Excitation spectra of heavy baryons in a quark-diquark model with relativistic corrections

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(Received 8 August 2023; accepted 5 October 2023; published 7 November 2023)

The excitation spectra of Λ_c and Λ_b baryons are investigated by using a quark-diquark model in which a single-heavy baryon is treated as the bound state of a heavy quark and a scalar diquark. We take two types of relativistic corrections into account for the quark-diquark potential. In the first type, we consider the one-gluon exchange between the heavy quark and one of the light quarks in the diquark. In the second, we consider the one-gluon exchange between a scalar particle and a heavy quark. We find that there is a large difference between the two types of corrections due to different treatments of the internal color structure of the diquark. The relativistic corrections are important for the solution to the string tension puzzle, particularly, the Darwin term makes a large contribution.

DOI: 10.1103/PhysRevD.108.094011

I. INTRODUCTION

Understanding the structure of hadrons is one of the most important topics in hadron physics. It is too complex to describe the properties of hadrons in quantum chromodynamics (QCD) directly since QCD is nonperturbative at low energies, so identifying the effective degree of freedom for hadrons is essential. The constituent quark plays a key role as the effective degree of freedom inside hadron, and, a diquark which is a two-quark pair correlation may also be one [1–4]. This is particularly true of a diquark with an antisymmetric color, flavor, and spin, also known as a good diquark, as it has the most attractive correlation [5]. The structure of hadrons is investigated in the viewpoint of diquark, for example, for baryons [1,2,6–18] and light scalars [19–22].

Single-heavy baryons (called heavy baryons hereafter) particularly show promise for reaching the properties of diquarks due to the mass difference between the light quarks and the heavy quark [9–15,17]. The $\Lambda_c(2595)$ and $\Lambda_c(2625)$ baryons are the lowest excited states and can be interpreted as the excitation of the λ-mode which is the relative motion between the heavy quark and the center of mass of light quarks, because these baryon can be regarded as spin-orbit (LS) partners of the rotational excitation of the heavy quark [23]. With this speculation the light quark component can be considered as a good diquark. The important point here is that the diquark correlation works as an effective degree of freedom in the heavy baryon.

The mass spectra of heavy baryons have been investigated in quark-diquark models in many previous works. In Ref. [24], the mass spectra of Λ_c and Σ_c baryons with $J^P = 1/2^+$ were investigated by using a quark-diquark approach. It reported that the light-heavy diquark component may be as important as the light-light diquark component. In Ref. [17], the masses of the ground state of Λ, Λ_c, and Λ_b baryons were calculated in the QCD sum rule in which the diquark was introduced as an elementary field. They estimated the constituent diquark mass to 0.4 GeV and found that the QCD sum rule works well for these baryons. In Ref. [11], the excitation spectra of Λ_c and Λ_b baryons were calculated in a quark-diquark model with the Coulomb-plus-linear-type potential. The diquark there is assumed to be a pointlike good diquark, and the heavy baryons are treated as the bound systems of the heavy quark and diquark. The confinement potential may depend only on the color charge. If the diquark in the heavy baryon has the same color as the antiquark, it would be reasonable to use the same potential for both quark-antiquark and quark-diquark systems. However, Ref. [11] found the possibility that the confinement force between the quark and the diquark was about half of that in the quark-antiquark system in order to reproduce the experimental value of the Λ_c $1p$ excitation energy. In Ref. [13], this puzzle, which we call a string tension puzzle hereafter, was tackled by considering the diquark size for the calculation of the excitation spectra of the Λ_c baryon. According to this work, the size effect of the diquark

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reduces the excitation energy, and the diquark size $\rho \simeq 1.1$ fm reproduces the Λ_c $1p$ excitation energy of 0.33 GeV. However, the spin-dependent force is not included in the potential.

In this paper, we consider relativistic corrections to the quark-diquark potential as an alternative approach to this puzzle. Assuming that one heavy quark and light scalar diquark compose the heavy baryon, we consider two types of relativistic corrections for the quark-diquark potential. The diquark inside the heavy baryon is assumed to have anti-color $\bar{\mathbf{3}}$ and spin singlet $S = 0$. One is derived by considering a one-gluon exchange between the heavy quark and the light quarks in the diquark and the other by considering a one-gluon exchange between the quark and the scalar diquark. In the former approach, the relativistic corrections are calculated so that the diquark is composed of two light quarks, and in the latter, the diquark is assumed to be a scalar particle not having an internal color structure. Since the confinement force mainly depends on only the color charge and the good diquark has the same color charge as the antiquark in quarkonia, we may determine the parameters of the potential in the quark-diquark system by quarkonium spectra. We will see that the Darwin term plays an important role to improve the model calculation. We will also calculate the Ξ_c excitation spectra to check the consistency of this model.

This article is organized as follows. In Sec. II, we calculate two types of the one-gluon exchange potential to introduce the relativistic corrections. In Sec. III, we determine model parameters to reproduce the mass spectra of the charmonium, and show the excitation spectra of the Λ_c, Λ_b, and Ξ_c baryons calculated with those parameters. Section IV is devoted to the summary.

II. FORMULATION

We describe the heavy baryon in a quark-diquark model. The diquark is assumed as the pointlike good diquark which has antisymmetric color, flavor, and spin under the exchange of quarks in the diquark.

We introduce two types of relativistic corrections whose differences come from the consideration of the color structure of the scalar diquark. One is that the color structure of the scalar diquark is considered by treating the diquark as a pair of two fermions. The relativistic corrections are derived from the interaction between the heavy quark and one of the light quarks inside the scalar diquark. We call this potential q-Q type potential. The other is that the internal color structure of the scalar diquark is not considered and the diquark is just treated as a scalar particle with color $\bar{\mathbf{3}}$. The relativistic corrections are derived from the interaction between the scalar particle and the heavy quark. We refer to this potential S-Q type potential.

We calculate the matrix elements of the considering quark-diquark potential V for each partial wave to obtain the effective potential V_{eff} as

$$V_{\mathrm{eff}}(r) = \langle {}^{2S+1}L_J | V | {}^{2S+1}L_J \rangle \tag{1}$$

which is calculated below in the following subsections. We write the angular momentum state as $|{}^{2S+1}L_J\rangle$ with the total spin S and the total angular momentum J. Then, we can get the Schrödinger equation describing this system as

$$\left[-\frac{1}{2\mu}\frac{1}{r}\frac{d^2}{dr^2}r + V_{\mathrm{eff}}(r) + V_0 + \frac{L(L+1)}{2\mu r^2}\right]R(r) = ER(r). \tag{2}$$

Here $R(r)$ is the radial wave function, $\mu = \frac{m_d m_Q}{m_d + m_Q}$ is the reduced mass with the diquark mass m_d and the heavy quark mass m_Q, and V_0 is a constant. The total energy of the system is given by E. Since we are interested in the excitation spectra of heavy baryons, the constant V_0 is irrelevant to this analysis.

A. q-Q type potential

In this subsection, we derive q-Q type quark-diquark potential with the QCD Breit-Fermi potential. We construct the two-body potential for the quark-diquark system by summing the quark-quark interactions between one of the light quarks and the heavy quark, and taking the distance between the two light quarks to zero ($\rho \to 0$) as in Fig. 1, for the pointlike diquark.

We start from the quark-quark potential. The scattering amplitude M_{qq} with the one-gluon exchange between quark-1 and quark-2 is expressed as

$$-iM_{qq} = 4\pi\frac{2}{3}\alpha_s \bar{u}(\vec{p}_1')\gamma^\mu u(\vec{p}_1) D_{\mu\nu}(\vec{q})\bar{u}(\vec{p}_2')\gamma^\nu u(\vec{p}_2), \tag{3}$$

where α_s is the strong fine structure constant and $D_{\mu\nu}$ is the gluon propagator. Here we consider a quark pair with color $\bar{\mathbf{3}}$ and factor 2/3 in Eq. (3) is a color factor for the $\bar{\mathbf{3}}$ quark pair. The gluon three-momentum $\vec{q}$ is given by $\vec{q} = \vec{p}_1' - \vec{p}_1 = \vec{p}_2 - \vec{p}_2'$ with momenta $\vec{p}_1$ and $\vec{p}_1'$ of quark-1 in the initial and final states and ones $\vec{p}_2$ and $\vec{p}_2'$ of quark-2 in the initial and final states. The quark spinor is given as

$$u(\vec{p}) = \sqrt{\frac{E+m}{2E}}\begin{pmatrix}\chi \\ \frac{\vec{\sigma}\cdot\vec{p}}{E+m}\chi\end{pmatrix}, \tag{4}$$

where χ denotes a two component spinor and $\vec{\sigma}$ is the Pauli matrix. Here, the normalization factor is determined to satisfy the normalization $u^\dagger u = 1$, because the wave function in the Schrödinger equation is normalized in such a way [25–27].

We reduce the amplitude of Eq. (3) in the small momentum expansion up to $O(\frac{\vec{p}^2}{m^2})$ as

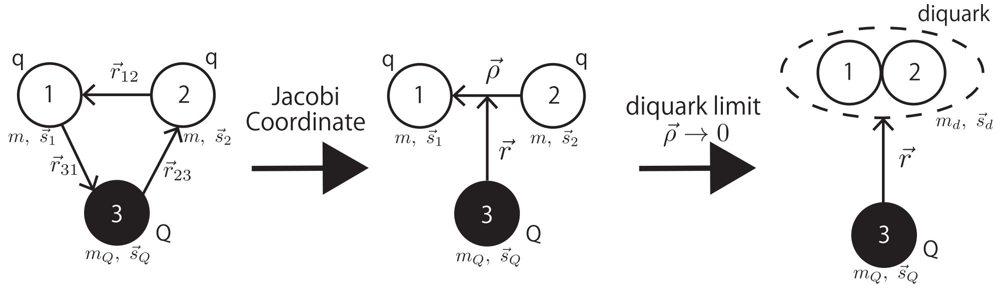

FIG. 1. Schematic diagrams of the heavy baryon in a quark-diquark model with the Jacobi coordinate. Quark-1 and quark-2 denote light quarks having masses m, and quark-3 denotes the heavy quark having mass m_Q. By taking the diquark limit $\vec{\rho} \to 0$, we treat the heavy baryon as the bound system of the diquark and heavy quark.

$$M_{qq} = 4\pi \frac{2\alpha_s}{3} \frac{-1}{\vec{q}^2} \chi'^{\dagger}_1 \chi^{\dagger}_1 \chi'^{\dagger}_2 \chi^{\dagger}_2 \Bigg[1 - \left(\frac{1}{8m_1^2} + \frac{1}{8m_2^2} \right) \vec{q}^2 + \frac{i\vec{\sigma}_1 \cdot (\vec{q} \times \vec{p}_1)}{4m_1^2} - \frac{i\vec{\sigma}_2 \cdot (\vec{q} \times \vec{p}_2)}{4m_2^2} - \frac{\vec{q}^2}{4m_1 m_2} \vec{\sigma}_1 \cdot \vec{\sigma}_2 + \frac{(\vec{\sigma}_1 \cdot \vec{q})(\vec{\sigma}_2 \cdot \vec{q})}{4m_1 m_2} - \frac{i\vec{\sigma}_1 \cdot (\vec{q} \times \vec{p}_2)}{2m_1 m_2} + \frac{i\vec{\sigma}_2 \cdot (\vec{q} \times \vec{p}_1)}{2m_1 m_2} - \frac{1}{m_1 m_2} \left(\vec{p}_1 \cdot \vec{p}_2 - \frac{(\vec{p}_1 \cdot \vec{q})(\vec{p}_2 \cdot \vec{q})}{\vec{q}^2} \right) \Bigg] \chi'_1 \chi_1 \chi'_2 \chi_2, \tag{5}$$

where m_1 and m_2 are masses of quark-1 and quark-2, respectively.

The quark-quark potential V_{qq} is defined by the Fourier transform of the scattering amplitude (5) as

$$V_{qq}(\vec{p}_1, \vec{p}_2, \vec{r}) = \int \frac{d^3 q}{(2\pi)^3} e^{-i\vec{r}\cdot\vec{q}} M_{qq}(\vec{p}_1, \vec{p}_2, \vec{q}), \tag{6}$$

where $\vec{r}$ is the relative coordinate between two quarks. Thus we obtain the quark-quark potential as

$$V_{qq}(\vec{p}_1, \vec{p}_2, \vec{r}) = -\frac{2}{3}\alpha_s \Bigg\{ \frac{1}{r} - \frac{8\pi}{3m_1 m_2} \vec{s}_1 \cdot \vec{s}_2 \delta^3(\vec{r}) - \frac{\pi}{2} \left(\frac{1}{m_1^2} + \frac{1}{m_2^2} \right) \delta^3(\vec{r}) - \frac{1}{m_1 m_2} \frac{1}{r^3} \left[\frac{3(\vec{s}_1 \cdot \vec{r})(\vec{s}_2 \cdot \vec{r})}{r^2} - \vec{s}_1 \cdot \vec{s}_2 \right] + \frac{1}{r^3} \left[\frac{\vec{s}_1 \cdot (\vec{r} \times \vec{p}_1)}{2m_1^2} - \frac{\vec{s}_2 \cdot (\vec{r} \times \vec{p}_2)}{2m_2^2} + \frac{\vec{s}_1 \cdot (\vec{r} \times \vec{p}_2) - \vec{s}_2 \cdot (\vec{r} \times \vec{p}_1)}{m_1 m_2} \right] - \frac{1}{2m_1 m_2} \frac{1}{r} \left(\vec{p}_1 \cdot \vec{p}_2 + \frac{(\vec{r} \cdot \vec{p}_1)(\vec{r} \cdot \vec{p}_2)}{r^2} \right) \Bigg\}. \tag{7}$$

Summing all interactions in the considering system, we obtain the so-called QCD Breit-Fermi potential.

Now, we derive the two-body potential for the quark-diquark system from the interactions between the heavy quark and the light quarks. The spacial coordinate of the center of mass $\vec{R}$ is written as

$$\vec{R} = \frac{m(\vec{r}_1 + \vec{r}_2) + m_Q \vec{r}_3}{2m + m_Q}. \tag{8}$$

Here, m is the light quark mass and m_Q is the heavy quark mass. The relative coordinate $\vec{\rho}$ between light quarks and the relative coordinate $\vec{r}$ between a heavy quark and the center of mass of light quarks are written as

$$\vec{\rho} = \vec{r}_1 - \vec{r}_2, \tag{9}$$

$$\vec{r} = \frac{1}{2}(\vec{r}_1 + \vec{r}_2) - \vec{r}_3. \tag{10}$$

After considering the interaction between the light quark and the heavy quark, we take the diquark limit $\vec{\rho} \to 0$.

The total spin $\vec{S}$ of the system is obtained by the sum of the heavy quark spin $\vec{s}_Q$ and the diquark spin $\vec{s}_d$ as

$$\vec{S} = \vec{s}_d + \vec{s}_Q, \tag{11}$$

where diquark spin is given by the sum of spins of two light quarks $\vec{s}_1$ and $\vec{s}_2$ as

$$\vec{s}_d = \vec{s}_1 + \vec{s}_2. \tag{12}$$

Since we consider a good diquark, the diquark spin is $\vec{s}_d = 0$. The total angular momentum $\vec{J}$ is written as

$$\vec{J} = \vec{L} + \vec{S}, \tag{13}$$

where the relative orbital angular momentum between a heavy quark and a diquark is $\vec{L}$. We can get the quark-diquark potential by the sum of the confinement part V_{conf} and the nonconfinement part obtained from the QCD Breit-Ferim potential. In the following, we calculate each term of the effective q-Q type potential $V_{Qd}^{(qQ)}$ separately. Here we allow to take an individual coupling constant for each term.

We first calculate the confinement term of the total potential. We consider a liner type confinement potential for a quark pair, which is given as $\frac{1}{2}kr$ with the string tension k and the distance between two quarks. The string tension is determined by the quark-antiquark system and factor $1/2$ is the relative color factor between the quark-quark and quark-antiquark systems. The confinement term is written as

$$\frac{1}{2}k\{|\vec{r}_{23}| + |\vec{r}_{31}|\} = k|\vec{r}| \tag{14}$$

by taking the diquark limit $\vec{\rho} \to 0$, that is, setting the distance between two light quarks to zero. Thus, the confinement potential V_{conf} is

$$V_{\mathrm{conf}} = kr. \tag{15}$$

The Coulomb term is written as

$$-\frac{2}{3}\alpha_{\mathrm{Coul}}\left\{\frac{1}{|\vec{r}_{23}|} + \frac{1}{|\vec{r}_{31}|}\right\} = -\frac{4}{3}\frac{\alpha_{\mathrm{Coul}}}{|\vec{r}|} \tag{16}$$

after taking the diquark limit $\vec{\rho} \to 0$. The effective Coulomb term V_{Coul} can be calculated as

$$V_{\mathrm{Coul}} = -\frac{4}{3}\frac{\alpha_{\mathrm{Coul}}}{r}. \tag{17}$$

We write the Darwin term for the effective q-Q potential by taking the diquark limit $\vec{\rho} \to 0$ as

$$\frac{2}{3}\alpha_{\mathrm{Dar}}\frac{\pi}{2}\left\{\delta^3(\vec{r}_{23})\left(\frac{1}{m^2} + \frac{1}{m_Q^2}\right) + \delta^3(\vec{r}_{31})\left(\frac{1}{m_Q^2} + \frac{1}{m^2}\right)\right\} = \frac{2\pi}{3}\alpha_{\mathrm{Dar}}\delta^3(\vec{r})\left(\frac{4}{m_d^2} + \frac{1}{m_Q^2}\right), \tag{18}$$

where we have assumed that the light quark mass is given by a half of the diquark mass m_d. The expectation value of the delta function can be evaluated by the value of the wave function at the origin as

$$\int d^3r\psi^*(\vec{r})\delta^3(\vec{r})\psi(\vec{r}) = \frac{2L+1}{4\pi}|R_{nL}(0)|^2. \tag{19}$$

Thus, we obtain the effective Darwin term V_{Dar} as

$$V_{\mathrm{Dar}} = \frac{2\pi}{3}\alpha_{\mathrm{Dar}}\left(\frac{1}{m_Q^2} + \frac{4}{m_d^2}\right)\frac{2L+1}{4\pi r^2}\delta(r). \tag{20}$$

Later we will regularize the delta function to reduce the singularity at the origin, and thus the Darwin term can contribute also to higher partial waves but its contributions are highly suppressed by the centrifugal barrier.

The hyperfine interaction is written as

$$\frac{1}{2}\frac{32\pi}{9}\frac{\alpha_{\mathrm{Hyp}}}{mm_Q}\{\delta^3(\vec{r}_{23})\vec{s}_2\cdot\vec{s}_Q + \delta^3(\vec{r}_{31})\vec{s}_Q\cdot\vec{s}_1\} = \frac{32\pi}{9}\alpha_{\mathrm{Hyp}}\frac{1}{m_d m_Q}\delta^3(\vec{r})\vec{s}_d\cdot\vec{s}_Q. \tag{21}$$

by taking the diquark limit $\vec{\rho} \to 0$. Since the good diquark has the spin $\vec{s}_d = 0$, the effective hyperfine interaction term V_{Hyp} for the scalar diquark is

$$V_{\mathrm{Hyp}} = 0. \tag{22}$$

The spin-orbit interaction of the q-Q type potential is expressed as

$$\begin{aligned}&\frac{1}{2}\frac{4}{3}\alpha_{\mathrm{LS}}\left\{\frac{1}{|\vec{r}_{23}|^3}\left[\left(\frac{1}{4m^2} + \frac{1}{4m_Q^2} + \frac{1}{mm_Q}\right)\vec{L}_{2Q}\cdot(\vec{s}_2 + \vec{s}_Q)\right.\right.\\&\left.+\left(\frac{1}{4m^2} - \frac{1}{4m_Q^2}\right)\vec{L}_{2Q}\cdot(\vec{s}_2 - \vec{s}_Q)\right]\\&+\frac{1}{|\vec{r}_{31}|^3}\left[\left(\frac{1}{4m_Q^2} + \frac{1}{4m^2} + \frac{1}{mm_Q}\right)\vec{L}_{Q1}\cdot(\vec{s}_Q + \vec{s}_1)\right.\\&\left.\left.+\left(\frac{1}{4m_Q^2} - \frac{1}{4m^2}\right)\vec{L}_{Q1}\cdot(\vec{s}_Q - \vec{s}_1)\right]\right\}\\&= \frac{4}{3}\frac{\alpha_{\mathrm{LS}}}{|\vec{r}|^3}\frac{m_d + m_Q}{m_d + 2m_Q}\left(\frac{1}{2m_Q^2} + \frac{4}{m_d m_Q}\right)\vec{L}\cdot\vec{s}_Q\end{aligned} \tag{23}$$

by taking the diquark limit $\vec{\rho} \to 0$, where the relative orbital angular momenta between quark-1 and the heavy quark and quark-2 and the heavy quark are written as $\vec{L}_{Q1}$ and $\vec{L}_{2Q}$, respectively. In the case of the pointlike diquark $\vec{\rho} \to 0$, they can be written as

$$\vec{L}_{Q1} = \vec{L}_{2Q} = \frac{m_d + m_Q}{m_d + 2m_Q}\vec{L}$$

with the relative orbital angular momentum $\vec{L}$ between the diquark and the heavy quark. The good diquark spin $\vec{s}_d = \vec{s}_1 + \vec{s}_2$ is zero. Thus integrating over a solid angle, we obtain the LS term V_{LS} of the effective potential as

$$V_{\mathrm{LS}} = \frac{4}{3}\frac{\alpha_{\mathrm{LS}}}{r^3}\frac{m_d + m_Q}{m_d + 2m_Q}\left(\frac{1}{2m_Q^2} + \frac{2}{m_d m_Q}\right) \times \frac{1}{2}\left(J(J+1) - L(L+1) - \frac{3}{4}\right). \tag{24}$$

The tensor term is obtained as

$$\begin{aligned} &\frac{1}{2}\left\{\frac{4}{3mm_Q}\frac{\alpha_{\mathrm{tens}}}{|\vec{r}_{23}|}\left(\frac{3(\vec{s}_2\cdot\vec{r}_{23})(\vec{s}_Q\cdot\vec{r}_{23})}{|\vec{r}_{23}|^2} - \vec{s}_2\cdot\vec{s}_Q\right) + \frac{4}{3mm_Q}\frac{\alpha_{\mathrm{tens}}}{|\vec{r}_{31}|}\left(\frac{3(\vec{s}_Q\cdot\vec{r}_{31})(\vec{s}_1\cdot\vec{r}_{31})}{|\vec{r}_{31}|^2} - \vec{s}_Q\cdot\vec{s}_1\right)\right\} \\ &= \frac{4}{3}\frac{\alpha_{\mathrm{tens}}}{|\vec{r}|^3}\frac{1}{m_d m_Q}\left(\frac{3(\vec{s}_d\cdot\vec{r})(\vec{s}_Q\cdot\vec{r})}{|\vec{r}|^2} + \vec{s}_d\cdot\vec{s}_Q\right) \end{aligned} \tag{25}$$

by taking the diquark limit $\vec{\rho}\to 0$ and replacing the quark mass to the diquak mass. Since the diquark spin is $\vec{s}_d = 0$, the part of the bracket in the final line vanishes. Therefore the effective tensor term V_{tens} is

$$V_{\mathrm{tens}} = 0. \tag{26}$$

The orbit-orbit interaction on the q-Q type potential is expressed as

$$\begin{aligned} &\frac{1}{2}\left\{\frac{2}{3mm_Q}\frac{\alpha_{\mathrm{oo}}}{|\vec{r}_{23}|}\left(\vec{p}_2\cdot\vec{p}_Q + \frac{(\vec{p}_2\cdot\vec{r}_{23})(\vec{p}_Q\cdot\vec{r}_{23})}{|\vec{r}_{23}|^2}\right) + \frac{2}{3mm_Q}\frac{\alpha_{\mathrm{oo}}}{|\vec{r}_{31}|}\left(\vec{p}_Q\cdot\vec{p}_1 + \frac{(\vec{p}_Q\cdot\vec{r}_{31})(\vec{p}_1\cdot\vec{r}_{31})}{|\vec{r}_{31}|^2}\right)\right\} \\ &= \frac{2}{3}\frac{1}{m_d m_Q}\frac{\alpha_{\mathrm{oo}}}{|\vec{r}|}\left[(\vec{p}_1+\vec{p}_2)\cdot\vec{p}_Q + \frac{((\vec{p}_1+\vec{p}_2)\cdot\vec{r})(\vec{p}_Q\cdot\vec{r})}{|\vec{r}|^2}\right] \end{aligned} \tag{27}$$

by taking the diquark limit $\vec{\rho}\to 0$ for the above expression and replacing the mass of the light quark to one of the diquark. The momentum of each quark can be rewritten with the total momentum $\vec{P}$ and the relative momentum $\vec{p}$ as

$$\vec{p}_1 + \vec{p}_2 = \frac{m_d}{m_d + m_Q}\vec{P} + \vec{p}$$

and

$$\vec{p}_Q = \frac{m_Q}{m_d + m_Q}\vec{P} - \vec{p}.$$

Thus, the orbit-orbit interaction V_{oo} is obtained as

$$V_{\mathrm{oo}} = -\frac{2}{3}\frac{1}{m_d m_Q}\frac{\alpha_{\mathrm{oo}}}{r}\left(-\frac{2}{r}\frac{d^2}{dr^2}r + \frac{L(L+1)}{r^2}\right). \tag{28}$$

Collecting the above results, we have the quark-diquark potential derived by the QCD Breit-Fermi potential as

$$V_{Qd}^{(qQ)} = V_{\mathrm{conf}} + V_{\mathrm{Coul}} + V_{\mathrm{Dar}} + V_{\mathrm{Hyp}} + V_{\mathrm{LS}} + V_{\mathrm{tens}} + V_{\mathrm{oo}}. \tag{29}$$

We regularize the singularities at the origin and obtain the q-Q type effective potential $V_{Qd}^{(qQ)}$ as

$$V_{Qd}^{(qQ)}(r) = kr - \frac{4}{3}\frac{\alpha_{\mathrm{Coul}}}{r} + \frac{2\pi}{3}\alpha_{\mathrm{Dar}}\left(\frac{1}{m_Q^2} + \frac{4}{m_d^2}\right)\frac{2L+1}{4\pi r^2}\Lambda e^{-\Lambda^2 r^2} + \frac{4}{3}\frac{\alpha_{\mathrm{LS}}}{r^3}\frac{m_d + m_Q}{m_d + 2m_Q}\left(\frac{1}{2m_Q^2} + \frac{2}{m_d m_Q}\right) \times \frac{(1-e^{-\Lambda r})^2}{2}\left(J(J+1) - L(L+1) - \frac{3}{4}\right) - \frac{2}{3}\frac{(1-e^{-\Lambda r})^2}{m_d m_Q}\frac{\alpha_{\mathrm{oo}}}{r}\left(-\frac{2}{r}\frac{d^2}{dr^2}r + \frac{L(L+1)}{r^2}\right) \tag{30}$$

with the regularization parameter Λ.

For later use, we also show the corresponding effective potential for the quark-antiquark $V_{Q\bar{Q}}$. According to the color factor the quark-antiquark interaction is twice as strong as the quark-quark potential. To this end, we take the factor 1 instead of 1/2 in Eq. (7). Each term of the regularized effective quark-antiquark potential for the heavy quark with mass m_Q is expressed as

$$V_{Q\bar{Q}} = V_{\mathrm{conf}} + V_{\mathrm{Coul}} + V_{\mathrm{Dar}} + V_{\mathrm{Hyp}} + V_{\mathrm{LS}} + V_{\mathrm{tens}} + V_{\mathrm{oo}} \tag{31}$$

with

$$V_{\mathrm{conf}} = kr, \tag{32}$$

$$V_{\mathrm{Coul}} = -\frac{4}{3}\frac{\alpha_{\mathrm{Coul}}}{r}, \tag{33}$$

$$V_{\mathrm{Dar}} = \frac{2\pi}{3}\alpha_{\mathrm{Dar}}\frac{2}{m_Q^2}\frac{2L+1}{4\pi r^2}\Lambda e^{-\Lambda^2 r^2}, \tag{34}$$

$$V_{\rm Hyp} = \frac{32\pi}{9m_Q^2}\alpha_{\rm Hyp}\frac{2L+1}{4\pi r^2}\Lambda e^{-\Lambda^2 r^2}\frac{1}{2}\left(S(S+1)-\frac{3}{2}\right), \quad (35)$$

$$V_{\rm LS} = \frac{4}{3}\frac{\alpha_{\rm LS}}{r^3}\frac{3}{2m_Q^2}(1-e^{-\Lambda r})^2\frac{1}{2}(J(J+1)-L(L+1) - S(S+1)), \quad (36)$$

$$V_{\rm tens} = \frac{4}{3m_Q^2}\frac{\alpha_{\rm tens}}{r^3}(1-e^{-\Lambda r})^2\alpha_{S,J}, \quad (37)$$

and

$$V_{\rm oo} = -\frac{2}{3m_Q^2}\frac{\alpha_{\rm oo}}{r}\left[-\frac{2}{r}\frac{d^2}{dr^2}r+\frac{L(L+1)}{r^2}\right](1-e^{-\Lambda r})^2, \quad (38)$$

where Λ is the regularization cutoff. The coefficient $\alpha_{S,J}$ appearing in Eq. (37) is listed in

$$\begin{aligned}\alpha_{0,J} &= 0,\\ \alpha_{1,L-1} &= -\frac{L+1}{2L-1},\\ \alpha_{1,L} &= 1,\\ \alpha_{1,L+1} &= -\frac{L}{2L+1}.\end{aligned} \quad (39)$$

The total spin $\vec{S}$ is the sum of heavy quark spins as $\vec{S} = \vec{s}_1 + \vec{s}_2$, and the orbital angular momentum is written as $\vec{L}$, so the total angular momentum is $\vec{J} = \vec{S} + \vec{L}$.

B. *S-Q* type potential

In this subsection, we derive the quark-diquark potential by treating the diquark as the scalar particle with color $\bar{\mathbf{3}}$. We consider the one-gluon exchange potential between the scalar particle and the remaining heavy quark [6]. The color structure of the diquark is not considered in this model, which is the difference from the *q-Q* type potential.

We call the obtained quark-diquark potential *S-Q* type potential. Since we assume that the scalar particle have anticolor $\bar{\mathbf{3}}$, we take the strength of the coupling constant for the scalar-quark system to be the same as the quark-antiquark system. Here we use again the Breit equation [25] to derive the effective interaction between the diquark and the heavy quark.

The scattering amplitude M_{SQ} for the system of a scalar diquark and a heavy quark is expressed as

$$-iM_{SQ} = 4\pi\frac{4}{3}\alpha_s\frac{(p_1'+p_1)^\mu}{\sqrt{2E_1'2E_1}}D_{\mu\nu}\bar{u}(\vec{p}_2')\gamma^\nu u(\vec{p}_2) \quad (40)$$

with the momenta $\vec{p}_1$ and $\vec{p}_1'$ of the scalar diquark in the initial and final states and those $\vec{p}_2$ and $\vec{p}_2'$ of the heavy quark in the initial and final states. E_1 and E_1' denote the energies of the scalar diquark in the initial and final states, respectively. Factor 4/3 in Eq. (40) is the color factor of one gluon exchange for the quark and $\bar{\mathbf{3}}$ diquark. The factor $\frac{1}{\sqrt{2E_1'2E_1}}$ is adapted as the normalization factor so that the time component of its current is unity. In fact, this causes the different expressions from the quark-diquark potential derived in Ref. [6]. We discuss this difference and importance in detail after completing this derivation.

We calculate the scattering amplitude in the nonrelativistic expansion as

$$\begin{aligned}M_{SQ} = 4\pi\frac{4\alpha_s}{3}\frac{-1}{\vec{q}^2}\chi'^\dagger\Bigg[&1-\frac{1}{8m_Q^2}\vec{q}^2 - \frac{i\vec{\sigma}\cdot(\vec{q}\times\vec{p}_2)}{4m_Q^2}+\frac{i\vec{\sigma}\cdot(\vec{q}\times\vec{p}_1)}{2m_d m_Q}\\ &-\frac{1}{m_d m_Q}\left(\vec{p}_1\cdot\vec{p}_2-\frac{(\vec{p}_1\cdot\vec{q})(\vec{p}_2\cdot\vec{q})}{\vec{q}^2}\right)\Bigg]\chi.\end{aligned} \quad (41)$$

Performing Fourier transform of the scattering amplitude

$$V_{SQ}(\vec{p}_1,\vec{p}_2,\vec{r}) = \int\frac{d^3q}{(2\pi)^3}e^{-i\vec{r}\cdot\vec{q}}M_{SQ}(\vec{p}_1,\vec{p}_2,\vec{q}), \quad (42)$$

we obtain the quark-diquark potential as

$$\begin{aligned}V_{SQ}(\vec{p},\vec{r}) = -\frac{4}{3}\alpha_s\Bigg[&\frac{1}{r}-4\pi\frac{1}{8m_Q^2}\delta^3(\vec{r}) - \frac{1}{2m_d m_Q r}\left(\vec{p}^2+\frac{(\vec{p}\cdot\vec{r})^2}{r^2}\right)\\ &-\frac{1}{r^3}\left(\frac{1}{2m_Q^2}+\frac{1}{m_d m_Q}\right)\vec{s}_Q\cdot\vec{L}\Bigg]\end{aligned} \quad (43)$$

with the relative momentum $\vec{p} = \vec{p}_1 = -\vec{p}_2$, the spin of the heavy quark $\vec{s}_Q$, and the relative orbital angular momentum $\vec{L}$ between the scalar particle and the heavy quark.

The *S-Q* type quark-diquark potential is given by the sum of the confinement term $V_{\rm conf}$ and the nonconfinement terms obtained from one-gluon exchange between the scalar particle and quark as

$$V_{Qd}^{(SQ)} = V_{\rm conf} + V_{\rm Coul} + V_{\rm Dar} + V_{\rm LS} + V_{\rm oo}. \quad (44)$$

The difference from the *q-Q* type potential is only the dependence of mass of the diquark. Thus the matrix elements can be obtained in the same way as the *q-Q* type potential with different coefficients. We obtain the effective potential as follows.

The effective confinement term $V_{\rm conf}$ can be obtained as

$$V_{\rm conf} = kr. \quad (45)$$

The effective Coulomb term V_{Coul} is

$$V_{\mathrm{Coul}} = -\frac{4}{3}\frac{\alpha_{\mathrm{Coul}}}{r}. \tag{46}$$

The effective Darwin term V_{Dar} is

$$V_{\mathrm{Dar}} = \frac{2\pi}{3}\alpha_{\mathrm{Dar}}\frac{1}{m_Q^2}\frac{2L+1}{4\pi r^2}\delta(r). \tag{47}$$

The effective spin-orbit interaction V_{LS} is

$$V_{\mathrm{LS}} = -\frac{4}{3}\frac{\alpha_{\mathrm{LS}}}{r^3}\left(\frac{1}{2m_Q^2}+\frac{1}{m_d m_Q}\right)\frac{1}{2}(J(J+1) - L(L+1) - s_Q(s_Q+1)). \tag{48}$$

The effective orbit-orbit interaction V_{oo} is

$$V_{\mathrm{oo}} = -\frac{4}{3}\frac{\alpha_{\mathrm{oo}}}{r}\frac{1}{2m_d m_Q}\left(-\frac{2}{r}\frac{d^2}{dr^2}r + \frac{L(L+1)}{r^2}\right). \tag{49}$$

Thus the regularized S-Q type effective potential $V_{Qd}^{(SQ)}$ is

$$\begin{aligned} V_{Qd}^{(SQ)}(r) = {} & kr - \frac{4}{3}\frac{\alpha_{\mathrm{Coul}}}{r} + \frac{16\pi}{3}\frac{\alpha_{\mathrm{Dar}}}{8m_Q^2}\frac{2L+1}{4\pi r^2}\Lambda e^{-\Lambda^2 r^2} \\ & + \frac{4}{3}\frac{\alpha_{\mathrm{LS}}}{r^3}\left(\frac{1}{2m_Q^2}+\frac{1}{m_d m_Q}\right)\frac{(1-e^{-\Lambda r})^2}{2}(J(J+1) \\ & - L(L+1) - s_Q(s_Q+1)) \\ & - \frac{4}{3}\frac{\alpha_{\mathrm{oo}}}{r}\frac{(1-e^{-\Lambda r})^2}{2m_d m_Q}\left(-\frac{2}{r}\frac{d^2}{dr^2}r + \frac{L(L+1)}{r^2}\right). \end{aligned} \tag{50}$$

The confinement term, Coulomb term, and the orbit-orbit interaction have the completely same forms as those on the q-Q type potential. The spin-orbit interaction and the Darwin term have different coefficients from those of the q-Q type potential.

Now, we discuss the Darwin term. The Darwin term in the S-Q type potential does not depends on the diquark mass m_d as in Eq. (47). This result for a scalar particle is consistent in Ref. [28]. According to Ref. [28], the Darwin term appears from the order of m^{-3} for a scalar particle. On the contrary, the potential between the scalar particle and the quark in Ref. [6] has the Darwin term containing the diquark mass in this order. Their scalar particle current is normalized so that the time component of its current is $2E$. Here we need to normalize the wave function of the scalar particle to satisfy that the time component of its current is unity because this is the way to normalize the wave function in Schrödinger equation. The orbit-orbit interaction in our result also has different form from that in Ref. [6] because of the difference of the normalization, and there are some missing terms.

III. NUMERICAL RESULTS

In this section, we discuss our results of the excitation energy spectra of quarkonia and heavy baryons. We determine the potential parameters to reproduce the experimental spectra of charmonium. After that, we show theoretical results of the Λ_c and Λ_b baryons calculated by using the q-Q type and S-Q type potentials. We also calculate the excitation spectrum of the Ξ_c baryon by using the q-Q type potential.

A. Model parameters

The diquark inside a heavy baryon has the same color as the antiquark inside a quarkonium. Since the confinement force depends only on the color charge and not on the flavor in the first approximation, it is reasonable to use the same potential for the quark-diquark system as for the quark-antiquark system. We determine the potential parameters to reproduce the experimental spectrum of charmonium.

The parameters appearing in the potential are the string tension k in the confinement part, the regularization parameter Λ, the fine structure constants α_s in the nonconfinement part, and the masses of the heavy quark and the diquark. The string tension k is fixed as $k = 0.9$ GeV/fm, which reproduces the global excitation spectra of the charmonium and bottonium [11], and the charm quark mass is set to be $m_c = 1.5$ GeV. We consider two parameter sets. In parameter set 1, we assume a common coupling constant α_s in the nonconfining potential. We determine α_s and Λ from the charmonium spectrum. In parameter set 2, we allow fine-tuning of the spectrum by introducing individual coupling constants for the nonconfinement terms. The coupling constants, $\alpha_{\mathrm{Coul}}, \alpha_{\mathrm{Dar}}, \alpha_{\mathrm{Hyp}}, \alpha_{\mathrm{LS}}, \alpha_{\mathrm{tens}}$, and α_{oo}, are determined together with Λ by the charmonium spectrum. The determined values are listed in Table I.

We show the excitation energy spectrum of charmonium calculated with the determined parameters in Fig. 2, where the excitation energies are measured from the lowest state. We also show the excitation energy spectrum of bottomonium obtained with the same parameter sets for comparison. The bottom quark mass is set as $m_b = 4.0$ GeV. As we

TABLE I. Parameter sets of the potentials determined to reproduce the experimental spectrum of the charmonium. For parameter set 1, the nonconfinement terms have a single parameter α_s. Parameter set 2 introduces individual coupling constants for the nonconfinement terms: $\alpha_{\mathrm{Coul}}, \alpha_{\mathrm{Hyp}}, \alpha_{\mathrm{Dar}}, \alpha_{\mathrm{tens}}, \alpha_{\mathrm{LS}}$ and α_{oo}.

	α_{Coul}	α_{Hyp}	α_{Dar}	α_{tens}	α_{LS}	α_{oo}	k [GeV/fm]	Λ [fm^{-1}]
Number 1			0.37				0.9	8.1
Number 2	0.40	0.32	0.22	0.54	0.46	0.65	0.9	6.4

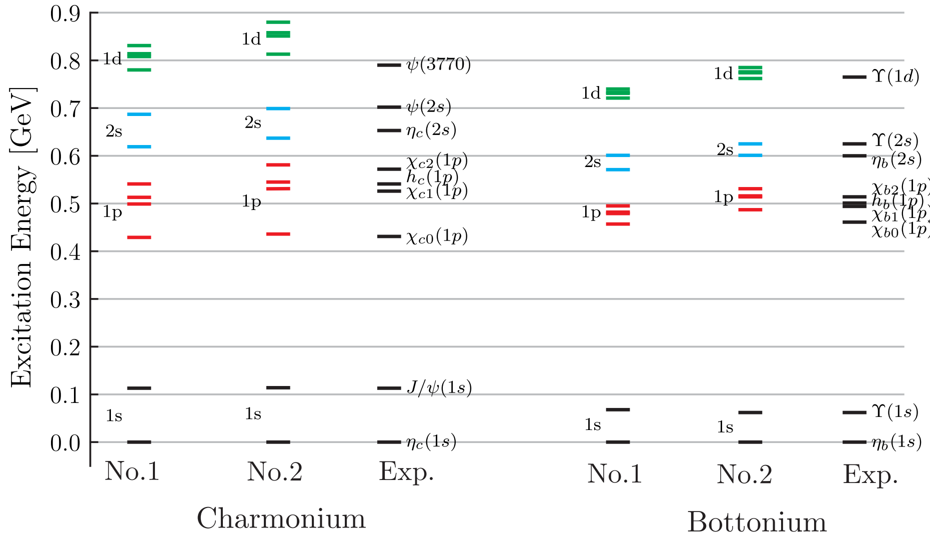

FIG. 2. Excitation spectra of the charmonium and bottomonium measured from the ground states. The masses of the charm bottom quarks are with $m_c = 1.5$ GeV, and $m_b = 4.0$ GeV, respectively. The parameters are determined to reproduce the experimental data of the charmonium. Two parameter sets are considered as shown in Table I. The experimental data are taken from the Particle Data Group [29].

can see, these parameters also reproduce the bottomonium spectrum well. The spectra calculated with parameter set 2 reproduce the experimental values better than those with parameter set 1.

B. Excitation spectra of Λ_c and Λ_b with q-Q type potential

In this subsection, we calculate the excitation spectra of the Λ_c and Λ_b baryons by using the q-Q type potential. In this article, we assume that the mass of the nonstrange scalar diquark is $m_d = 0.5$ GeV.

Figure 3 shows the excitation spectra of the Λ_c and Λ_b baryons calculated with the q-Q type potential by using the parameter sets in Table I. As we can see, the $1p$ excitation energies and LS splitting are reproduced the experimental data better than the previous work [11] but they are slightly overestimated. The higher excitation energies are not reproduced. The $\Lambda_c 1p$ excitation energies for parameter set 1 reproduce the experimental data better than those for parameter set 2, in contradiction to the results for the charmonium. This implies that it is hard to reproduce both spectra of the quarkonia and heavy hadrons by common parameters. In Ref. [11], the calculated excitation energy of

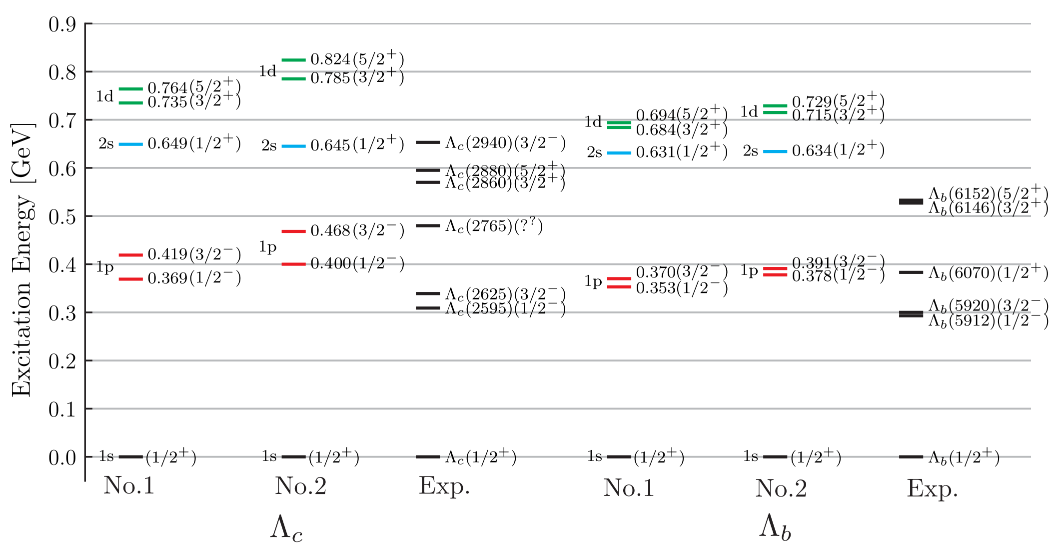

FIG. 3. Calculated spectra with the q-Q type potential for Λ_c and Λ_b systems. The parameter sets shown in Table I are used. The experimental data are taken from the Particle Data Group [29].

the $\Lambda_c 1p$ state was overestimated to be about 140 MeV if one uses the parameters $\alpha_s = 0.4$ and $k = 0.9$ GeV/fm which is determined so as to reproduce the quarkonia spectra. In contrast, in our work, the energy difference between the experimental and theoretical values is about 70–115 MeV. The calculated $\Lambda_c 1p$ excitation energies are improved roughly 20% or more by taking the relativistic corrections in the q-Q type potential model. The relativistic correction is significant for the heavy baryon spectra.

Next we discuss the LS splitting. We find that the LS splitting is slightly overestimated comparing the experimental data. We find that if we change the value of the parameter α_{LS} of the LS term, the LS splitting of the Λ_c baryon can be fitted simultaneously to the experimental data but $1p$ excitation energies cannot be done. Both of them can be fitted to the experimental data only by changing the strength of the string tension k, which is a similar situation with Ref. [11]. In Fig. 4, we show the excitation spectra of the Λ_c and Λ_b baryons calculated with the q-Q type potential by using parameters given in Table I but with different string tensions. The string tension is redetermined to reproduce the spin-weighted average of the excitation energies of the $\Lambda_c 1p$ states, 0.330 GeV. For parameter set 1, the string tension is found to be $k = 0.63$ GeV/fm, and for parameter set 2, it is to be $k = 0.50$ GeV/fm. As we can see, the LS splitting of the Λ_c baryon matches the experimental data as well as the $1p$ excitation energies. The string tension has to be reduced by 30–40% to reproduce the experimental data of both $1p$ excitation energies and the LS splitting quite well, although its reduction is smaller than that invented in Ref. [11].

Here, let us move on the discussion for other excitation states. In contrast to the $1p$ excitation states, higher excitation states do not match the experimental data and they are overestimated considerably. Many studies have found that the theoretical spectra of $\Lambda_c(2765)$ are overestimated as the $\Lambda_c 2s$ state against the experimental observation, which is suggested to appear as a Roper-like state. Similarly, our result for the $\Lambda_c 2s$ state is about 1.4 times larger than the experimental data, which is consistent with those works. As for the $\Lambda_c(2880)$, Ref. [30] evaluated the decay widths of charmed baryons from one-pion emission and found that the diquark inside $\Lambda_c(2880)$ potentially having spin 1 can explain the small decay ratio between $\Lambda_c^* \to \Sigma_c^*(2520)\pi$ and $\Lambda_c^* \to \Sigma_c(2455)\pi$. This model also implies that the $\Lambda_c(2880)$ does not have the spin 0 diquark.

Next we discuss the effect of each term of the potentials for the charmonium spectra and Λ_c spectra. Figure 5 shows the excitation spectra of the charmonium and Λ_c baryon calculated with the potentials as

$$
\begin{aligned}
&\text{(i)}\ V = V_{\mathrm{conf}} + V_{\mathrm{Coul}}\\
&\text{(ii)}\ V = V_{\mathrm{conf}} + V_{\mathrm{Coul}} + V_{\mathrm{Dar}}\\
&\text{(iii)}\ V = V_{\mathrm{conf}} + V_{\mathrm{Coul}} + V_{\mathrm{oo}}\\
&\text{(iv)}\ V = V_{\mathrm{conf}} + V_{\mathrm{Coul}} + V_{\mathrm{Dar}} + V_{\mathrm{oo}}
\end{aligned}
\tag{51}
$$

by using the $q - Q$ type potential for the Λ_c baryon. Here, we fix the parameters as $\alpha_s = 0.4$, $k = 0.9$ GeV/fm, and $\Lambda = 3.5$ fm^{-1}. The values of parameters α_s and k are used in Ref. [11] and the value of Λ is used in Ref. [23]. The common parameter in the nonconfinement part α_s is utilized to examine the effect of the potential term. We find that the effect of relativistic corrections for the charmonium spectra is rather small. From Fig. 5 (i) and (ii) on the charmonium

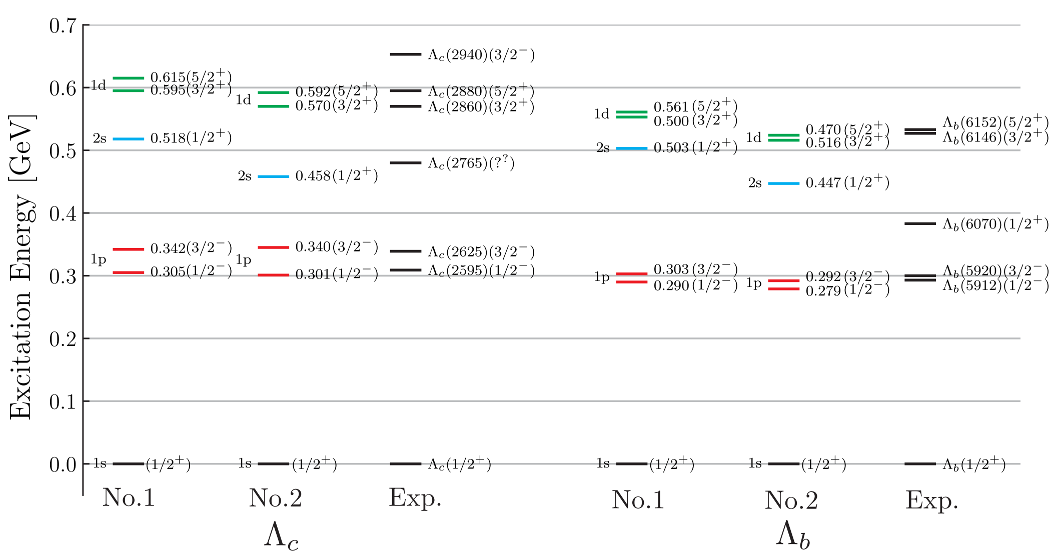

FIG. 4. Calculated spectra with $q - Q$ type potential for the Λ_c and Λ_b baryon. The string tensions are redetermined to reproduce the $\Lambda_c 1p$ excitation energies for each parameter set. For parameter set 1, the value of the string tension is $k = 0.63$ GeV/fm, and for parameter set 2, that is $k = 0.50$ GeV/fm. Experimental data are taken from the Particle Data Group [29].

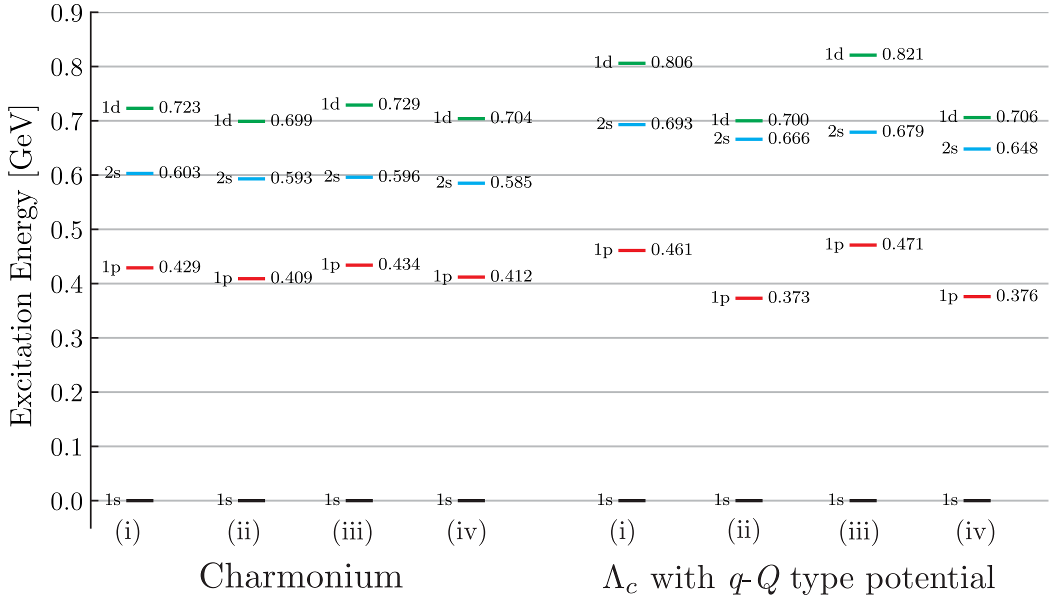

FIG. 5. Excitation energies calculated with potentials (i)–(iv) listed in Eq. (51). The charmonium spectra are on the left side, and the Λ_c spectra are on the right side. The Λ_c spectra are calculated with the $q-Q$ type potential. The potential parameters are fixed to $\alpha_s = 0.4$, $k = 0.9$ GeV/fm, and $\Lambda = 3.5\ \mathrm{fm}^{-1}$.

part, we find that the level spacing between $1s$ and $1p$ becomes smaller owing to the Darwin term, which reduces the $1p$ excitation energy by 20 MeV. This is because the Darwin term is repulsive for $1s$ state and lifts its energy level up. Looking at (iii) on the charmonium part in Fig. 5, the orbit-orbit interaction increases the $1p$ excitation energy of the charmonium by about 5 MeV, which is very small.

As for the Λ_c part, the effect of the Darwin term is large for the Λ_c spectra while that of the orbit-orbit interaction is small. The Darwin term reduces the $1p$ excitation energy by 88 MeV. The effect of the Darwin term for the Λ_c excitation spectra is thus more significant than that for the charmonium because of the different mass dependence of the Darwin term as

$$V_{\mathrm{Dar}} = \frac{2\pi}{3}\alpha_{\mathrm{Dar}}\frac{2}{m_Q^2}\frac{2L+1}{4\pi r^2}\Lambda e^{-\Lambda^2 r^2} \tag{52}$$

for the charmonium and

$$V_{\mathrm{Dar}} = \frac{2\pi}{3}\alpha_{\mathrm{Dar}}\left(\frac{1}{m_Q^2}+\frac{4}{m_d^2}\right)\frac{2L+1}{4\pi r^2}\Lambda e^{-\Lambda^2 r^2} \tag{53}$$

for the Λ_c baryon. We can see that the Darwin term for the Λ_c baryon contains the term depending on the diquark mass $1/m_d^2$ which remains finite even in the heavy quark limit ($m_h \to \infty$). This term provides large influence for the Λ_c spectra. The orbit-orbit interaction whose effect on the $\Lambda_c 1p$ excitation spectra is 10 MeV, has small effect on the excitation spectra of the Λ_c baryon with the q-Q type potential. Thus, the Darwin term is not so important for the charmonium, but important for the Λ_c.

In conclusion, the relativistic corrections are not enough large to reproduce the Λ_c $1p$ excitation energy, but thanks to the effect of the Darwin term, it is unnecessary to reduce the string tension in a quark-diquark system to as much as half of that in the quark-antiquark system to reproduce the $1p$ excitation energy of Λ_c baryon. The existence of the Darwin term related to the diquark mass can be one of the keys of the solution to the string tension puzzle.

C. Excitation spectra of Λ_c and Λ_b with S-Q type potential

In this subsection, we discuss the energy spectra of the Λ_c and Λ_b baryons calculated by using the S-Q type potential.

Figure 6 shows the Λ_c and Λ_b excitation spectra calculated with the S-Q type potential for parameter set 1. As we can see, all the calculated excitation energies are larger than the experimental values. Focusing on the $\Lambda_c 1p$ excited energies, we find that the energy difference between the calculated results and the experimental data is about 170 MeV, which is 1.2 times as large as that in Ref. [11].

Next we discuss each contribution of the relativistic correction. Figure 7 shows the excitation spectra of Λ_c baryon calculated by using the potentials (i)–(iv) listed in Eq. (51) whose terms are taken from the S-Q type potential, and we use parameters $\alpha_s = 0.4$, $k = 0.9$ GeV/fm, and $\Lambda = 3.5\ \mathrm{fm}^{-1}$ to see effects of each term for the excitation spectra of Λ_c. Effects of all terms of the potential are small, and they are 5 MeV. Comparing (ii) on the Λ_c part in Fig. 5

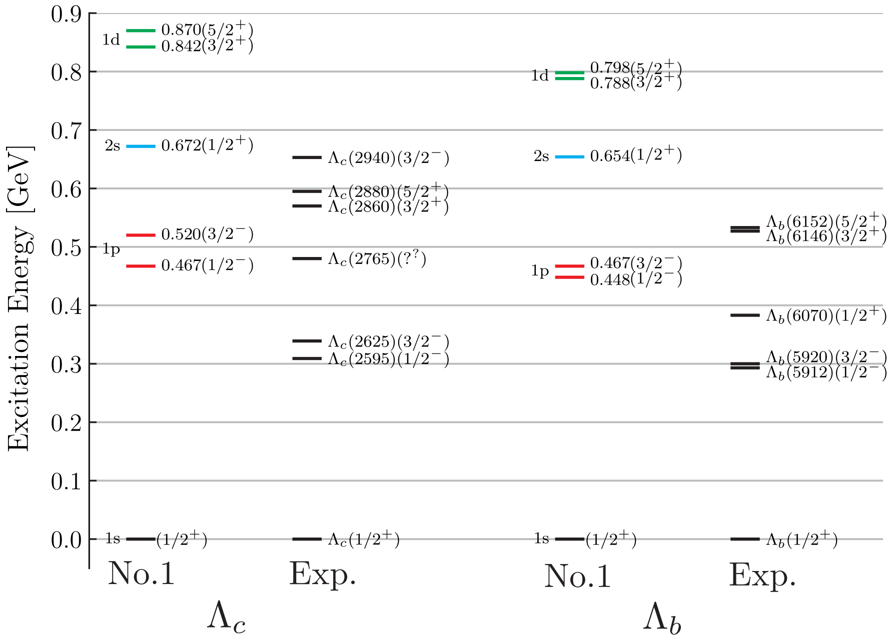

FIG. 6. Calculated spectra with the S-Q type potential for Λ_c and Λ_b systems. The experimental data are taken from the Particle Data Group [29]. Parameter set 1 shown in Table I is used.

to that in Fig. 7, in contrast to the q-Q type potential, the effect of the Darwin term on the S-Q type potential is much smaller. The term depending on the diquark mass in the Darwin term does not exist in the S-Q type potential (47), and this causes a large difference between the heavy baryon spectra with the q-Q and S-Q type potentials. Table II shows the expectation values of the orbit-orbit interaction for each state of the charmonium and the Λ_c baryon with the S-Q type potential. As we can see, since the orbit-orbit interaction is attractive and its effects for the s state are a bit larger, the energy differences between the $1s$ state and the $1p$ and $1d$ states become enhanced. Thus, the $\Lambda_c 1p$ excitation energies calculated with the S-Q type potential are larger than those calculated in Ref. [11]. Nevertheless, we find that the magnitudes of the relativistic corrections for the S-Q type potential are not so large.

As done in Ref. [11], we redetermine the string tension to reproduce the spin-weighted average of the excitation energies of the $\Lambda_c 1p$ states, 0.330 GeV. To reproduce the experimental data, we find the string tension k is needed to become smaller $k = 0.42$ GeV/fm for parameter set 1. Figure 8 shows the calculated excitation spectra with this string tension.

In conclusion, the relativistic corrections for the S-Q type potential are too small to solve the string tension puzzle, so we have to reduce the string tension k to reproduce the experimental spectrum with the S-Q type

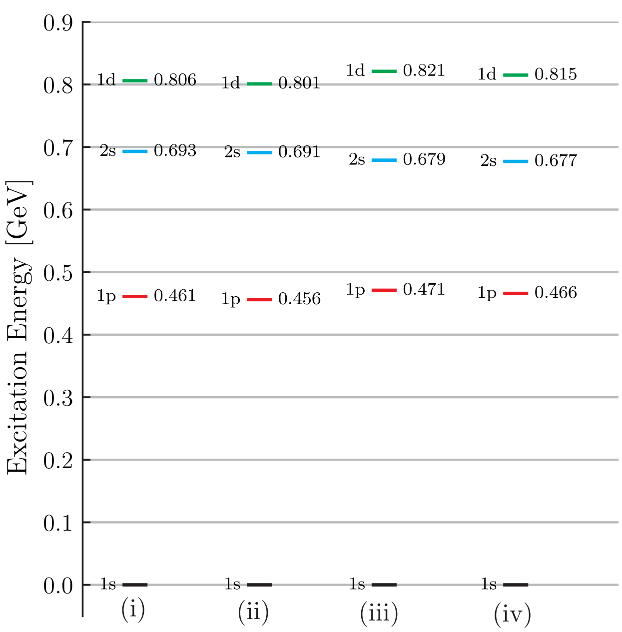

FIG. 7. Excitation energies of Λ_c calculated with potentials (i)–(iv) in Eq. (51) for the $S - Q$ type potential. The potential parameters are fixed to $\alpha_s = 0.4$, $k = 0.9$ GeV/fm, and $\Lambda = 3.5$ fm^{-1}.

TABLE II. Expectation values of the orbit-orbit interaction for each state of the charmonium and the Λ_c baryon calculated with S-Q type potential in unit of GeV. Parameters $\alpha_s = 0.4$, $k = 0.9$ GeV/fm, and $\Lambda = 3.5$ fm^{-1} are used.

	$1s$	$1p$	$2s$	$1d$
$c\bar{c}$	−0.030	−0.026	−0.036	−0.024
Λ_c with S-Q type	−0.051	−0.041	−0.064	−0.037

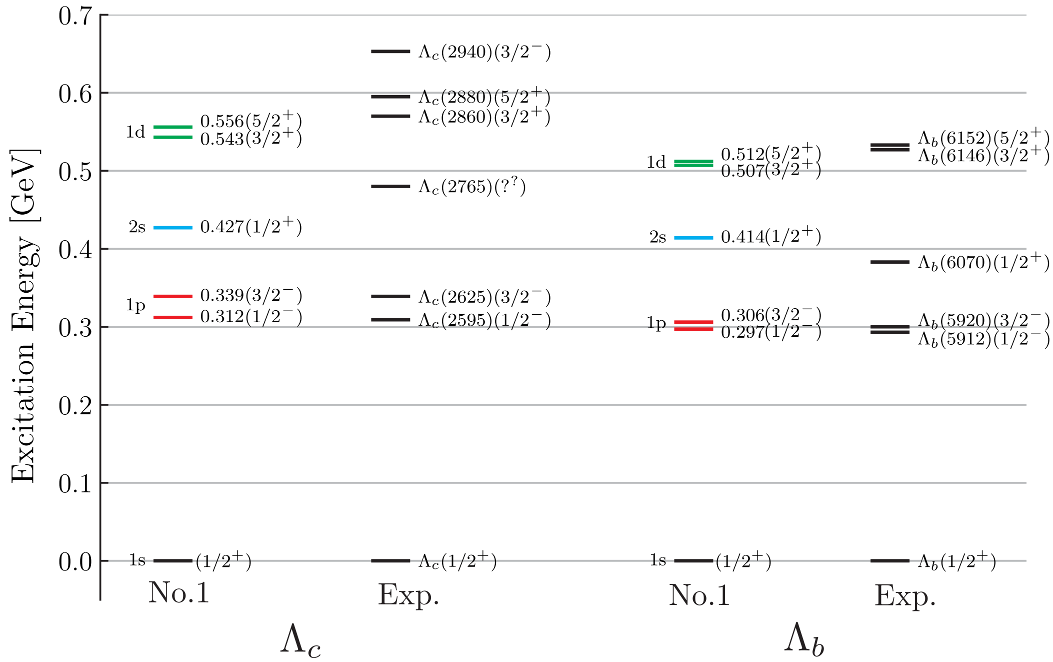

FIG. 8. Calculated spectra with the $S-q$ type potential for the Λ_c and Λ_b baryons. The string tension is redetermined to reproduce the $\Lambda_c 1p$ excitation energies. The value of the string tension is $k = 0.42$ GeV/fm. Experimental data are taken from the Particle Data Group [29].

potential. And then, the string tension puzzle still remains. We can see that there is a large difference between the two potential models stemming from treating the color structure of the diquark.

D. Excitation spectra of Ξ_c baryon

In this subsection, we show the calculated result of the Ξ_c excitation spectrum with the q-Q type potential. We consider that the Ξ_c baryon consists of a charm quark and a scalar strange diquark which is composed of a light quark and a strange quark.

We estimate the strange diquark mass m_{ds} by the difference of the ground state mass of Λ_c and the isospin averaged mass of Ξ_c, which is 0.18 GeV. The ground state mass of the Λ_c baryon is calculated by using the value of the diquark mass 0.5 GeV and parameter set 1. We determine the strange diquark mass by fitting the mass difference between the ground states of the Ξ_c and Λ_c baryons to the experimental value, and we find the strange diquark mass to be 0.81 GeV for parameter set 1.

Figure 9 shows the excitation spectrum of the Ξ_c baryon calculated with parameter set 1 listed in Table I. We find that the calculation a bit overestimates the $1p$ excitation energies and their differences between the calculation and the experiments is 45 MeV. Even though the average of the calculated $1p$ excitation energies is consistent with the experimental one, the LS splitting of the $1p$ states in the calculation is about twice larger than that of the experiments. Similarly to the result of the Λ_c excitation spectra with the q-Q type potential, in order to reproduce the $1p$ excitation energies and the LS splitting, we should reduce the string tension. For the higher excited states of the Ξ_c baryon, the calculation overestimates largely the excitation spectra. Similarly to the result of Λ_c excitation spectra with the q-Q type potential, if both of calculated results of the $1p$ excitation energies and the LS splitting reproduce experimental data, we should reduce the string tension. For the higher excitation states of the Ξ_c baryon, the calculated values are larger than the experimental data.

IV. SUMMARY

In this paper, we have calculated the excitation spectra of the heavy baryons in a quark-diquark model with relativistic corrections. The heavy baryon has been assumed to be composed of a heavy quark and a pointlike scalar diquark having anti-color $\bar{\mathbf{3}}$ and spin $S = 0$. We considered two types of relativistic corrections by depending on the internal color structure of the diquark. In the first approach, the diquark has been treated as an exact two-quark pair, and this model is called as q-Q type potential. In the second approach, the diquark has been treated as just a scalar particle having no internal structure, and this model is called as S-Q type potential. Our objective is to obtain a solution to the puzzle pointed out in Ref. [11], which we call as the string tension puzzle, by considering relativistic corrections for the quark-diquark potential.

For the q-Q type potential, we have found that the calculated $\Lambda_c 1p$ excitation energies are slightly larger than

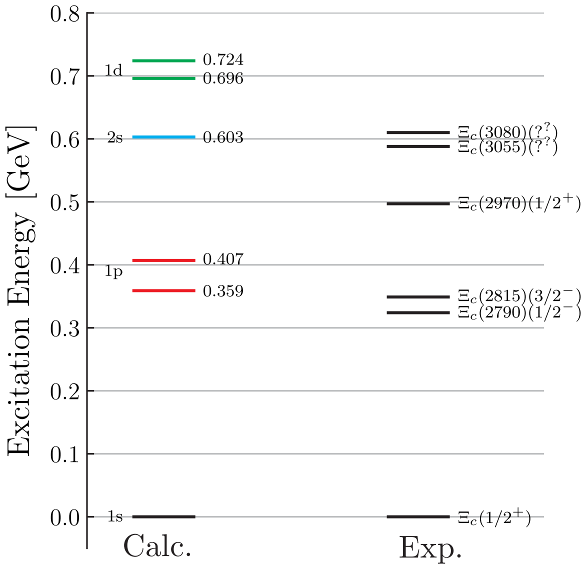

FIG. 9. Calculated excitation spectra with the q-Q type potential for the Ξ_c baryon. The q-Q type potential with parameter set 1 as in Table. I is used to calculate them. The experimental data are taken from the Particle Data Group [29].

the experimental data. We need to reduce the string tension to reproduce the experimental data of both $1p$ excitation energies and the LS splitting with good precision, but we do not need to reduce it by half of the strength in the quark-antiquark system. This potential has a Darwin term with the diquark mass. This originates from the quark structure of the diquark, which indicates that the Darwin effect is stronger for heavy baryons than for quarkonia. The energy difference between the $\Lambda_c 1s$ state and the $1p$ state becomes relatively small, as the energy of the $1s$ state increases due to the Darwin term. This demonstrates that considering the relativistic effects, especially the Darwin term, is important for solving the string tension puzzle.

The S-Q type potential in which the diquark has been treated as a scalar particle does not have a Darwin term with the diquark mass. This means that the string tension on the S-Q type potential should be taken much smaller than one in the quark-antiquark potential, which is qualitatively the same result as the previous work [11]. Our findings have shown that treating the internal color structure of the diquark causes large differences, and since the diquark is composed of two quarks, its structure should be carefully considered.

We have also calculated the Ξ_c excitation spectra by using the q-Q type potential. The Ξ_c baryon was considered as the bound system of the scalar strange diquark and the charm quark. We have found that the calculated $1p$ excitation energies reproduced the experimental data and the LS splitting was overestimated. The consistent results with the Λ_c baryon have been obtained.

In the future, it should be considered that the possible mixing with quark configurations other than the one considered here, such as the pion cloud and/or pentaquark systems, etc., should be taken into account in order to understand the full spectra of heavy baryons.

ACKNOWLEDGMENTS

This work was supported by JST, the establishment of university fellowships toward the creation of science technology innovation, Grant No. JPMJFS2127. The work of H. N. was supported by the Grants-in-Aid for Scientific Research [Grant No. JP21K03536], and the work of D. J. was partially supported by Grants-in-Aid for Scientific Research [No. JP21K03530, No. JP22H04917, No. JP23K03427].

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