) [58]. It can be found by modifying the entangler and going beyond the Bogoliubov transformation by adding the terms generate n-tuplet transformation in fields. Thus, the icMERA evolution is the combination of two Gaussian and non-Gaussian unitaries, exactly the same as RCMPS. However, for icMERA, the important point is the fact that, to date, we do not know for a given theory up to what n-tuplet interacting terms are exactly needed to capture the full nonperturbative structure of the theory. But in the case of RCMPS, the form of the ansatz is fixed for all the families of the relevant theories. On the other hand, there is freedom in choosing the entangling profile of the entangler operator of the cMERA. In particular, there is a specific choice that leads to another class of states called magic cMERA [77], which is already shown to have the same UV structure as the standard cMPS. Moreover, its entangler by itself has the continuous matrix product operator representation. Therefore, studying the connection between them might even help us for a better understanding of the interacting disentangler. In the end, we point out that, since cMERA is connected to AdS/CFT, it would be desirable to study the possible gravity dual of the states of the form of RCMPS.

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