Fully strange tetraquark states with the exotic quantum numbers $J^{PC} = 0^{+-}$ and 2^{+-}

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We study the fully strange tetraquark states with the exotic quantum numbers $J^{PC} = 0^{+-}$ and 2^{+-}. We construct their corresponding diquark-antidiquark interpolating currents and apply the QCD sum rule method to calculate both their diagonal and off-diagonal correlation functions. The obtained results are used to construct some mixing currents that are nearly noncorrelated, from which we extract the masses of the lowest-lying states to be $M_{0^{+-}} = 2.47^{+0.33}_{-0.44}$ and $M_{2^{+-}} = 3.07^{+0.25}_{-0.33}$ GeV. We apply the Fierz rearrangement to transform the diquark-antidiquark currents to be the combinations of meson-meson currents, and the obtained Fierz identities indicate that these two states may be searched for in the P-wave $\phi(1020)f_0(1710)/\phi(1020)f_2'(1525)(\to\phi K\bar{K}/\phi\pi\pi)$ channels.

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I. INTRODUCTION

Many candidates of exotic hadrons were observed in the past twenty years, which cannot be well explained in the traditional quark model as the $\bar{q}q$ mesons and qqq baryons [1]. However, most of them still have the "traditional" quantum numbers that the traditional hadrons can reach, making them not easy to be clearly identified as exotic hadrons. Interestingly, there exist some "exotic" quantum numbers that the traditional hadrons cannot reach, e.g., the spin-parity quantum numbers $J^{PC} = 0^{\pm-}/1^{-+}/2^{+-}/3^{-+}/4^{+-}/\cdots$ [2–14].

The hadrons with these exotic quantum numbers are definitely exotic hadrons, and their possible interpretations are compact multiquark states [15–21], hadronic molecules [22–29], glueballs [30–38], and hybrid states [39–44], etc. Especially, the light hybrid states with the exotic quantum numbers $J^{PC} = 1^{-+}$ have been extensively studied in the literature [45–75], since there is some experimental evidence of their existence [76–80]. We have also studied these exotic quantum numbers $J^{PC} = 1^{-+}$ in Refs. [81–85] through the QCD sum rule method. Additionally, the isoscalar $D^*\bar{D}_2^*$ molecular state of $J^{PC} = 3^{-+}$ was predicted in Ref. [22] through the one-boson-exchange model, and a narrow hadronic state with the exotic quantum numbers $J^{PC} = 0^{--}$ was predicted in Ref. [23] through the heavy quark spin symmetry.

In this paper, we shall investigate the fully strange tetraquark states with the exotic quantum numbers $J^{PC} = 0^{+-}$ and 2^{+-} through the method of QCD sum rules. Recently, we have applied this method to study the fully strange tetraquark states of $J^{PC} = 0^{-+}/1^{\pm\pm}/4^{+-}$ in Refs. [84–90]. In the present study we shall explicitly add the covariant derivative operator in order to construct the fully strange tetraquark currents of $J^{PC} = 0^{+-}$ and 2^{+-}. We shall construct some diquark-antidiquark interpolating currents and apply the QCD sum rule method to calculate both their diagonal and off-diagonal correlation functions. The obtained results will be used to construct some mixing currents that are nearly noncorrelated, from which we shall extract the masses of the lowest-lying states to be

$$M_{0^{+-}} = 2.47^{+0.33}_{-0.44}\ \text{GeV},$$
$$M_{2^{+-}} = 3.07^{+0.25}_{-0.33}\ \text{GeV}.$$

With a large amount of J/ψ sample, BESIII is carefully studying the physics happening around this energy region [91–97], as is Belle-II [98] and GlueX [99]. Therefore, the above fully strange tetraquark states of $J^{PC} = 0^{+-}$ and 2^{+-} are potential exotic hadrons to be observed in future experiments. The present study would provide not only complementary information for possible countercandidates in the charm sector $cc\bar{c}\,\bar{c}$ [100–102], but also systematic understanding of exotics in a wider flavor region.

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This paper is organized as follows. In Sec. II we construct the fully strange tetraquark currents with the exotic quantum numbers $J^{PC} = 0^{+-}$ and 2^{+-}. These currents are used in Sec. III to perform QCD sum rule analyses, where we calculate both their diagonal and off-diagonal correlation functions. Based on the obtained results, we use the diquark-antidiquark currents to perform numerical analyses in Sec. III A and use their mixing currents to perform numerical analyses in Sec. III B. Section IV is a summary.

II. TETRAQUARK CURRENTS

In this section, we construct the fully strange tetraquark currents with the exotic quantum numbers $J^{PC} = 0^{+-}$ and 2^{+-}. Note that these two quantum numbers cannot be simply reached by using one quark field and one antiquark field; neither can they be reached by only using two quark fields and two antiquark fields. Actually, we need two quark fields and two antiquark fields together with one or more derivatives to reach them.

We have systematically constructed three independent diquark-antidiquark currents in Ref. [90] using two quark fields and two antiquark fields together with two derivatives,

$$\begin{aligned}\eta^{\alpha_1\alpha_2\alpha_3\alpha_4}_{4^{+-};1} = {}& \epsilon^{abe}\epsilon^{cde} \\ & \times \mathcal{S}\{[s_a^T C\gamma_{\alpha_1}\overleftrightarrow{D}_{\alpha_3}\overleftrightarrow{D}_{\alpha_4}s_b](\bar{s}_c\gamma_{\alpha_2}C\bar{s}_d^T) \\ & - (s_a^T C\gamma_{\alpha_1}s_b)[\bar{s}_c\gamma_{\alpha_2}C\overleftrightarrow{D}_{\alpha_3}\overleftrightarrow{D}_{\alpha_4}\bar{s}_d^T]\},\end{aligned} \tag{1}$$

$$\begin{aligned}\eta^{\alpha_1\alpha_2\alpha_3\alpha_4}_{4^{+-};2} = {}& (\delta^{ac}\delta^{bd} + \delta^{ad}\delta^{bc}) \\ & \times \mathcal{S}\{[s_a^T C\gamma_{\alpha_1}\gamma_5\overleftrightarrow{D}_{\alpha_3}\overleftrightarrow{D}_{\alpha_4}s_b](\bar{s}_c\gamma_{\alpha_2}\gamma_5C\bar{s}_d^T) \\ & - (s_a^T C\gamma_{\alpha_1}\gamma_5s_b)[\bar{s}_c\gamma_{\alpha_2}\gamma_5C\overleftrightarrow{D}_{\alpha_3}\overleftrightarrow{D}_{\alpha_4}\bar{s}_d^T]\},\end{aligned} \tag{2}$$

$$\begin{aligned}\eta^{\alpha_1\alpha_2\alpha_3\alpha_4}_{4^{+-};3} = {}& \epsilon^{abe}\epsilon^{cde}g^{\mu\nu} \\ & \times \mathcal{S}\{[s_a^T C\sigma_{\alpha_1\mu}\overleftrightarrow{D}_{\alpha_3}\overleftrightarrow{D}_{\alpha_4}s_b](\bar{s}_c\sigma_{\alpha_2\nu}C\bar{s}_d^T) \\ & - (s_a^T C\sigma_{\alpha_1\mu}s_b)[\bar{s}_c\sigma_{\alpha_2\nu}C\overleftrightarrow{D}_{\alpha_3}\overleftrightarrow{D}_{\alpha_4}\bar{s}_d^T]\}.\end{aligned} \tag{3}$$

Here $a \cdots e$ are color indices, $[A\overleftrightarrow{D}_\alpha B] = A[D_\alpha B] - [D_\alpha A]B$ with the covalent derivative $D_\alpha = \partial_\alpha + ig_s A_\alpha$, and the symbol $\mathcal{S}$ denotes symmetrization and subtracting trace terms in the set $\{\alpha_1 \cdots \alpha_4\}$.

Following a similar procedure for $\eta^{\alpha_1\alpha_2\alpha_3\alpha_4}_{4^{+-};1/2/3}$, for $J^{PC} = 0^{+-}$ and 2^{+-} currents, we can use the spin-0 and spin-2 projection operators rather than the symmetrization operator $\mathcal{S}$,

$$\begin{aligned}\mathcal{P}^{\alpha_1\alpha_2\alpha_3\alpha_4}_{J=0} = {}& \frac{1}{2}g_{\mu_1\mu_3}g_{\mu_2\mu_4} \\ & \times \left(g^{\alpha_1\mu_1}g^{\alpha_2\mu_2} + g^{\alpha_1\mu_2}g^{\alpha_2\mu_1} - \frac{1}{2}g^{\alpha_1\alpha_2}g^{\mu_1\mu_2}\right) \\ & \times \left(g^{\alpha_3\mu_3}g^{\alpha_4\mu_4} + g^{\alpha_3\mu_4}g^{\alpha_4\mu_3} - \frac{1}{2}g^{\alpha_3\alpha_4}g^{\mu_3\mu_4}\right) \\ = {}& g^{\alpha_1\alpha_3}g^{\alpha_2\alpha_4} + g^{\alpha_1\alpha_4}g^{\alpha_2\alpha_3} - \frac{1}{2}g^{\alpha_1\alpha_2}g^{\alpha_3\alpha_4},\end{aligned} \tag{4}$$

$$\begin{aligned}\mathcal{P}^{\alpha_1\alpha_2\alpha_3\alpha_4}_{J=2;\beta_1\beta_2} = {}& g_{\mu_2\mu_4}\left(g_{\beta_1\mu_1}g_{\beta_2\mu_3} + g_{\beta_1\mu_3}g_{\beta_2\mu_1} - \frac{1}{2}g_{\beta_1\beta_2}g_{\mu_1\mu_3}\right)\left(g^{\alpha_1\mu_1}g^{\alpha_2\mu_2} + g^{\alpha_1\mu_2}g^{\alpha_2\mu_1} - \frac{1}{2}g^{\alpha_1\alpha_2}g^{\mu_1\mu_2}\right) \\ & \times \left(g^{\alpha_3\mu_3}g^{\alpha_4\mu_4} + g^{\alpha_3\mu_4}g^{\alpha_4\mu_3} - \frac{1}{2}g^{\alpha_3\alpha_4}g^{\mu_3\mu_4}\right) \\ = {}& g^{\alpha_1\alpha_2}g^{\alpha_3\alpha_4}g_{\beta_1\beta_2} - g^{\alpha_1\alpha_2}\delta^{\alpha_3}_{\beta_1}\delta^{\alpha_4}_{\beta_2} - g^{\alpha_1\alpha_2}\delta^{\alpha_3}_{\beta_2}\delta^{\alpha_4}_{\beta_1} - g^{\alpha_3\alpha_4}\delta^{\alpha_1}_{\beta_1}\delta^{\alpha_2}_{\beta_2} - g^{\alpha_3\alpha_4}\delta^{\alpha_1}_{\beta_2}\delta^{\alpha_2}_{\beta_1} - g^{\alpha_1\alpha_3}g^{\alpha_2\alpha_4}g_{\beta_1\beta_2} \\ & + g^{\alpha_1\alpha_3}\delta^{\alpha_2}_{\beta_1}\delta^{\alpha_4}_{\beta_2} + g^{\alpha_1\alpha_3}\delta^{\alpha_2}_{\beta_2}\delta^{\alpha_4}_{\beta_1} + g^{\alpha_2\alpha_4}\delta^{\alpha_1}_{\beta_1}\delta^{\alpha_3}_{\beta_2} + g^{\alpha_2\alpha_4}\delta^{\alpha_1}_{\beta_2}\delta^{\alpha_3}_{\beta_1} - g^{\alpha_1\alpha_4}g^{\alpha_2\alpha_3}g_{\beta_1\beta_2} - g^{\alpha_1\alpha_4}\delta^{\alpha_2}_{\beta_1}\delta^{\alpha_3}_{\beta_2} + g^{\alpha_1\alpha_4}\delta^{\alpha_2}_{\beta_2}\delta^{\alpha_3}_{\beta_1} \\ & + g^{\alpha_2\alpha_3}\delta^{\alpha_1}_{\beta_1}\delta^{\alpha_4}_{\beta_2} + g^{\alpha_2\alpha_3}\delta^{\alpha_1}_{\beta_2}\delta^{\alpha_4}_{\beta_1}.\end{aligned} \tag{5}$$

For $J^{PC} = 0^{+-}$, we can construct three independent diquark-antidiquark currents,

$$\eta_{0^{+-};1} = \mathcal{P}^{\alpha_1\alpha_2\alpha_3\alpha_4}_{J=0} \times \epsilon^{abe}\epsilon^{cde}[s_a^T C\gamma_{\alpha_1}\overleftrightarrow{D}_{\alpha_3}\overleftrightarrow{D}_{\alpha_4}s_b](\bar{s}_c\gamma_{\alpha_2}C\bar{s}_d^T) - (s_a^T C\gamma_{\alpha_1}s_b)[\bar{s}_c\gamma_{\alpha_2}C\overleftrightarrow{D}_{\alpha_3}\overleftrightarrow{D}_{\alpha_4}\bar{s}_d^T], \tag{6}$$

$$\eta_{0^{+-};2} = \mathcal{P}^{\alpha_1\alpha_2\alpha_3\alpha_4}_{J=0} \times (\delta^{ac}\delta^{bd} + \delta^{ad}\delta^{bc})[s_a^T C\gamma_{\alpha_1}\gamma_5\overleftrightarrow{D}_{\alpha_3}\overleftrightarrow{D}_{\alpha_4}s_b](\bar{s}_c\gamma_{\alpha_2}\gamma_5C\bar{s}_d^T) - (s_a^T C\gamma_{\alpha_1}\gamma_5s_b)[\bar{s}_c\gamma_{\alpha_2}\gamma_5C\overleftrightarrow{D}_{\alpha_3}\overleftrightarrow{D}_{\alpha_4}\bar{s}_d^T], \tag{7}$$

$$\eta_{0^{+-};3} = \mathcal{P}^{\alpha_1\alpha_2\alpha_3\alpha_4}_{J=0} \times \epsilon^{abe}\epsilon^{cde}g^{\mu\nu}[s_a^T C\sigma_{\alpha_1\mu}\overleftrightarrow{D}_{\alpha_3}\overleftrightarrow{D}_{\alpha_4}s_b](\bar{s}_c\sigma_{\alpha_2\nu}C\bar{s}_d^T) - (s_a^T C\sigma_{\alpha_1\mu}s_b)[\bar{s}_c\sigma_{\alpha_2\nu}C\overleftrightarrow{D}_{\alpha_3}\overleftrightarrow{D}_{\alpha_4}\bar{s}_d^T]. \tag{8}$$

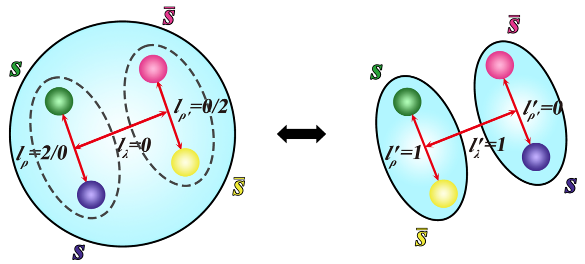

FIG. 1. Internal orbital angular momenta contained in the fully strange tetraquark currents $\eta^{\cdots}_{0/2/4^{+-};1/2/3}$ and $\xi^{\cdots}_{0/2/4^{+-};1/2/3}$. We use l_ρ and $l_{\rho'}$ to denote the momenta inside the diquark and antidiquark, respectively, and l_λ to denote the momentum between them. We use l'_ρ and $l'_{\rho'}$ to denote the momenta inside the two mesons and l'_λ to denote the momentum between them. The Fierz identities given in Eq. (23) indicate that the internal orbital angular momenta contained in the diquark-antidiquark currents $\eta^{\cdots}_{0/2/4^{+-};1/2/3}$, $\{L=2; l_\lambda=0, l_\rho=2/0, l_{\rho'}=0/2\}$, correspond to those contained in the meson-meson currents $\xi^{\cdots}_{0/2/4^{+-};1/2/3}$, $\{L=2; l'_\lambda=1, l'_\rho=1, l'_{\rho'}=0\}$.

For $J^{PC}=2^{+-}$, we can also construct three independent diquark-antidiquark currents,

$$\eta^{\beta_1\beta_2}_{2^{+-};1} = \mathcal{P}^{\alpha_1\alpha_2\alpha_3\alpha_4}_{J=2;\beta_1\beta_2} \times \epsilon^{abe}\epsilon^{cde} \times [s_a^T C\gamma_{\alpha_1} \overleftrightarrow{D}_{\alpha_3}\overleftrightarrow{D}_{\alpha_4} s_b](\bar{s}_c\gamma_{\alpha_2}C\bar{s}_d^T) - (s_a^T C\gamma_{\alpha_1} s_b)[\bar{s}_c\gamma_{\alpha_2}C\overleftrightarrow{D}_{\alpha_3}\overleftrightarrow{D}_{\alpha_4}\bar{s}_d^T], \tag{9}$$

$$\eta^{\beta_1\beta_2}_{2^{+-};2} = \mathcal{P}^{\alpha_1\alpha_2\alpha_3\alpha_4}_{J=2;\beta_1\beta_2} \times (\delta^{ac}\delta^{bd}+\delta^{ad}\delta^{bc}) \times [s_a^T C\gamma_{\alpha_1}\gamma_5 \overleftrightarrow{D}_{\alpha_3}\overleftrightarrow{D}_{\alpha_4} s_b](\bar{s}_c\gamma_{\alpha_2}\gamma_5 C\bar{s}_d^T) - (s_a^T C\gamma_{\alpha_1}\gamma_5 s_b)[\bar{s}_c\gamma_{\alpha_2}\gamma_5 C\overleftrightarrow{D}_{\alpha_3}\overleftrightarrow{D}_{\alpha_4}\bar{s}_d^T], \tag{10}$$

$$\eta^{\beta_1\beta_2}_{2^{+-};3} = \mathcal{P}^{\alpha_1\alpha_2\alpha_3\alpha_4}_{J=2;\beta_1\beta_2} \times \epsilon^{abe}\epsilon^{cde} g^{\mu\nu} \times [s_a^T C\sigma_{\alpha_1\mu} \overleftrightarrow{D}_{\alpha_3}\overleftrightarrow{D}_{\alpha_4} s_b](\bar{s}_c\sigma_{\alpha_2\nu}C\bar{s}_d^T) - (s_a^T C\sigma_{\alpha_1\mu} s_b)[\bar{s}_c\sigma_{\alpha_2\nu}C\overleftrightarrow{D}_{\alpha_3}\overleftrightarrow{D}_{\alpha_4}\bar{s}_d^T]. \tag{11}$$

TABLE I. QCD sum rule results for the fully strange tetraquark states with the exotic quantum numbers $J^{PC}=0/2/4^{+-}$, extracted from the diquark-antidiquark currents $\eta^{\cdots}_{0/2/4^{+-};1/2/3}$ as well as their mixing currents $J^{\cdots}_{0/2/4^{+-};1/2/3}$. The results for the exotic quantum numbers $J^{PC}=4^{+-}$ are taken from Ref. [90].

		Working regions			
Currents	$s_0^{\min}$ (GeV^2)	M_B^2 (GeV^2)	s_0 (GeV^2)	Pole (%)	Mass (GeV)
$\eta_{0^{+-};1}$	12.5	2.31–2.57	14 ± 3.0	40–51	$3.21^{+0.23}_{-0.28}$
$\eta_{0^{+-};2}$	21.2	2.91–3.31	23 ± 5.0	40–51	$4.41^{+0.36}_{-0.31}$
$\eta_{0^{+-};3}$	8.3	1.13–2.21	14 ± 3.0	40–94	$3.20^{+0.19}_{-0.30}$
$J_{0^{+-};1}$	8.3	1.33–1.42	9 ± 2.0	40–47	$2.47^{+0.33}_{-0.44}$
$J_{0^{+-};2}$	7.9	1.41–1.62	9 ± 2.0	40–54	$2.56^{+0.25}_{-0.35}$
$J_{0^{+-};3}$	…	…	…	…	…
$\eta^{\beta_1\beta_2}_{2^{+-};1}$	12.8	2.11–2.32	14 ± 3.0	40–50	$3.27^{+0.23}_{-0.28}$
$\eta^{\beta_1\beta_2}_{2^{+-};2}$	17.1	2.35–2.78	19 ± 4.0	40–55	$3.98^{+0.29}_{-0.25}$
$\eta^{\beta_1\beta_2}_{2^{+-};3}$	9.3	1.15–2.14	14 ± 3.0	40–91	$3.27^{+0.19}_{-0.23}$
$J^{\beta_1\beta_2}_{2^{+-};1}$	11.6	1.97–2.19	13 ± 3.0	40–51	$3.07^{+0.25}_{-0.33}$
$J^{\beta_1\beta_2}_{2^{+-};2}$	16.0	2.44–2.82	18 ± 4.0	40–54	$3.77^{+0.24}_{-0.30}$
$J^{\beta_1\beta_2}_{2^{+-};3}$	17.0	2.34–2.77	19 ± 4.0	40–55	$3.99^{+0.29}_{-0.25}$
$\eta^{\alpha_1\alpha_2\alpha_3\alpha_4}_{4^{+-};1}$	14.6	2.40–2.65	16 ± 3.0	40–50	$3.50^{+0.21}_{-0.25}$
$\eta^{\alpha_1\alpha_2\alpha_3\alpha_4}_{4^{+-};2}$	19.2	2.80–3.13	21 ± 4.0	40–51	$4.08^{+0.26}_{-0.31}$
$\eta^{\alpha_1\alpha_2\alpha_3\alpha_4}_{4^{+-};3}$	11.0	1.25–2.44	16 ± 3.0	40–91	$3.51^{+0.20}_{-0.20}$
$J^{\alpha_1\alpha_2\alpha_3\alpha_4}_{4^{+-};1}$	10.1	1.78–1.92	11 ± 2.0	40–48	$2.85^{+0.19}_{-0.22}$
$J^{\alpha_1\alpha_2\alpha_3\alpha_4}_{4^{+-};2}$	19.1	2.79–3.14	21 ± 4.0	40–51	$4.08^{+0.26}_{-0.31}$
$J^{\alpha_1\alpha_2\alpha_3\alpha_4}_{4^{+-};3}$	…	…	…	…	…

The internal orbital angular momenta contained in the above diquark-antidiquark currents are all

$$\eta^{\cdots}_{0/2/4^{+-};1/2/3}\colon\ L=2; l_\lambda=0, l_\rho=2/0, l_{\rho'}=0/2, \quad (12)$$

where L is the total orbital angular momentum, l_ρ and $l_{\rho'}$ are the momenta inside the diquark and antidiquark, respectively, and l_λ is the momentum between the diquark and antidiquark, as depicted in Fig. 1.

Among the above diquark-antidiquark currents, $\eta^{\cdots}_{0/2/4^{+-};1}$ and $\eta^{\cdots}_{0/2/4^{+-};3}$ have the antisymmetric color structure $[ss]_{\bar{\mathbf{3}}_C}[\bar{s}\,\bar{s}]_{\mathbf{3}_C}$, and $\eta^{\cdots}_{0/2/4^{+-};2}$ have the symmetric color structure $[ss]_{\mathbf{6}_C}[\bar{s}\,\bar{s}]_{\bar{\mathbf{6}}_C}$, so the internal structure of $\eta^{\cdots}_{0/2/4^{+-};1}$ and $\eta^{\cdots}_{0/2/4^{+-};3}$ is more stable than that of $\eta^{\cdots}_{0/2/4^{+-};2}$. Moreover, the currents $\eta^{\cdots}_{0/2/4^{+-};1}$ are constructed by using the S-wave diquark field $s_a^T C\gamma_\mu s_b$ of $J^P=1^+$, and the currents $\eta^{\cdots}_{0/2/4^{+-};3}$ are constructed by using the diquark field $s_a^T C\sigma_{\mu\nu} s_b$ of $J^P=1^\pm$ that contains both the S- and P-wave components, so they may lead to better QCD sum rule results. Oppositely, the currents $\eta^{\cdots}_{0/2/4^{+-};2}$ are constructed by using the P-wave diquark field $s_a^T C\gamma_\mu\gamma_5 s_b$ of $J^P=1^-$, so their predicted masses are probably larger. The results of Ref. [90] have partly verified these analyses, as summarized in Table I.

In addition to the diquark-antidiquark configuration, we can also investigate the meson-meson configuration. We have constructed three independent meson-meson currents of $J^{PC}=4^{+-}$ in Ref. [90],

$$\xi^{\alpha_1\alpha_2\alpha_3\alpha_4}_{4^{+-};1} = \mathcal{S}\{[\bar{s}_a\gamma_{\alpha_1}\overleftrightarrow{D}_{\alpha_3}s_a]\overleftrightarrow{D}_{\alpha_4}(\bar{s}_b\gamma_{\alpha_2}s_b)\}, \quad (13)$$

$$\xi^{\alpha_1\alpha_2\alpha_3\alpha_4}_{4^{+-};2} = \mathcal{S}\{[\bar{s}_a\gamma_{\alpha_1}\gamma_5\overleftrightarrow{D}_{\alpha_3}s_a]\overleftrightarrow{D}_{\alpha_4}(\bar{s}_b\gamma_{\alpha_2}\gamma_5 s_b)\}, \quad (14)$$

$$\xi^{\alpha_1\alpha_2\alpha_3\alpha_4}_{4^{+-};3} = g^{\mu\nu}\mathcal{S}\{[\bar{s}_a\sigma_{\alpha_1\mu}\overleftrightarrow{D}_{\alpha_3}s_a]\overleftrightarrow{D}_{\alpha_4}(\bar{s}_b\sigma_{\alpha_2\nu}s_b)\}. \quad (15)$$

We can similarly construct three independent meson-meson currents of $J^{PC}=0^{+-}$,

$$\xi_{0^{+-};1} = \mathcal{P}^{\alpha_1\alpha_2\alpha_3\alpha_4}_{J=0}[\bar{s}_a\gamma_{\alpha_1}\overleftrightarrow{D}_{\alpha_3}s_a]\overleftrightarrow{D}_{\alpha_4}(\bar{s}_b\gamma_{\alpha_2}s_b), \quad (16)$$

$$\xi_{0^{+-};2} = \mathcal{P}^{\alpha_1\alpha_2\alpha_3\alpha_4}_{J=0}[\bar{s}_a\gamma_{\alpha_1}\gamma_5\overleftrightarrow{D}_{\alpha_3}s_a]\overleftrightarrow{D}_{\alpha_4}(\bar{s}_b\gamma_{\alpha_2}\gamma_5 s_b), \quad (17)$$

$$\xi_{0^{+-};3} = \mathcal{P}^{\alpha_1\alpha_2\alpha_3\alpha_4}_{J=0}\times g^{\mu\nu}[\bar{s}_a\sigma_{\alpha_1\mu}\overleftrightarrow{D}_{\alpha_3}s_a]\overleftrightarrow{D}_{\alpha_4}(\bar{s}_b\sigma_{\alpha_2\nu}s_b), \quad (18)$$

and three independent meson-meson currents of $J^{PC}=2^{+-}$,

$$\xi^{\beta_1\beta_2}_{2^{+-};1} = \mathcal{P}^{\alpha_1\alpha_2\alpha_3\alpha_4}_{J=2;\beta_1\beta_2}[\bar{s}_a\gamma_{\alpha_1}\overleftrightarrow{D}_{\alpha_3}s_a]\overleftrightarrow{D}_{\alpha_4}(\bar{s}_b\gamma_{\alpha_2}s_b), \quad (19)$$

$$\xi^{\beta_1\beta_2}_{2^{+-};2} = \mathcal{P}^{\alpha_1\alpha_2\alpha_3\alpha_4}_{J=2;\beta_1\beta_2}[\bar{s}_a\gamma_{\alpha_1}\gamma_5\overleftrightarrow{D}_{\alpha_3}s_a]\overleftrightarrow{D}_{\alpha_4}(\bar{s}_b\gamma_{\alpha_2}\gamma_5 s_b), \quad (20)$$

$$\xi^{\beta_1\beta_2}_{2^{+-};3} = \mathcal{P}^{\alpha_1\alpha_2\alpha_3\alpha_4}_{J=2;\beta_1\beta_2}\times g^{\mu\nu}[\bar{s}_a\sigma_{\alpha_1\mu}\overleftrightarrow{D}_{\alpha_3}s_a]\overleftrightarrow{D}_{\alpha_4}(\bar{s}_b\sigma_{\alpha_2\nu}s_b). \quad (21)$$

As depicted in Fig. 1, the internal orbital angular momenta contained in the above meson-meson currents are all

$$\xi^{\cdots}_{0/2/4^{+-};1/2/3}\colon\ L=2; l'_\lambda=1, l'_\rho=1, l'_{\rho'}=0, \quad (22)$$

where l'_ρ and $l'_{\rho'}$ are the momenta inside the two mesons and l'_λ is the momentum between them.

We can apply the Fierz rearrangement to derive the relations between $\eta^{\cdots}_{0/2/4^{+-};1/2/3}$ and $\xi^{\cdots}_{0/2/4^{+-};1/2/3}$ to be

$$\begin{pmatrix}\eta^{\cdots}_{0/2/4^{+-};1}\\ \eta^{\cdots}_{0/2/4^{+-};2}\\ \eta^{\cdots}_{0/2/4^{+-};3}\end{pmatrix} = \begin{pmatrix}2 & -2 & -2\\ -2 & 2 & -2\\ -4 & -4 & 0\end{pmatrix}\begin{pmatrix}\xi^{\cdots}_{0/2/4^{+-};1}\\ \xi^{\cdots}_{0/2/4^{+-};2}\\ \xi^{\cdots}_{0/2/4^{+-};3}\end{pmatrix}. \quad (23)$$

We shall use these Fierz identities to study the decay behaviors at the end of this paper.

III. QCD SUM RULE ANALYSES

The QCD sum rule method has been successfully applied to study various conventional and exotic hadrons in the past fifty years [103–110]. In this section, we apply

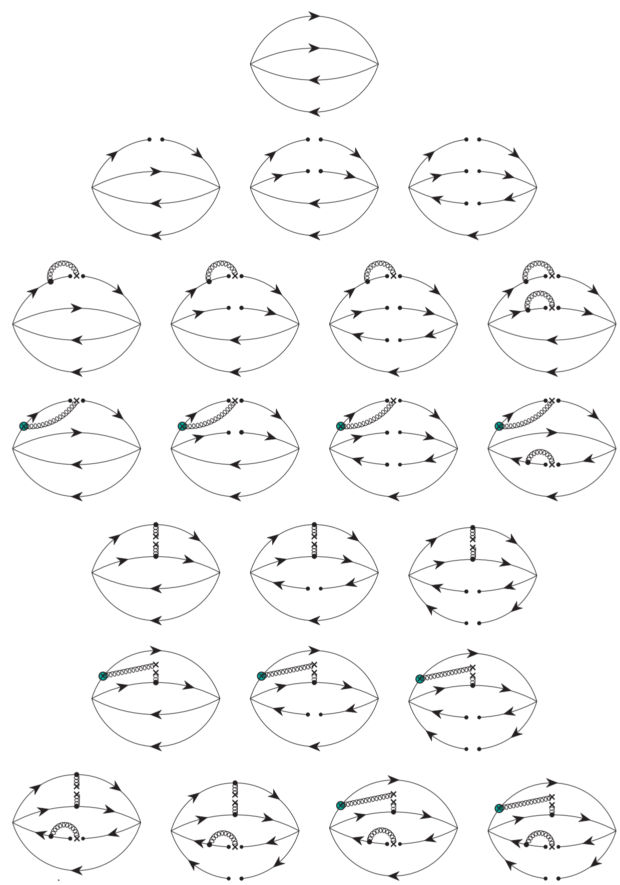

FIG. 2. Feynman diagrams calculated in the present study. The covariant derivative $D_\alpha = \partial_\alpha + ig_s A_\alpha$ contains two terms, and we use the green vertex to describe the latter term.

this nonperturbative method to study the fully strange tetraquark currents $\eta_{0^{+-};1/2/3}$ of $J^{PC}=0^{+-}$ and $\eta^{\beta_1\beta_2}_{2^{+-};1/2/3}$ of $J^{PC}=2^{+-}$.

We use the three currents $\eta_{0^{+-};1/2/3}$ of $J^{PC}=0^{+-}$ as examples and assume that they couple to the states X_n ($n=1\ldots N$) through

$$\langle 0|\eta_{0^{+-};i}|X_n\rangle = f_{\mathrm{in}}, \tag{24}$$

where X_n is the state to which the current $\eta_{0^{+-};i}$ can couple, N is the number of such states, and f_{in} is the $3\times N$ matrix for the coupling of the current $\eta_{0^{+-};i}$ to the state X_n. Then we can investigate the diagonal and off-diagonal correlation functions,

$$\Pi_{ij}(q^2) = i\int d^4x e^{iqx}\langle 0|\mathbf{T}[\eta_{0^{+-};i}(x)\eta^{\dagger}_{0^{+-};j}(0)]|0\rangle, \tag{25}$$

at both the hadron and quark-gluon levels.

At the hadron level, we use the dispersion relation to express $\Pi_{ij}(q^2)$ as

$$\Pi_{ij}(q^2) = \int_{s_<}^{\infty}\frac{\rho^{\mathrm{phen}}_{ij}(s)}{s-q^2-i\varepsilon}ds, \tag{26}$$

where $s_< = 16m_s^2$ is the physical threshold. The spectral density $\rho^{\mathrm{phen}}_{ij}(s)$ can be generally parametrized for the states X_n and a continuum as

$$\begin{aligned}\rho^{\mathrm{phen}}_{ij}(s) &= \sum_n \delta(s-M_n^2)\langle 0|\eta_{0^{+-};i}|X_n\rangle\langle X_n|\eta^{\dagger}_{0^{+-};j}|0\rangle + \cdots \\ &= \sum_n f_{\mathrm{in}}f_{jn}\delta(s-M_n^2) + \cdots,\end{aligned} \tag{27}$$

where M_n is the mass of the state X_n.

At the quark-gluon level, we apply the method of operator product expansion (OPE) to calculate $\Pi_{ij}(q^2)$, from which we can extract the OPE spectral density $\rho_{ij}(s)\equiv\rho^{\mathrm{OPE}}_{ij}(s)$. In the present study, we have calculated the Feynman diagrams depicted in Fig. 2, where we use the strangeness quark propagator as

$$\begin{aligned} iS_s^{ab}(x-y) &= \langle 0|\mathbf{T}[s^a(x)\bar{s}^b(y)]|0\rangle \\ &= \frac{i\delta^{ab}}{2\pi^2(x-y)^4}(\hat{x}-\hat{y}) - \frac{\delta^{ab}}{12}\langle\bar{s}s\rangle + \frac{i}{32\pi^2(x-y)^2}\frac{\lambda^n_{ab}}{2}g_sG^n_{\mu\nu}(\sigma^{\mu\nu}(\hat{x}-\hat{y})+(\hat{x}-\hat{y})\sigma^{\mu\nu}) \\ &\quad - \frac{1}{4\pi^2(x-y)^4}\frac{\lambda^n_{ab}}{2}g_sG^n_{\mu\nu}x^\mu y^\nu(\hat{x}-\hat{y}) + \frac{\delta^{ab}(x-y)^2}{192}\langle g_s\bar{s}\sigma Gs\rangle - \frac{m_s\delta^{ab}}{4\pi^2(x-y)^2} \\ &\quad + \frac{i\delta^{ab}}{48}m_s\langle\bar{s}s\rangle(\hat{x}-\hat{y}) + \frac{i\delta^{ab}}{8\pi^2(x-y)^2}m_s^2(\hat{x}-\hat{y}). \end{aligned} \tag{28}$$

We have considered the perturbative term, the quark condensate $\langle\bar{s}s\rangle$, the gluon condensate $\langle g_s^2GG\rangle$, the quark-gluon mixed condensate $\langle g_s\bar{s}\sigma Gs\rangle$, and their combinations. We have calculated all the diagrams proportional to $g_s^{N=0}$ and $g_s^{N=1}$, but we have only partly calculated the diagrams proportional to $g_s^{N\geq 2}$.

The OPE spectral densities $\rho_{ij}(s)\equiv\rho_{0^{+-};ij}(s)$ extracted from the three $J^{PC}=0^{+-}$ currents $\eta_{0^{+-};1/2/3}$ are

$$\begin{aligned}\rho_{0^{+-};11}(s) &= \frac{s^6}{1433600\pi^6} + \frac{7m_s^2s^5}{76800\pi^6} - \frac{3m_s\langle\bar{s}s\rangle}{640\pi^4}s^4 - \left(-\frac{7\langle g_s^2GG\rangle m_s^2}{12288\pi^6} - \frac{\langle\bar{s}s\rangle^2}{24\pi^2} + \frac{37m_s\langle g_s\bar{s}\sigma Gs\rangle}{2304\pi^4}\right)s^3 \\ &\quad - \left(\frac{43\langle g_s^2GG\rangle m_s\langle\bar{s}s\rangle}{2304\pi^4} + \frac{83m_s^2\langle\bar{s}s\rangle^2}{48\pi^2} - \frac{11\langle\bar{s}s\rangle\langle g_s\bar{s}\sigma Gs\rangle}{48\pi^2}\right)s^2 \\ &\quad - \left(-\frac{\langle g_s^2GG\rangle\langle\bar{s}s\rangle^2}{32\pi^2} + \frac{23\langle g_s^2GG\rangle m_s\langle g_s\bar{s}\sigma Gs\rangle}{768\pi^4} + \frac{97m_s^2\langle\bar{s}s\rangle\langle g_s\bar{s}\sigma Gs\rangle}{24\pi^2} - \frac{\langle g_s\bar{s}\sigma Gs\rangle^2}{8\pi^2}\right)s \\ &\quad + \frac{\langle g_s^2GG\rangle m_s^2\langle\bar{s}s\rangle^2}{72\pi^2} + \frac{\langle g_s^2GG\rangle\langle\bar{s}s\rangle\langle g_s\bar{s}\sigma Gs\rangle}{32\pi^2} - \frac{25m_s^2\langle g_s\bar{s}\sigma Gs\rangle^2}{24\pi^2},\end{aligned} \tag{29}$$

$$\rho_{0^{+-};22}(s) = \frac{s^6}{716800\pi^6} - \frac{m_s^2 s^5}{3840\pi^6} - \left(-\frac{\langle g_s^2 GG\rangle}{30720\pi^6} - \frac{11 m_s\langle\bar{s}s\rangle}{960\pi^4}\right)s^4 - \left(-\frac{3\langle g_s^2 GG\rangle m_s^2}{4096\pi^6} - \frac{77 m_s\langle g_s\bar{s}\sigma Gs\rangle}{2304\pi^4} + \frac{\langle\bar{s}s\rangle^2}{12\pi^2}\right)s^3$$
$$- \left(\frac{49\langle g_s^2 GG\rangle m_s\langle\bar{s}s\rangle}{2304\pi^4} - \frac{29 m_s^2\langle\bar{s}s\rangle^2}{24\pi^2} + \frac{19\langle\bar{s}s\rangle\langle g_s\bar{s}\sigma Gs\rangle}{48\pi^2}\right)s^2$$
$$- \left(\frac{23\langle g_s^2 GG\rangle m_s\langle g_s\bar{s}\sigma Gs\rangle}{768\pi^4} - \frac{\langle g_s^2 GG\rangle\langle\bar{s}s\rangle^2}{32\pi^2} + \frac{43 m_s^2\langle\bar{s}s\rangle\langle g_s\bar{s}\sigma Gs\rangle}{24\pi^2} + \frac{5\langle g_s\bar{s}\sigma Gs\rangle^2}{32\pi^2}\right)s$$
$$+ \frac{\langle g_s^2 GG\rangle\langle\bar{s}s\rangle\langle g_s\bar{s}\sigma Gs\rangle}{32\pi^2} - \frac{43 m_s^2\langle g_s\bar{s}\sigma Gs\rangle^2}{24\pi^2} + \frac{7\langle g_s^2 GG\rangle m_s^2\langle\bar{s}s\rangle^2}{144\pi^2}, \quad (30)$$

$$\rho_{0^{+-};33}(s) = \frac{s^6}{179200\pi^6} - \frac{m_s^2 s^5}{1600\pi^6} - \left(-\frac{\langle g_s^2 GG\rangle}{12288\pi^6} - \frac{m_s\langle\bar{s}s\rangle}{96\pi^4}\right)s^4 - \frac{5\langle g_s^2 GG\rangle m_s^2}{1536\pi^6}s^3 + \frac{\langle g_s^2 GG\rangle m_s\langle\bar{s}s\rangle}{36\pi^4}s^2 - \frac{5 m_s^2\langle\bar{s}s\rangle\langle g_s\bar{s}\sigma Gs\rangle}{\pi^2}s$$
$$- \frac{2 m_s^2\langle g_s\bar{s}\sigma Gs\rangle^2}{\pi^2} + \frac{31\langle g_s^2 GG\rangle m_s^2\langle\bar{s}s\rangle^2}{144\pi^2}, \quad (31)$$

$$\rho_{0^{+-};12}(s) = \frac{\langle g_s^2 GG\rangle s^4}{122880\pi^6} - \left(\frac{7\langle g_s^2 GG\rangle m_s^2}{6144\pi^6} - \frac{5 m_s\langle g_s\bar{s}\sigma Gs\rangle}{256\pi^4}\right)s^3 - \left(-\frac{\langle g_s^2 GG\rangle m_s\langle\bar{s}s\rangle}{768\pi^4} + \frac{\langle\bar{s}s\rangle\langle g_s\bar{s}\sigma Gs\rangle}{4\pi^2}\right)s^2$$
$$- \left(-\frac{9 m_s^2\langle\bar{s}s\rangle\langle g_s\bar{s}\sigma Gs\rangle}{8\pi^2} + \frac{3\langle g_s\bar{s}\sigma Gs\rangle^2}{8\pi^2}\right)s + \frac{\langle g_s^2 GG\rangle m_s^2\langle\bar{s}s\rangle^2}{32\pi^2} + \frac{15 m_s^2\langle g_s\bar{s}\sigma Gs\rangle^2}{32\pi^2}, \quad (32)$$

$$\rho_{0^{+-};13}(s) = -\frac{41 m_s^2 s^5}{153600\pi^6} + \frac{m_s\langle\bar{s}s\rangle}{96\pi^4}s^4 - \left(\frac{5\langle g_s^2 GG\rangle m_s^2}{9216\pi^6} - \frac{545 m_s\langle g_s\bar{s}\sigma Gs\rangle}{9216\pi^4}\right)s^3 - \left(-\frac{\langle g_s^2 GG\rangle m_s\langle\bar{s}s\rangle}{192\pi^4} - \frac{5 m_s^2\langle\bar{s}s\rangle^2}{3\pi^2}\right)s^2$$
$$- \left(-\frac{\langle g_s^2 GG\rangle m_s\langle g_s\bar{s}\sigma Gs\rangle}{128\pi^4} - \frac{\langle g_s^2 GG\rangle\langle\bar{s}s\rangle^2}{192\pi^2} - \frac{523 m_s^2\langle\bar{s}s\rangle\langle g_s\bar{s}\sigma Gs\rangle}{192\pi^2} + \frac{\langle g_s\bar{s}\sigma Gs\rangle^2}{384\pi^2}\right)s$$
$$+ \frac{\langle g_s^2 GG\rangle\langle\bar{s}s\rangle\langle g_s\bar{s}\sigma Gs\rangle}{144\pi^2} + \frac{m_s^2\langle g_s\bar{s}\sigma Gs\rangle^2}{48\pi^2}, \quad (33)$$

$$\rho_{0^{+-};23}(s) = \left(\frac{5\langle g_s^2 GG\rangle m_s^2}{3072\pi^6} - \frac{37 m_s\langle g_s\bar{s}\sigma Gs\rangle}{3072\pi^4}\right)s^3 - \frac{\langle g_s^2 GG\rangle m_s\langle\bar{s}s\rangle}{64\pi^4}s^2$$
$$- \left(\frac{3\langle g_s^2 GG\rangle m_s\langle g_s\bar{s}\sigma Gs\rangle}{128\pi^4} + \frac{\langle g_s^2 GG\rangle\langle\bar{s}s\rangle^2}{64\pi^2} + \frac{47 m_s^2\langle\bar{s}s\rangle\langle g_s\bar{s}\sigma Gs\rangle}{64\pi^2} - \frac{9\langle g_s\bar{s}\sigma Gs\rangle^2}{128\pi^2}\right)s$$
$$- \frac{\langle g_s^2 GG\rangle\langle\bar{s}s\rangle\langle g_s\bar{s}\sigma Gs\rangle}{48\pi^2} - \frac{m_s^2\langle g_s\bar{s}\sigma Gs\rangle^2}{8\pi^2}, \quad (34)$$

and the OPE spectral densities $\rho_{2^{+-};ij}(s)$ extracted from the three $J^{PC} = 2^{+-}$ currents $\eta_{2^{+-};1/2/3}^{\beta_1\beta_2}$ are

$$\rho_{2^{+-};11}(s) = \frac{341 s^6}{43545600\pi^6} - \frac{31 m_s^2 s^5}{44800\pi^6} - \left(\frac{41\langle g_s^2 GG\rangle}{1161216\pi^6} + \frac{31 m_s\langle\bar{s}s\rangle}{15120\pi^4}\right)s^4 - \left(-\frac{29\langle g_s^2 GG\rangle m_s^2}{69120\pi^6} - \frac{2\langle\bar{s}s\rangle^2}{9\pi^2} + \frac{2537 m_s\langle g_s\bar{s}\sigma Gs\rangle}{25920\pi^4}\right)s^3$$
$$- \left(\frac{7\langle g_s^2 GG\rangle m_s\langle\bar{s}s\rangle}{4320\pi^4} + \frac{5 m_s^2\langle\bar{s}s\rangle^2}{18\pi^2} - \frac{629\langle\bar{s}s\rangle\langle g_s\bar{s}\sigma Gs\rangle}{540\pi^2}\right)s^2$$
$$- \left(\frac{\langle g_s^2 GG\rangle\langle\bar{s}s\rangle^2}{36\pi^2} + \frac{17\langle g_s^2 GG\rangle m_s\langle g_s\bar{s}\sigma Gs\rangle}{576\pi^4} + \frac{155 m_s^2\langle\bar{s}s\rangle\langle g_s\bar{s}\sigma Gs\rangle}{18\pi^2} - \frac{221\langle g_s\bar{s}\sigma Gs\rangle^2}{216\pi^2}\right)s$$
$$- \frac{5\langle g_s^2 GG\rangle m_s^2\langle\bar{s}s\rangle^2}{81\pi^2} - \frac{\langle g_s^2 GG\rangle\langle\bar{s}s\rangle\langle g_s\bar{s}\sigma Gs\rangle}{81\pi^2} - \frac{116 m_s^2\langle g_s\bar{s}\sigma Gs\rangle^2}{27\pi^2}, \quad (35)$$

$$\begin{aligned}\rho_{2^{+-};22}(s) = {} & \frac{341s^6}{21772800\pi^6} - \frac{551m_s^2s^5}{201600\pi^6} - \left(-\frac{157\langle g_s^2GG\rangle}{829440\pi^6} - \frac{611m_s\langle\bar{s}s\rangle}{7560\pi^4}\right)s^4 \\ & - \left(\frac{241\langle g_s^2GG\rangle m_s^2}{41472\pi^6} + \frac{4\langle\bar{s}s\rangle^2}{9\pi^2} - \frac{4693m_s\langle g_s\bar{s}\sigma Gs\rangle}{25920\pi^4}\right)s^3 \\ & - \left(-\frac{\langle g_s^2GG\rangle m_s\langle\bar{s}s\rangle}{135\pi^4} - \frac{47m_s^2\langle\bar{s}s\rangle^2}{5\pi^2} + \frac{215\langle\bar{s}s\rangle\langle g_s\bar{s}\sigma Gs\rangle}{108\pi^2}\right)s^2 \\ & - \left(\frac{\langle g_s^2GG\rangle\langle\bar{s}s\rangle^2}{36\pi^2} + \frac{17\langle g_s^2GG\rangle m_s\langle g_s\bar{s}\sigma Gs\rangle}{576\pi^4} + \frac{m_s^2\langle\bar{s}s\rangle\langle g_s\bar{s}\sigma Gs\rangle}{2\pi^2} + \frac{349\langle g_s\bar{s}\sigma Gs\rangle^2}{216\pi^2}\right)s \\ & - \frac{\langle g_s^2GG\rangle m_s^2\langle\bar{s}s\rangle^2}{81\pi^2} - \frac{\langle g_s^2GG\rangle\langle\bar{s}s\rangle\langle g_s\bar{s}\sigma Gs\rangle}{81\pi^2} + \frac{28m_s^2\langle g_s\bar{s}\sigma Gs\rangle^2}{27\pi^2}, \end{aligned} \tag{36}$$

$$\begin{aligned}\rho_{2^{+-};33}(s) = {} & \frac{13s^6}{1036800\pi^6} - \frac{331m_s^2s^5}{201600\pi^6} - \left(-\frac{13\langle g_s^2GG\rangle}{161280\pi^6} - \frac{23m_s\langle\bar{s}s\rangle}{756\pi^4}\right)s^4 - \frac{5\langle g_s^2GG\rangle m_s^2}{10368\pi^6}s^3 \\ & - \left(\frac{2\langle g_s^2GG\rangle m_s\langle\bar{s}s\rangle}{135\pi^4} - \frac{49m_s^2\langle\bar{s}s\rangle^2}{15\pi^2}\right)s^2 - \frac{76m_s^2\langle\bar{s}s\rangle\langle g_s\bar{s}\sigma Gs\rangle}{9\pi^2}s - \frac{8\langle g_s^2GG\rangle m_s^2\langle\bar{s}s\rangle^2}{81\pi^2} - \frac{32m_s^2\langle g_s\bar{s}\sigma Gs\rangle^2}{9\pi^2}, \end{aligned} \tag{37}$$

$$\begin{aligned}\rho_{2^{+-};12}(s) = {} & -\frac{3\langle g_s^2GG\rangle s^4}{71680\pi^6} - \left(\frac{131\langle g_s^2GG\rangle m_s^2}{34560\pi^6} + \frac{31m_s\langle g_s\bar{s}\sigma Gs\rangle}{2160\pi^4}\right)s^3 - \left(-\frac{133\langle g_s^2GG\rangle m_s\langle\bar{s}s\rangle}{2880\pi^4} - \frac{2\langle\bar{s}s\rangle\langle g_s\bar{s}\sigma Gs\rangle}{45\pi^2}\right)s^2 \\ & - \left(\frac{2m_s^2\langle\bar{s}s\rangle\langle g_s\bar{s}\sigma Gs\rangle}{9\pi^2} - \frac{\langle g_s\bar{s}\sigma Gs\rangle^2}{18\pi^2}\right)s + \frac{7\langle g_s^2GG\rangle m_s^2\langle\bar{s}s\rangle^2}{27\pi^2}, \end{aligned} \tag{38}$$

$$\begin{aligned}\rho_{2^{+-};13}(s) = {} & -\frac{41m_s^2s^5}{80640\pi^6} + \frac{13m_s\langle\bar{s}s\rangle}{840\pi^4}s^4 - \left(\frac{59\langle g_s^2GG\rangle m_s^2}{207360\pi^6} - \frac{2\langle\bar{s}s\rangle^2}{15\pi^2} - \frac{53m_s\langle g_s\bar{s}\sigma Gs\rangle}{1296\pi^4}\right)s^3 \\ & - \left(-\frac{\langle g_s^2GG\rangle m_s\langle\bar{s}s\rangle}{480\pi^4} - \frac{44m_s^2\langle\bar{s}s\rangle^2}{15\pi^2} - \frac{481\langle\bar{s}s\rangle\langle g_s\bar{s}\sigma Gs\rangle}{270\pi^2}\right)s^2 \\ & - \left(\frac{\langle g_s^2GG\rangle\langle\bar{s}s\rangle^2}{216\pi^2} - \frac{\langle g_s^2GG\rangle m_s\langle g_s\bar{s}\sigma Gs\rangle}{72\pi^4} + \frac{199m_s^2\langle\bar{s}s\rangle\langle g_s\bar{s}\sigma Gs\rangle}{27\pi^2} - \frac{503\langle g_s\bar{s}\sigma Gs\rangle^2}{216\pi^2}\right)s \\ & - \frac{\langle g_s^2GG\rangle\langle\bar{s}s\rangle\langle g_s\bar{s}\sigma Gs\rangle}{27\pi^2} - \frac{140m_s^2\langle g_s\bar{s}\sigma Gs\rangle^2}{27\pi^2}, \end{aligned} \tag{39}$$

$$\begin{aligned}\rho_{2^{+-};23}(s) = {} & -\left(-\frac{59\langle g_s^2GG\rangle m_s^2}{69120\pi^6} - \frac{7m_s\langle g_s\bar{s}\sigma Gs\rangle}{1440\pi^4}\right)s^3 - \left(\frac{\langle g_s^2GG\rangle m_s\langle\bar{s}s\rangle}{160\pi^4} - \frac{\langle\bar{s}s\rangle\langle g_s\bar{s}\sigma Gs\rangle}{90\pi^2}\right)s^2 \\ & - \left(-\frac{\langle g_s^2GG\rangle\langle\bar{s}s\rangle^2}{72\pi^2} + \frac{\langle g_s^2GG\rangle m_s\langle g_s\bar{s}\sigma Gs\rangle}{24\pi^4} - \frac{5m_s^2\langle\bar{s}s\rangle\langle g_s\bar{s}\sigma Gs\rangle}{9\pi^2} + \frac{\langle g_s\bar{s}\sigma Gs\rangle^2}{24\pi^2}\right)s \\ & + \frac{\langle g_s^2GG\rangle\langle\bar{s}s\rangle\langle g_s\bar{s}\sigma Gs\rangle}{9\pi^2}. \end{aligned} \tag{40}$$

We take the following values for various QCD parameters contained in the above expressions [1,111–118]:

$$\begin{aligned} m_s(2\text{ GeV}) &= 93^{+11}_{-5}\text{ MeV} \\ \langle\bar{s}s\rangle &= -(0.8\pm 0.1)\times(0.240\text{ GeV})^3, \\ \langle g_s^2GG\rangle &= (0.48\pm 0.14)\text{ GeV}^4, \\ \langle g_s\bar{s}\sigma Gs\rangle &= -M_0^2\times\langle\bar{s}s\rangle, \\ M_0^2 &= (0.8\pm 0.2)\text{ GeV}^2. \end{aligned} \tag{41}$$

After performing the Borel transformation at both the hadron and quark-gluon levels, we approximate the continuum using $\rho_{ij}(s)$ above the threshold value s_0 to obtain

$$\Pi_{ij}(s_0, M_B^2) = \sum_n f_{in} f_{jn} e^{-M_n^2/M_B^2} = \int_{s_<}^{s_0} e^{-s/M_B^2} \rho_{ij}(s) ds. \tag{42}$$

We shall separately perform the single-channel and multichannel analyses in the following two subsections.

A. Single-channel analysis

In this subsection we perform the single-channel analysis by setting $\rho_{ij}(s)|_{i\neq j} = 0$. This assumption neglects the off-diagonal correlation functions to make the three currents $\eta_{0^{+-};1/2/3}$ "noncorrelated," i.e., any two of them cannot mainly couple to the same state X, otherwise,

$$\begin{aligned}\rho_{ij}(s) &= \sum_n \delta(s - M_n^2) \langle 0|\eta_{0^{+-};i}|X_n\rangle \langle X_n|\eta^\dagger_{0^{+-};j}|0\rangle + \cdots \\ &\approx \delta(s - M_X^2) \langle 0|\eta_{0^{+-};i}|X\rangle \langle X|\eta^\dagger_{0^{+-};j}|0\rangle + \cdots \\ &\neq 0.\end{aligned} \tag{43}$$

Accordingly, we further assume that there are three states $X_{1,2,3}$ separately corresponding to the three currents $\eta_{0^{+-};1/2/3}$ through

$$\langle 0|\eta_{0^{+-};i}|X_i\rangle = f_{ii}. \tag{44}$$

We parametrize the spectral density $\rho_{ii}(s)$ as one pole dominance for the single state X_i between the physical threshold $s_<$ and the threshold value s_0 as well as a continuum contribution above s_0. This simplifies Eq. (42) to be

$$\Pi_{ii}(s_0, M_B^2) = f_{ii}^2 e^{-M_i^2/M_B^2} = \int_{s_<}^{s_0} e^{-s/M_B^2} \rho_{ii}(s) ds, \tag{45}$$

and the mass M_i can be calculated through

$$M_i^2(s_0, M_B) = \frac{\int_{s_<}^{s_0} e^{-s/M_B^2} s \rho_{ii}(s) ds}{\int_{s_<}^{s_0} e^{-s/M_B^2} \rho_{ii}(s) ds}. \tag{46}$$

We use the spectral density $\rho_{11}(s)$ extracted from the current $\eta_{0^{+-};1}$ as an example to calculate the mass M_1 of the state X_1. As given in Eq. (46), the mass M_1 depends on two free parameters: the Borel mass M_B and the threshold value s_0. We consider three aspects to find their proper working regions: (a) the OPE convergence, (b) the one-pole-dominance assumption, and (c) the dependence of the mass M_1 on these two parameters.

First, we consider the OPE convergence (CVG) and require the $D = 12/10/8$ terms to be less than 5%/10%/20%, respectively,

$$\mathrm{CVG}_{12} = \left| \frac{\Pi_{11}^{D=12}(\infty, M_B^2)}{\Pi_{11}(\infty, M_B^2)} \right| \leq 5\%, \tag{47}$$

$$\mathrm{CVG}_{10} = \left| \frac{\Pi_{11}^{D=10}(\infty, M_B^2)}{\Pi_{11}(\infty, M_B^2)} \right| \leq 10\%, \tag{48}$$

$$\mathrm{CVG}_{8} = \left| \frac{\Pi_{11}^{D=8}(\infty, M_B^2)}{\Pi_{11}(\infty, M_B^2)} \right| \leq 20\%. \tag{49}$$

These conditions demand the Borel mass to be larger than $M_B^2 \geq 2.31$ GeV^2, as depicted in Fig. 3.

Second, we consider the one-pole-dominance assumption and require the pole contribution (PC) to be larger than 40%,

$$\mathrm{PC} = \left| \frac{\Pi_{11}(s_0, M_B^2)}{\Pi_{11}(\infty, M_B^2)} \right| \geq 40\%. \tag{50}$$

This condition demands the Borel mass to be smaller than $M_B^2 \leq 2.57$ GeV^2 when setting $s_0 = 14.0$ GeV^2, as depicted in Fig. 3.

Altogether, we determine the Borel window to be $2.31 \leq M_B^2 \leq 2.57$ GeV^2 for $s_0 = 14.0$ GeV^2. We redo the same procedures and find that the Borel windows exist as long as $s_0 \geq s_0^{\min} = 12.5$ GeV^2. Accordingly, we demand the threshold value s_0 to be slightly larger and choose its working region to be $11.0 \leq s_0 \leq 17.0$ GeV^2.

Third, we consider the dependence of the mass M_1 on M_B and s_0. As shown in Fig. 4, the mass M_1 is stable against M_B inside the Borel window $2.31 \leq M_B^2 \leq 2.57$ GeV^2, and its dependence on s_0 is acceptable inside the working region $11.0 \leq s_0 \leq 17.0$ GeV^2, where the mass M_1 is calculated to be

$$M_{0^{+-};1} = 3.21^{+0.23}_{-0.28} \text{ GeV}. \tag{51}$$

Its uncertainty comes from M_B and s_0 as well as various QCD parameters given in Eq. (42). Note that the mass M_1 has a

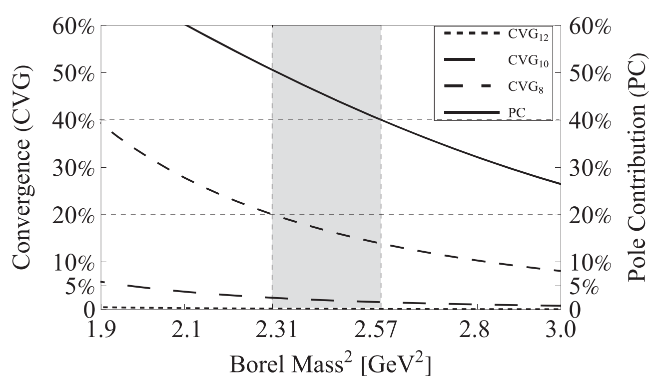

FIG. 3. $\mathrm{CVG}_{8/10/12}$ and PC with respect to the Borel mass M_B. These curves are obtained using the current $\eta_{0^{+-};1}$ by setting $s_0 = 14.0$ GeV^2.

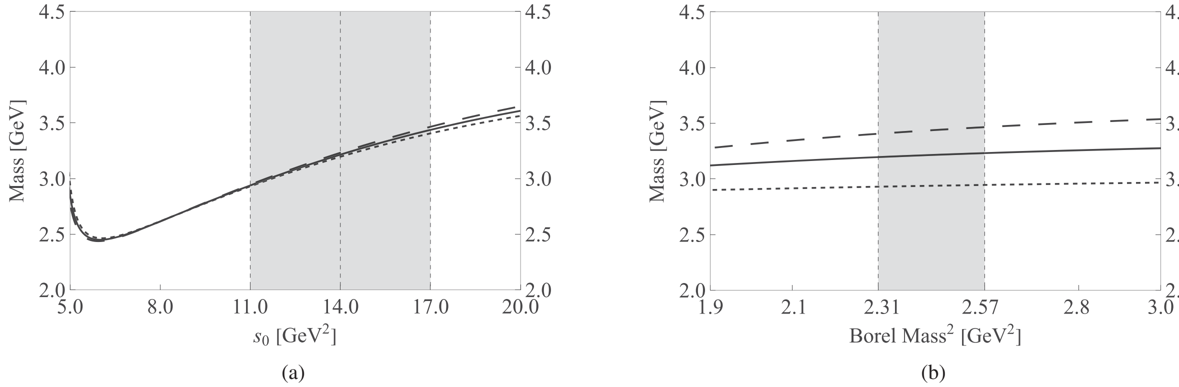

FIG. 4. The mass M_1 of the state X_1 with respect to (a) the threshold value s_0 and (b) the Borel mass M_B. In (a) the short-dashed, solid, and long-dashed curves are obtained by setting $M_B^2 = 2.31$, 2.44, and 2.57 GeV^2, respectively. In (b) the short-dashed, solid, and long-dashed curves are obtained by setting $s_0 = 11.0$, 14.0, and 17.0 GeV^2, respectively. These curves are obtained using the spectral density $\rho_{11}(s)$ extracted from the current $\eta_{0^{+-};1}$.

stability point at around $s_0 \sim 6.0$ GeV^2, as shown in Fig. 4(a). However, there does not exist a Borel window at this energy point.

We apply the same method to study the other two $J^{PC} = 0^{+-}$ currents $\eta_{0^{+-};2/3}$ and the three $J^{PC} = 2^{+-}$ currents $\eta^{\beta_1\beta_2}_{2^{+-};1/2/3}$. The obtained results are summarized in Table I.

B. Multichannel analysis

In this subsection we perform the multichannel analysis by taking into account the off-diagonal correlation functions that are actually nonzero, i.e., $\rho_{ij}(s)|_{i\neq j} \neq 0$. To see how large they are, we choose $s_0 = 9.0$ and $M_B^2 = 1.50$ GeV^2 to obtain

$$\Pi_{ij}(s_0, M_B^2) = \begin{pmatrix} 8.29 & 13.59 & -2.83 \\ 13.59 & -3.96 & -0.87 \\ -2.83 & -0.87 & 14.28 \end{pmatrix} \times 10^{-6}\ \text{GeV}^{14}. \tag{52}$$

This indicates that $\eta_{0^{+-};1}$ and $\eta_{0^{+-};2}$ are strongly correlated with each other, as depicted in Fig. 5.

To diagonalize the 3×3 matrix $\rho_{ij}(s)$, we construct three mixing currents $J_{0^{+-};1/2/3}$,

$$\begin{pmatrix} J_{0^{+-};1} \\ J_{0^{+-};2} \\ J_{0^{+-};3} \end{pmatrix} = \mathbb{T}_{0^{+-}} \begin{pmatrix} \eta_{0^{+-};1} \\ \eta_{0^{+-};2} \\ \eta_{0^{+-};3} \end{pmatrix}, \tag{53}$$

where $\mathbb{T}_{0^{+-}}$ is the transition matrix.

We use the method of operator product expansion to extract the spectral densities $\rho'_{ij}(s)$ from the mixing currents $J_{0^{+-};1/2/3}$. After choosing

$$\mathbb{T}_{0^{+-}} = \begin{pmatrix} -0.72 & -0.45 & 0.53 \\ 0.54 & -0.84 & 0.03 \\ 0.43 & 0.31 & 0.85 \end{pmatrix}, \tag{54}$$

we obtain

$$\Pi'_{ij}(s_0, M_B^2) = \begin{pmatrix} 12.52 & 0 & 0 \\ 0 & 18.85 & 0 \\ 0 & 0 & -12.77 \end{pmatrix} \times 10^{-6}\ \text{GeV}^{14}, \tag{55}$$

at $s_0 = 9.0$ and $M_B^2 = 1.50$ GeV^2. Therefore, the off-diagonal terms of $\rho'_{ij}(s)$ are negligible and the three mixing currents $J_{0^{+-};1/2/3}$ are nearly noncorrelated around here, as depicted in Fig. 5. Additionally, Eq. (55) indicates that the QCD sum rule result extracted from $J_{0^{+-};3}$ is nonphysical around here due to its negative correlation function.

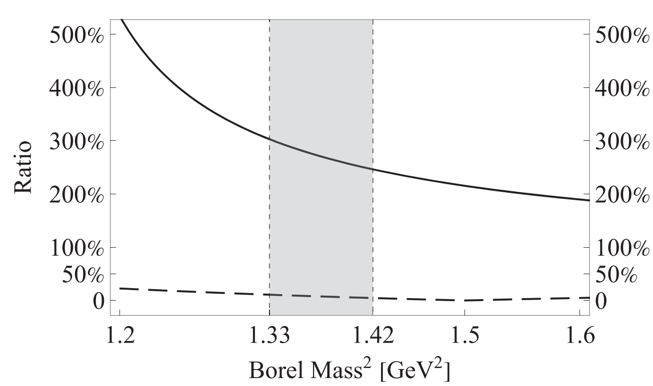

FIG. 5. The normalized off-diagonal correlation functions $|\Pi_{12}/\sqrt{\Pi_{11}\Pi_{22}}|$ (solid) and $|\Pi'_{12}/\sqrt{\Pi'_{11}\Pi'_{22}}|$ (dashed) with respect to the Borel mass M_B. These curves are obtained using the three currents $\eta_{0^{+-};1/2/3}$ and their mixing currents $J_{0^{+-};1/2/3}$ when setting $s_0 = 9.0$ GeV^2.

Since the off-diagonal terms of $\rho'_{ij}(s)$ are negligible around $s_0 = 9.0$ and $M_B^2 = 1.50\ \mathrm{GeV}^2$, the procedures used in the previous subsection can be applied to study the three mixing currents $J_{0^{+-};1/2/3}$. We summarize the obtained results in Table I. Especially, the mass extracted from the current $J_{0^{+-};1}$ is significantly reduced to be

$$M'_{0^{+-};1} = 2.47^{+0.33}_{-0.44}\ \mathrm{GeV}. \tag{56}$$

Similarly, we can investigate the three $J^{PC} = 2^{+-}$ currents $\eta^{\beta_1\beta_2}_{2^{+-};1/2/3}$. We construct three mixing currents $J^{\beta_1\beta_2}_{2^{+-};1/2/3}$ that are nearly noncorrelated at around $s_0 = 13.0$ and $M_B^2 = 2.0\ \mathrm{GeV}^2$,

$$\begin{pmatrix} J^{\cdots}_{2^{+-};1} \\ J^{\cdots}_{2^{+-};2} \\ J^{\cdots}_{2^{+-};3} \end{pmatrix} = \mathbb{T}_{2^{+-}} \begin{pmatrix} \eta^{\cdots}_{2^{+-};1} \\ \eta^{\cdots}_{2^{+-};2} \\ \eta^{\cdots}_{2^{+-};3} \end{pmatrix}, \tag{57}$$

where

$$\mathbb{T}_{2^{+-}} = \begin{pmatrix} -0.72 & 0.06 & 0.69 \\ -0.68 & 0.15 & -0.72 \\ -0.15 & -0.99 & -0.07 \end{pmatrix}. \tag{58}$$

We apply the QCD sum rule method to study the mixing currents $J^{\beta_1\beta_2}_{2^{+-};1/2/3}$, and the obtained results are summarized in Table I. Especially, the mass extracted from the current $J^{\beta_1\beta_2}_{2^{+-};1}$ is the lowest,

$$M'_{2^{+-};1} = 3.07^{+0.25}_{-0.33}\ \mathrm{GeV}. \tag{59}$$

For completeness, we also summarize in Table I the QCD sum rule results obtained in Ref. [90] using the three $J^{PC} = 4^{+-}$ currents $\eta^{\alpha_1\alpha_2\alpha_3\alpha_4}_{4^{+-};1/2/3}$ as well as their mixing currents $J^{\alpha_1\alpha_2\alpha_3\alpha_4}_{4^{+-};1/2/3}$ that are nearly noncorrelated at around $s_0 = 11.0$ and $M_B^2 = 1.85\ \mathrm{GeV}^2$,

$$\begin{pmatrix} J^{\cdots}_{4^{+-};1} \\ J^{\cdots}_{4^{+-};2} \\ J^{\cdots}_{4^{+-};3} \end{pmatrix} = \mathbb{T}_{4^{+-}} \begin{pmatrix} \eta^{\cdots}_{4^{+-};1} \\ \eta^{\cdots}_{4^{+-};2} \\ \eta^{\cdots}_{4^{+-};3} \end{pmatrix}, \tag{60}$$

where

$$\mathbb{T}_{4^{+-}} = \begin{pmatrix} 0.72 & -0.06 & -0.69 \\ 0.14 & 0.99 & 0.05 \\ 0.68 & -0.13 & 0.72 \end{pmatrix}. \tag{61}$$

IV. SUMMARY AND DISCUSSION

In this paper, we apply the QCD sum rule method to study the fully strange tetraquark states with the exotic quantum numbers $J^{PC} = 0^{+-}$ and 2^{+-}. We explicitly add the covariant derivative operator to construct some diquark-antidiquark interpolating currents and apply the method of operator product expansion to calculate both their diagonal and off-diagonal correlation functions. Based on the obtained results, we construct some mixing currents that are nearly noncorrelated.

We use both the diquark-antidiquark currents and their mixing currents to perform QCD sum rule analyses. The obtained results are summarized in Table I. Especially, we use the mixing currents $J_{0^{+-};1}$ and $J^{\beta_1\beta_2}_{2^{+-};1}$ to derive the masses of the lowest-lying $J^{PC} = 0^{+-}$ and 2^{+-} states to be

$$M_{0^{+-}} = 2.47^{+0.33}_{-0.44}\ \mathrm{GeV}, \tag{62}$$

$$M_{2^{+-}} = 3.07^{+0.25}_{-0.33}\ \mathrm{GeV}. \tag{63}$$

In this paper, we also construct some fully strange meson-meson currents of $J^{PC} = 0^{+-}$ and 2^{+-}, which are related to the diquark-antidiquark currents through the Fierz rearrangement. We can use these meson-meson currents and their mixing currents to perform QCD sum rule analyses. The results extracted from these mixing currents are the same.

We can apply Eq. (23) to transform the mixing currents $J_{0^{+-};1}$ and $J^{\beta_1\beta_2}_{2^{+-};1}$ to be

$$J_{0^{+-};1} = -2.7\xi_{0^{+-};1} - 1.6\xi_{0^{+-};2} + 2.3\xi_{0^{+-};3}, \tag{64}$$

$$J^{\cdots}_{2^{+-};1} = \quad -4.3\xi^{\cdots}_{2^{+-};1} - 1.2\xi^{\cdots}_{2^{+-};2} + 1.3\xi^{\cdots}_{2^{+-};3}. \tag{65}$$

For completeness, we compare these two combinations to the mixing current

$$J^{\cdots}_{4^{+-};1} = 4.3\xi^{\cdots}_{4^{+-};1} + 1.2\xi^{\cdots}_{4^{+-};2} - 1.3\xi^{\cdots}_{4^{+-};3}, \tag{66}$$

which was used in Ref. [90] to derive the mass of the lowest-lying $J^{PC} = 4^{+-}$ state to be

$$M_{4^{+-}} = 2.85^{+0.19}_{-0.22}\ \mathrm{GeV}. \tag{67}$$

The Fierz identity given in Eq. (64) indicates that the lowest-lying $J^{PC} = 0^{+-}$ state decays into the P-wave $\phi(1020)f_0(1710)(\to \phi K\bar{K}/\phi\pi\pi)$ channel through the meson-meson current $\xi_{0^{+-},1}$, and the Fierz identity given in Eq. (65) indicates that the lowest-lying $J^{PC} = 2^{+-}$ state decays into the P-wave $\phi(1020)f_0(1710)$ and $\phi(1020)f'_2(1525)$ channels through the meson-meson current $\xi^{\cdots}_{2^{+-},1}$. Accordingly, we propose to search for them in the $X_{0^{+-}} \to \phi(1020)f_0(1710) \to \phi K\bar{K}/\phi\pi\pi$ and $X_{2^{+-}} \to \phi(1020)f_0(1710)/\phi(1020)f'_2(1525) \to \phi K\bar{K}/\phi\pi\pi$ decay processes in the future Belle-II, BESIII, COMPASS, GlueX, J-PARC, and PANDA experiments.

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[1] P. A. Zyla *et al.*, Review of particle physics, Prog. Theor. Exp. Phys. **2020**, 083C01 (2020).
[2] H.-X. Chen, W. Chen, X. Liu, Y.-R. Liu, and S.-L. Zhu, An updated review of the new hadron states, Rep. Prog. Phys. **86**, 026201 (2023).
[3] E. Klempt and A. Zaitsev, Glueballs, hybrids, multiquarks: Experimental facts versus QCD inspired concepts, Phys. Rep. **454**, 1 (2007).
[4] C. A. Meyer and E. S. Swanson, Hybrid mesons, Prog. Part. Nucl. Phys. **82**, 21 (2015).
[5] C. Amsler and N. A. Tornqvist, Mesons beyond the naive quark model, Phys. Rep. **389**, 61 (2004).
[6] D. V. Bugg, Four sorts of meson, Phys. Rep. **397**, 257 (2004).
[7] C. A. Meyer and Y. Van Haarlem, Status of exotic-quantum-number mesons, Phys. Rev. C **82**, 025208 (2010).
[8] R. A. Briceno, J. J. Dudek, and R. D. Young, Scattering processes and resonances from lattice QCD, Rev. Mod. Phys. **90**, 025001 (2018).
[9] M. Aghasyan *et al.*, Light isovector resonances in $\pi^- p \to \pi^-\pi^-\pi^+ p$ at 190 GeV$/c$, Phys. Rev. D **98**, 092003 (2018).
[10] A. Rodas *et al.*, Determination of the pole position of the lightest hybrid meson candidate, Phys. Rev. Lett. **122**, 042002 (2019).
[11] B. Ketzer, B. Grube, and D. Ryabchikov, Light-meson spectroscopy with COMPASS, Prog. Part. Nucl. Phys. **113**, 103755 (2020).
[12] S. Jin and X. Shen, Highlights of light meson spectroscopy at the BESIII experiment, Natl. Sci. Rev. **8**, nwab198 (2021).
[13] L. Meng, B. Wang, G.-J. Wang, and S.-L. Zhu, Chiral perturbation theory for heavy hadrons and chiral effective field theory for heavy hadronic molecules, Phys. Rep. **1019**, 1 (2023).
[14] R. Albuquerque, S. Narison, and D. Rabetiarivony, Scrutinizing the light scalar quarkonia from LSR at higher orders, Nucl. Phys. **A1039**, 122743 (2023).
[15] C.-K. Jiao, W. Chen, H.-X. Chen, and S.-L. Zhu, Possible $J^{PC} = 0^{--}$ exotic state, Phys. Rev. D **79**, 114034 (2009).
[16] Z.-R. Huang, W. Chen, T. G. Steele, Z.-F. Zhang, and H.-Y. Jin, Investigation of the light four-quark states with exotic $J^{PC} = 0^{--}$, Phys. Rev. D **95**, 076017 (2017).
[17] H.-J. LEE, Discussion on the $J^{PC} = 0^{--}$ tetraquark of u and d quarks within the QCD sum rule, New Phys. Sae Mulli **70**, 836 (2020).
[18] M.-L. Du, W. Chen, X.-L. Chen, and S.-L. Zhu, The possible $J^{PC} = 0^{+-}$ exotic state, Chin. Phys. C **37**, 033104 (2013).
[19] Y.-C. Fu, Z.-R. Huang, Z.-F. Zhang, and W. Chen, Exotic tetraquark states with $J^{PC} = 0^{+-}$, Phys. Rev. D **99**, 014025 (2019).
[20] K. Azizi, S. S. Agaev, and H. Sundu, Light axial-vector and vector resonances $X(2100)$ and $X(2239)$, Nucl. Phys. **B948**, 114789 (2019).
[21] N. Su and H.-X. Chen, S- and P-wave fully strange tetraquark states from QCD sum rules, Phys. Rev. D **106**, 014023 (2022).
[22] W. Zhu, Y.-R. Liu, and T. Yao, Is $J^{PC} = 3^{-+}$ molecule possible?, Chin. Phys. C **39**, 023101 (2015).
[23] T. Ji, X.-K. Dong, F.-K. Guo, and B.-S. Zou, Prediction of a narrow exotic hadronic state with quantum numbers $J^{PC} = 0^{--}$, Phys. Rev. Lett. **129**, 102002 (2022).
[24] X. Zhang and J.-J. Xie, Prediction of possible exotic states in the $\eta\bar{K}K^*$ system, Chin. Phys. C **44**, 054104 (2020).
[25] X.-K. Dong, Y.-H. Lin, and B.-S. Zou, Interpretation of the $\eta_1(1855)$ as a $K\bar{K}_1(1400)+$ c.c. molecule, Sci. China Phys. Mech. Astron. **65**, 261011 (2022).
[26] F. Yang, H. Q. Zhu, and Y. Huang, Analysis of the $\eta_1(1855)$ as a $K\bar{K}_1(1400)$ molecular state, Nucl. Phys. **A1030**, 122571 (2023).
[27] B.-D. Wan, S.-Q. Zhang, and C.-F. Qiao, Possible structure of the newly found exotic state $\eta_1(1855)$, Phys. Rev. D **106**, 074003 (2022).
[28] X.-Y. Wang, F.-C. Zeng, and X. Liu, Production of the $\eta_1(1855)$ through kaon induced reactions under the assumptions that it is a molecular or a hybrid state, Phys. Rev. D **106**, 036005 (2022).
[29] Y. Yu, X. Zhuang, B.-C. Ke, Y. Teng, and Q.-S. Liu, Investigating $\eta_1'(1855)$ exotic states in $J/\psi \to \eta_1'(1855)\eta^{(\prime)}$ decays, Phys. Lett. B **842**, 137965 (2023).
[30] C.-F. Qiao and L. Tang, Finding the 0^{--} glueball, Phys. Rev. Lett. **113**, 221601 (2014).
[31] L. Tang and C.-F. Qiao, Mass spectra of 0^{+-}, 1^{-+}, and 2^{+-} exotic glueballs, Nucl. Phys. **B904**, 282 (2016).
[32] A. Pimikov, H.-J. Lee, N. Kochelev, P. Zhang, and V. Khandramai, Exotic glueball $0^{\pm-}$ states in QCD sum rules, Phys. Rev. D **96**, 114024 (2017).
[33] K. G. Wilson, Confinement of quarks, Phys. Rev. D **10**, 2445 (1974).
[34] Y. Chen *et al.*, Glueball spectrum and matrix elements on anisotropic lattices, Phys. Rev. D **73**, 014516 (2006).
[35] V. Mathieu, N. Kochelev, and V. Vento, The physics of glueballs, Int. J. Mod. Phys. E **18**, 1 (2009).
[36] H. B. Meyer, Glueball Regge trajectories, Ph.D. thesis, University of Oxford, 2004; arXiv:hep-lat/0508002.
[37] E. Gregory, A. Irving, B. Lucini, C. McNeile, A. Rago, C. Richards, and E. Rinaldi, Towards the glueball spectrum

from unquenched lattice QCD, J. High Energy Phys. 10 (2012) 170.
[38] A. Athenodorou and M. Teper, The glueball spectrum of $SU(3)$ gauge theory in $3+1$ dimensions, J. High Energy Phys. 11 (2020) 172.
[39] Z.-f. Zhang, H.-y. Jin, and T. G. Steele, Revisiting 1^{-+} and 0^{++} light hybrids from Monte-Carlo based QCD sum rules, Chin. Phys. Lett. **31**, 051201 (2014).
[40] Z.-R. Huang, H.-Y. Jin, and Z.-F. Zhang, New predictions on the mass of the 1^{-+} light hybrid meson from QCD sum rules, J. High Energy Phys. 04 (2015) 004.
[41] J. Ho, R. Berg, W. Chen, D. Harnett, and T. G. Steele, Mass calculations of light quarkonium, exotic $J^{PC}=0^{+-}$ hybrid mesons from Gaussian sum-rules, Phys. Rev. D **98**, 096020 (2018).
[42] H.-X. Chen, W. Chen, and S.-L. Zhu, New hadron configuration: The double-gluon hybrid state, Phys. Rev. D **105**, L051501 (2022).
[43] N. Su, H.-X. Chen, W. Chen, and S.-L. Zhu, Light double-gluon hybrid states from QCD sum rules, Phys. Rev. D **107**, 034010 (2023).
[44] N. Su, W.-H. Tan, H.-X. Chen, W. Chen, and S.-L. Zhu, Light double-gluon hybrid states with the exotic quantum numbers $J^{PC}=1^{-+}$ and 3^{-+}, Phys. Rev. D **107**, 114005 (2023).
[45] T. Barnes, Coloured quark and gluon constituents in the MIT bag model: A model of mesons, Nucl. Phys. **B158**, 171 (1979).
[46] P. Hasenfratz, R. R. Horgan, J. Kuti, and J. M. Richard, The effects of coloured glue in the QCD motivated bag of heavy quark-antiquark systems, Phys. Lett. B **95**, 299 (1980).
[47] M. S. Chanowitz and S. R. Sharpe, Hybrids: Mixed states of quarks and gluons, Nucl. Phys. **B222**, 211 (1983); Nucl. Phys. **B228**, 588(E) (1983).
[48] N. Isgur and J. E. Paton, A flux tube model for hadrons, Phys. Lett. **124B**, 247 (1983).
[49] F. E. Close and P. R. Page, The production and decay of hybrid mesons by flux-tube breaking, Nucl. Phys. **B443**, 233 (1995).
[50] P. R. Page, E. S. Swanson, and A. P. Szczepaniak, Hybrid meson decay phenomenology, Phys. Rev. D **59**, 034016 (1999).
[51] T. Burns and F. E. Close, Hybrid-meson properties in lattice QCD and flux-tube models, Phys. Rev. D **74**, 034003 (2006).
[52] L. Qiu and Q. Zhao, Towards the establishment of the light $J^{P(C)}=1^{-(+)}$ hybrid nonet, Chin. Phys. C **46**, 051001 (2022).
[53] A. P. Szczepaniak and E. S. Swanson, Coulomb gauge QCD, confinement, and the constituent representation, Phys. Rev. D **65**, 025012 (2001).
[54] F. Iddir and L. Semlala, Hybrid states from constituent glue model, Int. J. Mod. Phys. A **23**, 5229 (2008).
[55] P. Guo, A. P. Szczepaniak, G. Galata, A. Vassallo, and E. Santopinto, Gluelump spectrum from Coulomb gauge QCD, Phys. Rev. D **77**, 056005 (2008).
[56] C. Michael, Adjoint sources in lattice gauge theory, Nucl. Phys. **B259**, 58 (1985).
[57] C. McNeile, C. W. Bernard, T. A. DeGrand, C. E. DeTar, S. A. Gottlieb, U. M. Heller, J. Hetrick, R. Sugar, and D. Toussaint, Exotic meson spectroscopy from the clover action at $\beta=5.85$ and $\beta=6.15$, Nucl. Phys. B, Proc. Suppl. **73**, 264 (1999).
[58] K. J. Juge, J. Kuti, and C. Morningstar, Fine structure of the QCD string spectrum, Phys. Rev. Lett. **90**, 161601 (2003).
[59] P. Lacock, C. Michael, P. Boyle, and P. Rowland, Hybrid mesons from quenched QCD, Phys. Lett. B **401**, 308 (1997).
[60] C. W. Bernard *et al.*, Exotic mesons in quenched lattice QCD, Phys. Rev. D **56**, 7039 (1997).
[61] C. Bernard, T. Burch, E. B. Gregory, D. Toussaint, C. E. DeTar, J. Osborn, S. A. Gottlieb, U. M. Heller, and R. Sugar, Lattice calculation of 1^{-+} hybrid mesons with improved Kogut-Susskind fermions, Phys. Rev. D **68**, 074505 (2003).
[62] J. N. Hedditch, W. Kamleh, B. G. Lasscock, D. B. Leinweber, A. G. Williams, and J. M. Zanotti, 1^{-+} exotic meson at light quark masses, Phys. Rev. D **72**, 114507 (2005).
[63] J. J. Dudek, R. G. Edwards, M. J. Peardon, D. G. Richards, and C. E. Thomas, Highly excited and exotic meson spectrum from dynamical lattice QCD, Phys. Rev. Lett. **103**, 262001 (2009).
[64] J. J. Dudek, R. G. Edwards, M. J. Peardon, D. G. Richards, and C. E. Thomas, Toward the excited meson spectrum of dynamical QCD, Phys. Rev. D **82**, 034508 (2010).
[65] J. J. Dudek, R. G. Edwards, P. Guo, and C. E. Thomas, Toward the excited isoscalar meson spectrum from lattice QCD, Phys. Rev. D **88**, 094505 (2013).
[66] F. Chen, X. Jiang, Y. Chen, M. Gong, Z. Liu, C. Shi, and W. Sun, 1^{-+} hybrid in J/ψ radiative decays from lattice QCD, Phys. Rev. D **107**, 054511 (2023).
[67] I. I. Balitsky, D. Diakonov, and A. V. Yung, Exotic mesons with $J^{PC}=1^{-+}$ from QCD sum rules, Phys. Lett. B **112**, 71 (1982).
[68] J. Govaerts, F. de Viron, D. Gusbin, and J. Weyers, Exotic mesons from QCD sum rules, Phys. Lett. **128B**, 262 (1983); Phys. Lett. **136B**, 445(E) (1984).
[69] L. S. Kisslinger and Z. P. Li, Hybrid baryons via QCD sum rules, Phys. Rev. D **51**, R5986 (1995).
[70] K. G. Chetyrkin and S. Narison, Light hybrid mesons in QCD, Phys. Lett. B **485**, 145 (2000).
[71] H. Y. Jin, J. G. Korner, and T. G. Steele, Improved determination of the mass of the 1^{-+} light hybrid meson from QCD sum rules, Phys. Rev. D **67**, 014025 (2003).
[72] S. Narison, 1^{-+} light exotic mesons in QCD, Phys. Lett. B **675**, 319 (2009).
[73] Z.-R. Huang, H.-Y. Jin, T. G. Steele, and Z.-F. Zhang, Revisiting the $b_1\pi$ and $\rho\pi$ decay modes of the 1^{-+} light hybrid state with light-cone QCD sum rules, Phys. Rev. D **94**, 054037 (2016).
[74] S.-H. Li, Z.-S. Chen, H.-Y. Jin, and W. Chen, Mass of 1^{-+} four-quark-hybrid mixed states, Phys. Rev. D **105**, 054030 (2022).
[75] M. Alekseev *et al.*, Observation of a $J^{PC}=1^{-+}$ exotic resonance in diffractive dissociation of 190 GeV$/c$ π^- into $\pi^-\pi^-\pi^+$, Phys. Rev. Lett. **104**, 241803 (2010).

[76] D. Alde *et al.*, Evidence for a 1^{-+} exotic meson, Phys. Lett. B **205**, 397 (1988).
[77] G. S. Adams *et al.*, Observation of a new $J^{PC} = 1^{-+}$ exotic state in the reaction $\pi^- p \to \pi^+\pi^-\pi^- p$ at 18 GeV/c, Phys. Rev. Lett. **81**, 5760 (1998).
[78] J. Kuhn *et al.*, Exotic meson production in the $f_1(1285)\pi^-$ system observed in the reaction $\pi^- p \to \eta\pi^+\pi^-\pi^- p$ at 18 GeV/c, Phys. Lett. B **595**, 109 (2004).
[79] M. Ablikim *et al.*, Observation of an isoscalar resonance with exotic $J^{PC} = 1^{-+}$ quantum numbers in $J/\psi \to \gamma\eta\eta'$, Phys. Rev. Lett. **129**, 192002 (2022).
[80] M. Ablikim *et al.*, Partial wave analysis of $J/\psi \to \gamma\eta\eta'$, Phys. Rev. D **106**, 072012 (2022).
[81] P.-Z. Huang, H.-X. Chen, and S.-L. Zhu, Strong decay patterns of the 1^{-+} exotic hybrid mesons, Phys. Rev. D **83**, 014021 (2011).
[82] H.-X. Chen, Z.-X. Cai, P.-Z. Huang, and S.-L. Zhu, Decay properties of the 1^{-+} hybrid state, Phys. Rev. D **83**, 014006 (2011).
[83] H.-X. Chen, N. Su, and S.-L. Zhu, QCD axial anomaly enhances the $\eta\eta'$ decay of the hybrid candidate $\eta_1(1855)$, Chin. Phys. Lett. **39**, 051201 (2022).
[84] H.-X. Chen, A. Hosaka, and S.-L. Zhu, $I^G J^{PC} = 1^- 1^{-+}$ tetraquark states, Phys. Rev. D **78**, 054017 (2008).
[85] H.-X. Chen, A. Hosaka, and S.-L. Zhu, $I^G J^{PC} = 0^+ 1^{-+}$ tetraquark states, Phys. Rev. D **78**, 117502 (2008).
[86] H.-X. Chen, X. Liu, A. Hosaka, and S.-L. Zhu, $Y(2175)$ state in the QCD sum rule, Phys. Rev. D **78**, 034012 (2008).
[87] H.-X. Chen, C.-P. Shen, and S.-L. Zhu, Possible partner state of the $Y(2175)$, Phys. Rev. D **98**, 014011 (2018).
[88] E.-L. Cui, H.-M. Yang, H.-X. Chen, W. Chen, and C.-P. Shen, QCD sum rule studies of $ss\bar{s}\bar{s}$ tetraquark states with $J^{PC} = 1^{+-}$, Eur. Phys. J. C **79**, 232 (2019).
[89] R.-R. Dong, N. Su, H.-X. Chen, E.-L. Cui, and Z.-Y. Zhou, QCD sum rule studies on the $ss\bar{s}\bar{s}$ tetraquark states of $J^{PC} = 0^{-+}$, Eur. Phys. J. C **80**, 749 (2020).
[90] R.-R. Dong, N. Su, and H.-X. Chen, Highly excited and exotic fully-strange tetraquark states, Eur. Phys. J. C **82**, 983 (2022).
[91] J. Z. Bai *et al.*, Observation of a near-threshold enhancement in the $p\bar{p}$ mass spectrum from radiative $J/\psi \to \gamma p\bar{p}$ decays, Phys. Rev. Lett. **91**, 022001 (2003).
[92] M. Ablikim *et al.*, Observation of a resonance $X(1835)$ in $J/\psi \to \gamma\pi^+\pi^-\eta'$, Phys. Rev. Lett. **95**, 262001 (2005).
[93] BESIII Collaboration, Observation of a $p\bar{p}$ mass threshold enhancement in $\psi' \to \pi^+\pi^- J/\psi(J/\psi \to \gamma p\bar{p})$ decay, Chin. Phys. C **34**, 421 (2010).
[94] M. Ablikim *et al.*, Confirmation of the $X(1835)$ and observation of the resonances $X(2120)$ and $X(2370)$ in $J/\psi \to \gamma\pi^+\pi^-\eta'$, Phys. Rev. Lett. **106**, 072002 (2011).
[95] M. Ablikim *et al.*, Observation of pseudoscalar and tensor resonances in $J/\psi \to \gamma\phi\phi$, Phys. Rev. D **93**, 112011 (2016).
[96] M. Ablikim *et al.*, Observation of $X(2370)$ and search for $X(2120)$ in $J/\psi \to \gamma K\bar{K}\eta'$, Eur. Phys. J. C **80**, 746 (2020).
[97] M. Ablikim *et al.*, Observation of a resonant structure in $e^+e^- \to K^+K^-\pi^0\pi^0$, Phys. Rev. Lett. **124**, 112001 (2020).
[98] W. Altmannshofer *et al.*, The Belle II physics book, Prog. Theor. Exp. Phys. **2019**, 123C01 (2019); **2020**, 029201(E) (2020).
[99] A. Austregesilo, Light-meson spectroscopy at GlueX, Int. J. Mod. Phys. Conf. Ser. **46**, 1860029 (2018).
[100] LHCb Collaboration, Observation of structure in the J/ψ-pair mass spectrum, Sci. Bull. **65**, 1983 (2020).
[101] G. Aad *et al.*, Observation of an excess of dicharmonium events in the four-muon final state with the ATLAS detector, Phys. Rev. Lett. **131**, 151902 (2023).
[102] A. Hayrapetyan *et al.*, Observation of new structure in the $J/\psi J/\psi$ mass spectrum in proton-proton collisions at $\sqrt{s} = 13$ TeV, arXiv:2306.07164.
[103] M. A. Shifman, A. I. Vainshtein, and V. I. Zakharov, QCD and resonance physics. Theoretical foundations, Nucl. Phys. **B147**, 385 (1979).
[104] L. J. Reinders, H. Rubinstein, and S. Yazaki, Hadron properties from QCD sum rules, Phys. Rep. **127**, 1 (1985).
[105] S. Narison, *QCD as a Theory of Hadrons: From Partons to Confinement* (Oxford University Press, New York, 2005), Vol. 17.
[106] M. Nielsen, F. S. Navarra, and S. H. Lee, New charmonium states in QCD sum rules: A concise review, Phys. Rep. **497**, 41 (2010).
[107] H.-X. Chen, W. Chen, X. Liu, T. G. Steele, and S.-L. Zhu, Towards exotic hidden-charm pentaquarks in QCD, Phys. Rev. Lett. **115**, 172001 (2015).
[108] H.-X. Chen, W. Chen, R.-R. Dong, and N. Su, $X_0(2900)$ and $X_1(2900)$: Hadronic molecules or compact tetraquarks, Chin. Phys. Lett. **37**, 101201 (2020).
[109] H.-X. Chen, W. Chen, X. Liu, and X.-H. Liu, Establishing the first hidden-charm pentaquark with strangeness, Eur. Phys. J. C **81**, 409 (2021).
[110] H.-X. Chen, Hadronic molecules in B decays, Phys. Rev. D **105**, 094003 (2022).
[111] K.-C. Yang, W. Y. P. Hwang, E. M. Henley, and L. S. Kisslinger, QCD sum rules and neutron-proton mass difference, Phys. Rev. D **47**, 3001 (1993).
[112] V. Gimenez, V. Lubicz, F. Mescia, V. Porretti, and J. Reyes, Operator product expansion and quark condensate from lattice QCD in coordinate space, Eur. Phys. J. C **41**, 535 (2005).
[113] M. Jamin, Flavor-symmetry breaking of the quark condensate and chiral corrections to the Gell-Mann-Oakes-Renner relation, Phys. Lett. B **538**, 71 (2002).
[114] B. L. Ioffe and K. N. Zyablyuk, Gluon condensate in charmonium sum rules with three-loop corrections, Eur. Phys. J. C **27**, 229 (2003).
[115] A. A. Ovchinnikov and A. A. Pivovarov, QCD sum rule calculation of the quark gluon condensate, Sov. J. Nucl. Phys. **48**, 721 (1988).
[116] J. R. Ellis, E. Gardi, M. Karliner, and M. A. Samuel, Renormalization-scheme dependence of Padé summation in QCD, Phys. Rev. D **54**, 6986 (1996).
[117] S. Narison, Gluon condensates and precise $\bar{m}_{c,b}$ from QCD-moments and their ratios to order α_s^3 and $\langle G^4 \rangle$, Phys. Lett. B **706**, 412 (2012).
[118] S. Narison, QCD parameter correlations from heavy quarkonia, Int. J. Mod. Phys. A **33**, 1850045 (2018); **33**, 1892004(A) (2018).