Analytic results on the massive three-loop form factors: Quarkonic contributions

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The quarkonic contributions to the three-loop heavy-quark form factors for vector, axial-vector, scalar, and pseudoscalar currents are described by closed form difference equations for the expansion coefficients in the limit of small virtualities q^2/m^2. A part of the contributions can be solved analytically and expressed in terms of harmonic and cyclotomic harmonic polylogarithms and square-root valued iterated integrals. Other contributions obey equations which are not first-order factorizable. For them still infinite series expansions around the singularities of the form factors can be obtained by matching the expansions at intermediate points and using differential equations which are obeyed directly by the form factors and are derived by guessing algorithms. One may determine all expansion coefficients for $q^2/m^2 \to \infty$ analytically in terms of multiple zeta values. By expanding around the threshold and pseudothreshold, the corresponding constants are multiple zeta values supplemented by a finite amount of new constants, which can be computed at high precision. For a part of these coefficients, the infinite series in front of these constants may be even resummed into harmonic polylogarithms. In this way, one obtains a deeper analytic description of the massive form factors, beyond their pure numerical evaluation. The calculations of these analytic results are based on sophisticated computer algebra techniques. We also compare our results with numerical results in the literature.

I. INTRODUCTION

The massive form factors for the vector, axial-vector, scalar and pseudoscalar currents describe the decay of virtual bosons into two massive quarks in quantum chromodynamics (QCD). They are important building blocks to the virtual corrections to a given loop order for processes like bottom- and top-quark pair production at e^+e^- and pp colliders. In a similar way, they also contribute to the corresponding corrections in quantum electrodynamics (QED). These processes can be induced by virtual photons, Z-bosons and Higgs bosons in the Standard Model and its extensions. The form factors depend on the ratio of the current virtuality q^2 and the square of the heavy quark mass m.

The massive form factors were computed at two-loop order in Refs. [1–4] for the first time. Later in Ref. [5], the two-loop calculation was extended to include $O(\varepsilon)$ terms in the dimensional parameter $\varepsilon = (4 - D)/2$. In Ref. [6], the

$O(\varepsilon^2)$ terms were obtained. The further corrections in ε contribute to the respective higher order corrections. A first part of the analytic three-loop corrections has been obtained in Refs. [7–15]. Numerical results on the three-loop form factors have been presented in Refs. [16,17], where the first order differential equations for the master integrals were solved numerically in the whole region of the kinematic parameters expanding up to a certain order using formal Taylor series around a series of points to obtain a numerical representation with overlapping convergence radii. Very recently, also the anomaly contribution has been calculated in the same way [18].

The purpose of the present paper is to derive analytic results for the expansion coefficients of the quarkonic contributions to the form factors around the four character-istic kinematic points, namely, the low and high energy limits, as well as the two- and four-particle thresholds, by using the difference equations derived in [15] to very high order. The difference equations were obtained by using the method of arbitrary high moments [19] and its implemen-tation within the *Mathematica* package SolveCoupledSystem [15,20]. Our representations consist of (logarithmically modulated) Taylor series around the four critical points, which can be computed analytically to arbitrary high order.

The coefficients in these expansions are given in terms of multiple zeta values [21] and a series of new special constants, which we present at very high accuracy. For a part of these terms, we can even resum the complete series into special functions like classical polylogarithms [22]. The different expansions are found by using large scale computer algebra calculations. In particular, new difference and differential equation techniques based on the holonomic framework [23,24] and symbolic summation [25] are nontrivially used for this challenge. Our analytic representations allow one to perform detailed numerical comparisons with the work in Refs. [16,17].

The paper is organized as follows. In Sec. II, we describe the form factors in general and give a brief summary of the calculation done in [15], which is the basis of the calculations performed in the subsequent sections of the present paper. In Sec. III, we introduce our method for the calculation of power-log expansions at the singular points of the form factors and apply it to the nonsolvable parts described in [15] in the high energy limit. The method is based on the matching of expansions at different points (Sec. III A) and differential equations obeyed directly by the form factors (Sec. III B). We explain how to derive these differential equations, relying on guessing methods, cf. Refs. [23] and obtain numerical expansions. We show that the precision achieved for these expansions can be very high, allowing us to determine the expansion coefficients in terms of known constants (Sec. III C). For some of the sequences of rational numbers associated to these constants, recursion relations can be found and in some cases solved, leading to closed form solutions. In Sec. IV, we apply this method to the two- and four-particle thresholds, which required a transformation of the expansions used in [15] (Sec. IV A). In the case of the two-particle threshold (Sec. IV B), we found again that many coefficients in the expansions could be written in terms of known constants. However, this was not the case for all of them, which required the introduction of new constants that were computed with very high precision. Again, for some of the sequences of rational numbers associated to these constants, we were able to find recursions, and in some cases, closed form solutions. Section V contains graphs and numerical results, which we compare with results in the literature. Section VI contains the conclusions. In the appendices A-E, we present a series of analytic results. Ancillary files are provided in computer-readable form for analytic and numerical studies.

II. THE HEAVY QUARK FORM FACTORS

The decay of a virtual neutral boson into a pair of heavy quarks can be described by four different types of vertex amplitudes depending on the spin of the boson under consideration and the way it couples to the quarks. For a spin-1 boson, we may have vector and/or axial-vector couplings, while in the case of a spin-0 particle, we may deal with a scalar or a pseudoscalar. The spin-1 particles typically studied are the photon, which couples as a vector to the fermions, and the Z-boson, which has both types of couplings. In the latter case, the vertex Γ^μ_{cd} can be written in terms of form factors $F_{V,i}$ and $F_{A,i}$ (with $i = 1, 2$) as follows

$$
\begin{aligned}
\Gamma^\mu_{cd} &= \Gamma^\mu_{V,cd} + \Gamma^\mu_{A,cd} \\
&= -i\delta_{cd}\left[v_Q\left(\gamma^\mu F_{V,1}\left(\frac{q^2}{m^2}\right) + \frac{i}{2m}\sigma^{\mu\nu}q_\nu F_{V,2}\left(\frac{q^2}{m^2}\right) \right) \right. \\
&\quad \left. + a_Q\left(\gamma^\mu\gamma_5 F_{A,1}\left(\frac{q^2}{m^2}\right) + \frac{1}{2m}q^\mu\gamma_5 F_{A,2}\left(\frac{q^2}{m^2}\right) \right) \right],
\end{aligned}
$$

(2.1)

where $\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu]$, q is the momentum of the Z-boson, m the mass of the heavy quarks, c and d are their color indices, and v_Q and a_Q are the Standard Model vector and axial-vector couplings, respectively.

For scalar and pseudoscalar currents the generic vertex structure is given by the following, where F_S and F_P are the corresponding form factors

$$
\begin{aligned}
\Gamma_{cd} &= \Gamma_{S,cd} + \Gamma_{P,cd} \\
&= -\frac{m}{v}\delta_{cd}\left[s_Q F_S\left(\frac{q^2}{m^2}\right) + ip_Q\gamma_5 F_P\left(\frac{q^2}{m^2}\right) \right],
\end{aligned}
$$

(2.2)

where v is the vacuum expectation value of the Higgs field of the Standard Model, and s_Q and p_Q are the scalar and pseudoscalar couplings, respectively.

We introduce the variable x defined by

$$
\hat{s} = \frac{q^2}{m^2} := -\frac{(1-x)^2}{x},
$$

(2.3)

from which one obtains

$$
x = \frac{\sqrt{4-\hat{s}} - \sqrt{-\hat{s}}}{\sqrt{4-\hat{s}} + \sqrt{-\hat{s}}}.
$$

(2.4)

Our notation follows widely that of Ref. [15]. In Fig. 1, we can see how different regions of $\hat{s} \in \mathbb{R}$ are mapped to the complex plane in x. We will later present results both in the variables $\hat{s}$ and x.

To obtain the form factors $F_{I,i}$, $I = V, A$, we multiply the following projectors on Γ^μ_{cd} and perform a trace over the spinor and color indices
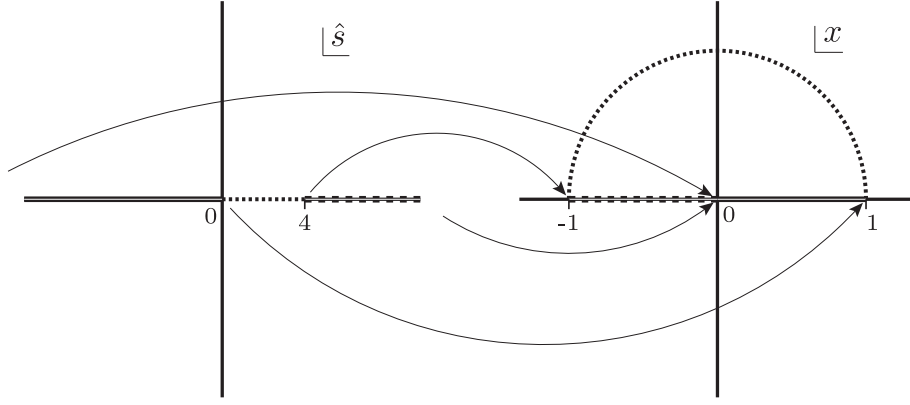
FIG. 1.   Mapping of regions from $\hat{s} \in \mathbb{R}$ to the complex plane in x. The region $\hat{s} \leq 0$ is mapped to the interval $x \in (0, 1]$, while the region $0 \leq \hat{s} \leq 4$ is mapped to the arc of unit radius, and the region $\hat{s} > 4$ is mapped to $x \in (-1, 0)$.

$$P_{V,i} = \frac{i}{v_Q} \frac{\delta_{cd}}{N_c} \frac{\slashed{q}_2 - m}{m} \left(\gamma_\mu g_{V,i}^{(1)} + \frac{1}{2m}(q_{2\mu} - q_{1\mu}) g_{V,i}^{(2)} \right)$$
$$\times \frac{\slashed{q}_1 + m}{m},$$

$$P_{A,i} = \frac{i}{a_Q} \frac{\delta_{cd}}{N_c} \frac{\slashed{q}_2 - m}{m} \left(\gamma_\mu \gamma_5 g_{A,i}^{(1)} + \frac{1}{2m}(q_{1\mu} + q_{2\mu}) \gamma_5 g_{A,i}^{(2)} \right)$$
$$\times \frac{\slashed{q}_1 + m}{m}, \tag{2.5}$$

where q_1 and q_2 are the momenta of the heavy quarks $(q_1 + q_2 = q)$ and N_c denotes the number of colors. In the present paper we relate all results to N_c as an explicit variable, i.e., the color factors of $\mathrm{SU}(N_C)$ are given by $C_A = N_c$, $C_F = (N_c^2 - 1)/(2N_c)$, $T_F = 1/2$. In the following we set $N_c = 3$. The factors $g_{I,i}^{(k)}$ are given by

$$g_{V,1}^{(1)} = \frac{x}{4(1-\varepsilon)(1+x)^2}, \qquad g_{V,1}^{(2)} = \frac{(3-2\varepsilon)x^2}{(1-\varepsilon)(1+x)^4}, \tag{2.6}$$

$$g_{V,2}^{(1)} = \frac{x^2}{(1-\varepsilon)(1-x^2)^2},$$
$$g_{V,2}^{(2)} = \frac{2x^2[-1+\varepsilon(1-x)^2+(4-x)x]}{(1-\varepsilon)(1-x)^2(1+x)^4}, \tag{2.7}$$

$$g_{A,1}^{(1)} = \frac{x}{4(1-\varepsilon)(1+x)^2}, \qquad g_{A,1}^{(2)} = \frac{x^2}{(1-\varepsilon)(1-x^2)^2}, \tag{2.8}$$

$$g_{A,2}^{(1)} = \frac{x^2}{(1-\varepsilon)(1-x^2)^2},$$
$$g_{A,2}^{(2)} = \frac{2x^2[1-\varepsilon(1+x)^2+x(4+x)]}{(1-\varepsilon)(1-x)^4(1+x)^2}. \tag{2.9}$$

We will later also refer to each separate term attached to the functions $g_{I,i}^{(k)}$'s after the projection is performed. We define $F_{v,i}(x)$ and $F_{a,i}(x)$ such that

$$F_{V,i}(x) = g_{V,i}^{(1)} F_{v,1}(x) + g_{V,i}^{(2)} F_{v,2}(x), \quad i = 1, 2. \tag{2.10}$$

$$F_{A,i}(x) = g_{A,i}^{(1)} F_{a,1}(x) + g_{A,i}^{(2)} F_{a,2}(x), \quad i = 1, 2. \tag{2.11}$$

The form factors F_S and F_P can be obtained from Γ_{cd} using the projectors given below and performing the trace over the bispinor and color indices

$$P_S = -\frac{v}{2ms_Q} \frac{\delta_{cd}}{N_c} \frac{x}{(1+x)^2} \frac{\slashed{q}_2 - m}{m} \frac{\slashed{q}_1 + m}{m}, \tag{2.12}$$

$$P_P = \frac{v}{2mp_Q} \frac{\delta_{cd}}{N_c} \frac{x}{(1-x)^2} \frac{\slashed{q}_2 - m}{m} i\gamma_5 \frac{\slashed{q}_1 + m}{m}. \tag{2.13}$$

For convenience, and in a similar way as in Eqs. (2.10) and (2.11), we extract the x-dependent part of the projectors by defining $F_s(x)$ and $F_p(x)$ as follows

$$F_S(x) = -\frac{x}{2(1+x)^2} F_s(x), \tag{2.14}$$

$$F_P(x) = \frac{x}{2(1-x)^2} F_p(x). \tag{2.15}$$

In the following, we will identify a generic form factor as $F_I(x)$, where the index I will denote either $I \in \{S, P\}$ or one of the pair of indices $I \in \{(V, 1), (V, 2), (A, 1), (A, 2)\}$. It may also denote any of the lower case versions $I \in \{(v, 1), (v, 2), (a, 1), (a, 2), s, p\}$.

We first summarize the results of our previous paper [15] on the quarkonic contributions to the three-loop form factor, i.e., for the contributions characterized by an overall factor of n_h or n_h^2, where n_h is the number of heavy quarks.
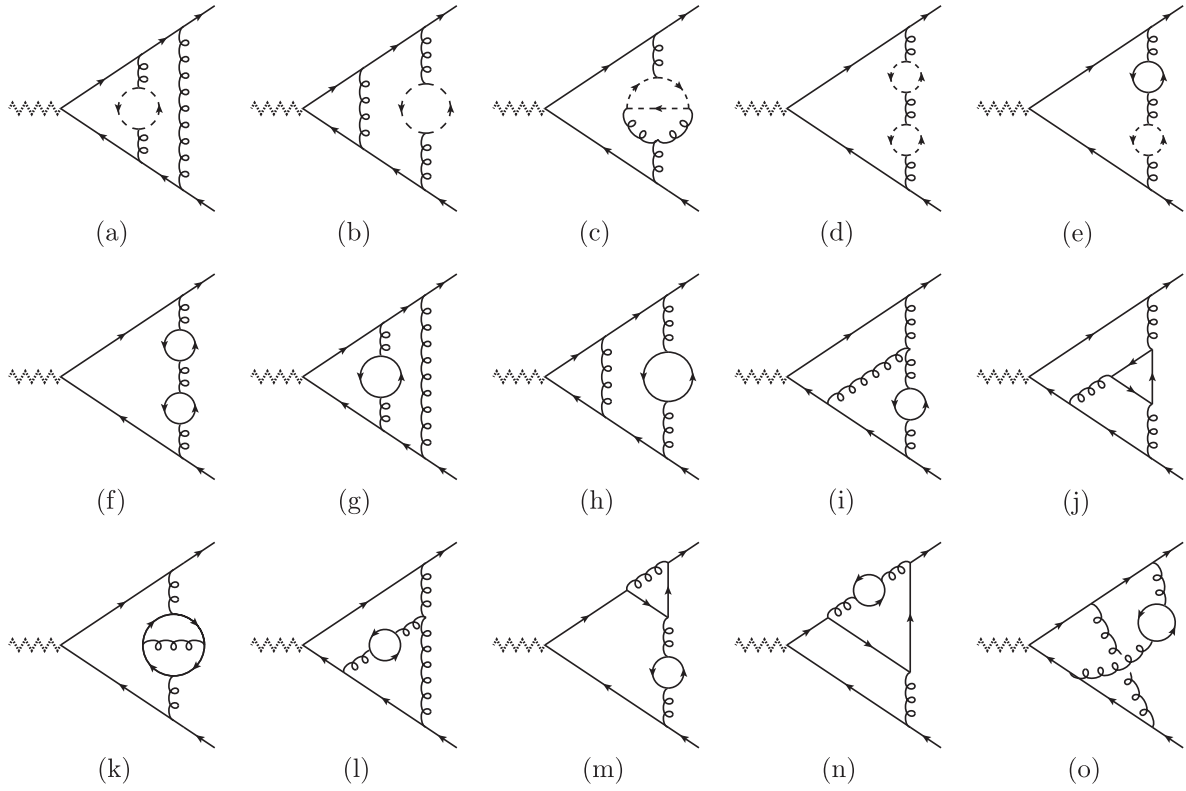
FIG. 2. Sample of diagrams required for the calculation of quarkonic contributions to three-loop massive form factors. Dashed arrow lines represent massless quarks, while solid arrow lines represent massive quarks. The dotted zigzag line represents any of the possible currents. Diagrams with closed massless quark loops, such as diagrams (a)–(e), were computed in Ref. [12]. Diagrams with closed heavy quark loops, such as (f)–(o), were first considered in Ref. [15]. Numerical expansions spanning all regions in $\hat{s}$ were calculated in Refs. [16,17]. In this paper, we use a different method to get very deep expansions in all regions of $\hat{s}$ with exact numerical representations around $x = 0$, and even analytic results in some cases.

The form factors $F_I(x)$ can be expanded perturbatively in QCD as follows

$$F_I(x) = \sum_{l=0}^{\infty} a_s^l F_I^{(l)}(x), \qquad (2.16)$$

where $a_s = g_s^2/(4\pi)^2$ is the strong coupling constant. In Ref. [15], four of the present authors computed all of the solvable parts of the $O(a_s^3)$ heavy fermion contributions. By "solvable parts" we mean all of the parts for which the corresponding differential equations or difference equations were factorizable to first order, and the corresponding solutions could be expressed in terms of harmonic polylogarithms (or any of their first-order factorizing generalizations) depending on the variable x, cf. Ref. [26] for a detailed discussion. The calculations were performed in $D = 4 - 2\varepsilon$ dimensions, which allows one to get the expansion in the dimensional parameter ε at each loop order as follows

$$F_I^{(l)}(x) = \sum_{k=-l}^{0} \varepsilon^k F_I^{(l,k)}(x) + O(\varepsilon). \qquad (2.17)$$

In general, if we want to obtain the $O(a_s^l)$ corrections, we need to compute the l-loop form factors up to $O(\varepsilon^0)$, while the j-loop form factors, with $j < l$, will be needed up to $O(\varepsilon^{l-j})$, in order to properly perform the corresponding renormalization procedure. This is the reason why one-loop and two-loop form factors were computed to higher orders in ε in [5,6]. In this paper, we are interested in three-loop corrections ($l = 3$), so in what follows we will drop the label l from the expansion terms in Eq. (2.17). In other words, from now on, $F_I^{(k)}(x)$ will denote the $O(\varepsilon^k)$ term in the ε expansion of the corresponding three-loop form factor, instead of the $O(a_s^k)$ term in the perturbative expansion.

In Ref. [15], the required Feynman diagrams with closed quark loops were generated using the program QGRAF [27], a sample of topologies is shown in Fig. 2. After introducing the projectors described in the previous section, the Lorentz algebra was performed using FORM [28,29] and the color algebra by COLOR [30]. After this, the results were expressed in terms of a linear combination of Feynman integrals, which were reduced by applying the integration-by-parts relations [31] using the package CRUSHER [32] to master integrals. A system of differential equations was

obtained for the corresponding master integrals $I_i(x)$, with $i = 1, 2 \ldots$

$$\frac{d}{dx} \begin{pmatrix} I_1(x, \varepsilon) \\ I_2(x, \varepsilon) \\ \vdots \end{pmatrix} = A(x, \varepsilon) \begin{pmatrix} I_1(x, \varepsilon) \\ I_2(x, \varepsilon) \\ \vdots \end{pmatrix}, \quad (2.18)$$

where $A(x, \varepsilon)$ is an invertible matrix with entries from the polynomial ring[1] $\mathbb{K}[x, \varepsilon]$. Taking advantage of the fact that at $q^2 = 0$ (i.e. $x = 1$), the nonsinglet massive 3-point Feynman integrals reduce to self-energies, it was possible to obtain initial conditions for this system of differential equations. In fact, the integrals turn out to be regular at $x = 1$, and a formal power series expansion around $y = 0$ (where $y = 1 - x$) can be performed

$$I_i(x, \varepsilon) = \sum_{n=0}^{\infty} c_i(n, \varepsilon) y^n. \quad (2.19)$$

Since the form factors are then linear combinations of the master integrals, they too can be expanded in the same way as in Eq. (2.19)

$$F_I(x, \varepsilon) = \sum_{n=0}^{\infty} c_I(n, \varepsilon) y^n, \quad (2.20)$$

or

$$F_I^{(k)}(x, \varepsilon) = \sum_{k=-3}^{0} \varepsilon^k \sum_{n=0}^{\infty} c_I^{(k)}(n) y^n. \quad (2.21)$$

Introducing these expansions in (2.18), and using the initial conditions, it was possible to generate thousands of expansion coefficients for all the different form factors. This is a highly nontrivial task, and it required the use of two different strategies described in Sec. 5 of Ref. [15] that have been implemented within the package SolveCoupledSystem [15,20]. For each color factor, the coefficients thus obtained are given in terms of linear combinations of the following constants

$$\{\zeta_2, \zeta_3, \zeta_4, \zeta_5, \zeta_2\zeta_3, \zeta_2 l_2, \zeta_2 l_2^2, l_2^4, a_4\} \quad (2.22)$$

where

$$l_2 = \ln(2), \qquad a_4 = \text{Li}_4\left(\frac{1}{2}\right), \quad (2.23)$$

and $\zeta_k = \zeta(k)$ is the Riemann ζ function evaluated at integer values $k \geq 2$. The numbers in front of these

constants in the linear combinations are just rational numbers and

$$\text{Li}_n(x) = \sum_{k=1}^{\infty} \frac{x^k}{k^n}, \qquad |x| \leq 1 \quad (2.24)$$

denote the classical polylogarithms [22].

Therefore, for each form factor and for each constant in (2.22), we get a sequence of rational numbers, which we can feed to the SAGE package ORE_ALGEBRA presented in [23] in order to guess a recursion relation satisfied by each one of the sequences. In several cases, the obtained recursion relations factorize at first order and can be solved even in terms of harmonic sums using the algorithms from [25,33] implemented in the *Mathematica* package SIGMA [34–36], which were then introduced in (2.20). The corresponding infinite sums can then be performed with the help of SIGMA and the *Mathematica* package HARMONICSUMS [37–46] and one obtains solutions in terms of harmonic polylogarithms. However, for a few of the color-factors/constants, the recursion relations turned out to be nonfirst order factorizable. In these cases no general techniques are available to obtain closed form solutions. Therefore, in Ref. [15] we had to limit ourselves to just giving the corresponding expansions around $y = 0$ in these cases.

In this paper, we go a step further and obtain thousands of expansion coefficients for the power-log expansion around $x = 0$ and other singular points, thus obtaining complete coverage of the whole kinematic range of $\hat{s}$. We decompose the unrenormalized form factors in terms of their first-order factorizable and nonfirst-order factorizable parts as follows,

$$F_I^{(0)}(x) = F_I^{(0),\text{sol}}(x) + n_h \Big(F_{I,1}^{(0)}(x) + \zeta_2 F_{I,2}^{(0)}(x)$$
$$+ \zeta_3 F_{I,3}^{(0)}(x) \Big). \quad (2.25)$$

The term $F_I^{(0),\text{sol}}(x)$ corresponds to the solvable parts. The results for this term were given in Ref. [15], together with the expressions for all the pole terms $F_I^{(k)}(x)$, $k = -3, -2, -1$, which were always solvable. The remaining terms, $F_{I,i}^{(0)}(x)$ with $i = 1, 2, 3$, are the nonsolvable ones. This decomposition indicates that all of the recursions associated with the constants in (2.22) were solvable except for the recursions coming from the factor n_h at $O(\varepsilon^0)$ associated with ζ_2, ζ_3 and the rational term[2] [i.e., the term arising from just rational numbers without any of the

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constants in (2.22)]. Following the decomposition in Eqs. (2.10) and (2.11) and the explicit form of the factors $g_{I,i}^{(k)}$ given in (2.6), we will also have

$$F_{V,1,i}^{(0)}(x) = \frac{x}{4(1+x)^2}F_{v,1,i}^{(0)}(x) + \frac{3x^2}{(1+x)^4}F_{v,2,i}^{(0)}(x) \quad (2.26)$$

$$F_{V,2,i}^{(0)}(x) = \frac{x^2}{(1-x^2)^2}F_{v,1,i}^{(0)}(x) + \frac{2x^2(4x-1-x^2)}{(1-x)^2(1+x)^4}F_{v,2,i}^{(0)}(x) \tag{2.27}$$

$$F_{A,1,i}^{(0)}(x) = \frac{x}{4(1+x)^2}F_{a,1,i}^{(0)}(x) + \frac{x^2}{(1-x^2)^2}F_{a,2,i}^{(0)}(x) \quad (2.28)$$

$$F_{A,2,i}^{(0)}(x) = \frac{x^2}{(1-x^2)^2}F_{a,1,i}^{(0)}(x) + \frac{2x^2(x^2+4x+1)}{(1-x)^4(1+x)^2}F_{a,2,i}^{(0)}(x). \tag{2.29}$$

Before describing the method we used to obtain power-log expansions around $x = 0$, i.e., in the high-energy limit $q^2 \to -\infty$, for all of the $F_{I,i}^{(0)}(x)$'s, we summarize here the ultraviolet renormalization procedure. We have followed a mixed scheme to renormalize all the fields and parameters appearing in the form factors. The mass and the wave function of the heavy quark have been renormalized using the on-shell scheme, while the strong coupling constant renormalization has been performed in the $\overline{\text{MS}}$ scheme. For the Yukawa coupling in the scalar and pseudoscalar form factors, we also used the $\overline{\text{MS}}$ renormalization scheme.[3] All the necessary renormalization constants are available in [47–63].

The ultraviolet renormalized form factors maintain a universal infrared (IR) structure. The IR singularities of the form factors can be factorized as a multiplicative renormalization factor [64,65], the structure of which is controlled by the renormalization group equation through the massive cusp anomalous dimension [66–69]. We note that one needs to consider the decoupling relation [70,71] of the strong coupling constant to obtain the complete (full-QCD) IR structure of the form factors. All final results for the form factors will thus be given in terms of $\alpha_s^{(n_l)}$.

III. THE HIGH ENERGY LIMIT

Unlike the expansions around $x = 1$, the expansions around singular points will involve logarithms $\ln^k(x)$. In particular, the expansions around $x = 0$ will be given in terms of power series multiplying various powers of $\ln(x)$. In principle, the coefficients of each power series could be obtained numerically by directly matching a truncated version of the expansion we want to determine with a

truncated version of the known expansion (in this case, the one at $x = 1$) at intermediate points $0 < x < 1$. However, this naive method will fail to produce precise enough coefficients in most cases, unless exact relations among the coefficients can be obtained, thus reducing the number of free coefficients to be determined. This can be achieved using differential equations, which can be derived from the large number of coefficients generated in Ref. [15] for the expansions around $x = 1$. In the next subsections, we will describe the matching of expansions at intermediate points, initially without relying on differential equations, and then we will explain how we derive those differential equations and how we use them in combination with the matching conditions to obtain the coefficients numerically. As we will see, the precision of the coefficients in the power–log expansions can be increased almost indefinitely by increasing the number of terms in both of the expansions to be matched.

A closely related method based also on differential equations was presented in Refs. [16,17] for the master integrals. There, however, numerical solutions were aimed at. In our case, we use guessing methods to obtain differential equations obeyed directly by the nonsolvable parts of the form factors themselves, determining the expansion coefficients analytically. This allows us to work with very deep expansions and also avoid many spurious singularities, and therefore, unlike the case of Refs. [16,17], in our case obtaining expansions at intermediate points is not strictly necessary, although it may still be useful for practical applications.

In this section we focus on the high energy limit $q^2 \to \pm\infty$. The same methods are applicable to all other singular points.

A. Matching at intermediate points

The form factors are regular at $x = 1$ and have an expansion of the form

$$f(x) = \sum_{j=0}^{\infty} a(j)(1-x)^j, \tag{3.1}$$

where $f(x)$ can be any of the nonsolvable contributions in (2.25), which were split into three terms so that the corresponding coefficients $a(j)$ contain only rational numbers.

We assume now that the function $f(x)$ in (3.1) has also a logarithmic expansion around $x = 0$

$$f(x) = \sum_{i=0}^{r} \sum_{j=\lambda_i}^{\infty} c(i,j)x^j \ln^i(x), \tag{3.2}$$

with $\lambda_i \le 0$. We want to determine as many coefficients as possible of the expansion (3.2). The two expansions (3.1) and (3.2) should be convergent and equal to each other in a

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[3]Note, that this is different than the procedure we used in [15].

region within $0 < x < 1$. Note that the expansion about $x = 1$ is convergent in $x \in (0, 1]$, but the expansion about $x = 0$ converges only in $x \in (-3 + 2\sqrt{2}, 3 - 2\sqrt{2})$ in the case of $F_{I,2}^{(0)}(x)$, due to a spurious singularity at $\hat{s} = -4$, while in the case of $F_{I,1}^{(0)}(x)$ and $F_{I,3}^{(0)}(x)$, the expansions about $x = 0$ converge in $x \in (4\sqrt{3} - 7, -4\sqrt{3} + 7)$, due to a four particle cut at $\hat{s} = 16$. Of course, since we are considering nonsolvable cases, we cannot perform the infinite sum in (3.1). We therefore consider the truncated expansion

$$\tilde{f}_1(x) = \sum_{j=0}^{l} a(j)(1 - x)^j, \qquad (3.3)$$

where l is large but finite. As we have already said, in all cases we have recursion relations for the coefficients $a(j)$ and enough initial values to be able to compute hundreds of thousands of expansion coefficients.

We want to use the truncated expression (3.3) in order to obtain approximate values for the coefficients $c(i, j)$ of the expansion (3.2). For this purpose, we also define a truncated expansion around $x = 0$,

$$\tilde{f}_0(x) = \sum_{i=0}^{r} \sum_{j=\lambda_i}^{\nu_i} \tilde{c}(i, j) x^j \ln^i(x). \qquad (3.4)$$

The truncated expansions have to be matched at values $x_i \in \chi$

$$\chi = \{x_k = x_0 + k\delta | k = 1, 2, ..., n_p\}, \qquad (3.5)$$

where $x_0 > 0$, δ is a real number, which in general should be small, and n_p is a positive integer such that $x_0 + n_p \delta < 1$. This leads to a system of equations

$$\frac{d^k}{dx^k} \tilde{f}_1(x_i) = \frac{d^k}{dx^k} \tilde{f}_0(x_i), \quad x_i \in \chi, \quad k = 0, ..., n_d, \qquad (3.6)$$

which consists of $n_p(n_d + 1)$ equations for the unknown coefficients $\tilde{c}(i, j)$.

To find a solution for all $\tilde{c}(i, j)$ the parameters ν_j, n_p, and n_d have to be chosen such that

$$\sum_{j=0}^{r} (\nu_j - \lambda_j + 1) = n_p(n_d + 1). \qquad (3.7)$$

In general, we try to distribute the coefficients evenly among the different powers of the logarithms in (3.4) by choosing

$$\nu_j = \left\lfloor \frac{n_p(n_d + 1)}{r + 1} + \frac{1}{2} \right\rfloor + \lambda_j - 1, \quad \text{for } j = 0, ..., r - 1,$$

$$(3.8)$$

$$\nu_r = n_p(n_d + 1) + \lambda_r - 1 - \sum_{j=0}^{r-1} (\nu_j - \lambda_j + 1). \qquad (3.9)$$

Our goal is to minimize the difference between the coefficients obtained by solving (3.6) and the actual yet unknown coefficients of the expansion around $x = 0$ given in (3.2), i.e., we want to minimize the quantities

$$\Delta_{i,j} = |c(i, j) - \tilde{c}(i, j)|. \qquad (3.10)$$

These differences between the actual coefficients and the ones obtained from the system of equations (3.6) will depend on the set of points in χ, the number of derivatives n_d and the number of coefficients l in the truncated series (3.3), i.e.,

$$\Delta_{i,j} = \Delta_{i,j}(x_0, \delta, n_p, n_d, l). \qquad (3.11)$$

Here the coefficients $\tilde{c}(i, j)$ will be numerical approximations of the coefficients $c(i, j)$, which we will obtain by evaluating $\tilde{f}_1(x)$ and $\tilde{f}_0(x)$ as well as their derivatives at different points in a region within $0 < x < 1$ where both series converge, and equating the corresponding results. This will then lead to a linear system of equations for the coefficients $\tilde{c}(i, j)$.

The task is then to find values for x_0, δ, n_p, n_d, and l such that $\Delta_{i,j} \sim 0$. This may seem impossible at first sight, given the fact that we do not know the coefficients $c(i, j)$ in (3.10) to begin with. Moreover, the values of r and λ_j in (3.4) are also not known *a priori*. As will be described in the next section, we know, however, that $r \leq 6$ and $\lambda_j \geq -2$, so if we choose r high enough or $\lambda_j < -2$, we should expect the corresponding coefficients $\tilde{c}_r(\lambda_j)$ to be close to zero, as long as the parameters in (3.11) are in the right region. We must therefore search for values of x_0, δ, n_p, n_d, and l such that these coefficients are as close to zero as possible. In general, higher values of l and n_d lead to better results, but determining good values for x_0, δ, and n_p is not as simple. In fact, we have observed that the optimal value for x_0, with δ and n_p fixed, shifts to the left as we increase l.

The number of points used ranged from $n_p = 400$ to $n_p = 600$. Such a large number of points makes it necessary to choose them very close to each other, in order to avoid moving too much to the right of the starting point x_0, where the evaluation of the expansion at $x = 0$ will be too imprecise. We used values of δ that ranged between

$$\delta = \left[\frac{1}{120000}, \frac{1}{20000} \right]. \qquad (3.12)$$

We proceeded in this way for all of the $F_{I,2}^{(0)}(x)$ terms, and found very high numerical precision for a relatively large

number of the coefficients in (3.4) and also applied this procedure to solvable cases and verified that it reproduces the coefficients obtained by directly expanding the corresponding exact solutions around $x = 0$.

Unfortunately, it is difficult to apply this method in the case of $F_{I,1}^{(0)}(x)$ and $F_{I,3}^{(0)}(x)$. This is due to the fact that, as we mentioned above, in the case of $F_{I,2}^{(0)}(x)$ the radius of convergence is $3 - 2\sqrt{2} \sim 0.1716$, while in the case of $F_{I,1}^{(0)}(x)$ and $F_{I,3}^{(0)}(x)$ it is $4\sqrt{3} - 7 \sim 0.0718$, which is closer to zero, making it difficult to find a good point to match the expansions without using a prohibitively large amount of coefficients for the expansions around $x = 1$ and $x = 0$. We will now see how the use of differential equations solves this problem.

B. Differential equations obeyed directly by the form factors

As we mentioned before, in Ref. [15] a large number of expansion coefficients around $x = 1$ ($y = 0$) were computed in order to obtain recursion relations satisfied by the sequence of rational numbers that multiply each constant in (2.22) in these coefficients. We can also use these sequences to derive the differential equations by using the method of guessing, cf., [23], such that the coefficients of the power expansion of the respective solutions match the corresponding sequences. This package has been used by us in many applications since 2008, and in calculating the three–loop massive form factor in Ref. [15] to guess the recursions. We obtained differential equations for all of the cases where the recursions did not factorize to first order. These differential equations have the following form

$$\sum_{k=0}^{n_{I,i}-1} \hat{p}_{I,i,k}(y) \frac{d^k}{dy^k} F_{I,i}^{(0)}(y) = 0,$$
$$I \in \{(v,1),(v,2),(a,1),(a,2),s,p\}, \quad i = 1,2,3,$$
$$(3.13)$$

where $n_{I,i} - 1$ is the order of the differential equation and $\hat{p}_{I,i,k}(y)$ is a polynomial in y. We can then perform the change of variables $y \to 1 - x$, which allows us to obtain differential equations in x,

$$\sum_{k=0}^{n_{I,i}} p_{I,i,k}(x) \frac{d^k}{dx^k} F_{I,i}^{(0)}(x) = 0. \qquad (3.14)$$

If we look for a power expansion solution of the form $x^\alpha \sum_{i=0}^{\infty} c_i x^i$, along the lines of what is done in the case of second order differential equations, we get an indicial equation where all the solutions in α are integers. For example, in the case of $F_{v,1,1}^{(0)}(x)$, the indicial equation turns out to be

TABLE I. Orders of the differential equations for the $F_{I,i}^{(0)}(x)$'s.

i	$F_{v,1,i}^{(0)}$	$F_{v,2,i}^{(0)}$	$F_{a,1,i}^{(0)}$	$F_{a,2,i}^{(0)}$	$F_{s,i}^{(0)}$	$F_{p,i}^{(0)}$
1	46	48	46	43	43	43
2	20	22	20	18	18	18
3	25	26	25	23	23	23

$$(\alpha + 2)(\alpha + 1)^6 \alpha^5 (\alpha - 1)^5 (\alpha - 2)^4 (\alpha - 7)^2$$
$$\times \Pi_{i=3}^{6}(\alpha - i)^3 \Pi_{j=8}^{18}(\alpha - j) = 0. \qquad (3.15)$$

Since $\alpha = -2$ is one of the solutions of this equation, we can expect our expansions to start from $1/x^2$ onward,[4] and since all roots differ by integer values, the solutions should involve powers of $\ln(x)$. The solution of this differential equation can then be expanded in a power-log series like the one in Eq. (3.2), i.e.,

$$F_{I,i}^{(0)}(x) = \sum_{j=0}^{r} \sum_{k=-2}^{\infty} c_{I,i}(j,k) x^k \ln^j(x). \qquad (3.16)$$

The ansatz (3.2) is thus justified for some integer r. The value of r depends on the differential equation under consideration; if we use an ansatz with a value of r that is too large, and then insert the ansatz in the differential equation, we will find that all of the coefficients for j larger than a certain value will vanish.

It is important to remark that the first few coefficients of the expansions obtained in [15] usually did not satisfy the guessed recursion relations, which were valid only after the second or third coefficient. This means that we cannot include these coefficients when guessing the differential equations, which leads to solutions where the sequence of expansion coefficients is shifted. We correct this by subtracting these first few terms from the expansion of the solution of the differential equation and performing a shift in the powers of y before doing the change to x. The resulting differential equation ends up with an inhomogeneous part, which can be removed at the expense of increasing the order of the differential equation by one. This is the reason why in Eq. (3.13) the order of the differential equation is $n_{I,i} - 1$, while in (3.14) it is $n_{I,i}$.

In Table I we summarize the orders of the differential equations obtained for each $F_{I,i}^{(0)}(x)$, while in Table II, the degree of the polynomial of highest degree in each differential equation is given. From the numbers in these tables, we can see that the guessed differential equations are quite formidable. Not only are the orders and especially the degrees of the polynomials in these equations quite large, but so are the integer coefficients in these polynomials. For

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[4]However, the final physical result will be free of negative powers of x.

TABLE II. Maximum degree of the polynomials in the differential equations for the $F_{I,i}^{(0)}(x)$'s.

i	$F_{v,1,i}^{(0)}$	$F_{v,2,i}^{(0)}$	$F_{a,1,i}^{(0)}$	$F_{a,2,i}^{(0)}$	$F_{s,i}^{(0)}$	$F_{p,i}^{(0)}$
1	2668	2734	2720	2347	2315	2347
2	593	637	605	511	505	511
3	875	920	895	759	753	759

example, the coefficient in front of x^{2734} in $p_{v,2,1,48}(x)$ is an integer with 358 digits. Nevertheless, the orders of the differential equations, albeit high, are still manageable. The highest of them is 48 [for $F_{v,2,1}^{(0)}(x)$], which means that only 48 initial conditions are needed to determine the solution. This is in sharp contrast with the method discussed in the previous section, where the expansions had to be evaluated at hundreds of different points in order to obtain precise enough coefficients. When we insert the power-log expansion (3.16) in (3.14), we therefore find that all of the coefficients can be written in terms of $n_{I,i}$ of them. We will refer to this subset of $n_{I,i}$ coefficients as the base

coefficients.[5] The relations between the base coefficients and the rest can be obtained by observing that Eq. (3.14) will be satisfied after inserting (3.16) only if the collected coefficients for each power of the log and each power in x vanish.

Notice the difference of the present method with the one used in Refs. [16,17], where the systems of differential equations obeyed by the master integrals were solved numerically in terms of power-log expansions. This required one initial condition per integral since the differential equations are of first order for each integral in the system. In our case, we solve differential equations which are obeyed directly by the form factors themselves. This allows us to get higher numerical precision and express our results in terms of rational numbers and a few known constants, as we will see in the next section.

In Table III we can see the list of base coefficients that are left to be determined by initial conditions in each case. In that Table, we use the following sets of pairs of indices (i, j) to identify the corresponding base coefficients $c_{I,k}(i, j)$,

$$W_1 = \{(5, -1), (4, -1), \ldots, (4, 1), (3, -1), \ldots (3, 2), (2, -1), \ldots, (2, 5), (1, -1), \ldots, (1, 6), (0, -2), \ldots, (0, 17)\},$$
$$W_2 = \{(3, -1), (2, -1), (2, 1), (1, -1), \ldots, (1, 4), (0, -2), \ldots, (0, 6)\},$$
$$W_3 = \{(4, -1), (3, -1), (2, -1), (2, 0), (2, 1), (1, -1), \ldots, (1, 5), (0, -2), \ldots, (0, 8)\}. \tag{3.17}$$

The following are some examples of the relations between the remaining coefficients of $F_{v,2,2}^{(0)}(x)$ and the ones shown in Table III,

$$c_{v,2,2}(4, 2) = -\frac{1}{4} c_{v,2,2}(3, -1), \tag{3.18}$$

$$c_{v,2,2}(3, -2) = 0, \tag{3.19}$$

$$c_{v,2,2}(3, 1) = \frac{27}{31} c_{v,2,2}(0, -2) + \frac{1499}{54} c_{v,2,2}(3, -1), \tag{3.20}$$

$$c_{v,2,2}(3, 2) = -\frac{14}{31} c_{v,2,2}(0, -2) - \frac{1}{3} c_{v,2,2}(2, -1) - \frac{521}{486} c_{v,2,2}(3, -1), \tag{3.21}$$

$$c_{v,2,2}(2, -2) = 0, \tag{3.22}$$

$$c_{v,2,2}(2, 3) = \frac{655}{186} c_{v,2,2}(0, -2) + \frac{329}{18} c_{v,2,2}(2, -1) + c_{v,2,2}(2, 1) + \frac{25307 c_{v,2,2}(3, -1)}{5832}, \tag{3.23}$$

$$c_{v,2,2}(1, -2) = 0, \tag{3.24}$$

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[5]We have some freedom in the choice of coefficients $c_{I,k}(i, j)$ in the basis, of course, but in general it is a good idea to choose them with the lowest possible values of j.

TABLE III. Base coefficients to be determined by initial conditions.

Coefficients	Values of i and j
$c_{v,1,1}(i,j),\ c_{a,1,1}(i,j)$	$(i,j) \in W_1 \cup \{(2,6),(1,7),(0,18)\}$
$c_{v,2,1}(i,j)$	$(i,j) \in W_1 \cup \{(5,0),(2,6),(1,7),(1,8),(0,18)\}$
$c_{a,2,1}(i,j),\ c_{s,1}(i,j),\ c_{p,1}(i,j)$	$(i,j) \in W_1$
$c_{v,1,2}(i,j),\ c_{a,1,2}(i,j)$	$(i,j) \in W_2 \cup \{(0,7),(0,8)\}$
$c_{v,2,2}(i,j)$	$(i,j) \in W_2 \cup \{(2,0),(0,7),(0,8),(0,9)\}$
$c_{a,2,2}(i,j),\ c_{s,2}(i,j),\ c_{p,2}(i,j)$	$(i,j) \in W_2$
$c_{v,1,3}(i,j),\ c_{a,1,3}(i,j)$	$(i,j) \in W_3 \cup \{(1,6),(0,9)\}$
$c_{v,2,3}(i,j)$	$(i,j) \in W_3 \cup \{(3,0),(1,6),(0,9)\}$
$c_{a,2,3}(i,j),\ c_{s,3}(i,j),\ c_{p,3}(i,j)$	$(i,j) \in W_3$

$$c_{v,2,2}(1,5) = -\frac{28281678020281 c_{v,2,2}(0,-2)}{158513905800} + \frac{994883 c_{v,2,2}(1,-1)}{1578195} + \frac{6235153 c_{v,2,2}(1,0)}{526065} + \frac{994883 c_{v,2,2}(1,1)}{1578195}$$
$$-\frac{11628593 c_{v,2,2}(1,2)}{1052130} + \frac{583312 c_{v,2,2}(1,3)}{1578195} + \frac{539344 c_{v,2,2}(1,4)}{105213} + \frac{576002152169 c_{v,2,2}(2,-1)}{7670027700}$$
$$-\frac{881799097 c_{v,2,2}(2,0)}{14203755} + \frac{16510826 c_{v,2,2}(2,1)}{14203755} - \frac{11311570189148899 c_{v,2,2}(3,-1)}{6212722437000}, \tag{3.25}$$

$$c_{v,2,2}(1,6) = -\frac{21830660599313 c_{v,2,2}(0,-2)}{36986578020} + \frac{2765144 c_{v,2,2}(1,-1)}{526065} + \frac{23030548 c_{v,2,2}(1,0)}{526065} + \frac{2765144 c_{v,2,2}(1,1)}{526065}$$
$$-\frac{7082437 c_{v,2,2}(1,2)}{175355} - \frac{2765144 c_{v,2,2}(1,3)}{526065} + \frac{10258102 c_{v,2,2}(1,4)}{526065} + \frac{1439537964281 c_{v,2,2}(2,-1)}{8948365650}$$
$$-\frac{181544612 c_{v,2,2}(2,0)}{946917} + \frac{51133604 c_{v,2,2}(2,1)}{4734585} - \frac{272690648223213781 c_{v,2,2}(3,-1)}{50737233235500}, \tag{3.26}$$

etc. In general, for a given value of i and k in $c_{I,k}(i,j)$, the numbers in front of the coefficients on the right-hand side of these relations keep increasing as we increase j. We can see this clearly in the case of Eqs. (3.25) and (3.26) by writing the numbers in front of the coefficients in floating point.

$$c_{v,2,2}(1,5) = -178.418 c_{v,2,2}(0,-2) + 0.630 c_{v,2,2}(1,-1) + 11.852 c_{v,2,2}(1,0) + 0.630 c_{v,2,2}(1,1) - 11.052 c_{v,2,2}(1,2)$$
$$+ 0.369 c_{v,2,2}(1,3) + 5.126 c_{v,2,2}(1,4) + 75.098 c_{v,2,2}(2,-1) - 62.082 c_{v,2,2}(2,0)$$
$$+ 1.162 c_{v,2,2}(2,1) - 1820.710 c_{v,2,2}(3,-1), \tag{3.27}$$

$$c_{v,2,2}(1,6) = -590.232 c_{v,2,2}(0,-2) + 5.256 c_{v,2,2}(1,-1) + 43.779 c_{v,2,2}(1,0) + 5.256 c_{v,2,2}(1,1) - 40.389 c_{v,2,2}(1,2)$$
$$- 5.256 c_{v,2,2}(1,3) + 19.499 c_{v,2,2}(1,4) + 160.872 c_{v,2,2}(2,-1) - 191.722 c_{v,2,2}(2,0)$$
$$+ 10.800 c_{v,2,2}(2,1) - 5374.567 c_{v,2,2}(3,-1). \tag{3.28}$$

Given that we will determine the base coefficients in Table III numerically with a certain precision, in principle we might lose precision on the coefficients on the left-hand side of relations like the ones given above, due to possible large cancellations. However, in general, we have observed that this does not happen. The coefficients on the left-hand side maintain the precision and are of a similar size to the largest rational number on the right-hand side of these relations.

Similar relations can be obtained for all other form factors using the differential equations. In principle, we can obtain relations as deep in the expansions around $x = 0$ as we want, only limited by computational power. We have obtained relations for the coefficients $c_{I,k}(i,j)$ as deep as $j = 3000$ in the most complicated cases.

Once we have these relations, all that remains to be done is to determine the base coefficients numerically using initial conditions. To do so, first we insert these relations in Eq. (3.16) in order to obtain expansions involving only these coefficients. As we will emphasize below, the more relations we insert, and therefore, the more coefficients of (3.16) we include, the more precision we gain for the base coefficients.

TABLE IV. Average number of correct decimal places for the base coefficients obtained using 3000, 5000, and 10000 coefficients for the truncated expansion around $x = 1$, and 500 and 1000 coefficients per power of the log for the expansion around $x = 0$.

		$x = 1$:								
		3000			5000			10000		
		$F_{I,1}^{(0)}$	$F_{I,2}^{(0)}$	$F_{I,3}^{(0)}$	$F_{I,1}^{(0)}$	$F_{I,2}^{(0)}$	$F_{I,3}^{(0)}$	$F_{I,1}^{(0)}$	$F_{I,2}^{(0)}$	$F_{I,3}^{(0)}$
$x = 0$:	500	2	90	28	22	134	57	70	205	110
	1000	5	120	36	38	185	74	110	300	150

After the relations are inserted, we can then proceed in a similar way as we did in the previous section, i.e., truncated versions of the expansions around $x = 0$ and around $x = 1$ (and their derivatives with respect to x) can be evaluated at different points in x, and the results can be equated as in Eq. (3.6). This produces a linear system of equations for the base coefficients that we can solve numerically, as long as the number of points and derivatives is chosen in such a way that the number of equations equals the order of the corresponding differential equation,[6] or equivalently, the number of base coefficients. In this way, we only need to determine $n_{I,i}$ coefficients as supposed to the hundreds of them that we needed to determine with the method of Sec. III A. This is a more efficient way of dealing with the problem that makes it easier to achieve very high precision. We remark that the entries of the corresponding matrix can be obtained with arbitrary precision, since they are given in terms of linear combinations of powers of x and of the log over rational numbers. The precision of the right-hand side of the equations can be increased dramatically by increasing the number of terms of the expansion around $x = 1$, which can be determined by the available recurrences.

As we will see later, we were able to obtain the base coefficients with a precision of more than a thousand decimal places. For physical applications, the precision needed is far below this. However, in the next subsection we will see that having such a high precision allows us to express our results in ways that go beyond what can be achieved with the method of Sec. III A or the method of

Refs. [16,17]. In fact, the results we will present may in a sense be considered as exact solutions.

In Table IV, we show the numerical precision that is achieved for the base coefficients using different configurations of the number of coefficients in the truncated expansions around $x = 1$ and $x = 0$. We took 3000, 5000, and 10000 coefficients for the expansion around $x = 1$ in combination with 500 and 1000 coefficients per power of the log for the expansion around $x = 0$. The precision of the base coefficients is quite similar for all of the base coefficients of any given $F_{I,k}^{(0)}(x)$, and also for different values of I with k fixed, so the numbers shown in Table IV are an average of those precisions and represent the number of correct decimal places that can be obtained in each case. We can see that the least precise base coefficients are those of $F_{I,1}^{(0)}(x)$, with a precision of only a few digits when 3000 and 500 (per power of the log) expansions coefficients are used for the expansions around $x = 1$ and $x = 0$, respectively. The precision increases as expected when we use deeper expansions. In all cases the base coefficients were obtained by evaluating directly at a few points in x without taking derivatives, choosing a set of points that optimizes the precision.[7] The precision could presumably be increased if derivatives were also used. However, even this straightforward evaluation at the points gives pretty good precision using a reasonably small number of coefficients for the expansions. We can see in Table IV that taking 5000 coefficients for the expansion around $x = 1$ and 1000 coefficients per power of the log around $x = 0$ leads to a precision of at least 38 decimal places for the base coefficients. This number of coefficients is much lower than the number of coefficients required to guess the differential equations themselves, and the precision achieved is more than enough for physical applications. After we combine the vector form factors $F_{v,1,i}^{(0)}(x)$ and $F_{v,2,i}^{(0)}(x)$ using Eqs. (2.26) and (2.27) to get $F_{V,1,i}^{(0)}(x)$, followed by inserting the results in Eq. (2.25), replacing x by $\hat{s}$, Eq. (2.3) subtracting the infrared divergences, and expanding everything, including the solvable part $F_{V,1}^{(0),\text{sol}}(x)$, in $\hat{s}$, we reproduce the result given in Refs. [16,17], and give more digits

$$
\begin{aligned}
F_{V,1}^{(0)}(x) = &-32.897722507313999578 n_h^2 - 87.77373705582837517 n_h n_l + 1147.4240120750982840 n_h \\
&+ (-27.400233959497856255 n_h^2 - 60.87855567202544303 n_h n_l + 820.7777180906884097 n_h) l_s \\
&+ (-8.632679058651395941 n_h^2 - 17.265358117302791882 n_h n_l + 186.38969938287483682 n_h) l_s^2
\end{aligned}
$$

---

[6]If we produce more equations, in principle the extra ones will be linearly dependent on the others and should not cause any harm, but we are looking for numerical solutions, and doing this may cause numerical instabilities.

[7]This is done by scanning different sets of points like those in Eq. (3.5) and choosing the ones that give the highest precision.

$$+ (-1.2510288065843621399 n_h^2 - 2.5020576131687242798 n_h n_l + 10.547828331577832952 n_h) l_s^3$$
$$+ (-0.098765432098765432210 n_h^2 - 0.19753086419753086420 n_h n_l - 1.6790123456790123457 n_h) l_s^4$$
$$- 0.39506172839506172840 n_h l_s^5 + [-253.82786033439203372 n_h^2 - 404.93106546060673261 n_h n_l$$
$$+ 6970.035736369961772 n_h + (-154.28901373485052021 n_h^2 - 185.93693993861074771 n_h n_l$$
$$+ 3125.4534231017452606 n_h) l_s + (-38.518518518518518519 n_h^2 - 53.33333333333333333 n_h n_l$$
$$+ 526.75432008601286 81 n_h) l_s^2 + (-5.9259259259259259926 n_h^2 - 7.1111111111111111111 n_h n_l$$
$$+ 49.976937394787551232 n_h) l_s^3 - 19.193415637860082305 n_h l_s^4$$
$$+ 0.049382716049382716049 n_h l_s^5] \frac{1}{\hat{s}} + O\left(\frac{1}{\hat{s}^2}\right), \tag{3.29}$$

where

$$l_s = \ln\left(-\frac{1}{\hat{s}}\right) \tag{3.30}$$

and n_l is the number of light quarks. In order to compare these numbers with the ones in [16,17], we need to replace the color factors for $SU(3)_c$ in that reference. We will perform a thorough numerical comparison of our results to the ones of [16,17] in Sec. V below. We can gain confidence in the precision of our results by increasing as much as possible the precision of the base coefficients, which can be done to almost arbitrary levels, and will be the topic of the next section.

C. Coefficients in terms of known constants

There is a total of 530 base coefficients, shown in Table III. This number corresponds to the sum of the orders of all the differential equations given in Table I. It turns out that all of these coefficients can be expressed in terms of linear combinations of known constants over rational numbers. To achieve this, we needed to substantially increase the precision of the base coefficients. Using over 3000 coefficients per power of the log for the expansion at $x = 0$, written in terms of the base coefficients through the corresponding relations, and using 500000 coefficients for the expansion around $x = 1$, we were able to obtain the base coefficients with over a thousand correct decimal places. Such a high precision allowed us to fully exploit the PSLQ algorithm [72] implemented in *Mathematica*. We found that all of the base coefficients in the case of $F_{I,2}^{(0)}(x)$ (for all values of I) can be expressed in terms of the following set of constants,

$$\{\zeta_2, \zeta_3, \zeta_4, l_2, l_2^2, \zeta_2 l_2\}. \tag{3.31}$$

Similarly, the base coefficients in the case of $F_{I,3}^{(0)}(x)$ can be written as linear combinations over rational numbers of the constants,

$$\left\{\zeta_2, \zeta_3, l_2, \zeta_2 l_2, \frac{1}{\pi^2}, \frac{\zeta_3}{\pi^2}, \frac{\zeta_5}{\pi^2}\right\}. \tag{3.32}$$

In the case of $F_{I,1}^{(0)}(x)$, the following nineteen constants were needed in order to express all base coefficients,

$$\left\{\zeta_2, \zeta_3, \zeta_4, \zeta_5, \zeta_6, \zeta_2\zeta_3, \zeta_3^2, l_2^4, \zeta_2 l_2^4, \zeta_2 l_2^2, \zeta_4 l_2^2, \zeta_3 l_2, \zeta_4 l_2, \zeta_2 \zeta_3 l_2, a_4, \zeta_2 a_4, \frac{\zeta_3}{\pi^2}, \frac{\zeta_3^2}{\pi^2}, \frac{\zeta_3 \zeta_5}{\pi^2}\right\}. \tag{3.33}$$

Since every coefficient in the expansions can be written in terms of the base coefficients, all coefficients can be written in terms of these constants. Here are some examples of the corresponding expressions,

$$c_{a,2,1}(5, -1) = \frac{352}{81}, \tag{3.34}$$

$$c_{p,1}(5, -1) = -\frac{352}{81}, \tag{3.35}$$

$$c_{s,3}(4, -1) = \frac{952}{81\pi^2}, \tag{3.36}$$

$$c_{p,3}(3, 1) = \frac{160}{9} - \frac{193424}{81\pi^2}, \tag{3.37}$$

$$c_{a,1,2}(0,-2) = -\frac{177152}{81}l_2, \quad (3.38)$$

$$c_{v,2,2}(3,0) = \frac{6592}{27} + 128l_2, \quad (3.39)$$

$$c_{a,1,1}(3,0) = \frac{10048}{27} + \frac{47872\zeta_2}{81} + \frac{98560\zeta_3}{81\pi^2}, \quad (3.40)$$

$$c_{a,2,2}(2,1) = \frac{58816}{27} - 192\zeta_2 - 256l_2, \quad (3.41)$$

$$c_{s,1}(3,1) = \frac{888016}{729} - \frac{10880}{81}\zeta_2 - \frac{992}{9}\zeta_3 + \frac{10192}{27\pi^2}\zeta_3, \quad (3.42)$$

$$c_{p,1}(2,1) = \frac{2709004}{729} + \frac{15536}{27}\zeta_2 - \frac{96}{5}\zeta_2^2 - \frac{24272}{27}\zeta_3$$
$$+ \frac{3925264}{243\pi^2}\zeta_3 - 896\frac{\zeta_3^2}{\pi^2}, \quad (3.43)$$

$$c_{p,1}(1,0) = -\frac{1551536}{243} - \frac{6656}{9}a_4 - \frac{832}{27}l_2^4 + \frac{399584}{243}\zeta_2$$
$$+ \frac{1664}{9}\zeta_2 l_2^2 + \frac{11776}{135}\zeta_2^2 + \frac{481120}{243}\zeta_3$$
$$- \frac{5824}{9}\zeta_3 l_2 - \frac{101696}{27\pi^2}\zeta_3 + \frac{17920}{3\pi^2}\zeta_3^2, \quad (3.44)$$

$$c_{a,1,1}(0,2) = -\frac{182732572}{2187} - \frac{544768a_4}{81} - \frac{68096l_2^4}{243}$$
$$- \frac{5075968\zeta_2}{243} + \frac{136192l_2^2\zeta_2}{81} - \frac{942976\zeta_2^2}{27}$$
$$+ \frac{2048l_2\zeta_2^2}{9} + \frac{32441728\zeta_3}{729} - \frac{318976l_2\zeta_3}{81}$$
$$+ \frac{174528536\zeta_3}{2187\pi^2} + \frac{626432\zeta_2\zeta_3}{405} - \frac{28672\zeta_3^2}{3\pi^2}$$
$$+ \frac{45440\zeta_5}{27}, \quad (3.45)$$

$$c_{s,1}(0,3) = \frac{3841800953}{6561000} - \frac{40960a_4}{81} - \frac{5120l_2^4}{243} + \frac{2843353\zeta_2}{729} + \frac{10240l_2^2\zeta_2}{81} - \frac{1903352\zeta_2^2}{405} - \frac{512}{9}l_2\zeta_2^2 + \frac{1664\zeta_2^3}{7}$$
$$+ \frac{47204788\zeta_3}{729} - \frac{17920l_2\zeta_3}{81} + \frac{55173839\zeta_3}{4374\pi^2} + \frac{2799872\zeta_2\zeta_3}{405} - 128\zeta_3^2 + \frac{1132768\zeta_3^2}{27\pi^2}$$
$$- \frac{175232\zeta_5}{9} + \frac{15008\zeta_3\zeta_5}{\pi^2}. \quad (3.46)$$

More examples can be found in Appendix A, where we give the explicit expressions for the first few coefficients $c_{I,k}(i,j)$, with $j = -2,\dots,3$ in the vector cases $I \in \{(v,1),(v,2)\}$.

The presence of constants with π^2 in the denominator, such as $\frac{\zeta_3}{\pi^2}$, $\frac{\zeta_5}{\pi^2}$, etc., is rather surprising, since such constants do not typically appear in these types of calculations. However, when we combine all the nonsolvable parts in Eq. (2.25), it turns out that these constants disappear. We illustrate this later in an explicit example.

Notice that in some cases the coefficients turned out to be quite simple, such as in the case of Eqs. (3.34) and (3.35), or even Eq. (3.36). In fact, it is precisely using this type of relations that we were able to estimate and improve the precision of the coefficients in the previous section. We can start by choosing some arbitrary intermediate set of points for the matching conditions, and then the simplicity of the coefficients under consideration will most likely allow us to determine them using the PSLQ algorithm, even if the choice of points is suboptimal. Once this is done, we can change the parameters of the matching conditions, as described in Sec. III A, to increase the precision systematically. In particular, we may scan different sets of intermediate points in the region $0 < x < 1$ until we find one that maximizes the precision.

The precision required to obtain relations such as the ones shown in Eqs. (3.34)–(3.46) depends on the coefficient under consideration. In general, the lower the values of i, and the higher the values of j in $c_{I,k}(i,j)$, the higher the precision is needed to obtain the coefficient. For example, Eqs. (3.34)–(3.36) require less than 13 digits in the PSLQ algorithm. Eq. (3.42) can be derived using just 20 digits, and Eq. (3.46) required 192 digits. Now consider the relation for the base coefficient $c_{v,2,1}(0,18)$, namely,

$$c_{v,2,1}(0,18) = 4096\zeta_2 a_4 + \frac{10923574219252432}{1527701175}a_4 - \frac{112576}{105}\zeta_2^3 - \frac{417009593483425474341751033}{167122870039125}\zeta_2^2$$
$$- 1024\zeta_2^2 l_2^2 + \frac{512}{9}\zeta_2^2 l_2 + \frac{25177667228688306512}{11486475}\zeta_2\zeta_3 + 3584\zeta_2\zeta_3 l_2 + \frac{512}{3}\zeta_2 l_2^4$$

$$
\begin{aligned}
&- \frac{1321519442493274234073574423 11}{2661099315365336289444000}\zeta_2 - \frac{2730893554813108}{1527701175}\zeta_2 l_2^2 + \frac{3971513592381793156831 9382}{682134163425\pi^2}\zeta_3^2 \\
&+ \frac{4096}{3}\zeta_3^2 - 21504\frac{\zeta_3\zeta_5}{\pi^2} - \frac{16565470270967358515914839837073806187910489333124 5527}{21136363518241796368643719716096000 00\pi^2}\zeta_3 \\
&+ \frac{33016255719498077965765879287030858 299}{46685952901146250692 00}\zeta_3 + \frac{2047720334163982}{654729075}\zeta_3 l_2 - \frac{36745784642748584464}{2297295}\zeta_5 \\
&+ \frac{1191970705684659284678506108933357745439542512230393299 47645441}{1227021800517714797366534263132719973286400000 0} \\
&+ \frac{1365446777406554}{4583103525}l_2^4.
\end{aligned} \tag{3.47}
$$

To obtain such a relation, we need to compute this coefficient with at least 1400 digits. A few hundred additional decimal places allowed us to make sure that the relation was stable. Another way we made sure that relations like these are not spurious was by introducing extra constants while applying the PSLQ algorithm and noticing that the corresponding coefficient vanishes (this is extremely unlikely unless the relation is correct).

Having the expansions in terms of the constants in (3.31)–(3.33), we can now insert them in Eqs. (2.26)–(2.29) in the case of the vector and axial-vector form factors, and in Eqs. (2.14) and (2.15) in the case of the scalar and pseudoscalar form factors, respectively, after which we can expand in x the terms coming from the projectors to obtain a full expansion of the nonsolvable parts in Eq. (2.25). In the case of $F_{V,1}^{(0)}(x)$, up to $O(x^3)$ we obtain

$$
\begin{aligned}
F_{V,1}^{(0)}(x) = F_{V,1}^{(0),\text{sol}}(x) + n_h \Bigg\{ &-\frac{44288}{81x}l_2\zeta_2 - 1024 l_2\zeta_2^2 - \frac{1120}{81}l_2^2\zeta_2 - \frac{1168}{243}l_2^4 - \frac{1640}{9}\zeta_2\zeta_3 + \frac{3964}{9}\zeta_5 + \frac{5068}{27}\zeta_2^2 - \frac{7226}{9} \\
&- \frac{9344}{81}a_4 + \frac{46324}{81}\zeta_3 + \frac{303056}{81}l_2\zeta_2 + \frac{1862569}{243}\zeta_2 + \left(\frac{64}{27}l_2^4 + \frac{512}{9}a_4 - \frac{512}{9}l_2^2\zeta_2 + \frac{15520}{27}l_2\zeta_2 + \frac{21728}{135}\zeta_2^2 \right. \\
&+ \frac{34660}{243}\zeta_2 - \frac{55676}{81}\zeta_3 + \left. \frac{670460}{243}\right)\ln(x) + \left(\frac{128}{9}l_2\zeta_2 - \frac{512}{9}\zeta_3 + \frac{2264}{81}\zeta_2 + \frac{18274}{27}\right)\ln^2(x) \\
&+ \left(60 - \frac{32}{3}\zeta_2\right)\ln^3(x) - \frac{172}{243}\ln^4(x) - \frac{176}{81}\ln^5(x) + \left[544\zeta_2\zeta_3 - 4688\zeta_5 - \frac{5056}{243}l_2^4 + \frac{36032}{81}l_2^2\zeta_2 - \frac{40448}{81}a_4 \right. \\
&+ \frac{41728}{9}l_2\zeta_2^2 - \frac{92176}{27}\zeta_3 + \frac{235288}{243} + \frac{355376}{135}\zeta_2^2 - \frac{722464}{81}l_2\zeta_2 - \frac{5251684}{243}\zeta_2 + \left(-800\zeta_3 + \frac{512}{27}l_2^4 \right. \\
&- \frac{640}{9}l_2^2\zeta_2 + \frac{4096}{9}a_4 + \frac{9376}{15}\zeta_2^2 - \frac{24272}{9}\zeta_2 - \frac{55168}{27}l_2\zeta_2 + \left.\frac{105760}{243}\right)\ln(x) + \left(\frac{1744}{9}\zeta_3 - \frac{3392}{9}l_2\zeta_2 \right. \\
&- \frac{27968}{81}\zeta_2 + \left.\frac{87068}{243}\right)\ln^2(x) + \left(-\frac{7312}{81}\zeta_2 + \frac{10496}{243}\right)\ln^3(x) + \frac{8480}{243}\ln^4(x) - \frac{44}{405}\ln^5(x)\bigg]x + \bigg[5120 a_4\zeta_2 \\
&- 11520 l_2\zeta_2^2 - 1280 l_2^2\zeta_2^2 - 304\zeta_3^2 + \frac{640}{3}l_2^4\zeta_2 + \frac{2768}{35}\zeta_2^3 - \frac{50704}{9}\zeta_2\zeta_3 + \frac{72704}{243}l_2^4 - \frac{249088}{81}l_2^2\zeta_2 \\
&+ \frac{328544}{9}\zeta_3 + \frac{581632}{81}a_4 + \frac{1230400}{27}l_2\zeta_2 + \frac{1581448}{27}\zeta_5 - \frac{2711600}{81}\zeta_2^2 + \frac{9890900}{81}\zeta_2 + \frac{116957693}{13122} \\
&+ \left(-128\zeta_2\zeta_3 + 1088\zeta_5 + \frac{640}{27}l_2^4 + \frac{5120}{9}a_4 - \frac{5120}{9}l_2^2\zeta_2 + \frac{21560}{3}\zeta_3 - \frac{68720}{9}\zeta_2^2 + \frac{241664}{27}l_2\zeta_2 \right. \\
&+ \frac{391184}{27}\zeta_2 + \left.\frac{47925197}{2187}\right)\ln(x) + \left(1280 a_4 - 320 l_2^2\zeta_2 + \frac{160}{3}l_2^4 - \frac{1512}{5}\zeta_2^2 + \frac{42368}{9}l_2\zeta_2 - \frac{79840}{27}\zeta_3 \right. \\
&+ \frac{121480}{81}\zeta_2 + \left.\frac{4100885}{729}\right)\ln^2(x) + \left(448 l_2\zeta_2 + 32\zeta_3 + \frac{87088}{81}\zeta_2 - \frac{761942}{729}\right)\ln^3(x) + \left(4\zeta_2 - \frac{56711}{243}\right)\ln^4(x)
\end{aligned}
$$

$$+\frac{154}{81}\ln^5(x)\Big]x^2 + \Big[-22528a_4\zeta_2 + 5632 l_2^2\zeta_2^2 - 448\zeta_3^2 - \frac{2816}{3}l_2^4\zeta_2 - \frac{147104}{35}\zeta_2^3 + \frac{222976}{9}l_2\zeta_2^2$$

$$-\frac{393184}{3}l_2\zeta_2 - \frac{557504}{243}l_2^4 + \frac{739936}{27}\zeta_2\zeta_3 + \frac{1400128}{81}l_2^2\zeta_2 - \frac{4460032}{81}a_4 - \frac{7842800}{27}\zeta_5 + \frac{7863616}{45}\zeta_2^2$$

$$-\frac{10205680}{81}\zeta_3 - \frac{84004240}{2187} - \frac{300606748}{729}\zeta_2 + \Big(640\zeta_2\zeta_3 - 12352\zeta_5 + \frac{4480}{9}l_2^2\zeta_2 + \frac{6784}{27}l_2^4 + \frac{21248}{27}l_2\zeta_2$$

$$+\frac{54272}{9}a_4 + \frac{672448}{15}\zeta_2^2 - \frac{1037608}{2187} - \frac{6858880}{81}\zeta_3 - \frac{11355152}{243}\zeta_2\Big)\ln(x) + \Big(-5632a_4 + 1408 l_2^2\zeta_2 - \frac{704}{3}l_2^4$$

$$+\frac{6672}{5}\zeta_2^2 - \frac{170816}{9}l_2\zeta_2 + \frac{184880}{27}\zeta_2 + \frac{390416}{27}\zeta_3 - \frac{1958972}{81}\Big)\ln^2(x) + \Big(-3200 l_2\zeta_2 - 256\zeta_3 - \frac{511504}{81}\zeta_2$$

$$+\frac{6621976}{729}\Big)\ln^3(x) + \Big(-8\zeta_2 + \frac{399020}{243}\Big)\ln^4(x) - \frac{40264}{405}\ln^5(x) - \frac{34}{15}\ln^6(x)\Big]x^3 + O(x^4)\Big\}. \tag{3.48}$$

Replacing x in terms of $\hat{s}$ using Eq. (2.4), and expanding everything (including the solvable part, as well as the renormalization and infrared subtraction terms) in $\hat{s} \to -\infty$ up to $O(1/\hat{s}^3)$, we obtain the following result for the term proportional to n_h [which we call $F_{V,1}^{n_h}(\hat{s})$],

$$F_{V,1}^{n_h}(\hat{s}) = \frac{64}{3}l_2\zeta_2^2 - \frac{464}{3}\zeta_2\zeta_3 - \frac{3808}{243}l_2^4 - \frac{4856}{9}\zeta_5 + \frac{5828}{27}\zeta_2^2 + \frac{7616}{81}l_2^2\zeta_2 - \frac{30464}{81}a_4 - \frac{52708}{81}\zeta_3 - \frac{107008}{81}l_2\zeta_2$$

$$+\frac{281176}{243}\zeta_2 + \frac{461144}{243} + \Big(-\frac{64}{81}l_2^4 + \frac{128}{27}l_2^2\zeta_2 + \frac{368}{27}\zeta_2^2 - \frac{512}{27}a_4 - \frac{1804}{9}\zeta_3 + \frac{47936}{243}\zeta_2 + \frac{171704}{243}\Big)l_s$$

$$+\Big(-\frac{160}{9}\zeta_3 + \frac{896}{81}\zeta_2 + \frac{46064}{243}\Big)l_s^2 + \Big(-\frac{416}{81}\zeta_2 + \frac{4616}{243}\Big)l_s^3 - \frac{136}{81}l_s^4 - \frac{32}{81}l_s^5 + \Big[-\frac{1664}{27}\zeta_2\zeta_3 + \frac{2560}{243}l_2^4$$

$$+\frac{4672}{9}\zeta_2^2 - \frac{5120}{81}l_2^2\zeta_2 - \frac{5632}{9}\zeta_5 + \frac{20480}{81}a_4 - \frac{54968}{27}\zeta_3 - \frac{70400}{3}l_2\zeta_2 + \frac{329392}{243} + \frac{5037584}{243}\zeta_2$$

$$+\Big(-1632\zeta_3 + \frac{928}{27}\zeta_2^2 + \frac{96304}{81}\zeta_2 + \frac{738352}{243}\Big)l_s + \Big(-\frac{16}{27}\zeta_3 + \frac{6896}{81}\zeta_2 + \frac{94144}{243}\Big)l_s^2 + \Big(\frac{64}{243} + \frac{272}{9}\zeta_2\Big)l_s^3$$

$$-\frac{4664}{243}l_s^4 + \frac{4}{81}l_s^5\Big]\frac{1}{\hat{s}} + \Big[-304\zeta_3^2 - \frac{128}{3}l_2\zeta_2^2 - \frac{14984}{27}\zeta_5 + \frac{22032}{35}\zeta_2^3 - \frac{45136}{27}\zeta_2\zeta_3 + \frac{85312}{243}l_2^4 - \frac{170624}{81}l_2^2\zeta_2$$

$$+\frac{682496}{81}a_4 - \frac{765568}{405}\zeta_2^2 - \frac{1017860}{81}\zeta_3 + \frac{3636101}{13122} - \frac{11654656}{81}l_2\zeta_2 + \frac{34490284}{243}\zeta_2 + \Big(1168\zeta_5 - 288\zeta_2\zeta_3$$

$$+\frac{128}{81}l_2^4 - \frac{256}{27}l_2^2\zeta_2 + \frac{1024}{27}a_4 - \frac{19456}{27}l_2\zeta_2 - \frac{24976}{135}\zeta_2^2 - \frac{1149688}{81}\zeta_3 + \frac{2440840}{243}\zeta_2 + \frac{57528233}{2187}\Big)l_s$$

$$-\Big(\frac{824}{5}\zeta_2^2 + \frac{1856}{3}\zeta_3 - \frac{126104}{81}\zeta_2 - \frac{2518829}{729}\Big)l_s^2 + \Big(32\zeta_3 + \frac{39152}{81}\zeta_2 - \frac{446750}{729}\Big)l_s^3 + \Big(-\frac{28079}{243} + 4\zeta_2\Big)l_s^4$$

$$+\frac{1306}{405}l_s^5\Big]\frac{1}{\hat{s}^2} + \Big[-768\zeta_3^2 - \frac{4096}{3}l_2\zeta_2^2 + \frac{14176}{9}\zeta_5 + \frac{268384}{35}\zeta_2^3 - \frac{306368}{27}\zeta_2\zeta_3 + \frac{879616}{243}l_2^4 - \frac{1759232}{81}l_2^2\zeta_2$$

$$-\frac{4436752}{81}\zeta_2^2 + \frac{7036928}{81}a_4 - \frac{13444000}{243}\zeta_3 - \frac{62817280}{81}l_2\zeta_2 - \frac{145774160}{6561} + \frac{615879368}{729}\zeta_2 + \Big(-3456\zeta_2\zeta_3$$

$$+18624\zeta_5 + \frac{4096}{81}l_2^4 - \frac{8192}{27}l_2^2\zeta_2 + \frac{32768}{27}a_4 - \frac{44288}{45}\zeta_2^2 - \frac{269312}{27}l_2\zeta_2 - \frac{8223584}{81}\zeta_3 + \frac{13513568}{243}\zeta_2$$

$$+\frac{354234328}{2187}\Big)l_s + \Big(-\frac{8784}{5}\zeta_2^2 - \frac{46528}{9}\zeta_3 + \frac{788512}{81}\zeta_2 + \frac{10524520}{729}\Big)l_s^2 + \Big(384\zeta_3 + \frac{361600}{81}\zeta_2$$

$$-\frac{5575000}{729}\Big)l_s^3 + \Big(24\zeta_2 - \frac{223328}{243}\Big)l_s^4 + \frac{1652}{27}l_s^5 + \frac{34}{15}l_s^6\Big]\frac{1}{\hat{s}^3} + O\Big(\frac{1}{\hat{s}^4}\Big), \tag{3.49}$$

which, of course, agrees numerically with Eq. (3.29). Similar expansions for the other form factors can be found in Appendix B. Notice that the constants with π^2 in the denominator are indeed not present in the final results (3.48) or (3.49). In fact, only the following constants appear in the final result of any of the form factors

$$\{1, \zeta_2, \zeta_3, \zeta_4, \zeta_5, \zeta_6, \zeta_2\zeta_3, \zeta_3^2, l_2^4, \zeta_2 l_2, \zeta_2 l_2^2, \zeta_2 l_2^4, \zeta_4 l_2,$$
$$\zeta_4 l_2^2, a_4, \zeta_2 a_4\}, \qquad (3.50)$$

so even the constants $\zeta_3 l_2$ and $\zeta_2\zeta_3 l_2$, which are present in (3.33), disappear from the final results. These cancellations give another strong indication that the guessed components of the individual terms using PSLQ are correct.

In general, the form factors can be written as follows,

$$F_I^{(0)}(x) = F_I^{(0),\text{sol}}(x) + n_h(F_{I,1}^{(0)}(x) + \zeta_2 F_{I,2}^{(0)}(x) + \zeta_3 F_{I,3}^{(0)}(x))$$

$$= F_I^{(0),\text{sol}}(x) + n_h \sum_{n=-1}^{\infty} \sum_{j=0}^{6} \sum_{i=1}^{16} r_{I;j,n}^{(C_i)} C_i x^n \ln^j(x),$$
$$(3.51)$$

where C_i is any of the constants in (3.50) and the $r_{I;j,n}^{(C_i)}$'s are rational numbers.

One may consider now the possibility of generating several thousand coefficients for the expansions around $x = 0$. These can be used to generate recursions by applying guessing methods on each of the series of rational numbers $r_{I;j,n}^{(C_i)}$ with i and j fixed. We did this and found that some of the recursions were solvable, allowing us to find closed form solutions in these cases. This is similar to what we did in the case of the expansions at $x = 1$ in Ref. [15]. The main difference now is the presence of powers of $\ln(x)$, but this does not represent a major difficulty. It merely means that the procedure applied in [15] needs to be repeated on each power series multiplying each power of the log.

Let us consider again the case of the nonsolvable parts of $F_{V,1}^{(0)}(x)$, and take as an example the sequence of rational numbers multiplying the term $\zeta_3 \ln^3(x)$, which is given by

$$\left\{ r_{V,1;3,n}^{(\zeta_3)} | n \ge 1 \right\} = \{0, 32, -256, 896, -2304, 4896,$$
$$- 9216, 15872, -25600, \ldots\}. \quad (3.52)$$

In other words, the rational number multiplying the term $\zeta_3 x^n \ln^3(x)$ in the expansion around $x = 0$ of the nonsolvable part of $F_{V,1}^{(0)}(x)$ with $n \ge 1$ is given by the nth term in the list (3.52). We can now use this list and the SAGE package ORE_ALGEBRA to obtain a recursion satisfied by this sequence. We get

$$(n+2)^2 f_1(n) - 4(n+1)f_1(n+1) - n^2 f_1(n+2) = 0,$$
$$(3.53)$$

where $f_1(n)$ is the nth term in the list (3.52). This recursion can be obtained using just the first 15 terms in that list. As another example, the list associated with the term $l_2\zeta_2$ $[= l_2\zeta_2 \ln^0(x)]$ is given by

$$\left\{ r_{V,1;0,n}^{(l_2\zeta_2)} | n \ge 1 \right\} = \left\{ -\frac{722464}{81}, \frac{1230400}{27}, -\frac{393184}{3}, \right.$$
$$\left. \frac{2295712}{9}, -\frac{260063392}{675}, \ldots \right\}. \quad (3.54)$$

We can verify that the nth term in the lists (3.52) and (3.54) agree with the corresponding rational numbers multiplying the corresponding factor in Eq. (3.48). A recursion of order 7 can also be found in the case of the list (3.54),

$$\sum_{i=0}^{7} p_i(n) f_2(n+i) = 0, \qquad (3.55)$$

where $f_2(n)$ is the nth term in (3.54), and the $p_i(n)$'s are polynomials in n of degree 35 with very large integer coefficients. In order to obtain the recursion (3.55), we must apply guessing algorithms using at least the first 142 terms in (3.54).

In the case of (3.53), a very simple solution can be found with SIGMA, namely

$$f_1(n) = -2 + 2(1 - 4n^2 + 2n^4)(-1)^n, \quad (3.56)$$

while in the case of the $l_2\zeta_2$ term, the solution found by SIGMA is somewhat more complicated

$$f_2(n) = -\frac{16T_1}{81n^2} - \frac{32(-1)^n T_2}{405n^2} + \frac{128}{9}n(112 + 13n^2)(-1)^n S_1$$
$$+ \left[-\frac{1168}{3} - \frac{32}{9}(-1)^n(413 + 264n^2 + 34n^4) \right] S_2$$
$$+ \frac{128}{9}n(-14 + 13n^2)(-1)^n S_{-1}$$
$$+ \left[-\frac{992}{9} - \frac{16}{9}(-1)^n(237 - 16n^2 + 14n^4) \right] S_{-2},$$
$$(3.57)$$

with

$$T_1 = 32n^5 + 405n^4 - 67n^3 - 432n^2 - 216n - 2052,$$
$$(3.58)$$

$$T_2 = 1032n^7 - 71100n^5 - 1080n^4 - 33847n^3 + 1080n^2$$
$$+ 1620n - 9990. \qquad (3.59)$$

TABLE V. Number of terms needed to derive recursions for the sequence of rational numbers associated to a given constant in the first column times a given power of $\ln(x)$ in the case of $F_{V,1}^{(0)}(x)$'s. A check mark indicates that the corresponding recursion was solvable. An empty cell means that the corresponding product of constant and power of the log is not present in this form factor. The interrogation sign in one of the entries indicates that no recursion could be found through guessing algorithms in this case.

	$\ln^6(x)$	$\ln^5(x)$	$\ln^4(x)$	$\ln^3(x)$	$\ln^2(x)$	$\ln^1(x)$	$\ln^0(x)$
1	9✓	136	512	3680	12100	23000	?
ζ_2		9✓	136	512	1440	2970
ζ_3			12✓	136	510	3560
$l_2\zeta_2$			15✓	18✓	70✓	140✓
l_2^4				15✓	28✓	88✓
$l_2^2\zeta_2$				15✓	28✓	88✓
a_4				15✓	28✓	88✓
ζ_4				15✓	136	510
ζ_5					15✓	136
$\zeta_2\zeta_3$					15✓	136
ζ_6						15✓
ζ_3^2						15✓
$l_2\zeta_4$						15✓
$l_2^2\zeta_4$						15✓
$l_2^4\zeta_2$						15✓
$\zeta_2 a_4$						15✓

Here the $S_k(n)$ are the single harmonics sums [39,40], which are defined as

$$S_k(n) = \sum_{i=1}^{n} \frac{(\text{sign}(k))^i}{i^{|k|}}; \tag{3.60}$$

for convenience we sometimes drop the n-dependence of the harmonic sums. We can now perform the sums. In the case of $f_1(n)$ we get

$$\sum_{n=1}^{\infty} f_1(n)x^n = \frac{32x^2(1-4x+x^2)}{(1-x)(1+x)^5}, \tag{3.61}$$

while in the case of $f_2(n)$ we obtain

$$\sum_{n=1}^{\infty} f_2(n)x^n = \frac{128\mathrm{H}_1(x)R_1}{9(1+x)^4} + \frac{128\mathrm{H}_{-1}(x)R_2}{9(1+x)^4}$$
$$- \frac{64\mathrm{H}_{0,1}(x)R_3}{9(1-x)(1+x)^5} + \frac{64\mathrm{H}_{0,-1}(x)R_4}{9(1-x)(1+x)^5}$$
$$- \frac{32xR_5}{81(1-x)^4(1+x)^6}, \tag{3.62}$$

with

$$R_1 = 3x^4 + 16x^3 + 62x^2 + 16x + 3, \tag{3.63}$$

$$R_2 = 9x^4 - 5x^3 + 98x^2 - 5x + 9, \tag{3.64}$$

$$R_3 = 57x^6 + 224x^5 + 423x^4 + 344x^3 + 423x^2$$
$$+ 224x + 57, \tag{3.65}$$

$$R_4 = 111x^6 + 104x^5 + 217x^4 - 368x^3 + 217x^2$$
$$+ 104x + 111, \tag{3.66}$$

$$R_5 = 22597x^8 - 78400x^7 + 43500x^6 + 179008x^5$$
$$- 327266x^4 + 179008x^3 + 43500x^2$$
$$- 78400x + 22597 \tag{3.67}$$

and $\mathrm{H}_{\vec{a}}(z)$ denotes a harmonic polylogarithm [42]. The term multiplying $\ln(2)\zeta_2$ in $F_{V,1}^{(0)}$ is thus given by

$$F_{\ln(2)\zeta_2,V,1}^{(0)} = -\frac{44288}{81x} + \frac{303056}{81} - \frac{44288}{81}x + \sum_{n=1}^{\infty} f_2(n)x^n. \tag{3.68}$$

We found recursions for all but one, cf. Table V, of the sequences of rational numbers multiplying all of the terms in the expansions around $x = 0$ of each form factor. In all cases where a solution could be found, we performed the corresponding sum as we did in (3.61) and (3.62), and added all of the results, leaving a remainder of nonsolvable parts. In the case of $F_{V,1}^{(0)}(x)$, we found that the formerly unsolvable parts at $x = 1$ on the right-hand side of $F_{V,1}(x)$ in Eq. (2.25) can now be resummed as follows

$$F_{V,1,1}^{(0)}(x) + \zeta_2 F_{V,1,2}^{(0)}(x) + \zeta_3 F_{V,1,3}^{(0)}(x)$$

$$= \ln(2)\left\{\frac{256\zeta_2^2 P_{50}}{9(x-1)(1+x)^5} + \left[\frac{128\ln(1+x)P_{17}}{9(1+x)^4} + \frac{64\text{Li}_2(x)P_{53}}{9(x-1)(1+x)^5} + \frac{64\text{Li}_2(-x)P_{59}}{9(x-1)(1+x)^5} - \frac{64\ln^2(x)P_{65}}{9(x-1)^2(1+x)^6}\right.\right.$$

$$- \frac{16P_{82}}{81(x-1)^4 x(1+x)^6} - \ln(1-x)\left(\frac{128P_9}{9(1+x)^4} - \frac{64\ln(x)P_{54}}{9(x-1)(1+x)^5}\right) + \ln(x)\left(\frac{64\ln(1+x)P_{58}}{9(x-1)(1+x)^5}\right.$$

$$\left.- \frac{32P_{78}}{27(x-1)^5(1+x)^5}\right) - \frac{64x^2(7-22x+7x^2)\ln^3(x)}{(x-1)(1+x)^5}\right]\zeta_2\right\} + \ln(2)^2\left\{\left[\frac{128\ln(x)P_{45}}{9(x-1)(1+x)^5} - \frac{32P_{73}}{81(x-1)^5(1+x)^5}\right.\right.$$

$$- \ln(1-x)\left(-\frac{128(1+x^2)(11-14x+11x^2)}{3(x-1)(1+x)^3} + \frac{256x^2(5-2x+5x^2)\ln(x)}{(x-1)(1+x)^5}\right) + \frac{64x^2(5-2x+5x^2)\ln^2(x)}{(x-1)(1+x)^5}$$

$$- \frac{128(1+x^2)(11-14x+11x^2)\ln(1+x)}{3(x-1)(1+x)^3} - \frac{256x^2(5-2x+5x^2)\text{Li}_2(x)}{(x-1)(1+x)^5}\right]\zeta_2 + \frac{256x^2(5-2x+5x^2)\zeta_2^2}{(x-1)(1+x)^5}\right\}$$

$$+ \ln^4(2)\left\{-\frac{64\ln(x)P_{43}}{27(x-1)(1+x)^5} - \frac{16P_{68}}{243(x-1)^5(1+x)^5} - \ln(1-x)\left[\frac{64(1+x^2)(11-14x+11x^2)}{9(x-1)(1+x)^3}\right.\right.$$

$$\left.- \frac{128x^2(5-2x+5x^2)\ln(x)}{3(x-1)(1+x)^5}\right] - \frac{128x^2(5-2x+5x^2)\zeta_2}{3(x-1)(1+x)^5} - \frac{32x^2(5-2x+5x^2)\ln^2(x)}{3(x-1)(1+x)^5}$$

$$\left.+ \frac{64(1+x^2)(11-14x+11x^2)\ln(1+x)}{9(x-1)(1+x)^3} + \frac{128x^2(5-2x+5x^2)\text{Li}_2(x)}{3(x-1)(1+x)^5}\right\} + \text{Li}_4\left(\frac{1}{2}\right)\left\{-\frac{512\ln(x)P_{43}}{9(x-1)(1+x)^5}\right.$$

$$- \frac{128P_{68}}{81(x-1)^5(1+x)^5} - \ln(1-x)\left[\frac{512(1+x^2)(11-14x+11x^2)}{3(x-1)(1+x)^3} - \frac{1024x^2(5-2x+5x^2)\ln(x)}{(x-1)(1+x)^5}\right]$$

$$\left.- \frac{256x^2(5-2x+5x^2)\ln^2(x)}{(x-1)(1+x)^5} + \frac{512(1+x^2)(11-14x+11x^2)\ln(1+x)}{3(x-1)(1+x)^3} + \frac{1024x^2(5-2x+5x^2)\text{Li}_2(x)}{(x-1)(1+x)^5}\right\}$$

$$+ \left\{-\frac{1024\text{Li}_4(\frac{1}{2})x^2(5-2x+5x^2)}{(x-1)(1+x)^5} + \frac{128x^2(1-x+x^2)\zeta_3\ln(x)}{(x-1)(1+x)^5} - \frac{4x^2\ln^4(x)}{(x-1)(1+x)^3}\right\}\zeta_2$$

$$+ \frac{16x^2(19+104x+19x^2)\zeta_3^2}{(x-1)(1+x)^5} - \frac{16x^2(173-8502x+173x^2)\zeta_2^3}{35(x-1)(1+x)^5} - \frac{64x^2(17-125x+17x^2)\zeta_5\ln(x)}{(x-1)(1+x)^5}$$

$$+ \frac{24x^2(63-26x+63x^2)\zeta_2^2\ln^2(x)}{5(x-1)(1+x)^5} - \frac{32x^2(1-4x+x^2)\zeta_3\ln^3(x)}{(x-1)(1+x)^5} + \frac{34x^3\ln^6(x)}{15(x-1)(1+x)^5} + \tilde{F}_{V,1}^{(0),\text{rest}}(x), \tag{3.69}$$

and the polynomials P_i are given in Appendix C. The function $\tilde{F}_{V,1}^{(0),\text{rest}}(x)$ includes all the terms in the expansion at $x = 0$ for which no closed form solutions could be found. Up to $O(x^3)$, we have

$$\tilde{F}_{V,1}^{(0),\text{rest}}(x) = \frac{5068}{27}\zeta_2^2 - \frac{1640}{9}\zeta_2\zeta_3 + \frac{1862569}{243}\zeta_2 + \frac{46324}{81}\zeta_3 + \frac{3964}{9}\zeta_5 - \frac{7226}{9} + \left(\frac{21728}{135}\zeta_2^2 + \frac{34660}{243}\zeta_2 - \frac{55676}{81}\zeta_3\right.$$

$$\left.+ \frac{670460}{243}\right)\ln(x) + \left(\frac{2264}{81}\zeta_2 - \frac{512}{9}\zeta_3 + \frac{18274}{27}\right)\ln^2(x) + \left(60 - \frac{32}{3}\zeta_2\right)\ln^3(x) - \frac{172}{243}\ln^4(x) - \frac{176}{81}\ln^5(x)$$

$$+ \left[\frac{355376}{135}\zeta_2^2 + 544\zeta_2\zeta_3 - \frac{5251684}{243}\zeta_2 - \frac{92176}{27}\zeta_3 - 4688\zeta_5 + \frac{235288}{243} + \left(\frac{9376}{15}\zeta_2^2 - \frac{24272}{9}\zeta_2 - 800\zeta_3\right.\right.$$

$$\left.+ \frac{105760}{243}\right)\ln(x) + \left(-\frac{27968}{81}\zeta_2 + \frac{1744}{9}\zeta_3 + \frac{87068}{243}\right)\ln^2(x) + \left(\frac{10496}{243} - \frac{7312}{81}\zeta_2\right)\ln^3(x) + \frac{8480}{243}\ln^4(x)$$

$$\left.- \frac{44}{405}\ln^5(x)\right]x + \left[-\frac{2711600}{81}\zeta_2^2 - \frac{50704}{9}\zeta_2\zeta_3 + \frac{9890900}{81}\zeta_2 + \frac{328544}{9}\zeta_3 + \frac{1581448}{27}\zeta_5 + \frac{116957693}{13122}\right.$$

$$+ \left(\frac{391184}{27}\zeta_2 - \frac{68720}{9}\zeta_2^2 + \frac{21560}{3}\zeta_3 + \frac{47925197}{2187}\right)\ln(x) + \left(\frac{121480}{81}\zeta_2 - \frac{79840}{27}\zeta_3 + \frac{4100885}{729}\right)\ln^2(x)$$

$$+ \left(\frac{87088}{81}\zeta_2 - \frac{761942}{729}\right)\ln^3(x) - \frac{56711}{243}\ln^4(x) + \frac{154}{81}\ln^5(x)\right]x^2 + \left[\frac{7863616}{45}\zeta_2^2 + \frac{739936}{27}\zeta_2\zeta_3\right.$$

$$- \frac{300606748}{729}\zeta_2 - \frac{10205680}{81}\zeta_3 - \frac{7842800}{27}\zeta_5 - \frac{84004240}{2187} + \left(\frac{672448}{15}\zeta_2^2 - \frac{11355152}{243}\zeta_2 - \frac{6858880}{81}\zeta_3\right.$$

$$- \frac{1037608}{2187}\right)\ln(x) + \left(\frac{184880}{27}\zeta_2 + \frac{390416}{27}\zeta_3 - \frac{1958972}{81}\right)\ln^2(x) + \left(\frac{6621976}{729} - \frac{511504}{81}\zeta_2\right)\ln^3(x)$$

$$+ \frac{399020}{243}\ln^4(x) - \frac{40264}{405}\ln^5(x)\right]x^3 + O(x^4). \tag{3.70}$$

We can see that the result obtained in (3.61) appears in the second to last line of Eq. (3.69) multiplying the corresponding factor $\zeta_3\ln^3(x)$, while the result obtained in (3.69) appears in the first and second lines of (3.69) multiplying the factor $\ln(2)\zeta_2$. The corresponding expressions for the other form factors can be found in Appendix C. To get the full solvable parts of the form factors at $x = 0$, we need to add the resummed expressions, like the one in Eq. (3.69), to the solvable parts arising from the expansion at $x = 1$, as shown in Eq. (2.25). Depending on the value of i and j in $r^{(C_i)}_{I;j,n}$, finding the corresponding recursion can be quite difficult, requiring sometimes the use of thousands of coefficients for the guessing algorithms. In Table V, we show the minimum number of terms required to guess the recursions for each of the constants listed in (3.50) that may appear in the power series multiplying each power of $\ln(x)$ in the case of $F^{(0)}_{V,1}(x)$. A check mark near a number indicates that a solution for the corresponding recursion was found (25 out of 42 cases). We can see that, for example, we needed 512 terms to get a recursion for the series of rational numbers defined by $r^{(\zeta_2)}_{V,1;2,n}$. The most complicated recursion that we found was the one associated to $r^{(1)}_{V,1;1,n}$, which required a sequence of 23000 terms. This was also the recursion of highest order that we derived in the vector case, with an order of 90. Finding this recursion, as well as similar ones in other form factors, pushed the limits of our computational resources, exhausting all 2 Tb of RAM memory in one of our machines. It was therefore not possible to find recursions for the sequence of numbers defined by $r^{(1)}_{I;0,n}$, which are even more demanding. This was the only case where no recursion was derived in this way.

The advantage of having all of these recursions, even in the cases where no solutions can be found, is that they can be used to calculate a very large number of expansion coefficients in a simple way compared with their direct computation from the differential equations as we did initially. We were able to compute up to 30000 coefficients for all of the constants in Table V, except the one where no recursion was found, in which case 20000 coefficients were derived by inserting the coefficients associated to the other constants to the recursion coming from the differential equations associated to this term.

IV. THRESHOLD EXPANSIONS

Power–log expansions around the two- and four–particle (pseudo)thresholds can also be found using the method presented in the previous section, provided an appropriate expansion parameter is used. The expansions around $\hat{s} = 4$ (two-particle threshold) and around $\hat{s} = 16$ (four-particle threshold), respectively, can be written in terms of the variables

$$z = \sqrt{4 - \hat{s}}, \tag{4.1}$$

$$\tilde{z} = \sqrt{16 - \hat{s}}. \tag{4.2}$$

In both cases the expansions will have the form

$$F^{(0)}_{I,i}(z) = \sum_{j=0}^{3}\sum_{k=k_{\min}}^{\infty}\tilde{c}_{I,i}(j,k)z^k\ln^j(z). \tag{4.3}$$

Notice that the variable $z = \sqrt{4 - \hat{s}}$ is similar to the velocity of the produced quarks $\beta = \sqrt{1 - 4/\hat{s}}$, but unlike the velocity, z is defined to be real for $\hat{s} < 4$, which is convenient since $F^{(0)}_{I,i}(z)$ is also real in this region, and therefore the corresponding expansion coefficients in (4.3) will be real.

The point $z = 0$ ($\hat{s} = 4$) corresponds to $x = -1$ in x-space. One might also think about finding expansions around $x = -1$ using the variable $\hat{y} = 1 + x$ instead of z. In this case, the expansions should involve powers of $\ln(\hat{y}) = \ln(1 + x)$. Deriving differential equations in $\hat{y}$ from the ones we already have would be simple, but after that we would also need to match the expansion at $\hat{y} = 0$ with the expansion at another point. So far, we have obtained the expansions at $x = 1$ and then at $x = 0$. As we can see in Fig. 1, the point $x = 1$ is directly connected to the point $x = -1$ through the unit arc. However, trying to match *directly* the expansions at $x = 1$ and $x = -1$ through this arc is not possible. The radius of convergence of the expansion around $x = 1$ is equal to one, which means that for $x = e^{i\phi}$ the expansion diverges for $\phi > \pi/3$. The radius of convergence of the expansion around $x = -1$ is even smaller, which means it diverges for $\phi < 2\pi/3$, and

therefore there is no common region along the unit arc where both series can be matched.[8]

On the other hand, to go from the expansion at $x = 0$ to the expansion at $x = -1$, we first need to go through the four-particle threshold at $x = 4\sqrt{3} - 7$ ($\hat{s} = 16$). This is certainly an option, and we indeed found expansions at this (pseudo)threshold by matching with the expansions at $x = 0$, as we will see later in Sec. IV C. However, we did not use the expansions at $\hat{s} = 16$ to obtain the expansions at $x = -1$ ($\hat{s} = 4$). Instead, we found expansions at $\hat{s} = 0$, derived the corresponding differential equations, and matched with the expansions at $\hat{s} = 4$ given by Eq. (4.3). We found this more convenient for several reasons. First of all, as we already mentioned, in the region $0 < \hat{s} < 4$, we deal with real numbers, while in the region $-1 < x < 4\sqrt{3} - 7$, the form factors will also have imaginary parts.[9] Second, with the expansions around $\hat{s} = 0$ and $\hat{s} = 4$ ($z = 0$), we cover the entire unit arc, which is something we would still have to do if we stay in x, as discussed above. Third, we can find much deeper expansions at $\hat{s} = 0$ than at $\hat{s} = 16$, and all the coefficients of the expansions at $\hat{s} = 0$ are given in terms of known constants, which allows us to evaluate these expansions at different points with very high precision. Finally, although it is possible to derive differential equations in $z = \sqrt{4 - \hat{s}}$ from the differential equations in x, the change of variables from x to z would lead to very clumsy expressions. From $\hat{s}$ to z, on the other hand, the replacement of the derivatives is a simple one, namely, $\frac{d}{d\hat{s}} \rightarrow -\frac{1}{2z}\frac{d}{dz}$, and the differential equations look similar to the ones in Eq. (3.14),

$$\sum_{k=0}^{o_{I,i}} \tilde{p}_{I,i,k}(z) \frac{d^k}{dz^k} F_{I,i}^{(0)}(z) = 0, \qquad (4.4)$$

where $o_{I,i}$ is the order of the differential equation and the $\tilde{p}_{I,i,k}(z)$'s are polynomials in z.

The expansions in $\hat{s}$ around $\hat{s} = 0$ can be obtained from the corresponding expansions at the level of the master integrals in the same way as we did in the case of the expansions in x around $x = 1$ in Ref. [15]. However, since this is very laborious, and since the expansions around $x = 1$ were already available from previous works, it was easier to use them to derive the ones at $\hat{s} = 0$. This is, nevertheless, harder than it sounds, since many thousands of coefficients are needed to find matching conditions that lead to high precision, and also to guess differential equations in $\hat{s}$. Simply replacing x by $\hat{s}$ using Eq. (2.4) in the expansions at $x = 1$ and expanding in $\hat{s}$ may take a

lot of time if done naively, for example, using directly the *Mathematica* function SERIES. In Sec. IV A, we will describe the computer algebra methods we used to derive arbitrary many coefficients of the expansions at $\hat{s} = 0$. These methods may find applications in other areas of physics or computer algebra where large expansions are needed. In Sec. IV B, we will describe the calculation of the threshold expansions at $\hat{s} = 4$ exploiting these ingredients. In Sec. IV C, we will discuss the four-particle expansions at $\hat{s} = 16$.

A. Computing the recurrences and differential equations in $\hat{s}$

In the following we suppose that $f(x)$ stands for any of the 18 possible nonsolvable parts of the form factors, namely, $F_{I,k}^{(0)}(x)$ with $I \in \{(v, 1), (v, 2), (a, 1), (a, 2), s, p\}$ and $k = 1, 2, 3$. Moreover, we assume that we are given the first $\nu + 1$ coefficients f_n in its power series expansion

$$f(x) = \sum_{n=0}^{\nu} f_n(1 - x)^n + O((1 - x)^{\nu+1}) \qquad (4.5)$$

around $x = 1$. In our concrete calculations we will set $\nu = 8000$. Then replacing x by the right-hand side of (2.4) we want to compute the first $\nu + 1$ coefficients $\hat{h}_n$ of

$$\hat{F}(\hat{s}) = f\left(\frac{\sqrt{4 - \hat{s}} - \sqrt{-\hat{s}}}{\sqrt{4 - \hat{s}} + \sqrt{-\hat{s}}}\right) = \sum_{n=0}^{\nu} \hat{h}_n \hat{s}^n + O(\hat{s}^{\nu+1}). \quad (4.6)$$

We proceed in two steps. First, expand the right-hand side of (2.4) in $\hat{s}$, and second plug this expansion into (4.5) and extract the expansion coefficients $\hat{h}_n$ on the right-hand side of (4.6). Given these coefficients, we are interested in a third step to derive a linear differential equation for $\hat{F}(\hat{s})$ and a linear recurrence for $\hat{h}_n$. This latter equation is particularly important to produce further coefficients $\hat{h}_n$ with $100000 \geq n \geq \nu = 8000$ that are needed in the calculations of Sec. IV B.

Step 1. As it turns out, the expansion of the right-hand side in (2.4) yields contributions in $\sqrt{\hat{s}}$ where the root contributions cancel in the final expansion given in (4.6). As a consequence, one actually needs $2\nu + 1$ coefficients in order to obtain $\nu + 1$ expansion coefficients in the final result. To avoid these roots and to work purely with polynomial expressions, we replace $\hat{s}$ by σ^2 and compute the expansion

$$g(\sigma) = f\left(\frac{\sqrt{4 - \sigma^2} - \sqrt{-\sigma^2}}{\sqrt{4 - \sigma^2} + \sqrt{-\sigma^2}}\right) = \sum_{n=0}^{2\nu} g_n \sigma^n + O(\sigma^{2\nu+1}).$$

More precisely, we compute the expansions of the numerator $p(\sigma) = \sqrt{4 - \sigma^2} - \sqrt{-\sigma^2} = \sum_{n=0}^{2\nu} p_n \sigma^n + O(\sigma^{2\nu+1})$

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and the denominator $q(\sigma) = \sqrt{4-\sigma^2} + \sqrt{-\sigma^2} = \sum_{n=0}^{2\nu} q_n \sigma^n + O(\sigma^{2\nu+1})$ with $q_0 \neq 0$, use afterwards the formula

$$q'_n(\sigma) = \begin{cases} \frac{1}{q_0} & \text{if } n = 0 \\ \frac{-1}{q_0} \sum_{j=1}^{n} q_j q'_{n-j} & \text{if } n \geq 1 \end{cases}$$

for $q^{-1}(\sigma) = \sum_{n=0}^{2\nu} q'_n \sigma^n + O(\sigma^{2\nu+1})$, and combine finally the two expansions of $p(\sigma)$ and $q^{-1}(\sigma)$ to obtain the coefficients g_n in the expansion of $g(\sigma)$ using the Cauchy product. This task can be accomplished in about 2 hours.

$$\texttt{FastPower}(h(\sigma), n) = \begin{cases} 1 & \text{if } n = 0 \\ \texttt{FastPower}\left(h(\sigma), \frac{n}{2}\right)^2 & \text{if } n \geq 2 \text{ is even} \\ \texttt{FastPower}\left(h(\sigma), \frac{n-1}{2}\right)^2 * h(\sigma) & \text{if } n \geq 1 \text{ is odd} \end{cases}$$

recursively; note that on the right-hand sides one or two polynomial multiplications have to be executed. Here one may use the standard formula of the Cauchy-product $P(\sigma) \cdot Q(\sigma)$. Alternatively, one can use the *Mathematica* command Expand that works extremely efficiently by parallelization mechanisms. However, starting with two truncated power series $P(\sigma)$ and $Q(\sigma)$, i.e., polynomials with degrees 2ν, such a naive expansion leads to a polynomial of degree $4\nu = 32000$ where the last $\nu = 16000$ coefficients are not needed (and are actually incorrect if one compares them with the underlying power series expansion). These unnecessary calculations in terms of very large rational numbers can be avoided by the following refinement which leads to the method[10] FastExpand. Here one truncates the polynomials $P(\sigma)$ and $Q(\sigma)$ with degrees 2ν to polynomials $P'(\sigma)$ and $Q'(\sigma)$ up to the degree ν, and denotes the tails by $\tilde{P}(\sigma)$ and $\tilde{Q}(\sigma)$, i.e., $P(\sigma) = P'(\sigma) + \tilde{P}(\sigma)$ and $Q(\sigma) = Q'(\sigma) + \tilde{Q}(\sigma)$. Then

$$P(\sigma) \cdot Q(\sigma) = P'(\sigma) \cdot Q'(\sigma) + P'(\sigma) \cdot \tilde{Q}(\sigma) + \tilde{P}(\sigma) \cdot Q'(\sigma) + \tilde{P}(\sigma) \cdot \tilde{Q}(\sigma), \quad (4.8)$$

and the essential observation is that one can drop the calculation of $\tilde{P}(\sigma) \cdot \tilde{Q}(\sigma)$ to get the coefficients correct up to degree 2ν. The expansion $P'(\sigma) \cdot Q'(\sigma)$ fully contributes to the desired output. However, the expansions in $P'(\sigma) \cdot \tilde{Q}(\sigma)$ and $\tilde{P}(\sigma) \cdot Q'(\sigma)$ can be refined further by applying the proposed method on these subterms recursively. In short, one only expands those expressions which actually

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[10]The method FastExpand is also utilized within the package SolveCoupledSystem [15,20] that implements the large moments method of power series expansions [15].

Step 2. To accomplish the second step, we first compute the expansions of

$$\tilde{H}_n(\sigma) = (1 - g(\sigma))^n, \quad n = 0, ..., 2\nu; \quad (4.7)$$

note that the constant term of $g(\sigma)$ is 1, thus the constant term of $1 - g(\sigma)$ is zero and therefore the compositions (4.7) can be carried out purely formally. The following speed-ups for these calculations are relevant. First, one can use a divide-and-conquer paradigm to compute $h(\sigma)^n$ for a truncated power series $h(\sigma)$ by applying the formula

contribute to the final result. If an intermediate expression turns out to be small enough (base case), one simply expands normally and truncates the unwanted terms having degrees larger than 2ν.

Using FastPower in combination with FastExpand we proceed as follows. We compute 80 supporting points equally distributed within the range $1 \leq n \leq 2\nu$, i.e., $n_i = 200i$ with $0 \leq i \leq 79$. E.g., computing the expansion of (4.7) for $n = 1000, 8000, 15000$ it takes 1131s, 1525s, 1297s, respectively. Note that for larger n, the lower bounds in the expansions (4.7) increase stepwise [since one can pull out x from $1 - g(\sigma)$]. Hence the execution of FastExpand gets simpler, but on the other side, the recursion steps in FastPower increase when n gets larger. Computing these 80 points in parallel with 10 kernels required around 5 hours. Finally, we compute for each segment the missing points $n_i \leq n < n_{i+1}$ by starting with the already computed expansion $\tilde{H}_{n_i} = (1 - g(\sigma))^{n_i}$ and incrementally computing the corresponding expansions $\tilde{H}_{n,i+j+1} = \tilde{H}_{n,i+j}(1 - g(\sigma))$ by using our FastExpand method. Doing this calculation with 15 kernels in parallel we needed 2 days and 5 hours to get all expansions (4.7) for $0 \leq n \leq 2\nu$. In principle 15 supporting points n_i would suffice for 15 kernels. However, in order to obtain a fair distribution to all available kernels, the generation of more jobs turned out to be appropriate.

Given the expansions in (4.7), one could carry out the calculation

$$\hat{F}(\sigma) = f\left(\frac{\sqrt{4-\sigma^2} - \sqrt{-\sigma^2}}{\sqrt{4-\sigma^2} + \sqrt{-\sigma^2}}\right)$$

$$= \sum_{n=0}^{2\nu} f_n \tilde{H}_n(\sigma) + O(\sigma^{2\nu+1}) = \sum_{k=0}^{2\nu} \hat{h}_n \sigma^n \quad (4.9)$$

for each of the 18 cases. But also this last step slows down considerably due to nontrivial rational number arithmetic. In order to streamline and speed up these last calculations, one can rearrange the calculation in (4.9) and obtain a matrix $A \in \mathbb{Q}^{(2\nu+1)\times(2\nu+1)}$ with $(2\nu+1)^2$ rational number entries such that

$$\begin{pmatrix} h_0 \\ \vdots \\ h_{2\nu} \end{pmatrix} = A \begin{pmatrix} f_0 \\ \vdots \\ f_{2\nu} \end{pmatrix} \qquad (4.10)$$

holds. Notice that the matrix A with $\nu = 8000$ requires about 22 Gb of memory within *Mathematica*. Given this compact transformation from the x-expansion to the s-expansion, we parallelized the matrix operation in (4.10) by carrying out the row multiplications of A with $(f_0, \dots, f_{2\nu})$ using 16 *Mathematica* kernels. As mentioned already above, in the output $(h_0, \dots, h_{2\nu})$ all entries h_{2n+1} with $0 \leq n \leq \nu - 1$ are zero, and one obtains the desired coefficients $\hat{h}_n = h_{2n}$ with $0 \leq n \leq \nu$ for (4.6). The calculation time of this matrix multiplication in its parallelized implementation ranged between 2000s and 5000s for the given 18 problems. In total one needs 12 hours to perform this last calculation step.

Summarizing, using all these techniques for steps 1 and 2, we needed around 3 days to obtain $\nu = 8000$ coefficients in (4.6) for all 18 cases.

Step 3. Given this data, one can now proceed to guess the 18 linear differential equations for the $\hat{F}(\hat{s})$ and the corresponding linear recurrences for their coefficients $\hat{h}(n)$ using the package ORE_ALGEBRA [23] in SAGE. E.g., for $f(x) = F_{s,1}^{(0)}(\hat{s})$ one obtains a linear differential equation of order 42 and maximal degree 622 for the coefficients in 40 minutes, and a linear recurrence for the power series coefficients of order 33 and maximal degree 569 in 17 minutes. We can then utilize the obtained recurrences up to order 36 to produce the required coefficients $\hat{h}_n$ with $0 \leq n \leq 100000$.

Remark. An alternative approach is to compute directly from the given differential equation for (4.5) a differential equation for (4.6) using holonomic closure properties [24]. E.g., consider the linear differential equation for $f(x) = F_{s,1}^{(0)}(x)$ of order 43 where the maximal degree of the coefficients is 2315. Then applying the package HolonomicFunctions [73] to this equation yields in 2 weeks a linear differential equation for (4.6); its order is 44 and the maximal degree of the coefficients is 645, which is of similar size to the equation that we obtained with the method described above (see details in Step 3). However, we need in addition a linear recurrence to extend the number of coefficients in (4.6) from 8000 to 100000. Thus we extract a linear recurrence of (4.6) for the coefficients $\hat{h}_n$

from the derived linear differential equation (using again the holonomic closure properties [24]). Performing this calculation one gets a recurrence of order 611 where the coefficients have maximal degree 44 (as expected by bounds in terms of the given order and degree of the input differential equation). The challenge is now to produce the first 100000 coefficients of $\hat{h}_n$ with this monster recurrence. It seems rather challenging to produce the required 611 initial values using, e.g., the plain *Mathematica* command SERIES, and one may opt to reuse at least in parts some of the tools described above. Using then this recurrence and the necessary initial values, one may produce 8000 coefficients and then proceed with our original approach: guess the minimal recurrence and produce the desired coefficients using this optimal recurrence. Summarizing, this approach seems to be far more time consuming and somehow throws one back to our original approach. One should mention that this second approach has to be applied stepwise to all 18 cases. Conversely, having the basis transformation (4.10), one can deal with all the cases (and problems that may arise in future calculations) straightforwardly within hours.

B. The two-particle threshold

Let us consider the expansions around $\hat{s} = 4$ now. Here the relation

$$z^2 = \frac{(1+x)^2}{x} \qquad (4.11)$$

holds, with x defined in (2.3). The indicial equation associated to the guessed differential equations allow us to determine the value of $k_{\min}$ in Eq. (4.3). In most cases $k_{\min} = -6$, with a few exceptions where $k_{\min} = -4$. Also, as in the previous section, the differential equations confirm that the highest power of the log is 3, as written in Eq. (4.3), since the insertion of an ansatz with higher powers of the log in (4.4) leads to equations showing that the corresponding coefficients vanish. Unsurprisingly, all differential equations guessed from the expansions around $\hat{s} = 0$ were of the same order as the ones guessed from the expansions around $x = 1$. However, the degrees of the polynomials in the differential equations were slightly smaller in $\hat{s}$ than in x. Interestingly, unlike what happened in the previous section, in the cases of $F_{I,1}^{(0)}(z)$ and $F_{I,2}^{(0)}(z)$, it was not necessary to remove inhomogeneous parts arising from the fact that the first few coefficients in the expansions in $\hat{s}$ must be ignored when guessing the differential equations. This had the welcomed effect of producing differential equations in z that were simpler than they would be otherwise.

As in the previous section, the insertion of the ansatz (4.3) in the differential equations (4.4) leads to relations among the coefficients $\tilde{c}_{I,i}(j,k)$, leaving only a few of them (as many as the order of the differential equation) to be determined by initial conditions. More precisely, these base

coefficients are numerically determined by matching truncated versions of the expansions around $\hat{s} = 0$ and $\hat{s} = 4$ as well as (possibly) their derivatives at some intermediate points. We used 100000 terms for the expansions at $\hat{s} = 0$, and 3000 coefficients per power of the log in Eq. (4.3), and matched at a set of equally spaced points,

$$\hat{s}_0 + (k-1)\delta, \quad k = 1, \ldots, o_{I,i}, \qquad (4.12)$$

without using derivatives and taking $\delta = 1/20000$. In a first attempt, we used the middle point $\hat{s}_0 = 2$, which allowed us to determine the simplest coefficients in terms of known constants using the PSLQ algorithm. For example, in the pseudoscalar case we found

$$\tilde{c}_{p,1}(2,-2) = \frac{2048}{3}\zeta_2, \qquad (4.13)$$

$$\tilde{c}_{p,2}(1,-6) = \frac{16384}{3}, \qquad (4.14)$$

$$\tilde{c}_{p,3}(0,-6) = -\frac{28672}{3}, \qquad (4.15)$$

and similar for other form factors. Simple coefficients $\tilde{c}_{I,i}(j,k)$ like these typically appear for higher values of j and/or lower values of k. In every case, such simple coefficients, determined numerically using the settings mentioned above, differ from their exact values by 10^{-300} to 10^{-400}. Such a precision is, of course, enough to determine a simple rational number (or a simple rational number times a simple constant such as ζ_2) through the PSLQ algorithm, but it is insufficient for more complicated coefficients involving many known constants multiplied by increasingly larger rational numbers. Therefore, we had to increase the numerical precision of our coefficients, which we did in the same way as in the case of the expansions around $x = 0$, that is, using the coefficients we have already determined, such as the ones in Eqs. (4.13)–(4.15), as an anchor to search for different values of δ and $\hat{s}_0$ such that the precision of the coefficients is as high as possible. We found that by maintaining the same value of δ and choosing $\hat{s}_0 = 381/100$, we could determine the base coefficients with a precision of around 1850 to 1950 decimal places. Notice that this value of $\hat{s}_0$ is very close $\hat{s} = 4$, which most certainly has to do with the fact that we are using many more coefficients for the expansions around $\hat{s} = 0$ than for the ones around $\hat{s} = 4$, and we are therefore better able to approximate the function away from its expansion point in the former case than in the latter. In principle, we could also increase the precision of our coefficients by increasing the number of expansion terms around $\hat{s} = 4$, which should shift the value of $\hat{s}_0$ back to the middle, but this can be computationally very expensive, and it is also unnecessary, since 1800 decimal places for the coefficients turn out to be sufficient for our purposes. This level of precision allowed

us to determine many of the remaining and more complicated coefficients using the PSLQ algorithm. In particular, we found that all coefficients $\tilde{c}_{I,i}(j,k)$ with $j > 1$ or $j = 1$ and $k < -1$, can be written in terms of known constants. For example,

$$\tilde{c}_{v,1,3}(2,-1) = \frac{4445056}{81\pi} - \frac{50176}{\pi}l_2, \qquad (4.16)$$

$$\tilde{c}_{v,1,2}(1,-2) = -\frac{3627008}{27} - \frac{26368}{3}\pi^2 + 4096 l_2, \qquad (4.17)$$

$$\tilde{c}_{a,1,1}(3,1) = \frac{22400}{9\pi}\zeta_3 - \frac{173824}{243}\pi, \qquad (4.18)$$

$$\tilde{c}_{a,2,1}(2,-1) = \frac{603008}{81\pi}\zeta_3 - \frac{50176}{3\pi}\zeta_3 l_2 + \frac{17536}{9}\pi + \frac{42880}{81}\pi^3, \qquad (4.19)$$

$$\tilde{c}_{s,1}(2,3) = -\frac{118790}{81\pi}\zeta_3 - \frac{1064}{\pi}\zeta_3 l_2 - \frac{21674}{243}\pi + \frac{1016}{27}\pi^3 + 256\pi l_2, \qquad (4.20)$$

$$\tilde{c}_{p,1}(1,-2) = -\frac{7168}{9}\zeta_3 + \frac{833536}{243}\pi^2 + \frac{896}{3}\pi^4. \qquad (4.21)$$

However, for $j \leq 1$, we were not able to express all base coefficients in terms of known constants, as we did in the case of the expansion around $x = 0$. In particular, for $F_{I,i}^{(0)}(x)$ with $i = 1, 3$ the first coefficients for which this happened were the $\tilde{c}_{I,i}(1,-1)$'s, while for $F_{I,2}^{(0)}(x)$ only the $\tilde{c}_{I,i}(0,-1)$'s were left undetermined. Of course, it is quite possible that we are missing some known constants in our PSLQ search that would allow to reveal their relation with the so far undetermined coefficients, but we have tried many different constants, including different (inverse) powers of π, different multiple zeta values and their products, different (powers of) logarithms and polylogarithms evaluated at different values, square roots of different integers and their products with the constants already mentioned, as well as constants such as Clausen functions [22,74] evaluated at $r\pi$, where r is a rational number.[11] All of these searches were unsuccessful. However, if we include more than one coefficient in a PSLQ search, we can find relations among them. For example,

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[11]These constants typically appear in calculations of Feynman graphs of elliptic nature, such as the banana and kite graphs, which are among the master integrals required for the calculation of three-loop form factors.

$$\tilde{c}_{a,1,3}(1,-1) = -\frac{31}{8}\tilde{c}_{v,2,3}(1,-1) - \frac{23128}{9}\pi - \frac{3644704}{81\pi}$$
$$-\frac{7168}{3\pi}l_2^2 - \frac{519232}{81\pi}l_2, \tag{4.22}$$

$$\tilde{c}_{a,2,3}(1,-1) = \frac{1}{8}\tilde{c}_{v,2,3}(1,-1) - \frac{45464}{9}\pi + \frac{747488}{81\pi}$$
$$-\frac{50176}{3\pi}l_2^2 + \frac{1226176}{81\pi}l_2, \tag{4.23}$$

$$\tilde{c}_{p,3}(1,-1) = -\tilde{c}_{a,2,3}(1,-1), \tag{4.24}$$

$$\tilde{c}_{s,3}(1,-1) = \frac{3}{2}\tilde{c}_{v,2,3}(1,-1) - \frac{11360}{9}\pi + \frac{2377088}{81\pi}$$
$$-\frac{7168}{\pi}l_2^2 + \frac{34048}{27\pi}l_2, \tag{4.25}$$

$$\tilde{c}_{v,1,3}(1,-1) = -\frac{59}{8}\tilde{c}_{v,2,3}(1,-1) + \frac{137032}{9}\pi - \frac{798112}{81\pi}$$
$$+\frac{50176}{\pi}l_2^2 - \frac{10079552}{81\pi}l_2. \tag{4.26}$$

We can also find relations involving more than two coefficients. For example,

$$\tilde{c}_{v,2,3}(0,-1) = \frac{2}{3}\tilde{c}_{s,3}(0,-1) - \frac{17}{12}\tilde{c}_{v,2,3}(1,-1) + \frac{10752}{\pi}a_4 + \frac{98224}{27\pi}\zeta_3 - \frac{3602}{45}\pi^3 - \frac{326464}{243}\pi - \frac{4332160}{243\pi} + \frac{448}{\pi}l_2^4$$
$$-\frac{14336}{9\pi}l_2^3 + 1472\pi l_2^2 + \frac{34048}{81\pi}l_2^2 - \frac{128512}{27}\pi l_2 + \frac{4068736}{243\pi}l_2, \tag{4.27}$$

$$\tilde{c}_{a,1,3}(0,-1) = -\frac{31}{12}\tilde{c}_{s,3}(0,-1) + \frac{25}{96}\tilde{c}_{v,2,3}(1,-1) - \frac{10752}{\pi}a_4 - \frac{150724}{9\pi}\zeta_3 + \frac{13529}{90}\pi^3 + \frac{3272722}{243}\pi - \frac{35407064}{729\pi}$$
$$-\frac{448}{\pi}l_2^4 + \frac{62720}{9\pi}l_2^3 - 1472\pi l_2^2 + \frac{42560}{27\pi}l_2^2 - \frac{56752}{27}\pi l_2 - \frac{7362320}{243\pi}l_2. \tag{4.28}$$

This suggests the idea of choosing some of these coefficients to define new constants and express all other coefficients in terms of them. We tried to reduce the number of new constants needed by finding as many relations among the base coefficients as possible. We ended up with 14 new constants, $\tilde{\kappa}_i$, with $i = 1, \ldots, 14$, defined as

$$\begin{aligned}
&\tilde{\kappa}_1 = \tilde{c}_{v,2,3}(1,-1), && \tilde{\kappa}_2 = \tilde{c}_{v,2,2}(0,-1), && \tilde{\kappa}_3 = \tilde{c}_{s,3}(0,-1), && \tilde{\kappa}_4 = \tilde{c}_{a,2,3}(0,0), \\
&\tilde{\kappa}_5 = \tilde{c}_{v,1,3}(0,0), && \tilde{\kappa}_6 = \tilde{c}_{a,1,3}(0,2), && \tilde{\kappa}_7 = \tilde{c}_{v,1,3}(0,2), && \tilde{\kappa}_8 = \tilde{c}_{s,3}(0,2), \\
&\tilde{\kappa}_9 = \tilde{c}_{v,2,1}(0,-1), && \tilde{\kappa}_{10} = \tilde{c}_{a,1,1}(0,0), && \tilde{\kappa}_{11} = \tilde{c}_{v,1,1}(0,0), && \tilde{\kappa}_{12} = \tilde{c}_{a,1,1}(0,2), \\
&\tilde{\kappa}_{13} = \tilde{c}_{v,1,1}(0,2), && \tilde{\kappa}_{14} = \tilde{c}_{s,1}(0,2).
\end{aligned} \tag{4.29}$$

In Appendix E, we show the numerical values of these constants with 60 digits. In an ancillary file, we provide these constants with 1800 digits. Of course, other choices of coefficients are also possible. One may even choose linear combinations of coefficients or even linear combinations of coefficients together with other known constants, but for the sake of simplicity, we chose the new constants to be directly equal to some of the $\tilde{c}_{I,i}(j,k)$'s with values of k as low as possible. Notice also that we chose coefficients from different types of form factors, that is, with different values of I, since this choice was simple enough and there is no particular reason to stick to a single value of[12] I.

After making the choice given in Eqs. (4.29) we were able to express all coefficients as linear combinations of the following constants over rational numbers,

$$\begin{aligned}
&\Big\{ \tilde{\kappa}_i, \pi^k, \pi^k l_2, \pi^j l_2^2, \pi^l l_2^3, \pi^n l_2^4, \pi l_2^5, \pi^n a_4, \pi l_2 a_4, \pi^j \zeta_3, \pi^n l_2 \zeta_3, \\
&\frac{l_2^2 \zeta_3}{\pi}, \pi l_2^2 \zeta_3, \frac{l_2^3 \zeta_3}{\pi}, \frac{\zeta_3^2}{\pi}, \pi \zeta_5, \tilde{\kappa}_1 \zeta_3, \pi^2 \tilde{\kappa}_2, \tilde{\kappa}_3 \zeta_3 \Big\},
\end{aligned} \tag{4.30}$$

where $i = 1, \ldots, 14$, $j = -1, \ldots, 3$, $k = -1, \ldots, 5$, $l = -1$, 1, 3, and $n = -1, 0, 1$. Notice the presence in the list above of some products of the new constants with known constants, such as $\tilde{\kappa}_1 \zeta_3$ and $\pi^2 \tilde{\kappa}_2$. Notice also the presence of odd powers of π in some of these constants. These are expected since they appear even in the expansions of solvable parts. Here are a few other examples of coefficients given in terms of the constants in (4.30),

---

[12]We have not checked whether such choice is even possible.

$$\tilde{c}_{a,2,1}(0,-5) = \frac{8}{3}\tilde{\kappa}_1\zeta_3 - \frac{381440}{243}\pi\zeta_3 + \frac{13432832}{729\pi}\zeta_3 + \frac{1347584}{81\pi}\zeta_3 l_2 + \frac{4096}{243}\pi^4 - \frac{75136}{243}\pi^3 + \frac{14336}{81}\pi^3 l_2, \qquad (4.31)$$

$$\tilde{c}_{p,2}(0,1) = \frac{\tilde{\kappa}_2}{64} - 15\pi\zeta_3 - \frac{92327}{162}\pi^3 + \frac{128}{9}\pi^2 - \frac{35974507}{1944}\pi + \frac{8}{3}\pi l_2^3 - \frac{4222}{27}\pi l_2^2 + \frac{22}{3}\pi^3 l_2 + \frac{776783}{81}\pi l_2, \qquad (4.32)$$

$$\tilde{c}_{s,1}(0,-1) = -\frac{\pi^2}{4}\tilde{\kappa}_2 - \tilde{\kappa}_3\zeta_3 + \frac{1024}{3}\pi a_4 + 80\pi\zeta_5 - 64\pi^3\zeta_3 + \frac{9472}{9}\pi\zeta_3 - \frac{276928}{1215}\pi^5 - \frac{1088}{81}\pi^4 - \frac{312302}{729}\pi^3 - \frac{52640}{27}\pi$$
$$+ \frac{128}{9}\pi l_2^4 - \frac{128}{3}\pi^3 l_2^3 - \frac{12064}{27}\pi^3 l_2^2 - \frac{352}{3}\pi^5 l_2 + \frac{1194896}{243}\pi^3 l_2, \qquad (4.33)$$

$$\tilde{c}_{v,1,1}(0,-1) = \frac{185}{144}\pi^2\tilde{\kappa}_2 + \frac{\tilde{\kappa}_9}{3} + \frac{50176}{81}\pi a_4 - \frac{4096}{3}\pi l_2 a_4 + \frac{905}{288}\tilde{\kappa}_1\zeta_3 + \frac{185}{36}\tilde{\kappa}_3\zeta_3 + 240\pi\zeta_5 - \frac{4025756}{81\pi}\zeta_3^2 + \frac{33097}{54}\pi^3\zeta_3$$
$$+ \frac{7207298}{729}\pi\zeta_3 - \frac{72875656}{729\pi}\zeta_3 + \frac{120064}{27\pi}\zeta_3 l_2^3 - 1152\pi\zeta_3 l_2^2 - \frac{722992}{81}\pi\zeta_3 l_2 - \frac{14331968}{243\pi}\zeta_3 l_2^2$$
$$+ \frac{107318512}{729\pi}\zeta_3 l_2 + \frac{2668418}{3645}\pi^5 + \frac{800}{9}\pi^4 - \frac{11892677}{1458}\pi^3 + \frac{420704}{405}\pi - \frac{512}{9}\pi l_2^5 + \frac{6272}{243}\pi l_2^4 + \frac{800}{3}\pi^3 l_2^3$$
$$+ \frac{294280}{243}\pi^3 l_2^2 + \frac{16408}{27}\pi^5 l_2 - \frac{1607788}{81}\pi^3 l_2, \qquad (4.34)$$

$$\tilde{c}_{v,2,3}(0,4) = -\frac{15531726824501}{297574847332017}\tilde{\kappa}_4 + \frac{2425509036937}{793532926218712}\tilde{\kappa}_5 + \frac{50909627017694}{495958078886695}\tilde{\kappa}_6 - \frac{37467581053953}{1983832315546780}\tilde{\kappa}_7$$
$$- \frac{8420376537572}{99191615777339}\tilde{\kappa}_8 + \frac{343394686678188704}{40172604389822295}\pi^2 - \frac{17742395132986623488}{66954340649703825} - \frac{180289931782198336}{4463622709980255}l_2. \qquad (4.35)$$

The last relation, Eq. (4.35), is particularly curious. The rational numbers multiplying the constants have a similarly large number of digits in the numerator and denominator, which might make one suspect that the relation is spurious. However, this particular relation can be obtained using the PSLQ algorithm with just 300 decimal places out of the more than 1800 available for the $\tilde{\kappa}_i$'s. The fact that the relation holds also when the full available precision is used gives us confidence that the relation is correct. Moreover, this relation is part of a pattern that is followed by the $\tilde{c}_{I,i}(0,k)$'s starting from $k \geq -1$, namely, for k even and $k \geq 2$, the base coefficients $\tilde{c}_{I,3}(0,k)$'s can be expressed as a linear combination of the constants

$$\{\tilde{\kappa}_4, \tilde{\kappa}_5, \tilde{\kappa}_6, \tilde{\kappa}_7, \tilde{\kappa}_8, \pi^2, l_2\}, \qquad (4.36)$$

for all values of I, while in the case of the $\tilde{c}_{I,1}(0,k)$'s, the following constants appear

$$\{\tilde{\kappa}_{10}, \tilde{\kappa}_{11}, \tilde{\kappa}_{12}, \tilde{\kappa}_{13}, \tilde{\kappa}_{14}, \pi^2, \pi^4, \pi^2 l_2, \pi^4 l_2, \pi^2 l_2^2, l_2^4, a_4,$$
$$\zeta_3, \pi^2\zeta_3, l_2\zeta_3\}. \qquad (4.37)$$

There are also patterns when k is odd. We find that the $\tilde{c}_{I,3}(0,k)$'s can be written in terms of

$$\left\{\tilde{\kappa}_1, \tilde{\kappa}_3, \frac{1}{\pi}, \pi, \pi^3, \frac{l_2}{\pi}, \pi l_2, \frac{l_2^2}{\pi}, \frac{l_2^3}{\pi}, \frac{\zeta_3}{\pi}\right\}, \qquad (4.38)$$

with $k \geq -1$ and $I \in \{(a,2), s, p\}$, while for $I \in \{(v,1), (v,2), (a,1)\}$ we have to include πl_2^2, $\frac{l_2^4}{\pi}$ and $\frac{a_4}{\pi}$ in the previous list. In the case of the $\tilde{c}_{I,1}(0,k)$'s (with k odd and $k \geq -1$) the list is

$$\left\{\pi^2\tilde{\kappa}_2, \tilde{\kappa}_1\zeta_3, \tilde{\kappa}_3\zeta_3, \pi, \pi^3, \pi^4, \pi^5, \pi l_2, \pi^3 l_2, \pi^5 l_2, \pi l_2^2, \pi^3 l_2^2,\right.$$
$$\pi l_2^3, \pi^3 l_2^3, \pi l_2^4, \pi a_4, \frac{\zeta_3}{\pi}, \pi\zeta_3, \pi^3\zeta_3, \frac{l_2\zeta_3}{\pi}, \pi l_2\zeta_3, \frac{l_2^2\zeta_3}{\pi},$$
$$\left.\frac{l_2^3\zeta_3}{\pi}, \frac{\zeta_3^2}{\pi}, \pi\zeta_5\right\}, \qquad (4.39)$$

for $I \in \{(a,2), s, p\}$, while for $I = (a,1)$ we have to add $\tilde{\kappa}_9$ to the previous list, and for $I \in \{(v,1), (v,2)\}$, we also have to add πl_2^5, $\pi l_2 a_4$ and $\pi l_2^2\zeta_3$. In all cases, the numerators and denominators of the rational numbers in the corresponding relations keep increasing as we increase k. For example, the last two base coefficients in the scalar case $F_{s,1}^{(0)}$ are

$$
\begin{aligned}
\tilde{c}_{s,1}(0,19) = & -\frac{5027881\pi^2\tilde{\kappa}_2}{281474976710656} + \frac{100060070143\tilde{\kappa}_1\zeta_3}{354658470655426560} - \frac{5027881\tilde{\kappa}_3\zeta_3}{70368744177664} \\
& -\frac{27599\pi a_4}{7247757312} + \frac{230945\pi\zeta_5}{17179869184} + \frac{111888465185\zeta_3^2}{118747255799808\pi} \\
& +\frac{67923409\pi^3\zeta_3}{35184372088832} - \frac{59887782087324315547\pi\zeta_3}{1116371213620875 3623040} \\
& -\frac{128605264002510928942244606345783 93\zeta_3}{182869352494433754483263520374784 0000\pi} - \frac{286229923\zeta_3 l_2^3}{618475290624\pi} \\
& -\frac{17309726854504 8371\zeta_3 l_2^2}{1079823346508287 18080\pi} + \frac{14447921549\pi\zeta_3 l_2}{29686813949952} \\
& +\frac{1663467490977652027449322769\zeta_3 l_2}{62843710295357808508403712000\pi} + \frac{125362825561\pi^5}{10687253021982720} \\
& +\frac{46211464633\pi^4}{56108078365409280} - \frac{8372674776503657147978188451\pi^3}{2111548665924022365882364723200 0} \\
& +\frac{24180987132697703983228181135504 5074601\pi}{70307069191793805111236142681607 372800000} - \frac{27599\pi l_2^4}{173946175488} \\
& -\frac{5027881\pi^3 l_2^3}{1649267441664} - \frac{185075295793\pi l_2^3}{1098453111275520} + \frac{4205606557\pi^3 l_2^2}{178120883699712} \\
& +\frac{5077431449402204323\pi l_2^2}{19178378386034487459840} - \frac{1867527173617354849\pi^3 l_2}{36282064442678449274880} \\
& -\frac{55306691\pi^5 l_2}{6597069766656} - \frac{2369609169473386648534800 17\pi l_2}{1395182437854261432520015872 00}
\end{aligned}
\tag{4.40}
$$

and

$$
\begin{aligned}
\tilde{c}_{s,1}(0,20) = & \frac{149525845164951428447942110051211 59757399663}{295874986039274182028353057002604 479465848913920}\tilde{\kappa}_{10} \\
& -\frac{878453210567550928560288639875393 97958571}{295874986039274182028353057002604 479465848913920}\tilde{\kappa}_{11} \\
& -\frac{118268786641370119341992739958834 46541836167}{845357102969354805795294448578869 9422704528261120}\tilde{\kappa}_{12} \\
& +\frac{183385316854768495836620588410641 3005446099}{281785700989784935265098149526289 9807568176087040}\tilde{\kappa}_{13} \\
& +\frac{392033991234856610040322693946026 06971865}{448295433392839669739928874246370 42393130074112}\tilde{\kappa}_{14} \\
& +\frac{265437110579464731874825963440076 9208778218233}{468083474007445483287042922211516 184251433128960}a_4 \\
& +\frac{377843701667024690186361039372089 880848543401}{480085614366610752089274792011437 557359121346560}\pi^2\zeta_3 \\
& -\frac{498066834545929606286213960496473 45829942814397941}{128388610013470761130160344377915 8724823250229657600}\zeta_3 \\
& +\frac{961991906094771402296164694348412 11096617849}{685836591952301074413249702873482 22479874478080}\zeta_3 l_2 \\
& +\frac{104089027227843224864458240755280 59119015945087}{732652394098610321666675878243541 66362196344627200}\pi^4 \\
& +\frac{173660487393931911831428045479960 975921805411478493}{754925026879208075445342824942145 3301960711350386688000}\pi^2
\end{aligned}
$$

$$+ \frac{50974594014764148007869743942673992785127104647414623}{11795703544987626178833481639721020784313611484979200000}$$
$$+ \frac{26543711057946473187482596344007692087782182333}{112340033761786915988890301330676388422034395095040} l_2^4$$
$$- \frac{37251700280342321185143792687343589255597721567}{28085008440446728997222575332669097105508598773760} \pi^2 l_2^2$$
$$+ \frac{1142631272386937273800530811457671017762103

1}{17457658704240390985064537891325002085786230784

0} \pi^4 l_2$$
$$+ \frac{6164392376155839301426663471148937126105465691

3}{800998458194559115785314091484323625112544706560

0} \pi^2 l_2. \tag{4.41}$$

In Eq. (4.40), the largest numerator or denominator has 41 digits, while the smallest one has only seven digits. Equation (4.41) looks a bit more cumbersome, and it is also a bit more uniform, with the largest numerator or denominator having 56 digits, and the smallest having 41. To get these relations, one needs a minimum of 1300 decimal places of precision in the case Eq. (4.40), and at least 1100 decimal places in the case of Eq. (4.41). These are the precisions required once we know exactly which constants are involved and then limit ourselves to such constants when applying the PSLQ algorithm, but of course, one does not usually know *a priori* which constants are going to be present, and therefore one needs much higher precision in order to be able to include more constants in the search. We did searches including all 1800 decimal places available for our base coefficients, and performed various checks of our results, including the introduction of spurious constants, like we did in the case of the expansion

at $x = 0$, to make sure that the PSLQ algorithm does in fact rule out such constants from the result, which is unlikely to happen unless the relation is not spurious.

As we did in the case of the expansions at $x = 0$, we can now obtain the form factors using Eq. (2.25), together with Eqs. (2.26)–(2.29) and Eqs. (2.14) and (2.15) rewritten in terms of z instead of x (using $z^2 = (1 + x)^2/x$). Since the expansions contain the unknown constants $\tilde{\kappa}_i$ we refrain from listing them here. The first 200 terms of the threshold expansion of all form factors are given in the ancillary files. After this we can find recursions obeyed by the sequences of rational numbers associated to the each combination of a given constant (including the known constants and the new ones, $\tilde{\kappa}_i$) and a power of the log. For example, the sequence of rational number associated to the term $\pi \ln^2(z)$ in the nonsolvable parts (at the expansion point $x = 1$) of $F_{V,1}^{(0)}(z)$ are given by

$$\left\{ 0, 0, 0, \frac{27008}{27}, 0, \frac{2752}{81}, 0, -\frac{339913}{1215}, 0, -\frac{27836749}{170100}, 0, -\frac{2883493363}{76204800}, 0, \right.$$
$$\left. -\frac{33749896783}{3772137600}, 0, -\frac{32996303383481}{15343379251200}, 0, -\frac{749728845867049}{1436140297912320}, 0, \ldots \right\}. \tag{4.42}$$

In other words, the nth term in the sequence above is the rational number multiplying $x^{-7+n}\pi\ln^2(z)$ in the expansion, starting from $n = 1$. As usual, the first few terms must be dropped in order to find a recursion. Notice that all terms with n even are equal to zero. This is a consequence of the patterns described above and requires us to find a sequence by considering the terms with n odd only. We did so, and found a recurrence of order five,

$$\sum_{i=0}^{5} \tilde{p}_i(n) f_3(n+i) = 0, \tag{4.43}$$

where $f_3(n)$ is the nth term of the sequence given above after dropping the first seven terms and all the zeroes, and the $\tilde{p}_i(n)$'s are polynomials in n. We found the following solution

$$f_3(n) = -\frac{2^{1-4n}T_3}{27(1+n)(2+n)(3+n)(1+2n)^2(3+2n)^2(5+2n)}\binom{2n}{n} - \frac{(349+1164n+612n^2)2^{5-2n}}{27(1+2n)(3+2n)}$$

$$-\frac{(41967+66774n+32884n^2+5768n^3)2^{1-4n}\binom{2n}{n}}{27(1+n)(2+n)(3+n)}S_{\{2,1,2\}}(n) - \frac{1}{27}2^{8-2n}\sum_{i_1=1}^{n}\frac{2^{-2i_1}\binom{2i_1}{i_1}}{(1+2i_1)^2}$$

$$+\left[-\frac{5}{27}2^{6-2n} - \frac{3(-7+4n+4n^2)2^{5-4n}\binom{2n}{n}}{(1+2n)(3+2n)} - \frac{5}{27}2^{6-2n}\sum_{i_1=1}^{n}\frac{2^{-2i_1}\binom{2i_1}{i_1}}{1+2i_1}\right]S_{\{2,1,1\}}(n)$$

$$-\frac{5(65+220n+116n^2)2^{5-2n}\sum_{i_1=1}^{n}\frac{2^{-2i_1}\binom{2i_1}{i_1}}{1+2i_1}}{27(1+2n)(3+2n)} - \frac{71}{27}2^{5-2n}\sum_{i_1=1}^{n}\frac{2^{-2i_1}\binom{2i_1}{i_1}S_{\{2,1,1\}}(i_1)}{1+2i_1}, \tag{4.44}$$

with

$$T_3 = 434624n^8 + 5219648n^7 + 26709296n^6$$
$$+ 75033264n^5 + 124288324n^4 + 121056044n^3$$
$$+ 64493993n^2 + 15127189n + 517179. \tag{4.45}$$

Here $S_{\{2,1,1\}}(n)$ and $S_{\{2,1,2\}}(n)$ are cyclotomic harmonic sums [37], which are defined as

$$S_{\{a,b,c\}}(n) = \sum_{k=1}^{n}\frac{(\text{sign}(c))^k}{(ak+b)^{|c|}}. \tag{4.46}$$

Resumming these expressions with the help of the HarmonicSums command Compute GeneratingFunction one obtains expressions depending on iterative integrals of the kind

$$G(\{f_1(\tau), f_2(\tau), ..., f_n(\tau)\}, z)$$
$$= \int_0^z d\tau_1 f_1(\tau_1) G(\{f_2(\tau), ..., f_n(\tau)\}, \tau_1), \tag{4.47}$$

with

$$F_3(x) = \sum_{i=0}^{\infty} f_3(i)x^i = -\frac{64(1266+43x)}{81x^2} - \frac{16G(\sqrt{4-\tau}\sqrt{\tau};x)R_6}{27x^{5/2}(4-x)} + \frac{4\sqrt{4-x}R_7}{27(4-x)x^2}$$

$$-48(-2+x)\sqrt{4-x}G\left(\frac{1}{4-\tau};x\right) - \frac{320(2-x)^2}{27x^{3/2}(4-x)}G\left(\frac{1}{4-\tau},\sqrt{4-\tau}\sqrt{\tau};x\right)$$

$$-\frac{2848(2-x)^2}{27x^{3/2}(4-x)}G\left(\frac{1}{\tau},\sqrt{4-\tau}\sqrt{\tau};x\right) - \frac{96(2-x)^2}{x^{3/2}(4-x)}G\left(\sqrt{4-\tau}\sqrt{\tau},\frac{1}{4-\tau};x\right)$$

$$-\frac{32(4836-1636x+19x^2+67x^3)\sqrt{4-x}}{27(4-x)x^3}G\left(\sqrt{4-\tau}\sqrt{\tau},\sqrt{4-\tau}\sqrt{\tau};x\right), \tag{4.48}$$

and

$$R_6 = 67x^5 - 383x^4 - 1202x^3 + 13712x^2 - 26888x + 1184, \tag{4.49}$$

$$R_7 = 67x^6 - 517x^5 - 104x^4 + 12936x^3 - 39828x^2 + 1928x + 13504. \tag{4.50}$$

Note that here the sum starts at $i = 0$, unlike the case for $f_{1(2)}(i)$ in Secton III. The corresponding expression is

$$F^{(0)}_{\pi\ln^2(z),V,1} = \frac{27008}{27z^3} + \frac{2752}{81z} + zF_3(z^2) \tag{4.51}$$

The G–functions are given as polynomials of rational terms and harmonic polylogarithms in the variable

$$u = \frac{1}{2}\left(2 + iz\sqrt{4-z^2} - z^2\right) = -\frac{1}{x}. \tag{4.52}$$

One obtains

$$G(\{\sqrt{4-\tau}\sqrt{\tau}\}, z^2) = i\left[-\frac{1}{2u^2} + \frac{u^2}{2} - 2H_0(u)\right], \tag{4.53}$$

$$G\left(\left\{\frac{1}{\tau}, \sqrt{4-\tau}\sqrt{\tau}\right\}, z^2\right), = i\left[-\frac{(1-u^2)(1+4u+u^2)}{4u^2} + 4H_0(u)H_1(u) - 4H_{0,1}(u) + 4\zeta_2 + H_0(u) + H_0^2(u)\right], \tag{4.54}$$

$$G\left(\left\{\frac{1}{4-\tau}, \sqrt{4-\tau}\sqrt{\tau}\right\}, z^2\right) = i\left[\frac{(1-u^2)(1-4u+u^2)}{4u^2} - H_0^2(u) - 4H_{0,-1}(u) + 2\zeta_2 + H_0(u)(-1+4H_{-1}(u))\right] \tag{4.55}$$

$$G\left(\left\{\sqrt{4-\tau}\sqrt{\tau}, \frac{1}{4-\tau}\right\}, z^2\right) = G\left(\left\{\frac{1}{4-\tau}\right\}, z^2\right)G(\{\sqrt{4-\tau}\sqrt{\tau}\}, z^2) - G\left(\left\{\frac{1}{4-\tau}, \sqrt{4-\tau}\sqrt{\tau}\right\}, z^2\right), \tag{4.56}$$

by shuffling [41], where

$$G\left(\left\{\frac{1}{4-\tau}\right\}, z^2\right) = -\ln\left[1 - \frac{z^2}{4}\right], \tag{4.57}$$

Iterative G-functions over root–valued alphabets were studied in [38,75].

After adding up all the terms for which the recurrences are solvable at the expansion point $\hat{s} = 4$, we find that the previously nonsolvable parts at the expansions point $x = 1$ in Eq. (2.25) can be resummed as follows,

$$
\begin{aligned}
F_{V,1,1}^{(0)}(z) + \zeta_2 F_{V,1,2}^{(0)}(z) + \zeta_3 F_{V,1,3}^{(0)}(z) = &-\frac{32\ln(2)\pi^3 \frac{1}{\sqrt{4-z^2}}\ln(1-\frac{z^2}{4})Q_{13}}{27z^5} + \frac{64\ln(2)\pi^3 \frac{1}{\sqrt{4-z^2}}\ln(z)Q_{24}}{27z^5} - \frac{4\ln^2(2)\pi\sqrt{4-z^2}Q_{42}}{27z} \\
&-\frac{8\ln(2)\pi\sqrt{4-z^2}\ln(z)Q_{42}}{9z} - \frac{4\pi^3 \frac{1}{\sqrt{4-z^2}}\ln(z)Q_{48}}{27z^3} - \frac{32\pi \frac{1}{\sqrt{4-z^2}}\ln^3(z)Q_{51}}{81z^3} \\
&-\frac{64\pi^2\ln^2(z)Q_{55}}{27(z-2)z^4(z+2)} - \frac{16\pi \frac{\zeta_3}{\sqrt{4-z^2}}Q_{64}}{729z^5} - \frac{2\ln(2)\pi^3 \frac{1}{\sqrt{4-z^2}}Q_{65}}{243z^5} + \frac{4\pi \frac{1}{\sqrt{4-z^2}}\ln^2(z)Q_{73}}{27z^3} \\
&-\frac{\pi^2\ln(z)Q_{80}}{486(z-2)z^2(z+2)} + \ln^2(2)\pi^3\left[-\frac{32(5981-3570z^2+300z^4)\frac{1}{\sqrt{4-z^2}}}{243z^3}\right. \\
&\left.-\frac{64(-12+5z^2)\frac{1}{\sqrt{4-z^2}}\ln(1-\frac{z^2}{4})}{3z^5}\right] + \ln^4(2)\pi\left[-\frac{128(85+24z^2+6z^4)\frac{1}{\sqrt{4-z^2}}}{243z^3}\right. \\
&\left.+\frac{64(-12+5z^2)\frac{1}{\sqrt{4-z^2}}\ln(1-\frac{z^2}{4})}{3z^5}\right] + \mathrm{Li}_4\left(\frac{1}{2}\right)\pi\left[-\frac{1024(85+24z^2+6z^4)\frac{1}{\sqrt{4-z^2}}}{81z^3}\right. \\
&\left.+\frac{512(-12+5z^2)\frac{1}{\sqrt{4-z^2}}\ln(1-\frac{z^2}{4})}{z^5}\right] + \pi^5\left[-\frac{4(20915-14010z^2+4836z^4)\frac{1}{\sqrt{4-z^2}}}{3645z^3}\right. \\
&\left.-\frac{8(12+43z^2)\frac{1}{\sqrt{4-z^2}}\ln(1-\frac{z^2}{4})}{45z^5}\right] + \left[-\frac{64\ln^3(2)\pi(z^2-2)^2}{9(z-2)z^2(z+2)} + \frac{128\pi^2 \frac{1}{\sqrt{4-z^2}}\ln^2(z)Q_8}{27z^5}\right. \\
&\left.-\frac{224\pi(z^2-2)\zeta_3 Q_{16}}{27(z-2)z^6(z+2)} - \frac{8\pi^3(z^2-2)\ln(z)Q_{29}}{27(z-2)z^6(z+2)} - \frac{16\ln(2)\pi^3 Q_{52}}{27(z-2)z^6(z+2)}\right.
\end{aligned}
$$

$$
+ \frac{16\ln^2(2)\pi Q_{56}}{27(z-2)z^4(z+2)} + \frac{32\ln(2)\pi\ln(z)Q_{56}}{9(z-2)z^4(z+2)} + \frac{16\pi\ln^2(z)Q_{57}}{27(z-2)z^4(z+2)} + \frac{\pi^2 \frac{1}{\sqrt{4-z^2}}\ln(z)Q_{62}}{81z^5}
$$

$$
- \frac{64\ln^2(2)\pi(z^2-2)^2\ln(z)}{(z-2)z^2(z+2)} + \frac{64\ln(2)\pi^2(-30+8z^2-3z^4+z^6)\frac{1}{\sqrt{4-z^2}}\ln(z)}{9z^5}
$$

$$
- \frac{192\ln(2)\pi(z^2-2)^2\ln^2(z)}{(z-2)z^2(z+2)} - \frac{10432\pi(z^2-2)^2\ln^3(z)}{81(z-2)z^2(z+2)} \Bigg] G(\sqrt{4-\tau}\sqrt{\tau};z^2)
$$

$$
+ \Bigg[- \frac{16\ln^2(2)\pi\frac{1}{\sqrt{4-z^2}}Q_{11}}{27z^5} - \frac{32\ln(2)\pi\frac{1}{\sqrt{4-z^2}}\ln(z)Q_{11}}{9z^5} - \frac{16\pi\frac{1}{\sqrt{4-z^2}}\ln^2(z)Q_{22}}{27z^5}
$$

$$
+ \frac{2\pi^2\ln(z)Q_{43}}{9(z-2)z^6(z+2)} + \frac{8\ln(2)\pi^3(-36+7z^2)\frac{1}{\sqrt{4-z^2}}}{z^5} \Bigg] G(\sqrt{4-\tau}\sqrt{\tau};z^2)^2
$$

$$
+ \Bigg[\frac{64\ln^2(2)\pi(z^2-2)^2}{27(z-2)z^2(z+2)} + \frac{128\ln(2)\pi(z^2-2)^2\ln(z)}{9(z-2)z^2(z+2)} + \frac{320\pi(z^2-2)^2\ln^2(z)}{27(z-2)z^2(z+2)}
$$

$$
+ \frac{64\pi^2(3+z^2)(6-4z^2+z^4)\frac{1}{\sqrt{4-z^2}}\ln(z)}{27z^5} \Bigg] G\left(\frac{1}{4-\tau},\sqrt{4-\tau}\sqrt{\tau};z^2\right)
$$

$$
+ \Bigg[\frac{32\ln^2(2)\pi(z^2-2)^2}{3(z-2)z^2(z+2)} - \frac{16\pi^2\frac{1}{\sqrt{4-z^2}}\ln(z)Q_{18}}{9z^5} + \frac{64\ln(2)\pi(z^2-2)^2\ln(z)}{(z-2)z^2(z+2)}
$$

$$
+ \frac{2848\pi(z^2-2)^2\ln^2(z)}{27(z-2)z^2(z+2)} \Bigg] G\left(\frac{1}{\tau},\sqrt{4-\tau}\sqrt{\tau};z^2\right) + \Bigg[\frac{32\ln^2(2)\pi(z^2-2)^2}{3(z-2)z^2(z+2)}
$$

$$
+ \frac{64\ln(2)\pi(z^2-2)^2\ln(z)}{(z-2)z^2(z+2)} + \frac{96\pi(z^2-2)^2\ln^2(z)}{(z-2)z^2(z+2)} \Bigg] G\left(\sqrt{4-\tau}\sqrt{\tau},\frac{1}{4-\tau};z^2\right)
$$

$$
+ \frac{4096\mathrm{Li}_4(\frac{1}{2})\ln(2)\pi\frac{1}{\sqrt{4-z^2}}}{3z^3} + \frac{512\ln(2)^5\pi\frac{1}{\sqrt{4-z^2}}}{9z^3} - \frac{512\ln^3(2)\pi^3\frac{1}{\sqrt{4-z^2}}}{9z^3} - \frac{1600\ln(2)\pi^5\frac{1}{\sqrt{4-z^2}}}{27z^3}
$$

$$
- \frac{\pi^2(-36+11z^2)\kappa_2\frac{1}{\sqrt{4-z^2}}}{36z^5} - \frac{(-36+11z^2)\kappa_9\frac{1}{\sqrt{4-z^2}}}{6z^5} + \frac{32}{9}\ln^3(2)\pi z(z^2-2)\sqrt{4-z^2}
$$

$$
+ \frac{248}{27}\ln^2(2)\pi z(4-z^2)^{3/2} + \frac{2166080(-36+11z^2)\frac{\zeta_3}{\sqrt{4-z^2}}}{729\pi z^5}
$$

$$
- \frac{1792\mathrm{Li}_4(\frac{1}{2})(-36+11z^2)\frac{\zeta_3}{\sqrt{4-z^2}}}{\pi z^5} - \frac{2034368\ln(2)(-36+11z^2)\frac{\zeta_3}{\sqrt{4-z^2}}}{729\pi z^5}
$$

$$
- \frac{17024\ln^2(2)(-36+11z^2)\frac{\zeta_3}{\sqrt{4-z^2}}}{243\pi z^5} + \frac{7168\ln^3(2)(-36+11z^2)\frac{\zeta_3}{\sqrt{4-z^2}}}{27\pi z^5} - \frac{224\ln^4(2)(-36+11z^2)\frac{\zeta_3}{\sqrt{4-z^2}}}{3\pi z^5}
$$

$$
+ \frac{64256\ln(2)\pi(-36+11z^2)\frac{\zeta_3}{\sqrt{4-z^2}}}{81z^5} - \frac{736\ln^2(2)\pi(-36+11z^2)\frac{\zeta_3}{\sqrt{4-z^2}}}{3z^5}
$$

$$+ \frac{\pi^3(-64836 + 25451z^2)\frac{\zeta_3}{\sqrt{4-z^2}}}{135z^5} + \frac{17(-36 + 11z^2)\kappa_1\frac{\zeta_3}{\sqrt{4-z^2}}}{72z^5} - \frac{(-36 + 11z^2)\kappa_3\frac{\zeta_3}{\sqrt{4-z^2}}}{9z^5}$$

$$- \frac{49112(-36 + 11z^2)\frac{\zeta_3^2}{\sqrt{4-z^2}}}{81\pi z^5} - \frac{640\pi^4(z^2-2)^2\ln(z)}{27(z-2)z^2(z+2)} - \frac{360\pi\frac{\zeta_5}{\sqrt{4-z^2}}}{z^3}$$

$$- \frac{32\ln(2)\pi^2(-16 + 10z^2 - 5z^4 + z^6)\ln(z)}{9z^2} + 32\ln^2(2)\pi z(z^2-2)\sqrt{4-z^2}\ln(z)$$

$$+ \frac{496}{9}\ln(2)\pi z(4-z^2)^{3/2}\ln(z) + \frac{4\pi^2(-12 + 5z^2)\frac{1}{\sqrt{4-z^2}}G(\sqrt{4-\tau}\sqrt{\tau}; z^2)^3\ln(z)}{3z^5}$$

$$+ 96\ln(2)\pi z(z^2-2)\sqrt{4-z^2}\ln^2(z) + \frac{488}{9}\pi z(4-z^2)^{3/2}\ln^2(z) - \frac{4096\pi^2(z^2-2)^2\ln^3(z)}{81(z-2)z^2(z+2)}$$

$$+ \frac{16}{3}\ln^2(2)\pi z(z^2-2)\sqrt{4-z^2}\ln\left(1 - \frac{z^2}{4}\right) + 32\ln(2)\pi z(z^2-2)\sqrt{4-z^2}\ln(z)\ln\left(1 - \frac{z^2}{4}\right)$$

$$+ 48\pi z(z^2-2)\sqrt{4-z^2}\ln^2(z)\ln\left(1 - \frac{z^2}{4}\right) + \tilde{F}_{V,1}^{(0),\text{rest}}(z). \tag{4.58}$$

The polynomials Q_i are given in Appendix D, where also the resummed expressions in z for the remaining form factors are presented.

C. The four-particle threshold

The differential equations in the case of the four-particle threshold were derived from the ones guessed using the expansions at $\hat{s} = 0$ by doing the change of variables $\tilde{z}$, after which we matched with the expansions at $\hat{s} \to +\infty$ ($x = 0$). In all cases the matching was done at a set of points given by Eq. (4.12) using $\delta = 1/2000$. No derivatives were used. Since the expansions at $\hat{s} = 16$ converge in the region $4 < \hat{s} < 28$, and the high energy expansion converges for $\hat{s} > 16$, we had to choose a value of $\hat{s}_0$ in Eq. (4.12) in the region $16 < \hat{s}_0 < 28$. We used $\hat{s}_0 = 20$ as the starting point.

We found agreement with two of the statements made in Refs. [16,17], namely, that these expansions contain no logarithms in the variable $\tilde{z}$, and that the first nonanalytic

terms (in other words, odd powers of $\tilde{z}$) start at order 7 in the case of the axial-vector and scalar form factors, and at order 9 in the case of the vector and pseudoscalar form factors. The authors of [16,17] did find powers of logarithms as high as $\ln^5(\tilde{z})$, together with powers of $\tilde{z}$ as low as $\tilde{z}^3$ at the level of the master integrals, which then disappear from the final results. Since we do not deal with master integrals in our method, at no point in our calculations do we find such terms, which makes matching with the high energy expansion easier in our case. We can even pinpoint from which of the form factors $F_{I,k}^{(0)}(\tilde{z})$ do the first nonanalytic terms come from. The nonsolvable parts $F_{I,2}^{(0)}(\tilde{z})$ turn out to be analytic for all values of I. In all other cases, the first nonanalytic terms start at $\tilde{z}^9$, with the exception of $F_{a,1,1}^{(0)}(\tilde{z})$, $F_{a,1,3}^{(0)}(\tilde{z})$, $F_{s,1}^{(0)}(\tilde{z})$, and $F_{s,3}^{(0)}(\tilde{z})$, where they start at order $\tilde{z}^7$. As examples, the nonsolvable (at $x = 1$) parts of the vector form factor, $F_{v,1,1}^{(0)}(\tilde{z})$ and $F_{v,1,3}^{(0)}(\tilde{z})$ have the following numerical expansions at $\tilde{z} = 0$,

$$F_{v,1,1}^{(0)}(\tilde{z}) = 708902.27306400652676 - 1166.85387707523430i$$
$$- (61456.423805594065279 - 15405.248917971629065i)\tilde{z}^2$$
$$+ (1716.6533220464368589 - 30.2703684431181752i)\tilde{z}^4$$
$$+ (11.2780421205283601446 + 4.36541807571570566768i)\tilde{z}^6$$
$$+ (0.74166620892322374519 + 0.87946094505024891071i)\tilde{z}^8$$
$$+ (0.00029237832702742120027 - 0.00015868965691879171216i)\tilde{z}^9$$
$$+ (0.063605792722744201987 + 0.086239707160403484962i)\tilde{z}^{10}$$
$$+ (0.000083577138055319044650 - 0.000045361868983569336325i)\tilde{z}^{11}$$
$$+ (0.0055226467252660130699 + 0.0075199442142108863165i)\tilde{z}^{12}$$

$$+ (0.00001670112578659215231 4 - 9.0646114169171 \times 10^{-6} i) \tilde{z}^{13}$$
$$+ (0.00047286013902545145990 + 0.00063297686865989569848 i) \tilde{z}^{14}$$
$$+ O(\tilde{z}^{15}), \tag{4.59}$$

and

$$F_{v,1,3}^{(0)}(\tilde{z}) = -(525309.50058104203958 + 59965.62896057212016 i)$$
$$+ (38742.883772468588201 + 5783.453693303291125 i) \tilde{z}^2$$
$$- (960.03909739184249814 + 38.76539648430706459 i) \tilde{z}^4$$
$$- (7.5313243403421680570 + 0.2607012992726984795 i) \tilde{z}^6$$
$$- (0.66891050212132794057 - 0.02251916817914504430 i) \tilde{z}^8$$
$$+ (5.98 \times 10^{-192} + 0.00013201509554304589965 4 i) \tilde{z}^9$$
$$- (0.056139396989125489861 - 0.0029113705169261656 54 i) \tilde{z}^{10}$$
$$+ (1.71 \times 10^{-192} + 0.0000377368732414714538 74 i) \tilde{z}^{11}$$
$$- (0.0046441851980261141902 - 0.0002618233295226892 918 i) \tilde{z}^{12}$$
$$+ (3.42 \times 10^{-193} + 7.5409170673126 \times 10^{-6} i) \tilde{z}^{13}$$
$$+ (-0.00038214879532394872786 + 0.0000210055684978355 8601 i) \tilde{z}^{14}$$
$$+ O(\tilde{z}^{15}). \tag{4.60}$$

Notice that in the case of $F_{v,1,3}^{(0)}(\tilde{z})$ the real parts of the coefficients of the odd powers in $\tilde{z}$ are very small, namely, of the order of 10^{-192} or lower. Clearly, the real parts of these coefficients are actually equal to zero, and the values obtained indicate that we are computing the coefficients with a precision of around 200 digits. To get the final form factors, we also need to combine these results using Eq. (2.25). It can be shown that the imaginary parts of the coefficients of the odd powers in $\tilde{z}$ of the combination $F_{v,1,1}^{(0)}(\tilde{z}) + \zeta_3 F_{v,1,3}^{(0)}(\tilde{z})$ are of the order of 10^{-166} or lower. Again, this indicates that these imaginary parts actually cancel in the combination, and it is also a strong indication that the calculation is correct, since such cancellations are unlikely to happen otherwise.

At this point, we could proceed in a similar way as we did with previous expansions and use this type of information about the coefficients to improve the precision by matching at better points (that is, choosing better values of $\hat{s}_0$ and δ). The main reason for doing this would be to try to determine the base coefficients in terms of known constants, as we have done in the case of the expansions at the other points. However, this turns out to be impractical in this case. Even the simplest coefficients of solvable parts are difficult to determine using the PSLQ algorithm, due to the large number of constants that would need to be included in a search. These constants involve all sorts of harmonic polylogarithms of weight up to 5 evaluated at

$$\xi = 7 - 4\sqrt{3}, \tag{4.61}$$

on top of the constants that appeared in previous expansions, such as ζ_k, a_4, l_2^k and their products. For example, the coefficient of $\tilde{z}^0$ of the term proportional to n_h^2 in the vector form factor $F_{v,1}^{(0)}(\tilde{z})$ (which is much simpler than any of the terms proportional to n_h) contains the following constants and their products,

$$\left\{ \mathrm{H}_{0,1}(\xi), l_2, \pi, \zeta_2, \zeta_3, \frac{1}{\sqrt{3}}, \mathrm{H}_0(\xi), \mathrm{H}_1(\xi), \mathrm{H}_{-1}(\xi), \mathrm{H}_{0,1}(\xi), \right.$$
$$\mathrm{H}_{0,-1}(\xi), \mathrm{H}_{0,0,1}(\xi), \mathrm{H}_{0,0,-1}(\xi), \mathrm{H}_{0,1,1}(\xi), \mathrm{H}_{0,0,0,1}(\xi),$$
$$\left. \mathrm{H}_{0,0,0,-1}(\xi), \mathrm{H}_{0,0,1,1}(\xi), \mathrm{H}_{0,1,1,1}(\xi) \right\}. \tag{4.62}$$

The nonsolvable parts of the form factors will most likely be much more complicated, containing these and new constants, and we would need to determine the coefficients numerically with an unattainable precision in order to be able to do a PSLQ search. Fortunately, the expansions at $\hat{s} = 16$, which are very important since they are the only ones that cover the region $8 < \hat{s} < 16$, are also very well behaved and rather smooth at the expansion point, and the numerical results we have obtained are enough for most applications.

With all the expansion thus far calculated, we cover the entire range of $\hat{s}$. In an ancillary file to this paper we provide the representations for the expansion around the four singular points for phenomenological use.

V. NUMERICAL RESULTS

In the following we illustrate nonsolvable constant parts, cf. Ref. [15], of the real part to the different form factors $F_{V,1}$, $F_{V,2}$, $F_{A,1}$, $F_{A,2}$, F_S and F_P as a function of $\hat{s} = q^2/m^2$ for the constant contribution at $O(a_s^3)$ numerically. Here we use the first 200 expansion terms around $x = 1$, $x = 0$, and the (pseudo) thresholds $\hat{s} = 16$ and 4. For this approach we estimated our numerical accuracy to be better than 10^{-40}. The numerical work is done by using *Mathematica*. For harmonic, generalized harmonic [76,77] and cyclotomic harmonic polylogarithms [77,78] there are also different other numerical implementations.

In Figs. 3–8 we use different rescalings in $\hat{s}$ in part to allow for better visibility. Figures 3, 4 show the results for the vector form factors, Figs. 5, 6 for the axial-vector form

factors, and Figs. 7 and 8 for the scalar and pseudo–scalar form factor, respectively. The left panels show the contributions as a function of $\hat{s}$, while in the right panel we illustrate the form factors in the region $x \in [0, 1]$. Similar numerical results are obtained for the corresponding imaginary parts in the nonsolvable case, which we are not illustrating. The singularities of the form factors in the variable $\hat{s}$ are indicated. In the above figures it is shown, that the series expansions used do significantly overlap, which allows for a continuous representation. Concerning the expansion around $x = 0$, the singularity left to this value at $x = 4 \cdot \sqrt{3} - 7 \approx -0.0718$ requires that the expansion around $x = 1$ has to be used even below $7 - 4 \cdot \sqrt{3}$. We can extend our series representations to higher orders, to meet even higher accuracy.

In Figs. 9–14 we illustrate the real and imaginary parts of the constant $O(a_s^3)$ contributions to the vector (Figs. 9, 10), axial-vector (Figs. 11, 12), scalar (Fig. 13), and pseudo-scalar form factor (Fig. 14) as functions of $\hat{s}$.

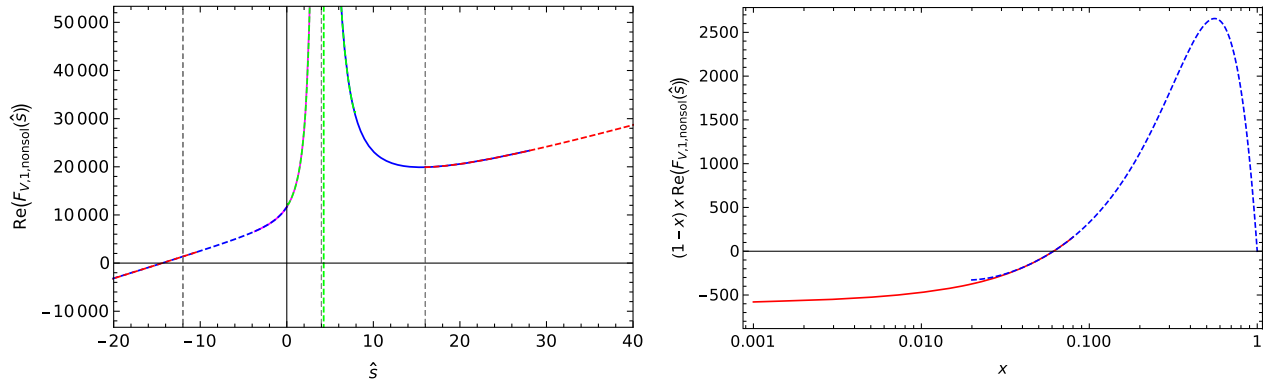Finally, we would like to discuss the numerical comparison of the present results with those of Ref. [16].

FIG. 3. Real part of the nonsolvable contribution for the vector form factor $F_{V,1}$ in dependence on $\hat{s}$ and x. The range of $x \in [0, 1]$ corresponds to $\hat{s} \in [-\infty, 0]$. The different line styles correspond to the various expansions. The vertical dashed lines indicate the (pseudo) thresholds. Left panel: the overlapping expansions are performed around $\hat{s} = -\infty, 0, 4, 16$ and $\hat{s} = \infty$. Right panel: full line: expansion around $x = 0$; dashed line: expansion around $x = 1$.
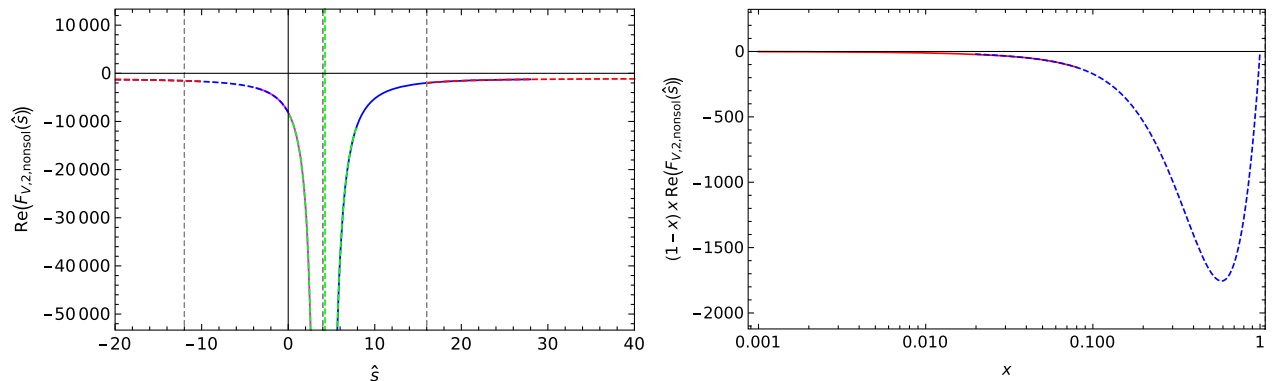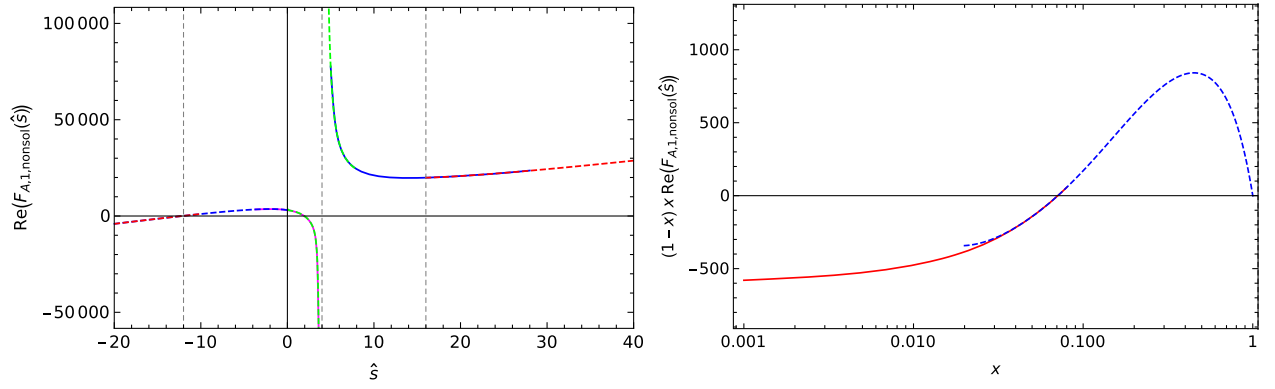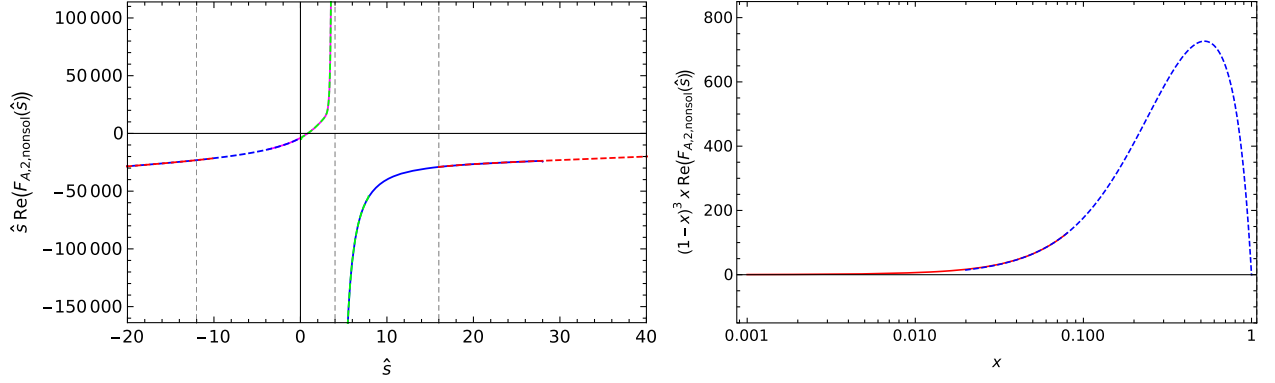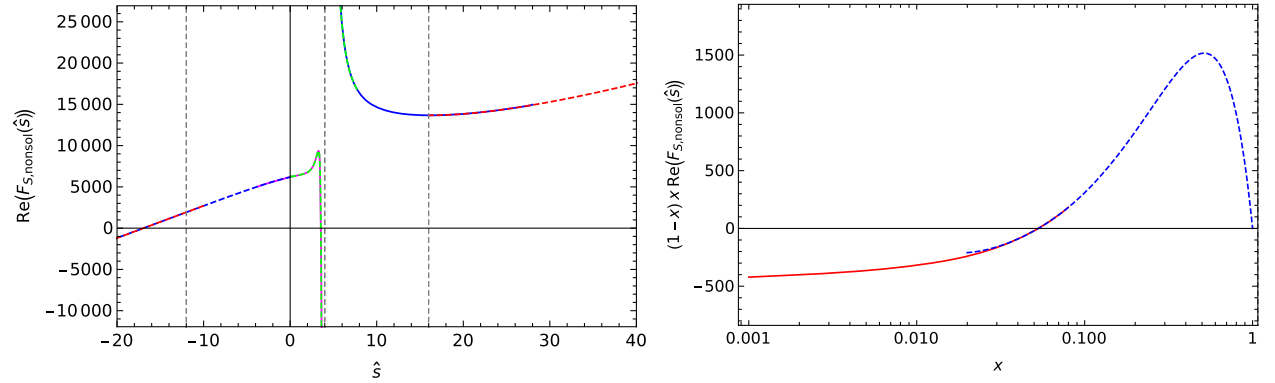

FIG. 4. The same as Fig. 3 for $F_{V,2}$.

FIG. 5. The same as Fig. 3 for $F_{A,1}$.

FIG. 6. The same as Fig. 3 for $F_{A,2}$.

FIG. 7. The same as Fig. 3 for F_S.

We compared the results at $\hat{s} = \{-300, -20, -10, -5, -0.1, -0.001, -0.0001, 0.00001, 0.1, 3.9, 3.9999, 10, 15.9999, 16.0001, 16.1, 20, 39.9999, 40\}$ using their *Mathematica* files. In general we found a relative agreement of $O(10^{-10})$ or better, except for $\hat{s} = 39.9999$, where the deviation was in the range $[6.2 \times 10^{-6}, 1.6 \times 10^{-9}]$.
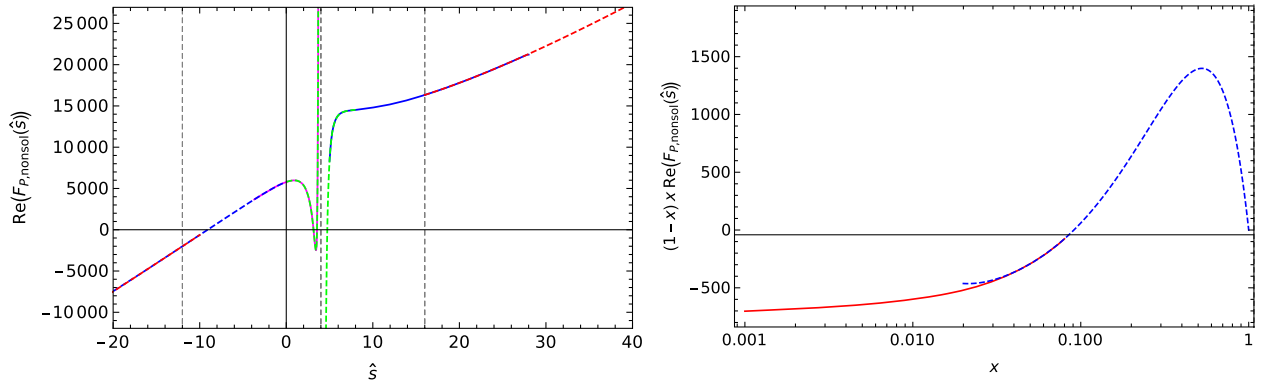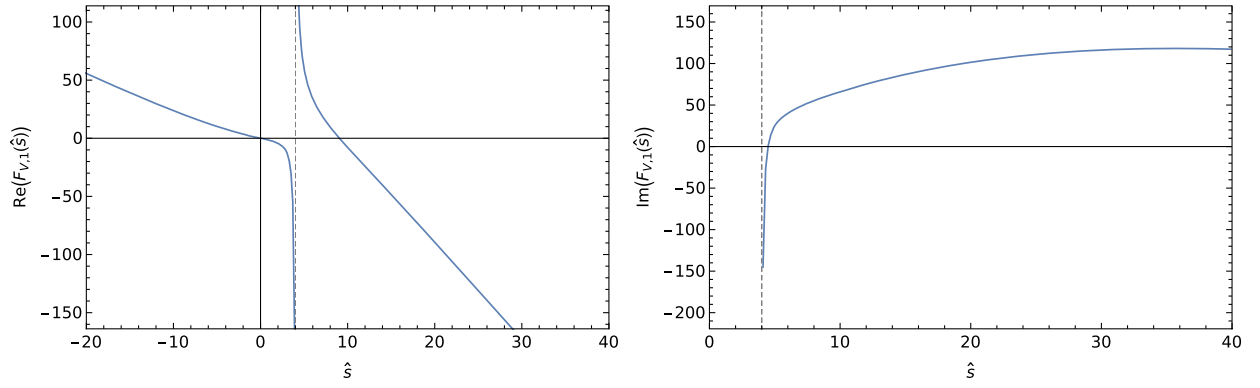
Furthermore, we compared the representations of [16], Eqs. (35)–(40), in the high energy limit $s \to -\infty$ to our

results for $n_h = 1$. Setting the color factors to those of QCD we obtain the following relative deviations

$$\left| \frac{F_I^{\text{KIT}} - F_I}{F_I} \right| \in [2.27 \times 10^{-7}, 1.13 \times 10^{-4}], \quad (5.1)$$

for $I = \{(V,1),(V,2),(A,1),(A,2),S,P\}$.

FIG. 8. The same as Fig. 3 for F_P.

FIG. 9. Real and imaginary part of the complete heavy-fermionic contributions to the vector form factor $F_{V,1}$ in dependence on $\hat{s}$.
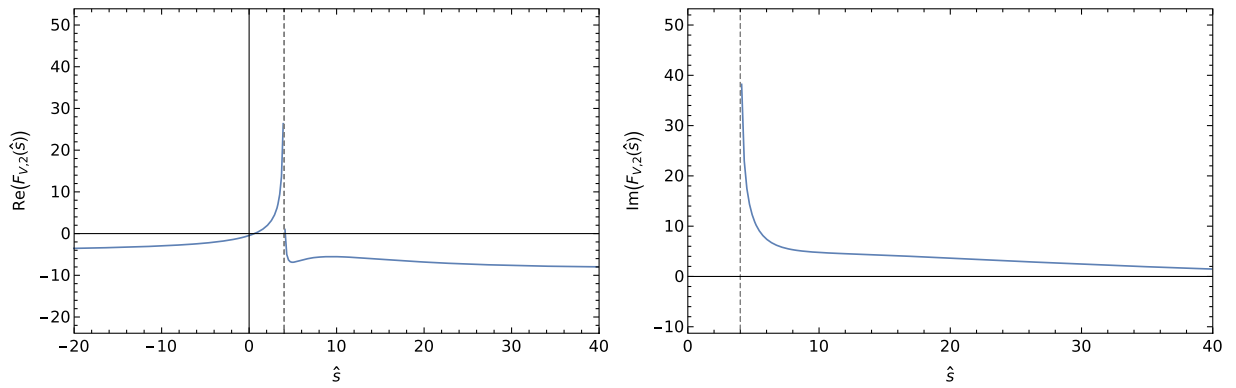
FIG. 10. Real and imaginary part of the complete heavy-fermonic contributions to the vector form factor $F_{V,2}$ in dependence on $\hat{s}$.
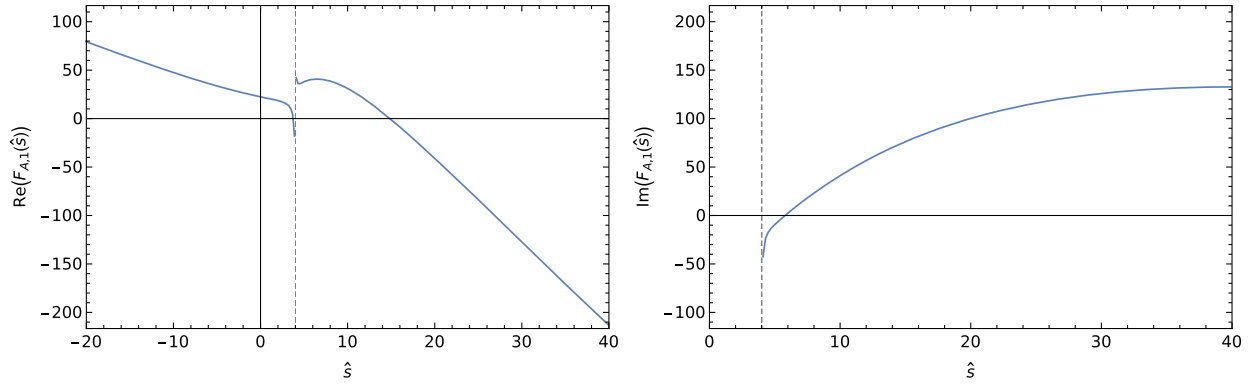
FIG. 11.   Real and imaginary part of the complete heavy-fermonic contributions to the axial-vector form factor $F_{A,1}$ in dependence on $\hat{s}$.
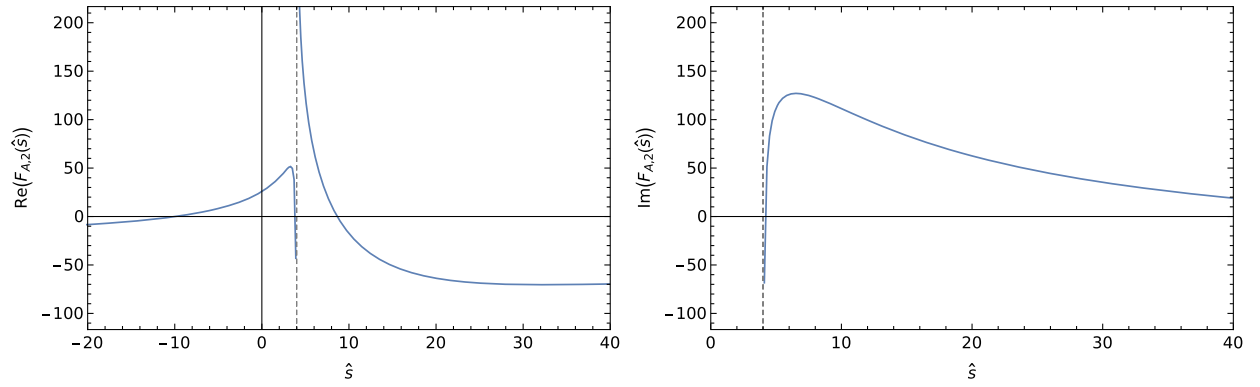

FIG. 12.   Real and imaginary part of the complete heavy-fermonic contributions to the axial-vector form factor $F_{A,2}$ in dependence on $\hat{s}$.
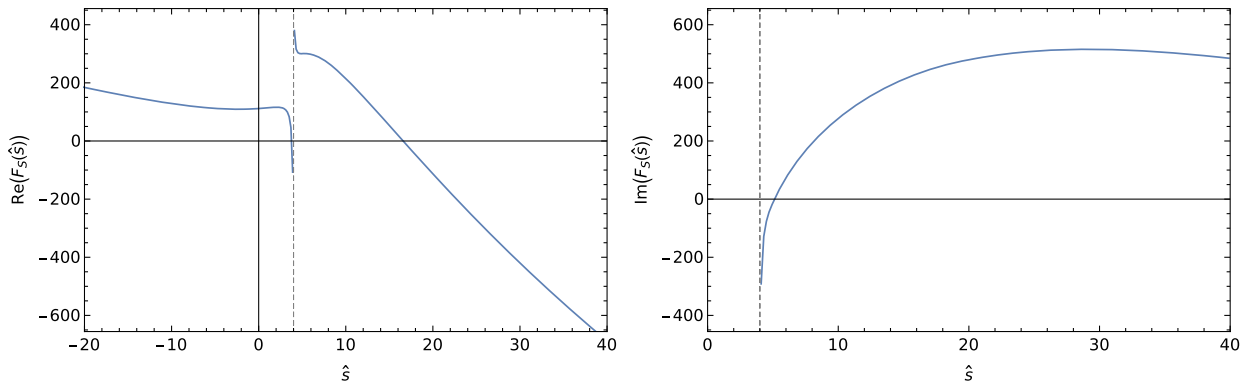

FIG. 13.   Real and imaginary part of the complete heavy-fermonic contributions to the scalar form factor F_S in dependence on $\hat{s}$.
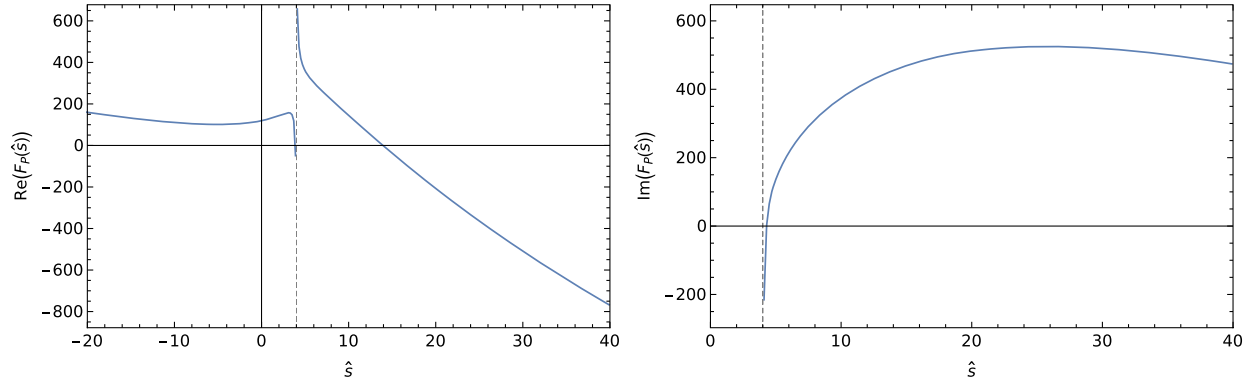
FIG. 14. Real and imaginary part of the complete heavy-fermonic contributions to the pseudo-scalar form factor F_P in dependence on $\hat{s}$.

We provide our numerical representation in *Mathematica* format at [79].

VI. CONCLUSIONS

We have calculated analytic representations for the quarkonic contributions to the massive three-loop form factors for different currents based on nontrivial computer algebra methods, including guessing algorithms to derive recursion relations and differential equation and to determine representations in terms of known constants. Up to two-loop order, the massive form factors can be represented in terms of harmonic polylogarithms. From three-loop order onward there are two principal contributions [15]. One part results from first-order factorizable recurrences which in x–space can be represented in terms of harmonic and cyclotomic harmonic polylogarithms. The second part is related to the remaining nonfirst order factorizing recurrences. We first consider their representation in terms of a series expansion around $x = 1$, which we then match with the symbolic series expansion around $x = 0$. The latter also contains a finite number of series–modulated logarithmic contributions. This series representation has a convergence radius of $x = 7 - 4\sqrt{3}$, due to the pseudo-threshold at $q^2/m^2 = 16$. Therefore, the matching of both series has to be performed for small positive values of x. We use a rather large number of expansion coefficients for this, at very high numerical precision. Thanks to this, the unknown expansion coefficients at $x = 0$ can be determined by using the PSLQ algorithm. In this way, we determine the series representations for all form factors around $x = 0$. The coefficients turn out to depend only on multiple zeta values linearly combined over $\mathbb{Q}$. Moreover, the x–series can be resummed for a series of constants into harmonic polylogarithms in x using the packages Sigma and HarmonicSums. This applies to cases in which the corresponding recurrence (guessed with the package ORE_ALGEBRA) factorizes to first order. We could determine all recurrences, which allows an even faster evaluation of the corresponding series, whenever needed.

This procedure was repeated for the other two singularities (thresholds) around $q^2/m^2 = 4$ and $q^2/m^2 = 16$. Here, however, unlike in the high energy limit $x = 0$, our analysis showed that multiple zeta values are not sufficient for the corresponding series representations, which was expected due to elliptic (or even higher order) master integrals. We determined the new constants $\tilde{\kappa}_i$ at high numerical precision for further special analyses. Our series expansions can be used in phenomenological and experimental applications. Still we provide analytic results in terms of series representations and in some cases also resumming the respective series into special functions. Our representations are given in terms of expansions around the four singular points to very high precision. We presented detailed numerical illustrations for the real and imaginary parts of the different form factors.

The present work resulted in a deeper analytic understanding of the massive three–loop form factors, beyond the numerical solutions given in Refs. [16,17]. We have compared to the numerical results given in [16,17] and found agreement, cf., Sec. V for details. References [16,17] estimated the numerical accuracy of the $O(\varepsilon^0)$ contribution based on that obtained for the pole terms, which are known analytically. This method is not always reliable. Ancillary files are provided for analytic and numerical comparisons. The ancillary files to the present paper are given at [79].

APPENDIX A: VECTOR COEFFICIENTS AT $x = 0$ IN TERMS OF KNOWN CONSTANTS

In this appendix we show the first few coefficients of the expansions at $x = 0$ of the vector form factors $F_{v,1,i}^{(0)}(x)$ and $F_{v,2,i}^{(0)}(x)$, with $i = 1, 2, 3$, as given in Eq. (3.2). The corresponding constants are given in Eqs. (3.31)–(3.33). In other words, we provide the first few coefficients $c_{I,i}(j, k)$ in Eq. (3.16) for $I \in \{(v, 1), (v, 2)\}$. The coefficients in the case of other form factors look similar. The coefficients of $F_{v,1,2}^{(0)}(x)$ are given by

$$c_{v,1,2}(4, -2) = 0, \tag{A1}$$

$$c_{v,1,2}(4, -1) = 0, \tag{A2}$$

$$c_{v,1,2}(4, 0) = 0, \tag{A3}$$

$$c_{v,1,2}(4, 1) = 0, \tag{A4}$$

$$c_{v,1,2}(4, 2) = 0, \tag{A5}$$

$$c_{v,1,2}(4, 3) = 0, \tag{A6}$$

$$c_{v,1,2}(3, -2) = 0, \tag{A7}$$

$$c_{v,1,2}(3, -1) = -128, \tag{A8}$$

$$c_{v,1,2}(3, 0) = -\frac{832}{9}, \tag{A9}$$

$$c_{v,1,2}(3, 1) = 256 l_2 + \frac{92416}{81}, \tag{A10}$$

$$c_{v,1,2}(3, 2) = \frac{41344}{27}, \tag{A11}$$

$$c_{v,1,2}(3, 3) = 256 l_2 + \frac{17920}{81}, \tag{A12}$$

$$c_{v,1,2}(2, -2) = 0, \tag{A13}$$

$$c_{v,1,2}(2, -1) = \frac{512}{9} l_2 + \frac{18464}{81}, \tag{A14}$$

$$c_{v,1,2}(2, 0) = -\frac{15616}{9} l_2 - \frac{97792}{81}, \tag{A15}$$

$$c_{v,1,2}(2, 1) = -256 \zeta_2 + 2048 l_2 - \frac{995296}{81}, \tag{A16}$$

$$c_{v,1,2}(2, 2) = -\frac{1024}{3} l_2 + \frac{579520}{27}, \tag{A17}$$

$$c_{v,1,2}(2, 3) = -256 \zeta_2 - \frac{7168}{9} l_2 - \frac{2448448}{81}, \tag{A18}$$

$$c_{v,1,2}(1, -2) = 0, \tag{A19}$$

$$c_{v,1,2}(1, -1) = \frac{14656}{27} \zeta_2 + \frac{62080}{27} l_2 - \frac{512}{3} l_2^2 + \frac{125968}{81}, \tag{A20}$$

$$c_{v,1,2}(1, 0) = \frac{256}{9} \zeta_2 - \frac{432128}{27} l_2 + \frac{2560}{3} l_2^2 - \frac{870272}{81}, \tag{A21}$$

$$c_{v,1,2}(1, 1) = -\frac{71360}{27} \zeta_2 + \frac{290944}{27} l_2 - 1536 l_2^2 - 512 \zeta_3 + \frac{9540560}{243}, \tag{A22}$$

$$c_{v,1,2}(1, 2) = -\frac{207872}{27} \zeta_2 - \frac{182272}{27} l_2 + \frac{5120}{3} l_2^2 - \frac{1943552}{27}, \tag{A23}$$

$$c_{v,1,2}(1, 3) = \frac{101888}{27} \zeta_2 - \frac{131584}{3} l_2 - \frac{5120}{3} l_2^2 - 512 \zeta_3 + \frac{18393728}{243}, \tag{A24}$$

$$c_{v,1,2}(0, -2) = -\frac{177152}{81} l_2, \tag{A25}$$

$$c_{v,1,2}(0, -1) = \frac{156256}{81} \zeta_2 + \frac{1429312}{81} l_2 - \frac{37120}{9} \zeta_2 l_2 - \frac{512}{3} l_2^2 - \frac{224}{3} \zeta_3 + \frac{2731388}{81}, \tag{A26}$$

$$c_{v,1,2}(0, 0) = \frac{166528}{27} \zeta_2 - \frac{6199040}{81} l_2 + \frac{149504}{9} \zeta_2 l_2 + \frac{2048}{3} l_2^2 + \frac{9856}{9} \zeta_3 - \frac{43085848}{243}, \tag{A27}$$

$$c_{v,1,2}(0, 1) = -\frac{1033504}{27} \zeta_2 - 128 \zeta_2^2 + \frac{1066304}{9} l_2 - 31488 \zeta_2 l_2 - \frac{6656}{3} l_2^2 + \frac{3296}{3} \zeta_3 + \frac{46115828}{243}, \tag{A28}$$

$$c_{v,1,2}(0, 2) = \frac{486400}{27} \zeta_2 - \frac{14123776}{81} l_2 + \frac{299008}{9} \zeta_2 l_2 + 4096 l_2^2 - \frac{14080}{27} \zeta_3 - \frac{159872992}{729}, \tag{A29}$$

$$c_{v,1,2}(0, 3) = -\frac{873344}{27} \zeta_2 - 128 \zeta_2^2 + \frac{5127808}{27} l_2 - \frac{320512}{9} \zeta_2 l_2 - 6144 l_2^2 - 1024 \zeta_3 + \frac{706205288}{3645}. \tag{A30}$$

The coefficients of $F_{v,2,2}^{(0)}(x)$ read

$$c_{v,2,2}(4,-2) = 0, \tag{A31}$$

$$c_{v,2,2}(4,-1) = 0, \tag{A32}$$

$$c_{v,2,2}(4,0) = 8, \tag{A33}$$

$$c_{v,2,2}(4,1) = 0, \tag{A34}$$

$$c_{v,2,2}(4,2) = 16, \tag{A35}$$

$$c_{v,2,2}(4,3) = 0, \tag{A36}$$

$$c_{v,2,2}(3,-2) = 0, \tag{A37}$$

$$c_{v,2,2}(3,-1) = -64, \tag{A38}$$

$$c_{v,2,2}(3,0) = 128l_2 + \frac{6592}{27}, \tag{A39}$$

$$c_{v,2,2}(3,1) = -512l_2 - \frac{47968}{27}, \tag{A40}$$

$$c_{v,2,2}(3,2) = 256l_2 + \frac{1888}{27}, \tag{A41}$$

$$c_{v,2,2}(3,3) = -512l_2 - \frac{79552}{27}, \tag{A42}$$

$$c_{v,2,2}(2,-2) = 0, \tag{A43}$$

$$c_{v,2,2}(2,-1) = \frac{256}{9}l_2 - \frac{320}{81}, \tag{A44}$$

$$c_{v,2,2}(2,0) = 64\zeta_2 + \frac{10880}{9}l_2 + \frac{113536}{81}, \tag{A45}$$

$$c_{v,2,2}(2,1) = -64\zeta_2 - \frac{8576}{9}l_2 + \frac{183632}{27}, \tag{A46}$$

$$c_{v,2,2}(2,2) = 128\zeta_2 + \frac{896}{9}l_2 - \frac{176864}{27}, \tag{A47}$$

$$c_{v,2,2}(2,3) = -64\zeta_2 - \frac{22528}{9}l_2 + \frac{174184}{27}, \tag{A48}$$

$$c_{v,2,2}(1,-2) = 0, \tag{A49}$$

$$c_{v,2,2}(1,-1) = \frac{7328}{27}\zeta_2 + \frac{27968}{27}l_2 - \frac{256}{3}l_2^2 + \frac{1240}{81}, \tag{A50}$$

$$c_{v,2,2}(1,0) = -\frac{5056}{9}\zeta_2 + \frac{8960}{3}l_2 - \frac{512}{3}l_2^2 + \frac{33008}{27}, \tag{A51}$$

$$c_{v,2,2}(1,1) = \frac{161696}{27}\zeta_2 + \frac{242368}{27}l_2 + \frac{512}{3}l_2^2 + 128\zeta_3 - \frac{63064}{27}, \tag{A52}$$

$$c_{v,2,2}(1,2) = \frac{18560}{27}\zeta_2 + \frac{63872}{9}l_2 - \frac{1024}{3}l_2^2 + \frac{861728}{81}, \tag{A53}$$

$$c_{v,2,2}(1,3) = \frac{203392}{27}\zeta_2 + \frac{319424}{27}l_2 + \frac{256}{3}l_2^2 + 128\zeta_3 - \frac{2598124}{243}, \tag{A54}$$

$$c_{v,2,2}(0,-2) = -\frac{15872}{27}l_2, \tag{A55}$$

$$c_{v,2,2}(0,-1) = \frac{82256}{81}\zeta_2 + \frac{367808}{81}l_2 - \frac{4736}{9}\zeta_2 l_2 + \frac{64}{3}l_2^2 - \frac{592}{3}\zeta_3 + \frac{3186380}{243}, \tag{A56}$$

$$c_{v,2,2}(0,0) = -6560\zeta_2 + \frac{2112}{5}\zeta_2^2 + \frac{103168}{9}l_2 + \frac{4352}{9}\zeta_2 l_2 - \frac{10496}{9}\zeta_3 + \frac{10523224}{243}, \tag{A57}$$

$$c_{v,2,2}(0,1) = \frac{3050672}{81}\zeta_2 - \frac{3008}{5}\zeta_2^2 + \frac{140480}{27}l_2 + \frac{16000}{9}\zeta_2 l_2 + \frac{64}{3}l_2^2 + \frac{7664}{9}\zeta_3 + \frac{59828}{243}, \tag{A58}$$

$$c_{v,2,2}(0,2) = -\frac{529600}{27}\zeta_2 + \frac{4224}{5}\zeta_2^2 - \frac{475072}{81}l_2 + \frac{8704}{9}\zeta_2 l_2 - \frac{23936}{27}\zeta_3 - \frac{18467960}{729}, \tag{A59}$$

$$c_{v,2,2}(0,3) = \frac{3651488}{81}\zeta_2 - \frac{3008}{5}\zeta_2^2 - \frac{151184}{27}l_2 + \frac{11264}{9}\zeta_2 l_2 + 1408\zeta_3 - \frac{29436937}{1215}. \tag{A60}$$

The first coefficients of $F_{v,1,3}^{(0)}(x)$ are

$$c_{v,1,3}(5,-2) = 0, \tag{A61}$$

$$c_{v,1,3}(5,-1) = 0, \tag{A62}$$

$$c_{v,1,3}(5,0) = 0, \tag{A63}$$

$$c_{v,1,3}(5,1) = \frac{224}{5\pi^2}, \tag{A64}$$

$$c_{v,1,3}(5,2) = 0, \tag{A65}$$

$$c_{v,1,3}(5,3) = \frac{224}{5\pi^2}, \tag{A66}$$

$$c_{v,1,3}(4,-2) = 0, \tag{A67}$$

$$c_{v,1,3}(4,-1) = -\frac{1904}{81\pi^2}, \tag{A68}$$

$$c_{v,1,3}(4,0) = -\frac{3248}{81\pi^2}, \tag{A69}$$

$$c_{v,1,3}(4,1) = \frac{3080}{27\pi^2}, \tag{A70}$$

$$c_{v,1,3}(4,2) = \frac{52640}{27\pi^2}, \tag{A71}$$

$$c_{v,1,3}(4,3) = \frac{44128}{81\pi^2}, \tag{A72}$$

$$c_{v,1,3}(3,-2) = 0, \tag{A73}$$

$$c_{v,1,3}(3,-1) = -\frac{70336}{243\pi^2}, \tag{A74}$$

$$c_{v,1,3}(3,0) = \frac{250880}{243\pi^2}, \tag{A75}$$

$$c_{v,1,3}(3,1) = -\frac{512}{9} - \frac{117712}{81\pi^2}, \tag{A76}$$

$$c_{v,1,3}(3,2) = \frac{5944064}{243\pi^2}, \tag{A77}$$

$$c_{v,1,3}(3,3) = -\frac{512}{9} - \frac{5339824}{243\pi^2}, \tag{A78}$$

$$c_{v,1,3}(2,-2) = 0, \tag{A79}$$

$$c_{v,1,3}(2,-1) = -\frac{17696}{81} - \frac{447104}{243\pi^2}, \tag{A80}$$

$$c_{v,1,3}(2,0) = \frac{2964416}{243\pi^2} - \frac{58240}{81}, \tag{A81}$$

$$c_{v,1,3}(2,1) = 896l_2 + \frac{896}{\pi^2}\zeta_3 + \frac{185888}{81} - \frac{11501560}{243\pi^2}, \tag{A82}$$

$$c_{v,1,3}(2,2) = \frac{143104}{81} + \frac{13919248}{81\pi^2}, \tag{A83}$$

$$c_{v,1,3}(2,3) = 896l_2 + \frac{896}{\pi^2}\zeta_3 + \frac{69952}{27} - \frac{28909132}{243\pi^2}, \tag{A84}$$

$$c_{v,1,3}(1,-2) = 0, \tag{A85}$$

$$c_{v,1,3}(1,-1) = -\frac{1792}{9}l_2 + \frac{60032}{27\pi^2}\zeta_3 + \frac{87056}{243} - \frac{1974784}{243\pi^2}, \tag{A86}$$

$$c_{v,1,3}(1,0) = -\frac{7168}{9}l_2 - \frac{12544}{\pi^2}\zeta_3 + \frac{11072}{9} + \frac{662144}{27\pi^2}, \tag{A87}$$

$$c_{v,1,3}(1,1) = \frac{9856}{5}\zeta_2 + \frac{490112}{27\pi^2}\zeta_3 - \frac{4687888}{243} - \frac{2968168}{243\pi^2}, \tag{A88}$$

$$c_{v,1,3}(1,2) = \frac{17920}{9}l_2 - \frac{272384}{9\pi^2}\zeta_3 + \frac{7113728}{243} + \frac{145353152}{729\pi^2}, \tag{A89}$$

$$c_{v,1,3}(1,3) = -\frac{10150048}{243} + \frac{14336l_2}{9} - \frac{115553998}{729\pi^2} + \frac{9856\zeta_2}{5} - \frac{1250816\zeta_3}{27\pi^2}, \tag{A90}$$

$$c_{v,1,3}(0,-2) = -\frac{221440}{243}, \tag{A91}$$

$$c_{v,1,3}(0,-1) = -\frac{14560}{81}\zeta_2 + \frac{26432}{81}l_2 + \frac{794752}{81\pi^2}\zeta_3 + \frac{11712400}{729} - \frac{4659760}{243\pi^2}, \tag{A92}$$

$$c_{v,1,3}(0,0) = -\frac{590848}{135}\zeta_2 - \frac{105728}{27}l_2 - \frac{160384}{3\pi^2}\zeta_3 + \frac{17780672}{243\pi^2} - \frac{71864320}{729}, \tag{A93}$$

$$c_{v,1,3}(0,1) = \frac{2887936}{405}\zeta_2 - \frac{688576}{81}l_2 + 3584\zeta_2 l_2 + 1344\zeta_3 + \frac{10114048}{81\pi^2}\zeta_3 + \frac{448}{\pi^2}\zeta_5 + \frac{120607088}{729} - \frac{3788428}{27\pi^2}, \tag{A94}$$

$$c_{v,1,3}(0,2) = -\frac{1532672}{405}\zeta_2 - \frac{476672}{81}l_2 - \frac{8866816}{81\pi^2}\zeta_3 + \frac{653277352}{2187\pi^2} - \frac{192621760}{729}, \tag{A95}$$

$$c_{v,1,3}(0,3) = -\frac{2014528}{405}\zeta_2 + \frac{68992}{3}l_2 + 3584\zeta_2 l_2$$
$$+ 1344\zeta_3 - \frac{905408}{81\pi^2}\zeta_3 + \frac{448}{\pi^2}\zeta_5$$
$$+ \frac{194264248}{729} - \frac{1214278177}{4374\pi^2}, \tag{A96}$$

and likewise for $F_{v,2,3}^{(0)}(x)$

$$c_{v,2,3}(5,-2) = 0, \tag{A97}$$

$$c_{v,2,3}(5,-1) = 0, \tag{A98}$$

$$c_{v,2,3}(5,0) = -\frac{112}{15\pi^2}, \tag{A99}$$

$$c_{v,2,3}(5,1) = \frac{476}{15\pi^2}, \tag{A100}$$

$$c_{v,2,3}(5,2) = -\frac{224}{15\pi^2}, \tag{A101}$$

$$c_{v,2,3}(5,3) = \frac{476}{15\pi^2}, \tag{A102}$$

$$c_{v,2,3}(4,-2) = 0, \tag{A103}$$

$$c_{v,2,3}(4,-1) = -\frac{952}{81\pi^2}, \tag{A104}$$

$$c_{v,2,3}(4,0) = \frac{6944}{81\pi^2}, \tag{A105}$$

$$c_{v,2,3}(4,1) = -\frac{14056}{27\pi^2}, \tag{A106}$$

$$c_{v,2,3}(4,2) = -\frac{728}{3\pi^2}, \tag{A107}$$

$$c_{v,2,3}(4,3) = -\frac{162400}{81\pi^2}, \tag{A108}$$

$$c_{v,2,3}(3,-2) = 0, \tag{A109}$$

$$c_{v,2,3}(3,-1) = -\frac{39200}{243\pi^2}, \tag{A110}$$

$$c_{v,2,3}(3,0) = \frac{320}{9} + \frac{7616}{81\pi^2}, \tag{A111}$$

$$c_{v,2,3}(3,1) = -\frac{1312}{9} - \frac{941248}{243\pi^2}, \tag{A112}$$

$$c_{v,2,3}(3,2) = \frac{640}{9} - \frac{790216}{243\pi^2}, \tag{A113}$$

$$c_{v,2,3}(3,3) = -\frac{1312}{9} - \frac{176512}{27\pi^2}, \tag{A114}$$

$$c_{v,2,3}(2,-2) = 0, \tag{A115}$$

$$c_{v,2,3}(2,-1) = -\frac{8848}{81} - \frac{322336}{243\pi^2}, \tag{A116}$$

$$c_{v,2,3}(2,0) = \frac{32560}{81} - 448l_2 + \frac{677152}{243\pi^2} + \frac{896\zeta_3}{\pi^2}, \tag{A117}$$

$$c_{v,2,3}(2,1) = -\frac{273584}{81} - \frac{3312008}{243\pi^2} - \frac{2688\zeta_3}{\pi^2} \tag{A118}$$

$$c_{v,2,3}(2,2) = -\frac{70624}{81} - 896l_2 - \frac{2675876}{81\pi^2} + \frac{1792\zeta_3}{\pi^2}, \tag{A119}$$

$$c_{v,2,3}(2,3) = -\frac{183808}{27} - \frac{1477532}{27\pi^2} - \frac{2688\zeta_3}{\pi^2}, \tag{A120}$$

$$c_{v,2,3}(1,-2) = 0, \tag{A121}$$

$$c_{v,2,3}(1,-1) = -\frac{126008}{243} - \frac{896l_2}{9} - \frac{1073632}{243\pi^2} + \frac{30016\zeta_3}{27\pi^2}, \tag{A122}$$

$$c_{v,2,3}(1,0) = \frac{130192}{243} - \frac{5824l_2}{9} - \frac{801472}{81\pi^2} + 64\zeta_2 + \frac{25088\zeta_3}{3\pi^2}, \tag{A123}$$

$$c_{v,2,3}(1,1) = \frac{1937720}{243} - \frac{30464l_2}{9} - \frac{13104616}{243\pi^2} - \frac{27232\zeta_2}{15} + \frac{169792\zeta_3}{27\pi^2}, \tag{A124}$$

$$c_{v,2,3}(1,2) = -\frac{423584}{81} - \frac{19712l_2}{9} - \frac{36706208}{729\pi^2} + 128\zeta_2 + \frac{186368\zeta_3}{9\pi^2}, \tag{A125}$$

$$c_{v,2,3}(1,3) = \frac{2046752}{243} - \frac{52864l_2}{9} - \frac{54828256}{729\pi^2} - \frac{27232\zeta_2}{15} + \frac{982016\zeta_3}{27\pi^2}, \tag{A126}$$

$$c_{v,2,3}(0,-2) = -\frac{19840}{81}, \tag{A127}$$

$$c_{v,2,3}(0,-1) = \frac{4465256}{729} + \frac{38752l_2}{81} - \frac{1908592}{243\pi^2} + \frac{124304\zeta_2}{405} + \frac{85568\zeta_3}{81\pi^2}, \tag{A128}$$

$$c_{v,2,3}(0,0) = \frac{13510576}{729} + \frac{72128l_2}{81} - \frac{2291072}{243\pi^2}$$
$$+ \frac{549968\zeta_2}{135} - 1792l_2\zeta_2 - \frac{2432\zeta_3}{3}$$
$$+ \frac{14336\zeta_3}{81\pi^2} + \frac{10752\zeta_5}{\pi^2}, \tag{A129}$$

$$c_{v,2,3}(0,1) = \frac{3356632}{729} + 7392l_2 + \frac{1120700}{243\pi^2}$$
$$- \frac{2580928\zeta_2}{405} - \frac{1184\zeta_3}{3} - \frac{2202368\zeta_3}{81\pi^2}$$
$$- \frac{6944\zeta_5}{\pi^2}, \tag{A130}$$

$$c_{v,2,3}(0,2) = \frac{5003536}{729} - \frac{26432l_2}{9} - \frac{110330920}{2187\pi^2}$$
$$+ \frac{2573792\zeta_2}{405} - 3584l_2\zeta_2 - \frac{4864\zeta_3}{3}$$
$$+ \frac{674240\zeta_3}{81\pi^2} + \frac{21504\zeta_5}{\pi^2}, \tag{A131}$$

$$c_{v,2,3}(0,3) = -\frac{4741880}{729} + \frac{363776l_2}{81} + \frac{102560675}{4374\pi^2}$$
$$- \frac{1971968\zeta_2}{405} - \frac{1184\zeta_3}{3} + \frac{151648\zeta_3}{81\pi^2}$$
$$- \frac{6944\zeta_5}{\pi^2}. \tag{A132}$$

The coefficients of $F_{v,1,1}^{(0)}(x)$ are

$$c_{v,1,1}(6,-2) = 0, \tag{A133}$$

$$c_{v,1,1}(6,-1) = 0, \tag{A134}$$

$$c_{v,1,1}(6,0) = 0, \tag{A135}$$

$$c_{v,1,1}(6,1) = 0, \tag{A136}$$

$$c_{v,1,1}(6,2) = 0, \tag{A137}$$

$$c_{v,1,1}(6,3) = 0, \tag{A138}$$

$$c_{v,1,1}(5,-2) = 0, \tag{A139}$$

$$c_{v,1,1}(5,-1) = -\frac{704}{81}, \tag{A140}$$

$$c_{v,1,1}(5,0) = \frac{13904}{405}, \tag{A141}$$

$$c_{v,1,1}(5,1) = -\frac{88}{45} - \frac{224}{5}\frac{\zeta_3}{\pi^2}, \tag{A142}$$

$$c_{v,1,1}(5,2) = \frac{6944}{81}, \tag{A143}$$

$$c_{v,1,1}(5,3) = -\frac{22384}{405} - \frac{224\zeta_3}{5\pi^2}, \tag{A144}$$

$$c_{v,1,1}(4,-2) = 0, \tag{A145}$$

$$c_{v,1,1}(4,-1) = -\frac{688}{243} + \frac{1904}{81}\frac{\zeta_3}{\pi^2}, \tag{A146}$$

$$c_{v,1,1}(4,0) = \frac{6944}{81} + \frac{3248}{81}\frac{\zeta_3}{\pi^2}, \tag{A147}$$

$$c_{v,1,1}(4,1) = -\frac{193676}{243} + 16\zeta_2 - \frac{3080}{27}\frac{\zeta_3}{\pi^2}, \tag{A148}$$

$$c_{v,1,1}(4,2) = \frac{63224}{81} - \frac{52640}{27}\frac{\zeta_3}{\pi^2}, \tag{A149}$$

$$c_{v,1,1}(4,3) = -\frac{190484}{81} + 16\zeta_2 - \frac{44128}{81}\frac{\zeta_3}{\pi^2}, \tag{A150}$$

$$c_{v,1,1}(3,-2) = 0, \tag{A151}$$

$$c_{v,1,1}(3,-1) = 240 + \frac{256}{3}\zeta_2 + \frac{70336}{243}\frac{\zeta_3}{\pi^2}, \tag{A152}$$

$$c_{v,1,1}(3,0) = -\frac{443584}{243} - \frac{7936}{81}\zeta_2 - \frac{250880}{243}\frac{\zeta_3}{\pi^2}, \tag{A153}$$

$$c_{v,1,1}(3,1) = \frac{4446616}{729} - \frac{16832}{81}\zeta_2 + \frac{512}{9}\zeta_3 + \frac{117712}{81}\frac{\zeta_3}{\pi^2}, \tag{A154}$$

$$c_{v,1,1}(3,2) = -\frac{4453616}{243} - \frac{8576}{9}\zeta_2 - \frac{5944064}{243}\frac{\zeta_3}{\pi^2}, \tag{A155}$$

$$c_{v,1,1}(3,3) = \frac{15976564}{729} + \frac{5632}{81}\zeta_2 + \frac{512}{9}\zeta_3 + \frac{5339824}{243}\frac{\zeta_3}{\pi^2}, \tag{A156}$$

$$c_{v,1,1}(2,-2) = 0, \tag{A157}$$

$$c_{v,1,1}(2,-1) = \frac{73096}{27} - \frac{3136}{27}\zeta_2 - \frac{736}{81}\zeta_3 + \frac{447104}{243}\frac{\zeta_3}{\pi^2}, \tag{A158}$$

$$c_{v,1,1}(2,0) = -\frac{3005056}{243} + \frac{13952}{27}\zeta_2 + \frac{194752}{81}\zeta_3$$
$$- \frac{2964416}{243}\frac{\zeta_3}{\pi^2}, \tag{A159}$$

$$c_{v,1,1}(2,1) = \frac{6205580}{729} + \frac{157280}{27}\zeta_2 + \frac{1184}{5}\zeta_2^2 + 256\zeta_2 l_2^2 - \frac{128}{3}l_2^4 - 1024a_4 - \frac{533408}{81}\zeta_3 + \frac{11501560}{243}\frac{\zeta_3}{\pi^2}$$
$$- 896\zeta_3 l_2 - 896\frac{\zeta_3^2}{\pi^2}, \tag{A160}$$

$$c_{v,1,1}(2,2) = -\frac{25654520}{729} - \frac{22144}{3}\zeta_2 + \frac{353408}{81}\zeta_3 - \frac{13919248}{81}\frac{\zeta_3}{\pi^2}, \tag{A161}$$

$$c_{v,1,1}(2,3) = \frac{10452319}{1215} + \frac{312752}{27}\zeta_2 + \frac{1184}{5}\zeta_2^2 + 256\zeta_2 l_2^2 - \frac{128}{3}l_2^4 - 1024a_4 - \frac{151616}{27}\zeta_3 + \frac{28909132}{243}\frac{\zeta_3}{\pi^2}$$
$$- 896\zeta_3 l_2 - 896\frac{\zeta_3^2}{\pi^2}, \tag{A162}$$

$$c_{v,1,1}(1,-2) = 0, \tag{A163}$$

$$c_{v,1,1}(1,-1) = \frac{2681840}{243} - \frac{239264}{243}\zeta_2 + \frac{4544}{45}\zeta_2^2 - \frac{512}{9}\zeta_2 l_2^2 + \frac{256}{27}l_2^4 + \frac{2048}{9}a_4 - \frac{755168}{243}\zeta_3 + \frac{1974784}{243}\frac{\zeta_3}{\pi^2} + \frac{1792}{9}\zeta_3 l_2$$
$$- \frac{60032}{27}\frac{\zeta_3^2}{\pi^2}, \tag{A164}$$

$$c_{v,1,1}(1,0) = -\frac{11847904}{243} + \frac{1762496}{243}\zeta_2 - \frac{13952}{135}\zeta_2^2 - \frac{2048}{9}\zeta_2 l_2^2 + \frac{1024}{27}l_2^4 + \frac{8192}{9}a_4 + \frac{960512}{81}\zeta_3 - \frac{662144}{27}\frac{\zeta_3}{\pi^2}$$
$$+ \frac{7168}{9}\zeta_3 l_2 + 12544\frac{\zeta_3^2}{\pi^2}, \tag{A165}$$

$$c_{v,1,1}(1,1) = \frac{16988900}{2187} - \frac{4210432}{243}\zeta_2 - \frac{6016}{3}\zeta_2^2 + \frac{6957088}{243}\zeta_3 + \frac{2968168}{243}\frac{\zeta_3}{\pi^2} - \frac{9856}{5}\zeta_2\zeta_3 - \frac{490112}{27}\frac{\zeta_3^2}{\pi^2} + 1280\zeta_5, \tag{A166}$$

$$c_{v,1,1}(1,2) = -\frac{23135480}{243} + \frac{9932032}{243}\zeta_2 - \frac{447872}{135}\zeta_2^2 + \frac{5120}{9}\zeta_2 l_2^2 - \frac{2560}{27}l_2^4 - \frac{20480}{9}a_4 - \frac{3832448}{243}\zeta_3 - \frac{145353152}{729}\frac{\zeta_3}{\pi^2}$$
$$- \frac{17920}{9}\zeta_3 l_2 + \frac{272384}{9}\frac{\zeta_3^2}{\pi^2}, \tag{A167}$$

$$c_{v,1,1}(1,3) = \frac{2962613437}{36450} - \frac{4402552}{81}\zeta_2 + \frac{51584}{45}\zeta_2^2 + \frac{4096}{9}\zeta_2 l_2^2 - \frac{2048}{27}l_2^4 - \frac{16384}{9}a_4 + \frac{15899680}{243}\zeta_3 + \frac{115553998}{729}\frac{\zeta_3}{\pi^2}$$
$$- \frac{9856}{5}\zeta_2\zeta_3 - \frac{14336}{9}\zeta_3 l_2 + \frac{1250816}{27}\frac{\zeta_3^2}{\pi^2} + 1280\zeta_5, \tag{A168}$$

$$c_{v,1,1}(0,-2) = \frac{221440}{243}\zeta_3, \tag{A169}$$

$$c_{v,1,1}(0,-1) = -\frac{28904}{9} - \frac{743888}{243}\zeta_2 - \frac{95440}{81}\zeta_2^2 + \frac{256}{9}\zeta_2^2 l_2 + \frac{9344}{81}\zeta_2 l_2^2 - \frac{4672}{243}l_2^4 - \frac{37376}{81}a_4 - \frac{10044736}{729}\zeta_3$$
$$+ \frac{4659760}{243}\frac{\zeta_3}{\pi^2} - \frac{38432}{81}\zeta_2\zeta_3 - \frac{26432}{81}\zeta_3 l_2 - \frac{794752}{81}\frac{\zeta_3^2}{\pi^2} + \frac{15856}{9}\zeta_5, \tag{A170}$$

$$c_{v,1,1}(0,0) = \frac{10612672}{243} + \frac{3300608}{243}\zeta_2 + \frac{349888}{135}\zeta_2^2 - \frac{1024}{9}\zeta_2^2 l_2 - \frac{268480}{9}\zeta_5 - \frac{25088}{27}\zeta_2 l_2^2 + \frac{12544}{81}l_2^4 + \frac{100352}{27}a_4$$
$$+ \frac{46482880}{729}\zeta_3 - \frac{17780672}{243}\frac{\zeta_3}{\pi^2} + \frac{1078528}{135}\zeta_2\zeta_3 + \frac{105728}{27}\zeta_3 l_2 + \frac{160384}{3}\frac{\zeta_3^2}{\pi^2}, \tag{A171}$$

$$c_{v,1,1}(0,1) = -\frac{641594606}{6561} + \frac{295424}{81}\zeta_2 + \frac{5199536}{405}\zeta_2^2 + \frac{63296}{35}\zeta_2^3 - \frac{256}{9}\zeta_2^2 l_2 - \frac{343936}{81}\zeta_2 l_2^2 + 1024\zeta_2^2 l_2^2 - 448\frac{\zeta_3\zeta_5}{\pi^2}$$
$$+ \frac{171968}{243}l_2^4 - \frac{512}{3}\zeta_2 l_2^4 + \frac{1375744}{81}a_4 + \frac{1642288}{27}\zeta_5 - 4096\zeta_2 a_4 - \frac{82803488}{729}\zeta_3 + \frac{3788428}{27}\frac{\zeta_3}{\pi^2}$$
$$- \frac{5779456}{405}\zeta_2\zeta_3 + \frac{688576}{81}\zeta_3 l_2 - 3584\zeta_2\zeta_3 l_2 - 1024\zeta_3^2 - \frac{10114048}{81}\frac{\zeta_3^2}{\pi^2},\tag{A172}$$

$$c_{v,1,1}(0,2) = \frac{285175364}{6561} + \frac{1200128 a_4}{81} + \frac{150016 l_2^4}{243} + \frac{5818816\zeta_2}{243} - \frac{300032}{81}l_2^2\zeta_2 - \frac{7099136\zeta_2^2}{405} - \frac{2048}{9}l_2\zeta_2^2 + \frac{186528256\zeta_3}{729}$$
$$+ \frac{476672 l_2\zeta_3}{81} - \frac{653277352\zeta_3}{2187\pi^2} + \frac{4719872\zeta_2\zeta_3}{405} + \frac{8866816\zeta_3^2}{81\pi^2} - \frac{653696\zeta_5}{9},\tag{A173}$$

$$c_{v,1,1}(0,3) = -\frac{1906863860207}{6561000} - \frac{185344 a_4}{3} - \frac{23168 l_2^4}{9} - \frac{15147034\zeta_2}{729} - 4096 a_4\zeta_2 + \frac{46336 l_2^2\zeta_2}{3} - \frac{512}{3}l_2^4\zeta_2 + \frac{8162192\zeta_2^2}{135}$$
$$+ 1024 l_2^2\zeta_2^2 + \frac{63296\zeta_2^3}{35} - \frac{169868872\zeta_3}{729} - \frac{68992 l_2\zeta_3}{3} + \frac{1214278177\zeta_3}{4374\pi^2} - \frac{1337792}{405}\zeta_2\zeta_3 - 3584 l_2\zeta_2\zeta_3$$
$$- 1024\zeta_3^2 + \frac{905408\zeta_3^2}{81\pi^2} + \frac{1681792\zeta_5}{27} - \frac{448\zeta_3\zeta_5}{\pi^2},\tag{A174}$$

and those of $F_{v,2,1}^{(0)}(x)$

$$c_{v,2,1}(6,-2) = 0,\tag{A175}$$

$$c_{v,2,1}(6,-1) = 0,\tag{A176}$$

$$c_{v,2,1}(6,0) = 0,\tag{A177}$$

$$c_{v,2,1}(6,1) = -\frac{34}{45},\tag{A178}$$

$$c_{v,2,1}(6,2) = 0,\tag{A179}$$

$$c_{v,2,1}(6,3) = -\frac{34}{45},\tag{A180}$$

$$c_{v,2,1}(5,-2) = 0,\tag{A181}$$

$$c_{v,2,1}(5,-1) = -\frac{352}{81},\tag{A182}$$

$$c_{v,2,1}(5,0) = -\frac{704}{81} + \frac{112}{15}\frac{\zeta_3}{\pi^2},\tag{A183}$$

$$c_{v,2,1}(5,1) = -\frac{17576}{405} - \frac{476}{15}\frac{\zeta_3}{\pi^2},\tag{A184}$$

$$c_{v,2,1}(5,2) = -\frac{6824}{405} + \frac{224}{15}\frac{\zeta_3}{\pi^2},\tag{A185}$$

$$c_{v,2,1}(5,3) = -\frac{31456}{405} - \frac{476}{15}\frac{\zeta_3}{\pi^2},\tag{A186}$$

$$c_{v,2,1}(4,-2) = 0,\tag{A187}$$

$$c_{v,2,1}(4,-1) = \frac{976}{243} + \frac{952}{81}\frac{\zeta_3}{\pi^2},\tag{A188}$$

$$c_{v,2,1}(4,0) = \frac{4784}{243} - 8\zeta_2 - \frac{6944}{81}\frac{\zeta_3}{\pi^2},\tag{A189}$$

$$c_{v,2,1}(4,1) = \frac{29620}{81} + \frac{14056}{27}\frac{\zeta_3}{\pi^2},\tag{A190}$$

$$c_{v,2,1}(4,2) = \frac{29512}{243} - 16\zeta_2 + \frac{728}{3}\frac{\zeta_3}{\pi^2},\tag{A191}$$

$$c_{v,2,1}(4,3) = \frac{113356}{243} + \frac{162400}{81}\frac{\zeta_3}{\pi^2},\tag{A192}$$

$$c_{v,2,1}(3,-2) = 0,\tag{A193}$$

$$c_{v,2,1}(3,-1) = \frac{5576}{27} + \frac{128}{3}\zeta_2 + \frac{39200}{243}\frac{\zeta_3}{\pi^2},\tag{A194}$$

$$c_{v,2,1}(3,0) = -\frac{31984}{81} - \frac{1888}{27}\zeta_2 - \frac{224}{9}\zeta_3 - \frac{7616}{81}\frac{\zeta_3}{\pi^2},\tag{A195}$$

$$c_{v,2,1}(3,1) = \frac{199544}{81} + \frac{58528}{81}\zeta_2 + \frac{928}{9}\zeta_3 + \frac{941248}{243}\frac{\zeta_3}{\pi^2},\tag{A196}$$

$$c_{v,2,1}(3,2) = -\frac{26168}{27} + \frac{9184}{81}\zeta_2 - \frac{448}{9}\zeta_3 + \frac{790216}{243}\frac{\zeta_3}{\pi^2},$$ (A197)

$$c_{v,2,1}(3,3) = \frac{1605320}{729} + \frac{101152}{81}\zeta_2 + \frac{928}{9}\zeta_3 + \frac{176512}{27}\frac{\zeta_3}{\pi^2},$$ (A198)

$$c_{v,2,1}(2,-2) = 0,$$ (A199)

$$c_{v,2,1}(2,-1) = \frac{43232}{27} - \frac{944}{27}\zeta_2 - \frac{368}{81}\zeta_3 + \frac{322336}{243}\frac{\zeta_3}{\pi^2},$$ (A200)

$$c_{v,2,1}(2,0) = \frac{130480}{27} - \frac{17872}{27}\zeta_2 - \frac{816}{5}\zeta_2^2 - 128\zeta_2 l_2^2 + \frac{64}{3}l_2^4 + 512a_4 - \frac{92944}{81}\zeta_3 - \frac{677152}{243}\frac{\zeta_3}{\pi^2} + 448\zeta_3 l_2 - 896\frac{\zeta_3^2}{\pi^2},$$ (A201)

$$c_{v,2,1}(2,1) = \frac{875360}{243} - \frac{86704}{27}\zeta_2 + \frac{544}{5}\zeta_2^2 + \frac{375056}{81}\zeta_3 + \frac{3312008}{243}\frac{\zeta_3}{\pi^2} + 2688\frac{\zeta_3^2}{\pi^2},$$ (A202)

$$c_{v,2,1}(2,2) = \frac{12501296}{729} + \frac{69008}{27}\zeta_2 - \frac{1632}{5}\zeta_2^2 - 256\zeta_2 l_2^2 + \frac{128}{3}l_2^4 + 1024a_4 - \frac{160448}{81}\zeta_3 + \frac{2675876}{81}\frac{\zeta_3}{\pi^2} + 896\zeta_3 l_2$$
$$- 1792\frac{\zeta_3^2}{\pi^2},$$ (A203)

$$c_{v,2,1}(2,3) = -\frac{9839219}{3645} - \frac{106528}{27}\zeta_2 + \frac{544}{5}\zeta_2^2 + \frac{51712}{9}\zeta_3 + \frac{1477532}{27}\frac{\zeta_3}{\pi^2} + 2688\frac{\zeta_3^2}{\pi^2},$$ (A204)

$$c_{v,2,1}(1,-2) = 0,$$ (A205)

$$c_{v,2,1}(1,-1) = \frac{1469552}{243} - \frac{128368}{243}\zeta_2 + \frac{2272}{45}\zeta_2^2 - \frac{256}{9}\zeta_2 l_2^2 + \frac{128}{27}l_2^4 + \frac{1024}{9}a_4 - \frac{315184}{243}\zeta_3 + \frac{1073632}{243}\frac{\zeta_3}{\pi^2} + \frac{896}{9}\zeta_3 l_2$$
$$- \frac{30016}{27}\frac{\zeta_3^2}{\pi^2},$$ (A206)

$$c_{v,2,1}(1,0) = \frac{4850800}{243} - \frac{242336}{243}\zeta_2 - \frac{65056}{135}\zeta_2^2 - \frac{1664}{9}\zeta_2 l_2^2 + \frac{832}{27}l_2^4 + \frac{6656}{9}a_4 - \frac{804832}{243}\zeta_3 + \frac{801472}{81}\frac{\zeta_3}{\pi^2} - 64\zeta_2\zeta_3$$
$$+ \frac{5824}{9}\zeta_3 l_2 - \frac{25088}{3}\frac{\zeta_3^2}{\pi^2} + 256\zeta_5,$$ (A207)

$$c_{v,2,1}(1,1) = \frac{32303216}{729} + \frac{24592}{243}\zeta_2 + \frac{261568}{135}\zeta_2^2 - \frac{8704}{9}\zeta_2 l_2^2 + \frac{4352}{27}l_2^4 + \frac{34816}{9}a_4 - \frac{7996304}{243}\zeta_3 + \frac{13104616}{243}\frac{\zeta_3}{\pi^2}$$
$$+ \frac{27232}{15}\zeta_2\zeta_3 + \frac{30464}{9}\zeta_3 l_2 - \frac{169792}{27}\frac{\zeta_3^2}{\pi^2} - 2880\zeta_5,$$ (A208)

$$c_{v,2,1}(1,2) = -\frac{3166864}{243} - \frac{1317808}{243}\zeta_2 - \frac{21664}{135}\zeta_2^2 - \frac{5632}{9}\zeta_2 l_2^2 + \frac{2816}{27}l_2^4 + \frac{22528}{9}a_4 + \frac{1241824}{81}\zeta_3 + \frac{36706208}{729}\frac{\zeta_3}{\pi^2}$$
$$- 128\zeta_2\zeta_3 + \frac{19712}{9}\zeta_3 l_2 - \frac{186368}{9}\frac{\zeta_3^2}{\pi^2} + 512\zeta_5,$$ (A209)

$$c_{v,2,1}(1,3) = \frac{2238816019}{54675} + \frac{397808}{81}\zeta_2 + \frac{607072}{135}\zeta_2^2 - \frac{15104}{9}\zeta_2 l_2^2 + \frac{7552}{27}l_2^4 + \frac{60416}{9}a_4 - \frac{11472656}{243}\zeta_3 + \frac{54828256}{729}\frac{\zeta_3}{\pi^2}$$
$$+ \frac{27232}{15}\zeta_2\zeta_3 + \frac{52864}{9}\zeta_3 l_2 - \frac{982016}{27}\frac{\zeta_3^2}{\pi^2} - 2880\zeta_5, \tag{A210}$$

$$c_{v,2,1}(0,-2) = \frac{19840}{81}\zeta_3, \tag{A211}$$

$$c_{v,2,1}(0,-1) = -\frac{936028}{243} - \frac{379792}{243}\zeta_2 - \frac{300856}{405}\zeta_2^2 + \frac{128}{9}\zeta_2^2 l_2 + \frac{10952}{9}\zeta_5 + \frac{11200}{81}\zeta_2 l_2^2 - \frac{5600}{243}l_2^4 - \frac{44800}{81}a_4$$
$$- \frac{2901776}{729}\zeta_3 + \frac{1908592}{243}\frac{\zeta_3}{\pi^2} - \frac{179024}{405}\zeta_2\zeta_3 - \frac{38752}{81}\zeta_3 l_2 - \frac{85568}{81}\frac{\zeta_3^2}{\pi^2}, \tag{A212}$$

$$c_{v,2,1}(0,0) = \frac{102440}{27} - \frac{1816256}{243}\zeta_2 - \frac{47008}{405}\zeta_2^2 - \frac{56288}{105}\zeta_2^3 + \frac{256}{9}\zeta_2^2 l_2 + \frac{10112}{81}\zeta_2 l_2^2 - 512\zeta_2^2 l_2^2 - \frac{5056}{243}l_2^4 + \frac{256}{3}\zeta_2 l_2^4$$
$$- \frac{40448}{81}a_4 + 2048a_4\zeta_2 - \frac{6183424}{729}\zeta_3 + \frac{2291072}{243}\frac{\zeta_3}{\pi^2} - \frac{627088}{135}\zeta_2\zeta_3 - \frac{72128}{81}\zeta_3 l_2 + 1792\zeta_2\zeta_3 l_2 + \frac{2048}{3}\zeta_3^2$$
$$- \frac{14336}{81}\frac{\zeta_3^2}{\pi^2} + \frac{375584}{27}\zeta_5 - 10752\frac{\zeta_3\zeta_5}{\pi^2}, \tag{A213}$$

$$c_{v,2,1}(0,1) = \frac{19646248}{2187} - \frac{826384}{81}\zeta_2 - \frac{6101096}{405}\zeta_2^2 - \frac{102272}{105}\zeta_2^3 + \frac{1408}{9}\zeta_2^2 l_2 + \frac{32192}{9}\zeta_2 l_2^2 - \frac{16096}{27}l_2^4 + 6944\frac{\zeta_3\zeta_5}{\pi^2}$$
$$- \frac{128768}{9}a_4 - \frac{6590224}{729}\zeta_3 - \frac{1120700}{243}\frac{\zeta_3}{\pi^2} - \frac{786136}{27}\zeta_5 + \frac{3236608}{405}\zeta_2\zeta_3 - 7392\zeta_3 l_2 - \frac{640}{3}\zeta_3^2 + \frac{2202368}{81}\frac{\zeta_3^2}{\pi^2}, \tag{A214}$$

$$c_{v,2,1}(0,2) = \frac{777783418}{6561} + \frac{65024a_4}{9} + \frac{8128l_2^4}{27} - \frac{1764008\zeta_2}{243} + 4096a_4\zeta_2 - \frac{16256}{9}l_2^2\zeta_2 + \frac{512l_2^4\zeta_2}{3} + \frac{163568\zeta_2^2}{27}$$
$$+ \frac{512l_2\zeta_2^2}{9} - 1024l_2^2\zeta_2^2 - \frac{112576\zeta_2^3}{105} - \frac{29704720\zeta_3}{729} + \frac{26432l_2\zeta_3}{9} + \frac{110330920\zeta_3}{2187\pi^2} - \frac{3101792}{405}\zeta_2\zeta_3$$
$$+ 3584l_2\zeta_2\zeta_3 + \frac{4096\zeta_3^2}{3} - \frac{674240\zeta_3^2}{81\pi^2} + \frac{762688\zeta_5}{27} - \frac{21504\zeta_3\zeta_5}{\pi^2} \tag{A215}$$

$$c_{v,2,1}(0,3) = \frac{8943651091}{729000} - \frac{481280a_4}{81} - \frac{60160l_2^4}{243} - \frac{8824004\zeta_2}{729} + \frac{120320l_2^2\zeta_2}{81} - \frac{3384272\zeta_2^2}{135} + \frac{512l_2\zeta_2^2}{3} - \frac{102272\zeta_2^3}{105}$$
$$- \frac{9461092\zeta_3}{729} - \frac{363776l_2\zeta_3}{81} - \frac{102560675\zeta_3}{4374\pi^2} + \frac{2475008\zeta_2\zeta_3}{405} - \frac{640\zeta_3^2}{3} - \frac{151648\zeta_3^2}{81\pi^2} - \frac{479104\zeta_5}{27}$$
$$+ \frac{6944\zeta_3\zeta_5}{\pi^2}. \tag{A216}$$

APPENDIX B: EXPANSIONS AT $x = 0$ ($\hat{s} \to -\infty$)

In Eq. (3.48), we showed the expansion up to $O(x^3)$ of the nonsolvable parts of $F_{V,1}^{(0)}(x)$. In this appendix, we show the expansions up to $O(x^3)$ of the remaining form factors,

$$F_{V,2}^{(0)}(x) = F_{V,2}^{(0),\text{sol}}(x) + n_h\Bigg\{-\frac{81920}{81}l_2\zeta_2 + \Bigg[-3072l_2\zeta_2^2 - 64\zeta_2\zeta_3 - 672\zeta_5 + \frac{2176}{81}l_2^4 + \frac{9248}{45}\zeta_2^2 - \frac{10112}{27}l_2^2\zeta_2$$
$$+ \frac{17408}{27}a_4 - \frac{18016}{9}\zeta_3 + \frac{40768}{27}l_2\zeta_2 + \frac{1091648}{243} + \frac{1837100}{243}\zeta_2 + \Bigg(\frac{2048}{9}l_2\zeta_2 + \frac{7936}{9}\zeta_3$$
$$+ \frac{14368}{9}\zeta_2 - \frac{257264}{243}\Bigg)\ln(x) + \Bigg(-\frac{4456}{9} + \frac{5120}{27}\zeta_2\Bigg)\ln^2(x) - \frac{4672}{27}\ln^3(x) - \frac{880}{81}\ln^4(x)\Bigg]x$$

$$+ \left[-4096 a_4 \zeta_2 + 1024 l_2^2 \zeta_2^2 + 256 \zeta_3^2 - \frac{512}{3} l_2^4 \zeta_2 - \frac{19456}{243} l_2^4 + \frac{23872}{105} \zeta_2^3 + \frac{83968}{9} l_2 \zeta_2^2 \right.$$

$$+ \frac{113216}{27} \zeta_2 \zeta_3 + \frac{114944}{81} l_2^2 \zeta_2 - \frac{155648}{81} a_4 - \frac{295936}{9} l_2 \zeta_2 - \frac{1162336}{27} \zeta_5 - \frac{2363776}{81} \zeta_3 - \frac{2463584}{243}$$

$$- \frac{7840040}{81} \zeta_2 + \frac{10280288}{405} \zeta_2^2 + \left(-512 \zeta_5 - \frac{256}{9} l_2^2 \zeta_2 + \frac{896}{27} l_2^4 + \frac{7168}{9} a_4 + \frac{29376}{5} \zeta_2^2 - \frac{254848}{81} \zeta_3 \right.$$

$$\left. - \frac{257792}{27} l_2 \zeta_2 - \frac{817856}{81} \zeta_2 - \frac{1304960}{81} \right) \ln(x) + \left(-1024 a_4 + 256 l_2^2 \zeta_2 - \frac{128}{3} l_2^4 + \frac{992}{5} \zeta_2^2 \right.$$

$$\left. - \frac{34304}{9} l_2 \zeta_2 + \frac{48896}{27} \zeta_3 - \frac{71200}{27} \zeta_2 - \frac{684640}{243} \right) \ln^2(x) + \left(-256 l_2 \zeta_2 - \frac{64}{3} \zeta_3 \right.$$

$$\left. - \frac{64384}{81} \zeta_2 + \frac{350528}{243} \right) \ln^3(x) + \frac{22976}{243} \ln^4(x) - \frac{176}{405} \ln^5(x) \right] x^2 + \left[20480 a_4 \zeta_2 - 5120 l_2^2 \zeta_2^2 + \frac{2560}{3} l_2^4 \zeta_2 \right.$$

$$- \frac{207872}{9} l_2 \zeta_2^2 + \frac{355520}{3} l_2 \zeta_2 + \frac{364096}{105} \zeta_2^3 + \frac{548480}{243} l_2^4 + \frac{766304}{3} \zeta_5 - \frac{647488}{27} \zeta_2 \zeta_3$$

$$- \frac{1356160}{81} l_2^2 \zeta_2 + \frac{4387840}{81} a_4 + \frac{10194944}{81} \zeta_3 - \frac{12707392}{81} \zeta_2^2 + \frac{92640452}{243} \zeta_2$$

$$+ \frac{204738082}{6561} + \left(-768 \zeta_2 \zeta_3 + 10112 \zeta_5 - \frac{1792}{27} l_2^4 - \frac{13312}{9} l_2^2 \zeta_2 - \frac{14336}{9} a_4 + \frac{114944}{27} l_2 \zeta_2 \right.$$

$$\left. - \frac{5501312}{135} \zeta_2^2 + \frac{5769280}{81} \zeta_3 + \frac{10840352}{243} \zeta_2 + \frac{25002020}{2187} \right) \ln(x) + \left(5120 a_4 - 1280 l_2^2 \zeta_2 \right.$$

$$\left. + \frac{640}{3} l_2^4 - \frac{6496}{5} \zeta_2^2 + \frac{160000}{9} l_2 \zeta_2 - \frac{279616}{81} \zeta_2 - \frac{351296}{27} \zeta_3 + \frac{14492732}{729} \right) \ln^2(x)$$

$$+ \left(2816 l_2 \zeta_2 + \frac{640}{3} \zeta_3 + \frac{152384}{27} \zeta_2 - \frac{6464984}{729} \right) \ln^3(x)$$

$$\left. + \left(16 \zeta_2 - \frac{342692}{243} \right) \ln^4(x) + \frac{6872}{81} \ln^5(x) + \frac{68}{45} \ln^6(x) \right] x^3 + O(x^4) \right\}, \tag{B1}$$

$$F_{A,1}^{(0)}(x) = F_{A,1}^{(0),\text{sol}}(x) + n_h \left\{ -\frac{44288}{81 x} l_2 \zeta_2 - 1024 l_2 \zeta_2^2 - \frac{1120}{81} l_2^2 \zeta_2 - \frac{1168}{243} l_2^4 - \frac{1640}{9} \zeta_2 \zeta_3 + \frac{3964}{9} \zeta_5 + \frac{5068}{27} \zeta_2^2 - \frac{7226}{9} \right.$$

$$- \frac{9344}{81} a_4 + \frac{46324}{81} \zeta_3 + \frac{255952}{81} l_2 \zeta_2 + \frac{1862569}{243} \zeta_2 + \left(\frac{64}{27} l_2^4 + \frac{512}{9} a_4 - \frac{512}{9} l_2^2 \zeta_2 \right.$$

$$\left. + \frac{15520}{27} l_2 \zeta_2 + \frac{21728}{135} \zeta_2^2 + \frac{34660}{243} \zeta_2 - \frac{55676}{81} \zeta_3 + \frac{670460}{243} \right) \ln(x)$$

$$+ \left(\frac{128}{9} l_2 \zeta_2 - \frac{512}{9} \zeta_3 + \frac{2264}{81} \zeta_2 + \frac{18274}{27} \right) \ln^2(x) + \left(60 - \frac{32}{3} \zeta_2 \right) \ln^3(x) - \frac{172}{243} \ln^4(x) - \frac{176}{81} \ln^5(x)$$

$$+ \left[\frac{1600}{243} l_2^4 + \frac{12352}{81} l_2^2 \zeta_2 + \frac{12800}{81} a_4 + \frac{16000}{27} \zeta_2 \zeta_3 + \frac{27392}{9} l_2 \zeta_2^2 \right.$$

$$- \frac{132320}{27} \zeta_5 + \frac{194336}{81} \zeta_2^2 - \frac{442960}{81} \zeta_3 - \frac{850912}{81} l_2 \zeta_2 + \frac{1637576}{243}$$

$$- \frac{2210888}{81} \zeta_2 + \left(\frac{640}{27} l_2^4 - \frac{1664}{9} l_2^2 \zeta_2 + \frac{2912}{5} \zeta_2^2 + \frac{5120}{9} a_4 - \frac{13664}{81} \zeta_3 - \frac{53248}{27} l_2 \zeta_2 \right.$$

$$\left. - \frac{167312}{81} \zeta_2 - \frac{1355456}{243} \right) \ln(x) + \left(-\frac{1904}{9} \zeta_2 - \frac{3520}{9} l_2 \zeta_2 + \frac{4432}{27} \zeta_3 - \frac{86036}{81} \right) \ln^2(x)$$

$$+ \left(-\frac{7664}{81}\zeta_2 - \frac{13088}{81} \right)\ln^3(x) + \frac{17200}{243}\ln^4(x) + \frac{44}{405}\ln^5(x) \right]x + \left[1024a_4\zeta_2 + 26304l_2\zeta_2 - 256l_2^2\zeta_2^2 - 48\zeta_3^2 \right.$$

$$+ \frac{128}{3}l_2^4\zeta_2 - \frac{24320}{3}l_2\zeta_2^2 + \frac{32176}{105}\zeta_2^3 + \frac{36608}{243}l_2^4 - \frac{87664}{27}\zeta_2\zeta_3 - \frac{100864}{81}l_2^2\zeta_2 - \frac{129970051}{13122} + \frac{292864}{81}a_4$$

$$+ \frac{1359904}{81}\zeta_3 + \frac{4874924}{81}\zeta_2 - \frac{5137136}{405}\zeta_2^2 + \frac{249592}{9}\zeta_5 + \left(6320\zeta_2 - 128\zeta_2\zeta_3 + 576\zeta_5 - \frac{1280}{27}l_2^4 \right.$$

$$+ \frac{3328}{9}l_2^2\zeta_2 - \frac{10240}{9}a_4 - \frac{36608}{27}l_2\zeta_2 - \frac{138448}{45}\zeta_2^2 + \frac{520232}{81}\zeta_3 + \frac{22789421}{2187} \right)\ln(x)$$

$$+ \left(256a_4 - 64l_2^2\zeta_2 - 104\zeta_2^2 + \frac{32}{3}l_2^4 + \frac{12032}{9}l_2\zeta_2 - \frac{31048}{27}\zeta_2 - \frac{43904}{27}\zeta_3 + \frac{4494101}{729} \right)\ln^2(x)$$

$$+ \left(192l_2\zeta_2 + \frac{32}{3}\zeta_3 + \frac{43696}{81}\zeta_2 + \frac{91018}{729} \right)\ln^3(x) + \left(4\zeta_2 - \frac{52199}{243} \right)\ln^4(x) + \frac{242}{405}\ln^5(x) \right]x^2$$

$$+ \left[-2048a_4\zeta_2 + 512l_2^2\zeta_2^2 - 512\zeta_3^2 - \frac{256}{3}l_2^4\zeta_2 - \frac{18656}{15}\zeta_2^3 - \frac{125888}{243}l_2^4 + \frac{136960}{9}l_2\zeta_2^2 + \frac{162944}{27}\zeta_2\zeta_3 + \frac{298432}{81}l_2^2\zeta_2 \right.$$

$$- \frac{468128}{9}l_2\zeta_2 - \frac{1007104}{81}a_4 - \frac{1501472}{27}\zeta_5 - \frac{2243680}{81}\zeta_3 + \frac{2364736}{81}\zeta_2^2 - \frac{47478772}{6561} - \frac{80144464}{729}\zeta_2$$

$$+ \left(-640\zeta_2\zeta_3 - 1728\zeta_5 + \frac{5632}{27}l_2^4 - \frac{13952}{9}l_2^2\zeta_2 + \frac{45056}{9}a_4 + \frac{206204}{243} + \frac{213248}{27}l_2\zeta_2 + \frac{1104608}{135}\zeta_2^2 \right.$$

$$- \frac{2051936}{81}\zeta_3 - \frac{5104624}{243}\zeta_2 \right)\ln(x) + \left(-512a_4 + 128l_2^2\zeta_2 - \frac{64}{3}l_2^4 + \frac{176}{5}\zeta_2^2 - \frac{21824}{9}l_2\zeta_2 + \frac{75472}{27}\zeta_3 \right.$$

$$+ \frac{493280}{81}\zeta_2 - \frac{9056684}{729} \right)\ln^2(x) + \left(\frac{64}{3}\zeta_3 - 384l_2\zeta_2 - \frac{35312}{81} - \frac{111536}{81}\zeta_2 \right)\ln^3(x)$$

$$+ \left(\frac{5900}{9} - 8\zeta_2 \right)\ln^4(x) - \frac{3712}{405}\ln^5(x) - \frac{34}{45}\ln^6(x) \right]x^3 + O(x^4) \right\}, \tag{B2}$$

$$F_{A,2}^{(0)}(x) = F_{A,2}^{(0),\text{sol}}(x) + n_h \left\{ \frac{13312}{27}l_2\zeta_2 + \left[-\frac{128}{3}l_2^2\zeta_2 - \frac{128}{9}l_2^4 - \frac{22528}{135}\zeta_2^2 - \frac{1024}{3}a_4 + \frac{96032}{27}\zeta_3 + \frac{97984}{27}l_2\zeta_2 \right. \right.$$

$$+ \frac{273376}{81} + \frac{1417540}{81}\zeta_2 + \left(\frac{2816}{9}l_2\zeta_2 - \frac{5888}{27}\zeta_3 + \frac{18016}{27}\zeta_2 + \frac{1883216}{243} \right)\ln(x)$$

$$+ \left(\frac{992}{27}\zeta_2 + \frac{347672}{243} \right)\ln^2(x) - \frac{2368}{81}\ln^3(x) - \frac{1360}{27}\ln^4(x) \right]x + \left[-\frac{5888}{81}l_2^2\zeta_2 + \frac{13312}{243}l_2^4 \right.$$

$$+ \frac{14144}{27}\zeta_2\zeta_3 + \frac{81920}{9}l_2\zeta_2^2 + \frac{106496}{81}a_4 - \frac{397408}{27}\zeta_5 - \frac{579328}{27}l_2\zeta_2 - \frac{920960}{81}\zeta_3$$

$$+ \frac{1591712}{81} + \frac{3029984}{405}\zeta_2^2 - \frac{13386440}{243}\zeta_2 + \left(\frac{2048}{3}\zeta_2^2 + \frac{3712}{27}l_2^4 - \frac{8960}{9}l_2^2\zeta_2 + \frac{29696}{9}a_4 \right.$$

$$- \frac{41216}{9}l_2\zeta_2 + \frac{45824}{81}\zeta_3 - \frac{49856}{81} - \frac{387008}{81}\zeta_2 \right)\ln(x) + \left(\frac{1280}{27}\zeta_3 - \frac{14080}{9}l_2\zeta_2 - \frac{79136}{81}\zeta_2 - \frac{351616}{243} \right)\ln^2(x)$$

$$+ \left(\frac{5248}{27} - \frac{24896}{81}\zeta_2 \right)\ln^3(x) + \frac{22624}{243}\ln^4(x) + \frac{176}{405}\ln^5(x) \right]x^2$$

$$+ \left[4096a_4\zeta_2 + 53824l_2\zeta_2 - 1024l_2^2\zeta_2^2 - 13696\zeta_2\zeta_3 - 1408\zeta_3^2 \right.$$

$$
\begin{aligned}
&+\frac{512}{3}l_2^4\zeta_2-\frac{3776}{105}\zeta_2^3-\frac{53248}{9}l_2\zeta_2^2+\frac{108416}{243}l_2^4-\frac{206464}{81}l_2^2\zeta_2+\frac{867328}{81}a_4+\frac{1905088}{27}\zeta_5+\frac{4172704}{81}\zeta_3\\
&-\frac{5531966}{2187}+\frac{7887212}{81}\zeta_2-\frac{12842176}{405}\zeta_2^2+\left(-2304\zeta_2\zeta_3+1152\zeta_5-\frac{1024}{3}l_2^2\zeta_2+\frac{1280}{9}l_2^4+\frac{10240}{3}a_4\right.\\
&\left.-\frac{345856}{27}l_2\zeta_2-\frac{2236736}{135}\zeta_2^2+\frac{2694272}{81}\zeta_3+\frac{4301152}{243}\zeta_2+\frac{24011348}{729}\right)\ln(x)+\left(1024a_4-256l_2^2\zeta_2\right.\\
&\left.+\frac{128}{3}l_2^4-\frac{3808}{5}\zeta_2^2+\frac{15872}{9}l_2\zeta_2-\frac{48256}{9}\zeta_3-\frac{100384}{27}\zeta_2+\frac{126532}{9}\right)\ln^2(x)\\
&+\left(768l_2\zeta_2+128\zeta_3-\frac{76184}{243}+\frac{144704}{81}\zeta_2\right)\ln^3(x)+\left(16\zeta_2-\frac{66932}{81}\right)\ln^4(x)\\
&\left.+\frac{184}{15}\ln^5(x)-\frac{68}{45}\ln^6(x)\right]x^3+O(x^4)\bigg\},
\end{aligned}
\tag{B3}
$$

$$
\begin{aligned}
F_S^{(0)}(x)=&\,F_S^{(0),\text{sol}}(x)+n_h\bigg\{-\frac{34304}{81x}l_2\zeta_2-1024l_2\zeta_2^2-\frac{256}{81}l_2^2\zeta_2-\frac{304}{243}l_2^4-\frac{1640}{9}\zeta_2\zeta_3-\frac{2432}{81}a_4+\frac{3964}{9}\zeta_5\\
&+\frac{10324}{45}\zeta_2^2+\frac{107668}{81}\zeta_3+\frac{209600}{81}l_2\zeta_2-\frac{1676170}{243}+\frac{1828108}{243}\zeta_2+\left(\frac{64}{27}l_2^4+\frac{512}{9}a_4-\frac{512}{9}l_2^2\zeta_2\right.\\
&-\frac{5876}{243}\zeta_2+\frac{13408}{27}l_2\zeta_2+\frac{21728}{135}\zeta_2^2-\frac{51260}{81}\zeta_3+\left.\frac{22184}{27}\right)\ln(x)+\left(\frac{128}{9}l_2\zeta_2-\frac{512}{9}\zeta_3\right.\\
&\left.+\frac{1520}{81}\zeta_2+\frac{77548}{243}\right)\ln^2(x)+\left(-\frac{32}{3}\zeta_2+\frac{5452}{81}\right)\ln^3(x)+\frac{2888}{243}\ln^4(x)-\frac{176}{81}\ln^5(x)\\
&+\left[2304l_2\zeta_2^2-\frac{1472}{81}l_2^4+\frac{8128}{27}l_2^2\zeta_2-\frac{11776}{27}a_4+\frac{13328}{27}\zeta_2\zeta_3-\frac{23896}{27}\zeta_5+\frac{62168}{405}\zeta_2^2-\frac{99776}{81}\zeta_3\right.\\
&-\frac{534016}{81}l_2\zeta_2+\frac{1087216}{243}-\frac{4558496}{243}\zeta_2+\left(64l_2^2\zeta_2-256a_4-\frac{32}{3}l_2^4+\frac{6176}{15}\zeta_2^2-\frac{40912}{27}\zeta_2\right.\\
&\left.-\frac{25408}{27}l_2\zeta_2-\frac{44704}{81}\zeta_3-\frac{2297624}{243}\right)\ln(x)-\left(\frac{2344}{81}\zeta_2+\frac{138080}{81}-\frac{4112}{27}\zeta_3\right)\ln^2(x)\\
&\left.-\left(\frac{160}{9}\zeta_2+\frac{9088}{27}\right)\ln^3(x)+\frac{5456}{81}\ln^4(x)\right]x+\left[-272\zeta_3^2-\frac{2144}{35}\zeta_2^3-\frac{24320}{3}l_2\zeta_2^2+\frac{34816}{243}l_2^4\right.\\
&-\frac{41776}{27}\zeta_2\zeta_3-\frac{100736}{81}l_2^2\zeta_2+\frac{259208}{27}\zeta_5+\frac{278528}{81}a_4+\frac{510944}{27}l_2\zeta_2+\frac{1004896}{81}\zeta_3\\
&-\frac{1411504}{405}\zeta_2^2+\frac{11143964}{243}\zeta_2-\frac{62619035}{6561}+\left(-192\zeta_2\zeta_3-544\zeta_5+\frac{512}{27}l_2^4-\frac{640}{9}l_2^2\zeta_2+\frac{4096}{9}a_4+\frac{12032}{9}l_2\zeta_2\right.\\
&\left.-\frac{19904}{27}\zeta_2^2+\frac{68696}{27}\zeta_3+\frac{963968}{243}\zeta_2+\frac{15324602}{2187}\right)\ln(x)+\left(\frac{4736}{9}l_2\zeta_2-\frac{432}{5}\zeta_2^2-\frac{20480}{27}\zeta_3-\frac{29176}{81}\zeta_2\right.\\
&\left.+\frac{4544258}{729}\right)\ln^2(x)-\left(\frac{32}{3}\zeta_3+\frac{2324}{729}-\frac{4544}{27}\zeta_2\right)\ln^3(x)+\left(8\zeta_2-\frac{9028}{243}\right)\ln^4(x)-\frac{3032}{405}\ln^5(x)-\frac{17}{45}\ln^6(x)\right]x^2\\
&+\left[544\zeta_3^2+\frac{4288}{35}\zeta_2^3+\frac{43264}{3}l_2\zeta_2^2+\frac{82448}{27}\zeta_2\zeta_3-\frac{2633152}{81}\zeta_3+\frac{218560}{81}l_2^2\zeta_2-\frac{424736}{9}l_2\zeta_2-\frac{431096}{27}\zeta_5\right.\\
&+\frac{652576}{81}\zeta_2^2-\frac{687616}{81}a_4-\frac{85952}{243}l_2^4+\frac{29263327}{2187}-\frac{72816460}{729}\zeta_2+\left(384\zeta_2\zeta_3+1088\zeta_5-\frac{64}{3}l_2^2\zeta_2\right.
\end{aligned}
$$

$$
\begin{aligned}
&\left.- \frac{352}{9} l_2^4 - \frac{2176}{3} l_2\zeta_2 - \frac{2816}{3} a_4 + \frac{530096}{135}\zeta_2^2 - \frac{873344}{81}\zeta_3 - \frac{1344104}{81}\zeta_2 - \frac{50224357}{2187}\right) \ln(x) \\
&+ \left(\frac{864}{5}\zeta_2^2 - \frac{3008}{3} l_2\zeta_2 + \frac{9584}{9}\zeta_3 + \frac{142976}{81}\zeta_2 - \frac{2236508}{243}\right) \ln^2(x) + \left(\frac{64}{3}\zeta_3 - \frac{41008}{81}\zeta_2\right. \\
&\left.\left. - \frac{1157750}{729}\right) \ln^3(x) - \left(16\zeta_2 - \frac{115906}{243}\right) \ln^4(x) + \frac{722}{135} \ln^5(x) + \frac{34}{45}\ln^6(x)\right] x^3 + O(x^4)\right\},
\end{aligned}
\tag{B4}
$$

$$
\begin{aligned}
F_P^{(0)}(x) = F_P^{(0),\text{sol}}(x) + n_h &\left\{ -\frac{54272}{81x} l_2\zeta_2 - 1024 l_2\zeta_2^2 - \frac{256}{81} l_2^2\zeta_2 - \frac{304}{243} l_2^4 - \frac{1640}{9}\zeta_2\zeta_3 - \frac{2432}{81} a_4 + \frac{3964}{9}\zeta_5 \right. \\
&+ \frac{10324}{45}\zeta_2^2 + \frac{202432}{81} l_2\zeta_2 + \frac{107668}{81}\zeta_3 - \frac{1676170}{243} + \frac{1828108}{243}\zeta_2 + \left(\frac{64}{27} l_2^4 + \frac{512}{9} a_4 - \frac{512}{9} l_2^2\zeta_2 - \frac{5876}{243}\zeta_2\right. \\
&+ \frac{13408}{27} l_2\zeta_2 + \frac{21728}{135}\zeta_2^2 + \frac{22184}{27} - \frac{51260}{81}\zeta_3\bigg) \ln(x) + \left(\frac{128}{9} l_2\zeta_2 - \frac{512}{9}\zeta_3 + \frac{1520}{81}\zeta_2 + \frac{77548}{243}\right) \ln^2(x) \\
&+ \left(-\frac{32}{3}\zeta_2 + \frac{5452}{81}\right) \ln^3(x) + \frac{2888}{243} \ln^4(x) - \frac{176}{81} \ln^5(x) + \left[768 l_2\zeta_2^2 - \frac{128}{9} l_2^4 + \frac{448}{3} l_2^2\zeta_2 - \frac{1024}{3} a_4 \right. \\
&+ \frac{12464}{27}\zeta_2\zeta_3 - \frac{32968}{27}\zeta_5 - \frac{68672}{81}\zeta_3 + \frac{180392}{405}\zeta_2^2 - \frac{279424}{81} l_2\zeta_2 + \frac{853856}{243} - \frac{1159744}{243}\zeta_2 \\
&+ \left(-256 a_4 + 64 l_2^2\zeta_2 - \frac{32}{3} l_2^4 + \frac{6176}{15}\zeta_2^2 - \frac{14512}{27}\zeta_2 - \frac{18112}{27} l_2\zeta_2 - \frac{33952}{81}\zeta_3 - \frac{376456}{243}\right) \ln(x) \\
&+ \left(\frac{3632}{243} + \frac{4112}{27}\zeta_3 + \frac{4136}{81}\zeta_2\right) \ln^2(x) + \left(-\frac{160}{9}\zeta_2 - \frac{18208}{81}\right) \ln^3(x) + \frac{1808}{81} \ln^4(x)\bigg] x \\
&+ \left[304\zeta_3^2 + \frac{2208}{7}\zeta_2^3 + \frac{3952}{9}\zeta_2\zeta_3 + \frac{8200}{3}\zeta_5 + \frac{10912}{9} l_2\zeta_2 + \frac{17024}{243} l_2^4 - \frac{18688}{9} l_2\zeta_2^2 - \frac{43856}{45}\zeta_2^2 \right. \\
&- \frac{51328}{81} l_2^2\zeta_2 + \frac{136192}{81} a_4 - \frac{215776}{81}\zeta_3 + \frac{952988}{243}\zeta_2 - \frac{1907579}{6561} + \left(448\zeta_2\zeta_3 + 288\zeta_5 - \frac{128}{3} l_2^2\zeta_2 - \frac{128}{9} l_2^4\right. \\
&- \frac{1024}{3} a_4 - \frac{14080}{27} l_2\zeta_2 + \frac{37984}{27}\zeta_2^2 - \frac{126008}{81}\zeta_3 - \frac{129382}{2187} - \frac{160576}{243}\zeta_2\bigg) \ln(x) + \left(\frac{432}{5}\zeta_2^2 + \frac{1024}{9} l_2\zeta_2\right. \\
&- \frac{7072}{27}\zeta_3 - \frac{58168}{81}\zeta_2 + \frac{1143650}{729}\bigg) \ln^2(x) + \left(-\frac{64}{3}\zeta_3 - \frac{4928}{81}\zeta_2 + \frac{224332}{729}\right) \ln^3(x) \\
&+ \frac{4124}{81} \ln^4(x) - \frac{304}{135} \ln^5(x) + \frac{17}{45} \ln^6(x)\bigg] x^2 + \left[608\zeta_3^2 + \frac{4416}{7}\zeta_2^3 + \frac{6400}{9} l_2\zeta_2^2 - \frac{6560}{3} l_2\zeta_2 + \frac{8320}{81} l_2^4\right. \\
&- \frac{11456}{27} l_2^2\zeta_2 - \frac{19048}{27}\zeta_5 + \frac{33328}{27}\zeta_2\zeta_3 + \frac{66560}{27} a_4 - \frac{120800}{81}\zeta_2^2 + \frac{275728}{81}\zeta_3 + \frac{7061252}{729}\zeta_2 \\
&+ \frac{57246137}{6561} + \left(896\zeta_2\zeta_3 + 576\zeta_5 - \frac{1312}{27} l_2^4 + \frac{1856}{9} l_2^2\zeta_2 - \frac{10496}{9} a_4 - \frac{25472}{9} l_2\zeta_2 + \frac{113296}{135}\zeta_2^2\right. \\
&- \frac{187424}{81}\zeta_3 + \frac{336520}{243}\zeta_2 + \frac{529369}{729}\bigg) \ln(x) + \left(\frac{864}{5}\zeta_2^2 - \frac{1328}{3}\zeta_2 + \frac{1600}{9} l_2\zeta_2 + \frac{2384}{9}\zeta_3 + \frac{579628}{729}\right) \ln^2(x) \\
&\left.\left. + \left(\frac{400}{9}\zeta_2 - \frac{128}{3}\zeta_3 - \frac{42146}{243}\right) \ln^3(x) + \frac{15370}{243} \ln^4(x) + \frac{238}{405} \ln^5(x) + \frac{34}{45} \ln^6(x)\right] x^3 + O(x^4)\right\}.
\end{aligned}
\tag{B5}
$$

As we did in the case of Eq. (3.49) for $F_{V,1}^{n_h}(x)$, we can also perform the corresponding expansions in $\hat{s}$ up to $O(\hat{s}^{-3})$ for the other form factors. These expansions include the solvable parts, as well as the renormalization and infrared subtraction terms. One obtains

$$
\begin{aligned}
F_{V,2}^{n_h}(\hat{s}) = &\left[832\zeta_5 + \frac{256}{27} l_2^4 - \frac{512}{9} l_2^2\zeta_2 + \frac{2048}{9} a_4 - \frac{32896}{27}\zeta_2^2 + \frac{47776}{27}\zeta_3 + \frac{114688}{9} l_2\zeta_2 - \frac{535360}{243} - \frac{2382464}{243}\zeta_2 \right. \\
&\left. + \left(\frac{8192}{27}\zeta_3 - \frac{54592}{81}\zeta_2 - \frac{176576}{243}\right) l_s + \left(-\frac{1696}{27}\zeta_2 - \frac{1808}{243}\right) l_s^2 + \frac{1664}{81} l_s^3 + \frac{320}{81} l_s^4 \right] \frac{1}{\hat{s}} \\
&+ \left[-\frac{35264}{105}\zeta_2^3 - \frac{36352}{243} l_2^4 + \frac{72704}{81} l_2^2\zeta_2 + \frac{1091648}{81}\zeta_3 - \frac{116192}{81}\zeta_2^2 - \frac{6016}{27}\zeta_5 + \frac{7808}{9}\zeta_2\zeta_3 \right. \\
&- \frac{290816}{81} a_4 - \frac{481984}{81} + \frac{2732032}{27} l_2\zeta_2 + 256\zeta_3^2 - \frac{23493952}{243}\zeta_2 + \left(\frac{724352}{81}\zeta_3 - \frac{508864}{81}\zeta_2 - \frac{1299712}{81}\right. \\
&\left. + \frac{768}{5}\zeta_2^2 - 512\zeta_5\right) l_s + \left(\frac{288}{5}\zeta_2^2 + \frac{896}{3}\zeta_3 - \frac{37792}{27}\zeta_2 - \frac{389248}{243}\right) l_s^2 + \left(-\frac{64}{3}\zeta_3 - \frac{8320}{27}\zeta_2\right. \\
&\left.\left. + \frac{159104}{243}\right) l_s^3 + \frac{11648}{243} l_s^4 - \frac{16}{81} l_s^5 \right] \frac{1}{\hat{s}^2} + \left[1024 l_2\zeta_2^2 + 1024\zeta_3^2 + \frac{65920}{9}\zeta_2\zeta_3 - \frac{88768}{27}\zeta_5 - \frac{202432}{35}\zeta_2^3 \right. \\
&- \frac{226304}{81} l_2^4 + \frac{452608}{27} l_2^2\zeta_2 - \frac{1810432}{27} a_4 + \frac{4540000}{81}\zeta_3 - \frac{5931872}{9}\zeta_2 - \frac{7876978}{6561} + \frac{15570016}{405}\zeta_2^2 \\
&+ \frac{16488448}{27} l_2\zeta_2 + \left(2304\zeta_2\zeta_3 - 13440\zeta_5 - \frac{1024}{27} l_2^4 + \frac{2048}{9} l_2^2\zeta_2 - \frac{8192}{9} a_4 + \frac{57472}{45}\zeta_2^2 + \frac{212992}{27} l_2\zeta_2\right. \\
&\left. + \frac{6275776}{81}\zeta_3 - \frac{10957184}{243}\zeta_2 - \frac{262638740}{2187}\right) l_s + \left(\frac{6432}{5}\zeta_2^2 + \frac{104896}{27}\zeta_3 - \frac{772160}{81}\zeta_2 - \frac{5223476}{729}\right) l_s^2 \\
&\left. + \left(-\frac{896}{3}\zeta_3 - \frac{291136}{81}\zeta_2 + \frac{5233208}{729}\right) l_s^3 + \left(-16\zeta_2 + \frac{161668}{243}\right) l_s^4 - \frac{16984}{405} l_s^5 - \frac{68}{45} l_s^6 \right] \frac{1}{\hat{s}^3} + O\left(\frac{1}{\hat{s}^4}\right), \quad (B6)
\end{aligned}
$$

$$
\begin{aligned}
F_{A,1}^{n_h}(\hat{s}) = &\frac{64}{3} l_2\zeta_2^2 - \frac{464}{3}\zeta_2\zeta_3 - \frac{3808}{243} l_2^4 - \frac{4856}{9}\zeta_5 + \frac{5828}{27}\zeta_2^2 + \frac{7616}{81} l_2^2\zeta_2 - \frac{30464}{81} a_4 - \frac{52708}{81}\zeta_3 - \frac{107008}{81} l_2\zeta_2 \\
&+ \frac{281176}{243}\zeta_2 + \frac{461144}{243} + \left(-\frac{64}{81} l_2^4 + \frac{128}{27} l_2^2\zeta_2 + \frac{368}{27}\zeta_2^2 - \frac{512}{27} a_4 - \frac{1804}{9}\zeta_3 + \frac{47936}{243}\zeta_2 + \frac{171704}{243}\right) l_s \\
&+ \left(-\frac{160}{9}\zeta_3 + \frac{896}{81}\zeta_2 + \frac{46064}{243}\right) l_s^2 + \left(-\frac{416}{81}\zeta_2 + \frac{4616}{243}\right) l_s^3 - \frac{136}{81} l_s^4 - \frac{32}{81} l_s^5 + \left[-\frac{1600}{27}\zeta_5 + \frac{3584}{243} l_2^4\right. \\
&- \frac{2240}{9}\zeta_2\zeta_3 - \frac{7168}{81} l_2^2\zeta_2 + \frac{14224}{81}\zeta_2^2 + \frac{28672}{81} a_4 - \frac{77576}{81}\zeta_3 - \frac{207760}{243} - \frac{625408}{27} l_2\zeta_2 + \frac{5245840}{243}\zeta_2 \\
&+ \left(\frac{1696}{45}\zeta_2^2 + \frac{86320}{81}\zeta_2 - \frac{118048}{81}\zeta_3 + \frac{296048}{81}\right) l_s + \left(\frac{32}{9}\zeta_2 + \frac{112}{9}\zeta_3 + \frac{54928}{81}\right) l_s^2 + \left(\frac{848}{27}\zeta_2 + \frac{1088}{81}\right) l_s^3 \\
&\left. - \frac{5896}{243} l_s^4 - \frac{4}{81} l_s^5 \right] \frac{1}{\hat{s}} + \left[-48\zeta_3^2 - \frac{128}{3} l_2\zeta_2^2 + \frac{16832}{81} l_2^4 + \frac{30832}{105}\zeta_2^3 - \frac{33664}{27} l_2^2\zeta_2 \right. \\
&- \frac{36560}{27}\zeta_2\zeta_3 - \frac{58952}{27}\zeta_5 - \frac{70276}{81}\zeta_3 + \frac{134656}{27} a_4 - \frac{498592}{405}\zeta_2^2 - \frac{8287744}{81} l_2\zeta_2 + \frac{23853932}{243}\zeta_2 - \frac{78457723}{13122} \\
&+ \left(-288\zeta_2\zeta_3 + 656\zeta_5 + \frac{128}{81} l_2^4 - \frac{256}{27} l_2^2\zeta_2 + \frac{1024}{27} a_4 + \frac{7024}{45}\zeta_2^2 - \frac{19456}{27} l_2\zeta_2 - \frac{658360}{81}\zeta_3\right. \\
&\left. + \frac{1181320}{243}\zeta_2 + \frac{40251689}{2187}\right) l_s + \left(-\frac{536}{5}\zeta_2^2 + \frac{3272}{27}\zeta_2 - \frac{9632}{27}\zeta_3 + \frac{2847341}{729}\right) l_s^2 \\
&\left. + \left(\frac{32}{3}\zeta_3 + \frac{21296}{81}\zeta_2 - \frac{66110}{729}\right) l_s^3 + \left(4\zeta_2 - \frac{9493}{81}\right) l_s^4 + \frac{986}{405} l_s^5 \right] \frac{1}{\hat{s}^2} + \left[320\zeta_3^2 - \frac{1024}{3} l_2\zeta_2^2 + \frac{83744}{35}\zeta_2^3 \right.
\end{aligned}
$$

$$
\begin{aligned}
&- \frac{105920}{9}\zeta_5 + \frac{108032}{81}l_2^4 - \frac{151808}{27}\zeta_2\zeta_3 - \frac{216064}{27}l_2^2\zeta_2 + \frac{591248}{243}\zeta_3 + \frac{864256}{27}a_4 - \frac{5388448}{405}\zeta_2^2 \\
&- \frac{36299776}{81}l_2\zeta_2 - \frac{180817228}{6561} + \frac{320235272}{729}\zeta_2 + \left(-1280\zeta_2\zeta_3 + 4992\zeta_5 + \frac{1024}{81}l_2^4 - \frac{2048}{27}l_2^2\zeta_2 \right. \\
&+ \frac{2336}{5}\zeta_2^2 + \frac{8192}{27}a_4 - \frac{54272}{9}l_2\zeta_2 - \frac{3323392}{81}\zeta_3 + \frac{5432992}{243}\zeta_2 + \left. \frac{7229564}{81} \right)l_s \\
&+ \left(-\frac{416}{3}\zeta_2 - \frac{2352}{5}\zeta_2^2 - \frac{53728}{27}\zeta_3 + \frac{4294144}{243} \right)l_s^2 + \left(\frac{126080}{81}\zeta_2 + \frac{64}{3}\zeta_3 - \frac{296848}{243} \right)l_s^3 \\
&+ \left(24\zeta_2 - \frac{141088}{243} \right)l_s^4 + \frac{5324}{405}l_s^5 + \frac{34}{45}l_s^6 \Big] \frac{1}{\hat{s}^3} + O\left(\frac{1}{\hat{s}^4} \right),
\end{aligned} \tag{B7}
$$

$$
\begin{aligned}
F_{A,2}^{n_h}(\hat{s}) = &\left[384\zeta_2\zeta_3 - 320\zeta_5 - \frac{256}{81}l_2^4 + \frac{512}{27}l_2^2\zeta_2 - \frac{2048}{27}a_4 - \frac{4256}{27}\zeta_3 - \frac{67936}{135}\zeta_2^2 + \frac{111296}{243} + \frac{327680}{27}l_2\zeta_2 \right. \\
&- \frac{329024}{27}\zeta_2 + \left(\frac{512}{3}\zeta_3 - \frac{13312}{27}\zeta_2 - \frac{637504}{243} \right)l_s + \left(\frac{256}{3}\zeta_2 - \frac{49168}{81} \right)l_s^2 + \frac{832}{81}l_s^3 + \frac{320}{27}l_s^4 \Big] \frac{1}{\hat{s}} \\
&+ \left[\frac{5504}{27}\zeta_2\zeta_3 - \frac{9280}{3} - \frac{15872}{243}l_2^4 + \frac{31744}{81}l_2^2\zeta_2 - \frac{46720}{27}\zeta_5 - \frac{119584}{405}\zeta_2^2 - \frac{126976}{81}a_4 + \frac{506560}{81}\zeta_3 + \frac{2017280}{27}l_2\zeta_2 \right. \\
&- \frac{17172416}{243}\zeta_2 + \left(\frac{306176}{81}\zeta_3 - \frac{70592}{135}\zeta_2^2 - \frac{110272}{81}\zeta_2 - \frac{2675584}{243} \right)l_s + \left(-\frac{7904}{81}\zeta_2 - \frac{11648}{27}\zeta_3 - \frac{576256}{243} \right)l_s^2 \\
&+ \left(-\frac{3520}{27}\zeta_2 + \frac{6848}{27} \right)l_s^3 + \frac{15904}{243}l_s^4 + \frac{16}{81}l_s^5 \Big] \frac{1}{\hat{s}^2} + \left[1408\zeta_3^2 + \frac{1024}{3}l_2\zeta_2^2 + \frac{9152}{105}\zeta_2^3 + \frac{93952}{27}\zeta_2\zeta_3 \right. \\
&- \frac{126976}{243}l_2^4 + \frac{212224}{27}\zeta_5 + \frac{251008}{135}\zeta_2^2 + \frac{253952}{81}l_2^2\zeta_2 + \frac{614656}{81}\zeta_3 - \frac{1015808}{81}a_4 + \frac{8947712}{27}l_2\zeta_2 \\
&+ \frac{17539406}{2187} - \frac{79538816}{243}\zeta_2 + \left(2048\zeta_2\zeta_3 - 512\zeta_5 - \frac{1024}{81}l_2^4 + \frac{2048}{27}l_2^2\zeta_2 + \frac{7616}{135}\zeta_2^2 - \frac{8192}{27}a_4 \right. \\
&+ \frac{623488}{27}\zeta_3 - \frac{3769600}{243}\zeta_2 - \left. \frac{48018980}{729} \right)l_s + \left(\frac{3872}{5}\zeta_2^2 + \frac{13952}{27}\zeta_3 - \frac{174656}{81}\zeta_2 - \frac{2940964}{243} \right)l_s^2 \\
&+ \left(-128\zeta_3 - \frac{99008}{81}\zeta_2 + \frac{324088}{243} \right)l_s^3 - \left(16\zeta_2 - \frac{115612}{243} \right)l_s^4 + \frac{824}{405}l_s^5 + \frac{68}{45}l_s^6 \Big] \frac{1}{\hat{s}^3} + O\left(\frac{1}{\hat{s}^4} \right),
\end{aligned} \tag{B8}
$$

$$
\begin{aligned}
F_S^{n_h}(\hat{s}) = &\frac{64}{3}\zeta_2^2 l_2 - \frac{107008}{81}\zeta_2 l_2 + \frac{7616}{81}\zeta_2 l_2^2 - \frac{3808}{243}l_2^4 + \frac{29116}{135}\zeta_2^2 - \frac{464}{3}\zeta_2\zeta_3 + \frac{177760}{243}\zeta_2 - \frac{37876}{81}\zeta_3 \\
&- \frac{4856}{9}\zeta_5 + \frac{200698}{243} - \frac{30464}{81}a_4 + \left(-\frac{512}{27}a_4 + \frac{128}{27}\zeta_2 l_2^2 - \frac{64}{81}l_2^4 + \frac{368}{27}\zeta_2^2 + \frac{20288}{243}\zeta_2 - \frac{1420}{9}\zeta_3 \right. \\
&+ \left. \frac{19496}{243} \right)l_s + \left(\frac{1472}{81}\zeta_2 - \frac{160}{9}\zeta_3 + \frac{16256}{243} \right)l_s^2 + \left(-\frac{416}{81}\zeta_2 + \frac{5624}{243} \right)l_s^3 + \frac{104}{81}l_s^4 - \frac{32}{81}l_s^5 \\
&+ \left[\frac{256}{9}\zeta_2 l_2^2 + \frac{78008}{81}\zeta_2^2 - \frac{128}{27}l_2^4 - \frac{152320}{9}\zeta_2 l_2 - \frac{10528}{27}\zeta_2\zeta_3 + \frac{131744}{9}\zeta_2 - \frac{5080}{81}\zeta_3 \right. \\
&- \frac{20192}{27}\zeta_5 - \frac{4432}{3} - \frac{1024}{9}a_4 + \left(\frac{108416}{81}\zeta_2 - \frac{2512}{27}\zeta_2^2 - \frac{59552}{81}\zeta_3 + \frac{203600}{81} \right)l_s \\
&+ \left(-\frac{2600}{81}\zeta_2 - \frac{2576}{27}\zeta_3 + \frac{46400}{81} \right)l_s^2 + \left(-\frac{32}{27}\zeta_2 + \frac{8464}{81} \right)l_s^3 - \frac{1280}{81}l_s^4 \Big] \frac{1}{\hat{s}} + \left[-\frac{22226279}{6561} + \frac{128}{3}\zeta_2^2 l_2 \right.
\end{aligned}
$$

$$-\frac{6406144}{81}\zeta_2 l_2 - \frac{44672}{81}\zeta_2 l_2^2 + \frac{22336}{243}l_2^4 - \frac{2144}{35}\zeta_2^3 + \frac{1023424}{405}\zeta_2^2 - \frac{43408}{27}\zeta_2\zeta_3 + \frac{17358044}{243}\zeta_2$$

$$+\frac{203516}{81}\zeta_3 - \frac{105608}{27}\zeta_5 + \frac{178688}{81}a_4 - 272\zeta_3^2 + \left(\frac{256}{27}\zeta_2 l_2^2 - \frac{1024}{27}a_4 - \frac{19456}{27}\zeta_2 l_2 - \frac{128}{81}l_2^4\right.$$

$$-\frac{61216}{135}\zeta_2^2 + \frac{475480}{81}\zeta_2 - \frac{328664}{81}\zeta_3 - 288\zeta_2\zeta_3 - 624\zeta_5 + \frac{25638926}{2187}\bigg)l_s + \left(-\frac{432}{5}\zeta_2^2 + \frac{11944}{81}\zeta_2\right.$$

$$-\frac{3616}{9}\zeta_3 + \frac{2984282}{729}\bigg)l_s^2 + \left(-\frac{32}{3}\zeta_3 + \frac{1664}{27}\zeta_2 + \frac{236572}{729}\right)l_s^3 + \left(8\zeta_2 - \frac{1424}{27}\right)l_s^4 - \frac{832}{405}l_s^5 - \frac{17}{45}l_s^6\bigg]\frac{1}{\hat{s}^2}$$

$$+\bigg[-\frac{95621945}{6561} + \frac{512}{3}\zeta_2^2 l_2 - \frac{28773376}{81}\zeta_2 l_2 - \frac{324608}{81}\zeta_2 l_2^2 + \frac{162304}{243}l_2^4 - \frac{12864}{35}\zeta_2^3 + \frac{565208}{81}\zeta_2^2$$

$$+\frac{241177964}{729}\zeta_2 - \frac{179552}{27}\zeta_2\zeta_3 + \frac{4438880}{243}\zeta_3 - \frac{59792}{3}\zeta_5 - 1632\zeta_3^2 + \frac{1298432}{81}a_4$$

$$+\left(-\frac{4096}{27}a_4 - \frac{152576}{27}\zeta_2 l_2 + \frac{1024}{27}\zeta_2 l_2^2 - \frac{512}{81}l_2^4 - \frac{120976}{45}\zeta_2^2 + \frac{7481048}{243}\zeta_2 - \frac{1568096}{81}\zeta_3\right.$$

$$\left.- 1728\zeta_2\zeta_3 - 3744\zeta_5 + \frac{138016885}{2187}\right)l_s + \left(-\frac{2592}{5}\zeta_2^2 + \frac{68104}{81}\zeta_2 - \frac{15776}{9}\zeta_3 + \frac{14330044}{729}\right)l_s^2$$

$$+\left(\frac{43600}{81}\zeta_2 - 64\zeta_3 + \frac{870614}{729}\right)l_s^3 + \left(48\zeta_2 - \frac{83306}{243}\right)l_s^4 - \frac{2342}{135}l_s^5 - \frac{34}{15}l_s^6\bigg]\frac{1}{\hat{s}^3} + O\left(\frac{1}{\hat{s}^4}\right), \tag{B9}$$

$$F_P^{n_h}(\hat{s}) = \frac{64}{3}\zeta_2^2 l_2 - \frac{107008}{81}\zeta_2 l_2 + \frac{7616}{81}\zeta_2 l_2^2 - \frac{3808}{243}l_2^4 + \frac{29116}{135}\zeta_2^2 - \frac{464}{3}\zeta_2\zeta_3 + \frac{177760}{243}\zeta_2 - \frac{37876}{81}\zeta_3$$

$$-\frac{4856}{9}\zeta_5 + \frac{200698}{243} - \frac{30464}{81}a_4 + \left(-\frac{512}{27}a_4 + \frac{128}{27}\zeta_2 l_2^2 - \frac{64}{81}l_2^4 + \frac{368}{27}\zeta_2^2 + \frac{20288}{243}\zeta_2 - \frac{1420}{9}\zeta_3\right.$$

$$\left.+ \frac{19496}{243}\right)l_s + \left(\frac{1472}{81}\zeta_2 - \frac{160}{9}\zeta_3 + \frac{16256}{243}\right)l_s^2 + \left(-\frac{416}{81}\zeta_2 + \frac{5624}{243}\right)l_s^3 + \frac{104}{81}l_s^4 - \frac{32}{81}l_s^5$$

$$+\bigg[\frac{256}{27}\zeta_2 l_2^2 - \frac{121088}{27}\zeta_2 l_2 - \frac{405008}{243} - \frac{128}{81}l_2^4 + \frac{41224}{405}\zeta_2^2 - \frac{5344}{27}\zeta_2\zeta_3 + \frac{984992}{243}\zeta_2 + \frac{49064}{81}\zeta_3$$

$$-\frac{13280}{27}\zeta_5 - \frac{1024}{27}a_4 + \left(-\frac{2512}{27}\zeta_2^2 + \frac{5504}{9}\zeta_2 - \frac{41504}{81}\zeta_3 + \frac{261968}{243}\right)l_s + \left(-\frac{1688}{81}\zeta_2 - \frac{2576}{27}\zeta_3 + \frac{25328}{243}\right)l_s^2$$

$$+\left(-\frac{32}{27}\zeta_2 + \frac{6224}{81}\right)l_s^3 - \frac{640}{81}l_s^4\bigg]\frac{1}{\hat{s}} + \bigg[-\frac{25532903}{6561} + \frac{128}{3}\zeta_2^2 l_2 - \frac{1576960}{81}\zeta_2 l_2 - \frac{37504}{81}\zeta_2 l_2^2 + \frac{18752}{243}l_2^4$$

$$+\frac{2208}{7}\zeta_2^3 - \frac{287296}{405}\zeta_2^2 - \frac{13072}{27}\zeta_2\zeta_3 + \frac{1327348}{81}\zeta_2 + \frac{27796}{27}\zeta_3 - \frac{5896}{27}\zeta_5 + \frac{150016}{81}a_4 + 304\zeta_3^2$$

$$+\left(-\frac{1024}{27}a_4 - \frac{19456}{27}\zeta_2 l_2 + \frac{256}{27}\zeta_2 l_2^2 - \frac{128}{81}l_2^4 + \frac{22976}{135}\zeta_2^2 + \frac{62744}{81}\zeta_2 - \frac{190744}{81}\zeta_3 + 224\zeta_2\zeta_3\right.$$

$$\left.+ 528\zeta_5 + \frac{5114702}{2187}\right)l_s + \left(\frac{432}{5}\zeta_2^2 - \frac{36152}{81}\zeta_2 - \frac{2048}{9}\zeta_3 + \frac{701306}{729}\right)l_s^2 + \left(-\frac{128}{3}\zeta_2 - \frac{64}{3}\zeta_3 + \frac{179260}{729}\right)l_s^3$$

$$+\frac{424}{243}l_s^4 + \frac{1192}{405}l_s^5 + \frac{17}{45}l_s^6\bigg]\frac{1}{\hat{s}^2} + \bigg[-\frac{85198093}{6561} + \frac{512}{3}\zeta_2^2 l_2 - \frac{5825536}{81}\zeta_2 l_2 - \frac{26624}{27}\zeta_2 l_2^2 + \frac{13312}{81}l_2^4$$

$$+ \frac{4416}{7}\zeta_2^3 + \frac{101672}{405}\zeta_2^2 - \frac{44192}{27}\zeta_2\zeta_3 + \frac{38327036}{729}\zeta_2 + \frac{364016}{243}\zeta_3 + \frac{4400}{9}\zeta_5 + 608\zeta_3^2 + \frac{106496}{27}a_4$$

$$+ \left(1056\zeta_5 - \frac{4096}{27}a_4 - \frac{5120}{3}\zeta_2l_2 + \frac{1024}{27}\zeta_2l_2^2 - \frac{512}{81}l_2^4 + \frac{9232}{9}\zeta_2^2 + \frac{2008}{243}\zeta_2 - \frac{532288}{81}\zeta_3 + 448\zeta_2\zeta_3\right.$$

$$\left. + \frac{5075447}{729}\right)l_s + \left(\frac{864}{5}\zeta_2^2 - \frac{36776}{27}\zeta_2 - \frac{25792}{27}\zeta_3 + \frac{965428}{243}\right)l_s^2 + \left(-\frac{17584}{81}\zeta_2 - \frac{128}{3}\zeta_3 + \frac{235522}{243}\right)l_s^3$$

$$+ \frac{10862}{243}l_s^4 + \frac{6254}{405}l_s^5 + \frac{34}{45}l_s^6\bigg]\frac{1}{\hat{s}^3} + O\left(\frac{1}{\hat{s}^4}\right). \tag{B10}$$

APPENDIX C: PARTIALLY RESUMMED EXPANSIONS AT $x = 0$

In Eq. (3.69) we showed the result for the partially resummed expansion of $F_{V,1,1}^{(0)}(x) + \zeta_2 F_{V,1,2}^{(0)}(x) + \zeta_3 F_{V,1,3}^{(0)}(x)$. In this appendix, we show the corresponding results for the other form factors,

$$F_{V,2,1}^{(0)}(x) + \zeta_2 F_{V,2,2}^{(0)}(x) + \zeta_3 F_{V,2,3}^{(0)}(x)$$

$$= \ln(2)\Bigg\{\frac{1024x\zeta_2^2 P_{48}}{9(x-1)^3(1+x)^5} + \Bigg[\frac{256x\ln(1+x)P_{18}}{9(x-1)^2(1+x)^4} + \frac{512x\text{Li}_2(-x)P_{46}}{9(x-1)^3(1+x)^5}$$

$$- \frac{256x^2\ln^2(x)P_{49}}{9(x-1)^3(1+x)^6} - \frac{64P_{66}}{81(x-1)^2(1+x)^6} - \ln(1-x)\left(\frac{1024x^2\ln(x)P_1}{9(x-1)^3(1+x)^5} - \frac{1024xP_4}{9(x-1)^2(1+x)^4}\right)$$

$$+ \ln(x)\left(\frac{512x\ln(1+x)P_{47}}{9(x-1)^3(1+x)^5} - \frac{256xP_{55}}{27(x-1)^3(1+x)^5}\right) + \frac{256x^2(1-7x+x^2)\ln^3(x)}{(x-1)(1+x)^5}$$

$$- \frac{1024x^2(1-12x+x^2)(1-x+x^2)\text{Li}_2(x)}{9(x-1)^3(1+x)^5}\Bigg]\zeta_2\Bigg\} + \ln^2(2)\Bigg\{\Bigg[\frac{256x^2\ln(x)P_5}{9(x-1)^3(1+x)^5}$$

$$- \frac{128xP_{60}}{81(x-1)^3(1+x)^5} - \ln(1-x)\left(-\frac{1024x(1-x+x^2)}{(x-1)(1+x)^3} - \frac{1024x^2(1-x+x^2)\ln(x)}{(x-1)(1+x)^5}\right)$$

$$- \frac{256x^2(1-x+x^2)\ln^2(x)}{(x-1)(1+x)^5} - \frac{1024x(1-x+x^2)\ln(1+x)}{(x-1)(1+x)^3} + \frac{1024x^2(1-x+x^2)\text{Li}_2(x)}{(x-1)(1+x)^5}\Bigg]\zeta_2$$

$$- \frac{1024x^2(1-x+x^2)\zeta_2^2}{(x-1)(1+x)^5}\Bigg\} + \text{Li}_4\left(\frac{1}{2}\right)\Bigg\{-\frac{1024x^2\ln(x)P_{11}}{9(x-1)^3(1+x)^5} + \frac{1024xP_{56}}{81(x-1)^3(1+x)^5}$$

$$- \ln(1-x)\left[\frac{4096x(1-x+x^2)}{(x-1)(1+x)^3} + \frac{4096x^2(1-x+x^2)\ln(x)}{(x-1)(1+x)^5}\right]$$

$$+ \frac{1024x^2(1-x+x^2)\ln^2(x)}{(x-1)(1+x)^5} + \frac{4096x(1-x+x^2)\ln(1+x)}{(x-1)(1+x)^3} - \frac{4096x^2(1-x+x^2)\text{Li}_2(x)}{(x-1)(1+x)^5}\Bigg\}$$

$$+ \ln^4(2)\Bigg\{-\frac{128x^2\ln(x)P_{11}}{27(x-1)^3(1+x)^5} + \frac{128xP_{56}}{243(x-1)^3(1+x)^5} - \ln(1-x)\Bigg[\frac{512x(1-x+x^2)}{3(x-1)(1+x)^3}$$

$$+ \frac{512x^2(1-x+x^2)\ln(x)}{3(x-1)(1+x)^5}\Bigg] + \frac{512x^2(1-x+x^2)\zeta_2}{3(x-1)(1+x)^5} + \frac{128x^2(1-x+x^2)\ln^2(x)}{3(x-1)(1+x)^5}$$

$$+ \frac{512x(1-x+x^2)\ln(1+x)}{3(x-1)(1+x)^3} - \frac{512x^2(1-x+x^2)\text{Li}_2(x)}{3(x-1)(1+x)^5}\Bigg\} + \Bigg\{\frac{4096\text{Li}_4(\frac{1}{2})x^2(1-x+x^2)}{(x-1)(1+x)^5}$$

$$+ \frac{768x^3(1+x^2)\zeta_3 \ln(x)}{(x-1)^3(1+x)^5} - \frac{16x^3\ln^4(x)}{(x-1)^3(1+x)^3}\bigg\}\zeta_2 + \frac{64x^2(1-4x+x^2)^2\zeta_3\ln^3(x)}{3(x-1)^3(1+x)^5}$$

$$- \frac{68x^3(1-4x+x^2)\ln^6(x)}{45(x-1)^3(1+x)^5} - \frac{128x^2\zeta_3^2 P_8}{(x-1)^3(1+x)^5} + \frac{128x^2\zeta_5 \ln(x)P_{10}}{(x-1)^3(1+x)^5} - \frac{32x^2\zeta_2^2\ln^2(x)P_{24}}{5(x-1)^3(1+x)^5}$$

$$- \frac{64x^2\zeta_2^3 P_{38}}{105(x-1)^3(1+x)^5} + \tilde{F}_{V,2}^{(0),\text{rest}}(x), \tag{C1}$$

$$F_{A,1,1}^{(0)}(x) + \zeta_2 F_{A,1,2}^{(0)}(x) + \zeta_3 F_{A,1,3}^{(0)}(x)$$

$$= \ln(2)\bigg\{ \frac{256(1+x^2)\zeta_2^2 P_{27}}{9(x-1)^3(1+x)^3} + \bigg[\frac{128\ln(1+x)P_{15}}{9(x-1)^2(1+x)^2} + \frac{64(1+x^2)\text{Li}_2(-x)P_{36}}{9(x-1)^3(1+x)^3}$$

$$+ \frac{64\text{Li}_2(x)P_{51}}{9(x-1)^3(1+x)^3} - \frac{64\ln^2(x)P_{61}}{9(x-1)^3(1+x)^4}$$

$$- \frac{16P_{83}}{81(x-1)^4x(1+x)^6} - \ln(1-x)\bigg(\frac{128(3-2x^2+3x^4)}{9(x-1)^2(1+x)^2}$$

$$- \frac{64\ln(x)P_{52}}{9(x-1)^3(1+x)^3}\bigg) + \ln(x)\bigg(\frac{64\ln(1+x)P_{57}}{9(x-1)^3(1+x)^3} - \frac{32P_{79}}{27(x-1)^5(1+x)^5}\bigg)$$

$$- \frac{192x^2\ln^3(x)}{(x-1)(1+x)^3}\bigg]\zeta_2\bigg\} + \ln^2(2)\bigg\{ \bigg[\frac{128\ln(x)P_{44}}{9(x-1)^3(1+x)^3} - \frac{32P_{74}}{81(x-1)^5(1+x)^5}$$

$$- \ln(1-x)\bigg(-\frac{128(1+x^2)(11-2x+11x^2)}{3(x-1)(1+x)^3} + \frac{256x^2\ln(x)}{(x-1)(1+x)^3}\bigg) + \frac{64x^2\ln^2(x)}{(x-1)(1+x)^3}$$

$$- \frac{128(1+x^2)(11-2x+11x^2)\ln(1+x)}{3(x-1)(1+x)^3} - \frac{256x^2\text{Li}_2(x)}{(x-1)(1+x)^3}\bigg]\zeta_2 + \frac{256x^2\zeta_2^2}{(x-1)(1+x)^3}\bigg\}$$

$$+ \ln^4(2)\bigg\{ -\frac{64\ln(x)P_{42}}{27(x-1)^3(1+x)^3} - \frac{16P_{67}}{243(x-1)^5(1+x)^5} - \ln(1-x)$$

$$\times \bigg[\frac{64(1+x^2)(11-2x+11x^2)}{9(x-1)(1+x)^3} - \frac{128x^2\ln(x)}{3(x-1)(1+x)^3}\bigg] - \frac{128x^2\zeta_2}{3(x-1)(1+x)^3}$$

$$- \frac{32x^2\ln^2(x)}{3(x-1)(1+x)^3} + \frac{64(1+x^2)(11-2x+11x^2)\ln(1+x)}{9(x-1)(1+x)^3} + \frac{128x^2\text{Li}_2(x)}{3(x-1)(1+x)^3}\bigg\}$$

$$+ \text{Li}_4\bigg(\frac{1}{2}\bigg)\bigg\{ -\frac{512\ln(x)P_{42}}{9(x-1)^3(1+x)^3} - \frac{128P_{67}}{81(x-1)^5(1+x)^5} - \ln(1-x)$$

$$\times \bigg[\frac{512(1+x^2)(11-2x+11x^2)}{3(x-1)(1+x)^3} - \frac{1024x^2\ln(x)}{(x-1)(1+x)^3}\bigg] - \frac{256x^2\ln^2(x)}{(x-1)(1+x)^3}$$

$$+ \frac{512(1+x^2)(11-2x+11x^2)\ln(1+x)}{3(x-1)(1+x)^3} + \frac{1024x^2\text{Li}_2(x)}{(x-1)(1+x)^3}\bigg\} + \bigg\{ -\frac{1024\text{Li}_4(\frac{1}{2})x^2}{(x-1)(1+x)^3}$$

$$+ \frac{128x^2(1+5x+x^2)\zeta_3 \ln(x)}{(x-1)^3(1+x)^3} - \frac{4x^2\ln^4(x)}{(x-1)(1+x)^3}\bigg\}\zeta_2 + \frac{16x^2(3+32x+3x^2)\zeta_3^2}{(x-1)^3(1+x)^3}$$

$$- \frac{16x^2(2011-8162x+2011x^2)\zeta_2^3}{105(x-1)^3(1+x)^3} - \frac{576x^2(1-3x+x^2)\zeta_5 \ln(x)}{(x-1)^3(1+x)^3}$$

$$+ \frac{8x^2(65-22x+65x^2)\zeta_2^2\ln^2(x)}{5(x-1)^3(1+x)^3} - \frac{32x^2\zeta_3\ln^3(x)}{3(x-1)^3(1+x)} + \frac{34x^3\ln^6(x)}{45(x-1)^3(1+x)^3} + \tilde{F}_{A,1}^{(0),\text{rest}}(x), \tag{C2}$$

$$F_{A,2,1}^{(0)}(x) + \zeta_2 F_{A,2,2}^{(0)}(x) + \zeta_3 F_{A,2,3}^{(0)}(x)$$

$$= \ln(2)\left\{-\frac{2048x^2\zeta_2^2 P_{29}}{9(x-1)^5(1+x)^3} + \left[\frac{1024x^2\mathrm{Li}_2(x)P_2}{9(x-1)^5(1+x)^3} + \frac{256x\ln(1+x)P_{14}}{9(x-1)^4(1+x)^2}\right.\right.$$

$$- \frac{512x^2\mathrm{Li}_2(-x)P_{32}}{9(x-1)^5(1+x)^3} + \frac{256x^2\ln^2(x)P_{39}}{9(x-1)^5(1+x)^4} + \frac{64P_{75}}{27(x-1)^4(1+x)^6}$$

$$- \ln(1-x)\left(\frac{1024x^2(1+4x+x^2)}{9(x-1)^4(1+x)^2} - \frac{1024x^2\ln(x)P_3}{9(x-1)^5(1+x)^3}\right) + \ln(x)$$

$$\times \left(-\frac{512x^2\ln(1+x)P_{33}}{9(x-1)^5(1+x)^3} - \frac{256xP_{64}}{27(x-1)^5(1+x)^5}\right) - \frac{768x^3\ln^3(x)}{(x-1)^3(1+x)^3}\bigg]\zeta_2\right\}$$

$$+ \ln^2(2)\left\{\frac{256x^2\ln(x)P_{26}}{9(x-1)^5(1+x)^3} - \frac{128xP_{63}}{81(x-1)^5(1+x)^5} - \ln(1-x)\right.$$

$$\times \left[\frac{1024x^2(3-2x+3x^2)}{(x-1)^3(1+x)^3} + \frac{1024x^3\ln(x)}{(x-1)^3(1+x)^3}\right] + \frac{256x^3\ln^2(x)}{(x-1)^3(1+x)^3}$$

$$+ \frac{1024x^2(3-2x+3x^2)\ln(1+x)}{(x-1)^3(1+x)^3} - \frac{1024x^3\mathrm{Li}_2(x)}{(x-1)^3(1+x)^3}\bigg]\zeta_2 + \frac{1024x^3\zeta_2^2}{(x-1)^3(1+x)^3}\right\}$$

$$+ \ln^4(2)\left\{-\frac{128x^2\ln(x)P_{23}}{27(x-1)^5(1+x)^3} - \frac{128xP_{62}}{243(x-1)^5(1+x)^5} - \ln(1-x)\right.$$

$$\times \left[-\frac{512x^2(3-2x+3x^2)}{3(x-1)^3(1+x)^3} - \frac{512x^3\ln(x)}{3(x-1)^3(1+x)^3}\right]$$

$$- \frac{512x^3\zeta_2}{3(x-1)^3(1+x)^3} - \frac{128x^3\ln^2(x)}{3(x-1)^3(1+x)^3} - \frac{512x^2(3-2x+3x^2)\ln(1+x)}{3(x-1)^3(1+x)^3}$$

$$+ \frac{512x^3\mathrm{Li}_2(x)}{3(x-1)^3(1+x)^3}\right\} + \mathrm{Li}_4\left(\frac{1}{2}\right)\left\{-\frac{1024x^2\ln(x)P_{23}}{9(x-1)^5(1+x)^3} - \frac{1024xP_{62}}{81(x-1)^5(1+x)^5}\right.$$

$$- \ln(1-x)\left[-\frac{4096x^2(3-2x+3x^2)}{(x-1)^3(1+x)^3} - \frac{4096x^3\ln(x)}{(x-1)^3(1+x)^3}\right] - \frac{1024x^3\ln^2(x)}{(x-1)^3(1+x)^3}$$

$$- \frac{4096x^2(3-2x+3x^2)\ln(1+x)}{(x-1)^3(1+x)^3} + \frac{4096x^3\mathrm{Li}_2(x)}{(x-1)^3(1+x)^3}\right\}$$

$$+ \left\{-\frac{4096\mathrm{Li}_4(\frac{1}{2})x^3}{(x-1)^3(1+x)^3} + \frac{768x^3(3+8x+3x^2)\zeta_3\ln(x)}{(x-1)^5(1+x)^3} - \frac{16x^3\ln^4(x)}{(x-1)^3(1+x)^3}\right\}\zeta_2$$

$$+ \frac{64x^3(59+12302x+59x^2)\zeta_2^3}{105(x-1)^5(1+x)^3} + \frac{128x^3(11+35x+11x^2)\zeta_3^2}{(x-1)^5(1+x)^3}$$

$$- \frac{1152x^3(1-8x+x^2)\zeta_5\ln(x)}{(x-1)^5(1+x)^3} + \frac{32x^3(119+86x+119x^2)\zeta_2^2\ln^2(x)}{5(x-1)^5(1+x)^3}$$

$$- \frac{128x^3\zeta_3\ln^3(x)}{(x-1)^5(1+x)} + \frac{68x^3(1+4x+x^2)\ln^6(x)}{45(x-1)^5(1+x)^3} + \tilde{F}_{A,2}^{(0),\mathrm{rest}}(x), \tag{C3}$$

$$
F_{S,1}^{(0)}(x) + \zeta_2 F_{S,2}^{(0)}(x) + \zeta_3 F_{S,3}^{(0)}(x)
$$

$$
= \ln^4(2)\left\{ -\frac{32\ln(x)P_7}{27(x-1)(1+x)^3} - \frac{64\ln(1-x)P_{19}}{9(x-1)(1+x)^3} + \frac{64\ln(1+x)P_{19}}{9(x-1)(1+x)^3} \right.
$$

$$
\left. + \frac{16P_{71}}{243(x-1)^5(1+x)^5} \right\} + \mathrm{Li}_4\left(\frac{1}{2}\right)\left\{ -\frac{256\ln(x)P_7}{9(x-1)(1+x)^3} - \frac{512\ln(1-x)P_{19}}{3(x-1)(1+x)^3} \right.
$$

$$
\left. + \frac{512\ln(1+x)P_{19}}{3(x-1)(1+x)^3} + \frac{128P_{71}}{81(x-1)^5(1+x)^5} \right\} + \ln(2)\left\{ \frac{256\zeta_2^2 P_{20}}{3(x-1)(1+x)^3} \right.
$$

$$
+ \left[\frac{64\mathrm{Li}_2(x)P_{31}}{9(x-1)(1+x)^3} + \frac{64\mathrm{Li}_2(-x)P_{37}}{9(x-1)(1+x)^3} - \frac{64\ln^2(x)P_{41}}{9(x-1)^2(1+x)^3} \right.
$$

$$
- \frac{64P_{80}}{81(x-1)^4 x(1+x)^6} - \ln(1-x)\left(\frac{128(3+8x+3x^2)}{9(1+x)^2} - \frac{64\ln(x)P_{22}}{3(x-1)(1+x)^3} \right)
$$

$$
\left. + \ln(x)\left(\frac{64\ln(1+x)P_{25}}{3(x-1)(1+x)^3} - \frac{32P_{76}}{27(x-1)^5(1+x)^5} \right) + \frac{64(9-32x+9x^2)\ln(1+x)}{9(1+x)^2} \right]\zeta_2 \right\}
$$

$$
+ \ln(2)^2\left\{ \frac{64\ln(x)P_{13}}{9(x-1)(1+x)^3} + \frac{128\ln(1-x)P_{19}}{3(x-1)(1+x)^3} - \frac{128\ln(1+x)P_{19}}{3(x-1)(1+x)^3} \right.
$$

$$
\left. - \frac{64P_{69}}{81(x-1)^5(1+x)^5} \right\}\zeta_2 + \left\{ \frac{192x^2\zeta_3\ln(x)}{(x-1)(1+x)^3} - \frac{8x^2\ln^4(x)}{(x-1)(1+x)^3} \right\}\zeta_2
$$

$$
+ \frac{2144x^2\zeta_2^3}{35(x-1)(1+x)^3} + \frac{272x^2\zeta_3^2}{(x-1)(1+x)^3} + \frac{544x^2\zeta_5\ln(x)}{(x-1)(1+x)^3} + \frac{432x^2\zeta_2^2\ln^2(x)}{5(x-1)(1+x)^3}
$$

$$
+ \frac{32x^2\zeta_3\ln^3(x)}{3(x-1)(1+x)^3} + \frac{17x^2\ln^6(x)}{45(x-1)(1+x)^3} + \tilde{F}_S^{(0),\mathrm{rest}}(x), \tag{C4}
$$

$$
F_{P,1}^{(0)}(x) + \zeta_2 F_{P,2}^{(0)}(x) + \zeta_3 F_{P,3}^{(0)}(x)
$$

$$
= \ln(2)\left\{ \frac{256\zeta_2^2 P_{28}}{9(x-1)^3(1+x)} + \left[\frac{64\ln(1+x)P_{16}}{9(x-1)^2(1+x)^2} + \frac{64\mathrm{Li}_2(x)P_{21}}{3(x-1)^3(1+x)} + \frac{64\mathrm{Li}_2(-x)P_{35}}{9(x-1)^3(1+x)} \right. \right.
$$

$$
- \frac{64\ln^2(x)P_{40}}{9(x-1)^3(1+x)^2} - \frac{64P_{81}}{81(x-1)^4 x(1+x)^6} - \ln(1-x)\left(\frac{128(3-8x+3x^2)}{9(x-1)^2} \right.
$$

$$
\left. - \frac{64\ln(x)P_{30}}{9(x-1)^3(1+x)} \right) + \ln(x)\left(\frac{64\ln(1+x)P_{34}}{9(x-1)^3(1+x)} - \frac{32P_{77}}{27(x-1)^5(1+x)^5} \right)\left. \left. \right]\zeta_2 \right\}
$$

$$
+ \ln^4(2)\left\{ -\frac{32\ln(x)P_6}{27(x-1)^3(1+x)} + \frac{16P_{72}}{243(x-1)^5(1+x)^5} - \frac{64(11-6x+11x^2)\ln(1-x)}{9(x-1)(1+x)} \right.
$$

$$
\left. + \frac{64(11-6x+11x^2)\ln(1+x)}{9(x-1)(1+x)} \right\} + \mathrm{Li}_4\left(\frac{1}{2}\right)\left\{ -\frac{256\ln(x)P_6}{9(x-1)^3(1+x)} + \frac{128P_{72}}{81(x-1)^5(1+x)^5} \right.
$$

$$
\left. - \frac{512(11-6x+11x^2)\ln(1-x)}{3(x-1)(1+x)} + \frac{512(11-6x+11x^2)\ln(1+x)}{3(x-1)(1+x)} \right\}
$$

$$
+ \ln^2(2)\left\{ \frac{64\ln(x)P_{12}}{9(x-1)^3(1+x)} - \frac{64P_{70}}{81(x-1)^5(1+x)^5} + \frac{128(11-6x+11x^2)\ln(1-x)}{3(x-1)(1+x)} \right.
$$

$$
\left. - \frac{128(11-6x+11x^2)\ln(1+x)}{3(x-1)(1+x)} \right\}\zeta_2 - \frac{2208x^2\zeta_2^3}{7(x-1)^3(1+x)} - \frac{304x^2\zeta_3^2}{(x-1)^3(1+x)}
$$

$$
- \frac{448x^2\zeta_2\zeta_3\ln(x)}{(x-1)^3(1+x)} - \frac{288x^2\zeta_5\ln(x)}{(x-1)^3(1+x)} - \frac{432x^2\zeta_2^2\ln^2(x)}{5(x-1)^3(1+x)} + \frac{64x^2\zeta_3\ln^3(x)}{3(x-1)^3(1+x)}
$$

$$
- \frac{17x^2\ln^6(x)}{45(x-1)^3(1+x)} + \tilde{F}_P^{(0),\mathrm{rest}}(x). \tag{C5}
$$

The polynomials are given by

$$P_1 = x^4 - 21x^3 + 34x^2 - 21x + 1, \tag{C6}$$

$$P_2 = x^4 - 5x^3 - 10x^2 - 5x + 1, \tag{C7}$$

$$P_3 = x^4 - x^3 - 6x^2 - x + 1, \tag{C8}$$

$$P_4 = x^4 + 5x^3 - 10x^2 + 5x + 1, \tag{C9}$$

$$P_5 = x^4 + 90x^3 + 46x^2 + 90x + 1, \tag{C10}$$

$$P_6 = 2x^4 - 13x^3 + 6x^2 - 13x + 2, \tag{C11}$$

$$P_7 = 2x^4 - 5x^3 - 2x^2 - 5x + 2, \tag{C12}$$

$$P_8 = 2x^4 + 4x^3 - 29x^2 + 4x + 2, \tag{C13}$$

$$P_9 = 3x^4 + 16x^3 + 62x^2 + 16x + 3, \tag{C14}$$

$$P_{10} = 4x^4 - 71x^3 + 168x^2 - 71x + 4, \tag{C15}$$

$$P_{11} = 7x^4 + 36x^3 + 106x^2 + 36x + 7, \tag{C16}$$

$$P_{12} = 8x^4 - 25x^3 + 24x^2 - 25x + 8, \tag{C17}$$

$$P_{13} = 8x^4 + 7x^3 - 8x^2 + 7x + 8, \tag{C18}$$

$$P_{14} = 9x^4 - 32x^3 + 34x^2 - 32x + 9, \tag{C19}$$

$$P_{15} = 9x^4 - 27x^3 + 34x^2 - 27x + 9, \tag{C20}$$

$$P_{16} = 9x^4 - 22x^3 + 34x^2 - 22x + 9, \tag{C21}$$

$$P_{17} = 9x^4 - 5x^3 + 98x^2 - 5x + 9, \tag{C22}$$

$$P_{18} = 11x^4 - 56x^3 + 118x^2 - 56x + 11, \tag{C23}$$

$$P_{19} = 11x^4 + 4x^3 + 34x^2 + 4x + 11, \tag{C24}$$

$$P_{20} = 12x^4 - 3x^3 + 41x^2 - 3x + 12, \tag{C25}$$

$$P_{21} = 19x^4 - 38x^3 + 42x^2 - 38x + 19, \tag{C26}$$

$$P_{22} = 19x^4 + 38x^3 + 34x^2 + 38x + 19, \tag{C27}$$

$$P_{23} = 29x^4 - 28x^3 + 94x^2 - 28x + 29, \tag{C28}$$

$$P_{24} = 31x^4 - 141x^3 + 112x^2 - 141x + 31, \tag{C29}$$

$$P_{25} = 33x^4 + 12x^3 + 98x^2 + 12x + 33, \tag{C30}$$

$$P_{26} = 35x^4 - 58x^3 + 106x^2 - 58x + 35, \tag{C31}$$

$$P_{27} = 36x^4 - 107x^3 + 141x^2 - 107x + 36, \tag{C32}$$

$$P_{28} = 36x^4 - 99x^3 + 127x^2 - 99x + 36, \tag{C33}$$

$$P_{29} = 40x^4 - 106x^3 + 135x^2 - 106x + 40, \tag{C34}$$

$$P_{30} = 57x^4 - 114x^3 + 118x^2 - 114x + 57, \tag{C35}$$

$$P_{31} = 57x^4 + 114x^3 + 110x^2 + 114x + 57, \tag{C36}$$

$$P_{32} = 73x^4 - 192x^3 + 262x^2 - 192x + 73, \tag{C37}$$

$$P_{33} = 79x^4 - 214x^3 + 282x^2 - 214x + 79, \tag{C38}$$

$$P_{34} = 99x^4 - 252x^3 + 310x^2 - 252x + 99, \tag{C39}$$

$$P_{35} = 111x^4 - 276x^3 + 338x^2 - 276x + 111, \tag{C40}$$

$$P_{36} = 111x^4 - 200x^3 + 170x^2 - 200x + 111, \tag{C41}$$

$$P_{37} = 111x^4 + 60x^3 + 274x^2 + 60x + 111, \tag{C42}$$

$$P_{38} = 373x^4 + 6435x^3 - 14420x^2 + 6435x + 373, \tag{C43}$$

$$P_{39} = 28x^5 - 42x^4 + 35x^3 + 29x^2 - 117x + 55, \tag{C44}$$

$$P_{40} = 70x^5 - 97x^4 + 13x^3 + 12x^2 - 2x + 2, \tag{C45}$$

$$P_{41} = 70x^5 - 11x^4 + 71x^3 - 70x^2 - 2x - 2, \tag{C46}$$

$$P_{42} = x^6 + 10x^5 - 23x^4 + 40x^3 - 23x^2 + 10x + 1, \tag{C47}$$

$$P_{43} = x^6 + 12x^5 + 47x^4 + 96x^3 + 47x^2 + 12x + 1, \tag{C48}$$

$$P_{44} = 4x^6 + 13x^5 - 38x^4 + 52x^3 - 38x^2 + 13x + 4, \tag{C49}$$

$$P_{45} = 4x^6 + 21x^5 + 80x^4 + 60x^3 + 80x^2 + 21x + 4, \tag{C50}$$

$$P_{46} = 27x^6 - 35x^5 + 85x^4 - 114x^3 + 85x^2 - 35x + 27, \tag{C51}$$

$$P_{47} = 27x^6 - 29x^5 + 15x^4 - 14x^3 + 15x^2 - 29x + 27, \tag{C52}$$

$$P_{48} = 27x^6 - 28x^5 - 15x^4 + 38x^3 - 15x^2 - 28x + 27, \tag{C53}$$

$$P_{49} = 27x^6 + 136x^5 - 245x^4 - 221x^3 + 202x^2 + 223x - 134, \tag{C54}$$

$$P_{50} = 36x^6 - 19x^5 - 67x^4 - 66x^3 - 67x^2 - 19x + 36, \tag{C55}$$

$$P_{51} = 57x^6 + 4x^5 - 65x^4 - 16x^3 - 65x^2 + 4x + 57, \tag{C56}$$

$$P_{52} = 57x^6 + 4x^5 - 57x^4 - 16x^3 - 57x^2 + 4x + 57, \tag{C57}$$

$$P_{53} = 57x^6 + 224x^5 + 423x^4 + 344x^3 + 423x^2 + 224x + 57, \tag{C58}$$

$$P_{54} = 57x^6 + 224x^5 + 431x^4 + 216x^3 + 431x^2 + 224x + 57, \tag{C59}$$

$$P_{55} = 69x^6 - 905x^5 - 1667x^4 + 6264x^3 - 1856x^2 - 797x + 24, \tag{C60}$$

$$P_{56} = 75x^6 - 58x^5 - 2637x^4 + 5456x^3 - 2907x^2 - 598x - 51, \tag{C61}$$

$$P_{57} = 99x^6 - 212x^5 + 333x^4 - 448x^3 + 333x^2 - 212x + 99, \tag{C62}$$

$$P_{58} = 99x^6 + 68x^5 + 77x^4 - 96x^3 + 77x^2 + 68x + 99, \tag{C63}$$

$$P_{59} = 111x^6 + 104x^5 + 217x^4 - 368x^3 + 217x^2 + 104x + 111, \tag{C64}$$

$$P_{60} = 285x^6 - 440x^5 - 4707x^4 + 10912x^3 - 6381x^2 - 872x - 237, \tag{C65}$$

$$P_{61} = 70x^7 + 15x^6 - 69x^5 - 94x^4 + 6x^3 + 127x^2 - 53x + 2, \tag{C66}$$

$$P_{62} = 27x^8 - 328x^7 - 1934x^6 - 2456x^5 - 7952x^4 - 2024x^3 - 962x^2 + 104x - 27, \tag{C67}$$

$$P_{63} = 27x^8 + 494x^7 + 3382x^6 + 4750x^5 + 15904x^4 + 4210x^3 + 2410x^2 - 46x - 27, \tag{C68}$$

$$P_{64} = 60x^8 - 537x^7 - 2164x^6 + 5223x^5 + 8561x^4 + 5277x^3 - 2002x^2 - 483x + 33, \tag{C69}$$

$$P_{65} = 70x^8 + 173x^7 - 342x^6 + 245x^5 + 860x^4 + 241x^3 - 458x^2 + 45x - 2, \tag{C70}$$

$$P_{66} = 1280x^8 + 3209x^7 + 41288x^6 + 2239x^5 - 120320x^4 + 2239x^3 + 41288x^2 + 3209x + 1280, \tag{C71}$$

$$P_{67} = 17x^{10} + 1828x^9 - 2261x^8 - 2392x^7 - 7422x^6 + 24304x^5 - 6162x^4 - 3688x^3 - 2531x^2 + 2476x - 73, \tag{C72}$$

$$P_{68} = 17x^{10} + 2276x^9 - 2885x^8 - 24696x^7 + 91794x^6 - 126000x^5 + 85278x^4 - 20808x^3 - 2867x^2 + 2060x - 73, \tag{C73}$$

$$P_{69} = 32x^{10} - 1617x^9 + 1830x^8 + 1114x^7 + 1234x^6 - 9448x^5 + 5734x^4 - 506x^3 + 390x^2 - 807x - 4, \tag{C74}$$

$$P_{70} = 32x^{10} - 1377x^9 + 82x^8 + 2442x^7 - 1938x^6 - 728x^5 - 1110x^4 + 1686x^3 - 134x^2 - 999x - 4, \tag{C75}$$

$$P_{71} = 37x^{10} - 2748x^9 + 2823x^8 + 1256x^7 + 5114x^6 - 18896x^5 + 8822x^4 - 40x^3 + 1617x^2 - 2100x + 19, \tag{C76}$$

$$P_{72} = 37x^{10} - 2592x^9 - 241x^8 + 4560x^7 - 2526x^6 - 1456x^5 - 3570x^4 + 3696x^3 + 137x^2 - 2160x + 19, \tag{C77}$$

$$P_{73} = 91x^{10} - 3086x^9 + 5585x^8 + 19836x^7 - 87474x^6 + 126000x^5 - 89598x^4 + 25668x^3 + 167x^2 - 1250x - 35, \tag{C78}$$

$$P_{74} = 91x^{10} - 2314x^9 + 2585x^8 + 3364x^7 + 5910x^6 - 24304x^5 + 7674x^4 + 2716x^3 + 2207x^2 - 1990x - 35, \tag{C79}$$

$$P_{75} = 208x^{10} + 1947x^9 - 6722x^8 + 104x^7 + 33106x^6 + 61370x^5 + 33106x^4 + 104x^3 - 6722x^2 + 1947x + 208, \tag{C80}$$

$$P_{76} = 437x^{10} - 866x^9 - 1849x^8 + 6412x^7 - 7694x^6 + 5232x^5 - 5930x^4 + 5764x^3 - 2407x^2 - 542x + 419, \tag{C81}$$

$$P_{77} = 437x^{10} - 422x^9 - 3057x^8 + 148x^7 + 3786x^6 - 3312x^5 + 4686x^4 - 68x^3 - 3327x^2 - 314x + 419, \tag{C82}$$

$$P_{78} = 557x^{10} - 2012x^9 + 4515x^8 + 10256x^7 - 65160x^6 + 103636x^5 - 68040x^4 + 13496x^3 + 2715x^2 - 1472x + 485, \quad (C83)$$

$$P_{79} = 557x^{10} - 1736x^9 - 5117x^8 + 18176x^7 - 4312x^6 - 16556x^5 - 3304x^4 + 17528x^3 - 5333x^2 - 1412x + 485, \quad (C84)$$

$$P_{80} = 536x^{12} - 2203x^{11} + 807x^{10} - 2747x^9 + 13604x^8 - 42x^7 - 16838x^6 - 42x^5 + 13604x^4 - 2747x^3 + 807x^2$$
$$- 2203x + 536, \quad (C85)$$

$$P_{81} = 848x^{12} - 1467x^{11} - 3883x^{10} + 10033x^9 + 11148x^8 - 4342x^7 - 21602x^6 - 4342x^5 + 11148x^4 + 10033x^3$$
$$- 3883x^2 - 1467x + 848, \quad (C86)$$

$$P_{82} = 2768x^{12} - 13405x^{11} + 1776x^{10} - 116585x^9 + 235760x^8 + 331206x^7 - 870752x^6 + 331206x^5 + 235760x^4$$
$$- 116585x^3 + 1776x^2 - 13405x + 2768, \quad (C87)$$

$$P_{83} = 2768x^{12} - 10461x^{11} + 15692x^{10} - 6353x^9 - 33936x^8 - 3154x^7 + 83176x^6 - 3154x^5 - 33936x^4 - 6353x^3$$
$$+ 15692x^2 - 10461x + 2768. \quad (C88)$$

APPENDIX D: PARTIALLY RESUMMED EXPANSIONS AT $\hat{s} = 4$

In the main text, we showed the partially resummed expansion of $F_{V,1,1}^{(0)}(z) + \zeta_2 F_{V,1,2}^{(0)}(z) + \zeta_3 F_{V,1,3}^{(0)}(z)$. We give now the corresponding results for the remaining factors. The *G*-functions appearing in these expressions are given in Eqs. (4.53)–(4.56).

$$F_{V,2,1}^{(0)}(z) + \zeta_2 F_{V,2,2}^{(0)}(z) + \zeta_3 F_{V,2,3}^{(0)}(z)$$
$$= \frac{64\pi^2 \frac{1}{(4-z^2)^{3/2}} G(\frac{1}{\tau}, \sqrt{4-\tau}\sqrt{\tau}; z^2) \ln(z) Q_5}{9z^5} + \frac{128 \ln(2)\pi^3 \frac{1}{(4-z^2)^{3/2}} \ln(1 - \frac{z^2}{4}) Q_{14}}{27z^5}$$
$$+ \frac{512\pi^2 \ln^2(z) Q_{19}}{27(z-2)z^4(z+2)} - \frac{16\pi \frac{1}{(4-z^2)^{3/2}} \ln^2(z) Q_{60}}{27z^3} + \frac{2 \ln(2)\pi^3 \frac{1}{\sqrt{4-z^2}} Q_{63}}{243z^5}$$
$$- \frac{2\pi^2 \ln(z) Q_{68}}{81(z-2)z^2(z+2)} + \ln(2)\pi^3 \ln(z) \left[-\frac{4 \frac{1}{\sqrt{4-z^2}} Q_3}{27z^5} - \frac{4}{9} z^3 \frac{1}{(4-z^2)^{3/2}} \right]$$
$$+ \text{Li}_4\left(\frac{1}{2}\right)\pi \left[\frac{256}{3} z^3 \frac{1}{(4-z^2)^{3/2}} + \frac{256(340 + 108z^2 + 27z^4) \frac{1}{\sqrt{4-z^2}}}{81z^3} \right.$$
$$\left. - \frac{2048(z^2 - 3) \frac{1}{\sqrt{4-z^2}} \ln(1 - \frac{z^2}{4})}{z^5} \right] + \ln^4(2)\pi \left[\frac{32}{9} z^3 \frac{1}{(4-z^2)^{3/2}} \right.$$
$$\left. + \frac{32(340 + 108z^2 + 27z^4) \frac{1}{\sqrt{4-z^2}}}{243z^3} - \frac{256(z^2 - 3) \frac{1}{\sqrt{4-z^2}} \ln(1 - \frac{z^2}{4})}{3z^5} \right]$$

$$+\ln^2(2)\pi^3\left[-\frac{14}{3}z^3\frac{1}{(4-z^2)^{3/2}}-\frac{2(-95696+33372z^2+567z^4)\frac{1}{\sqrt{4-z^2}}}{243z^3}\right.$$

$$+\frac{256(z^2-3)\frac{1}{\sqrt{4-z^2}}\ln(1-\frac{z^2}{4})}{3z^5}\right]+\pi^5\left[-\frac{16(-20915+6761z^2)\frac{1}{(4-z^2)^{3/2}}}{3645z^3}\right.$$

$$\left.+\frac{32(3+11z^2)\frac{1}{\sqrt{4-z^2}}\ln(1-\frac{z^2}{4})}{45z^5}\right]+\left[\frac{128\ln^2(2)\pi(-36-4z^2+5z^4)}{27(z-2)z^4(z+2)}\right.$$

$$+\frac{64\pi\ln^2(z)Q_{33}}{27(z-2)z^4(z+2)}+\frac{32\ln(2)\pi^3Q_{35}}{3(z-2)z^6(z+2)}-\frac{4\pi^2\frac{1}{(4-z^2)^{3/2}}\ln(z)Q_{69}}{27z^5}-\frac{3584\pi(z^2-2)\zeta_3}{3(z-2)z^6(z+2)}$$

$$+\frac{1024\pi^3(z^2-2)\ln(z)}{9(z-2)z^6(z+2)}+\frac{256\ln(2)\pi^2(-10+z^2)(z^2-3)\frac{1}{(4-z^2)^{3/2}}\ln(z)}{9z^5}$$

$$\left.+\frac{256\ln(2)\pi(-36-4z^2+5z^4)\ln(z)}{9(z-2)z^4(z+2)}+\frac{512\pi^2(1758-855z^2+104z^4)\frac{1}{(4-z^2)^{3/2}}\ln^2(z)}{27z^5}\right]$$

$$\times G(\sqrt{4-\tau}\sqrt{\tau};z^2)+\left[\frac{8\pi^2\ln(z)Q_{36}}{3(z-2)z^6(z+2)}-\frac{32\ln(2)\pi^3(-3+z)(3+z)\frac{1}{\sqrt{4-z^2}}}{z^5}\right.$$

$$\left.-\frac{512\ln(2)\pi\frac{1}{\sqrt{4-z^2}}\ln(z)}{3z^5}-\frac{256\ln^2(2)\pi\frac{1}{\sqrt{4-z^2}}}{9z^5}+\frac{64\pi(1612-814z^2+103z^4)\frac{1}{(4-z^2)^{3/2}}\ln^2(z)}{9z^5}\right]$$

$$\times G(\sqrt{4-\tau}\sqrt{\tau};z^2)^2+\frac{778}{81}\ln(2)\pi^3z^3\frac{1}{(4-z^2)^{3/2}}-\frac{4096\text{Li}_4(\frac{1}{2})\ln(2)\pi\frac{1}{\sqrt{4-z^2}}}{3z^3}$$

$$-\frac{512\ln(2)^5\pi\frac{1}{\sqrt{4-z^2}}}{9z^3}+\frac{512\ln^3(2)\pi^3\frac{1}{\sqrt{4-z^2}}}{9z^3}+\frac{1600\ln(2)\pi^5\frac{1}{\sqrt{4-z^2}}}{27z^3}+\frac{\pi^2(-9+2z^2)\kappa_2\frac{1}{\sqrt{4-z^2}}}{9z^5}$$

$$+\frac{2(-9+2z^2)\kappa_9\frac{1}{\sqrt{4-z^2}}}{3z^5}-\frac{128\ln^2(2)\pi(9+z^2)\sqrt{4-z^2}}{27z}+\pi^3\left[\frac{\frac{1}{\sqrt{4-z^2}}Q_{27}}{270z^5}+\frac{1}{2}z^3\frac{1}{(4-z^2)^{3/2}}\right]\zeta_3$$

$$-\frac{8664320(-9+2z^2)\frac{\zeta_3}{\sqrt{4-z^2}}}{729\pi z^5}+\frac{7168\text{Li}_4(\frac{1}{2})(-9+2z^2)\frac{\zeta_3}{\sqrt{4-z^2}}}{\pi z^5}$$

$$+\frac{8137472\ln(2)(-9+2z^2)\frac{\zeta_3}{\sqrt{4-z^2}}}{729\pi z^5}+\frac{68096\ln^2(2)(-9+2z^2)\frac{\zeta_3}{\sqrt{4-z^2}}}{243\pi z^5}$$

$$-\frac{28672\ln^3(2)(-9+2z^2)\frac{\zeta_3}{\sqrt{4-z^2}}}{27\pi z^5}+\frac{896\ln^4(2)(-9+2z^2)\frac{\zeta_3}{\sqrt{4-z^2}}}{3\pi z^5}$$

$$-\frac{257024\ln(2)\pi(-9+2z^2)\frac{\zeta_3}{\sqrt{4-z^2}}}{81z^5}+\frac{2944\ln(2)^2\pi(-9+2z^2)\frac{\zeta_3}{\sqrt{4-z^2}}}{3z^5}$$

$$-\frac{128\pi(-45909-4864z^2+7533z^4)\frac{\zeta_3}{\sqrt{4-z^2}}}{729z^5}-\frac{17(-9+2z^2)\kappa_1\frac{\zeta_3}{\sqrt{4-z^2}}}{18z^5}$$

$$+ \frac{4(-9+2z^2)\kappa_3 \frac{\zeta_3}{\sqrt{4-z^2}}}{9z^5} + \frac{196448(-9+2z^2)\frac{\zeta_3^2}{\sqrt{4-z^2}}}{81\pi z^5} + \pi\left[\frac{5}{2}z^3\frac{1}{(4-z^2)^{3/2}}\right.$$

$$+ \left.\frac{5(144+4z^2+z^4)\frac{1}{\sqrt{4-z^2}}}{2z^3}\right]\zeta_5 + \frac{128\ln(2)\pi^2(z^2-6)\ln(z)}{9z^2}$$

$$+ \frac{704\pi^3(z^2-2)\frac{1}{\sqrt{4-z^2}}\ln(z)}{9z^3} - \frac{256\ln(2)\pi(9+z^2)\sqrt{4-z^2}\ln(z)}{9z}$$

$$- \frac{16\pi^2(z^2-3)\frac{1}{\sqrt{4-z^2}}G(\sqrt{4-\tau}\sqrt{\tau};z^2)^3\ln(z)}{3z^5}$$

$$- \frac{256\pi^2(z^2-6)(z^2-3)\frac{1}{(4-z^2)^{3/2}}G(\frac{1}{4-\tau},\sqrt{4-\tau}\sqrt{\tau};z^2)\ln(z)}{27z^5}$$

$$+ \frac{32}{9}\pi z^3\frac{1}{(4-z^2)^{3/2}}\ln^2(z) - \frac{512\pi(z^2-2)\frac{1}{\sqrt{4-z^2}}\ln^3(z)}{9z^3} + \tilde{F}_{V,2}^{(0),\text{rest}}(z), \tag{D1}$$

$$F_{A,1,1}^{(0)}(z) + \zeta_2 F_{A,1,2}^{(0)}(z) + \zeta_3 F_{A,1,3}^{(0)}(z)$$

$$= -\frac{8}{9}\ln(2)\pi z\sqrt{4-z^2}\ln(z)Q_9 - \frac{4}{27}\ln^2(2)\pi z\sqrt{4-z^2}Q_{10} + \frac{32\ln(2)\pi^3\frac{1}{(4-z^2)^{3/2}}\ln(1-\frac{z^2}{4})Q_{12}}{27z^3}$$

$$- \frac{64\ln(2)\pi^3\frac{1}{(4-z^2)^{3/2}}\ln(z)Q_{23}}{27z^3} - \frac{64\pi^2\ln^2(z)Q_{37}}{27(z-2)z^2(z+2)} - \frac{4\pi^3\frac{1}{\sqrt{4-z^2}}\ln(z)Q_{47}}{27z^3} - \frac{16\pi\frac{\zeta_3}{\sqrt{4-z^2}}Q_{54}}{243z^3}$$

$$- \frac{32\pi\frac{1}{\sqrt{4-z^2}}\ln^3(z)Q_{51}}{81z^3} + \frac{4\pi\frac{1}{\sqrt{4-z^2}}\ln^2(z)Q_{74}}{27z^3} + \frac{2\ln(2)\pi^3\frac{1}{(4-z^2)^{3/2}}Q_{81}}{81z^5}$$

$$- \frac{\pi^2\ln(z)Q_{82}}{486(z-2)z^2(z+2)} + \ln^2(2)\pi^3\left[\frac{32\frac{1}{(4-z^2)^{3/2}}Q_{25}}{81z^3} - \frac{64\frac{1}{\sqrt{4-z^2}}\ln(1-\frac{z^2}{4})}{3z^3}\right]$$

$$+ \pi^5\left[\frac{4\frac{1}{(4-z^2)^{3/2}}Q_{34}}{1215z^3} + \frac{8\frac{1}{\sqrt{4-z^2}}\ln(1-\frac{z^2}{4})}{45z^3}\right] + \ln^4(2)\pi\left[\frac{128\frac{1}{(4-z^2)^{3/2}}Q_2}{81z^3} + \frac{64\frac{1}{\sqrt{4-z^2}}\ln(1-\frac{z^2}{4})}{3z^3}\right]$$

$$+ \text{Li}_4\left(\frac{1}{2}\right)\pi\left[\frac{1024\frac{1}{(4-z^2)^{3/2}}Q_2}{27z^3} + \frac{512\frac{1}{\sqrt{4-z^2}}\ln(1-\frac{z^2}{4})}{z^3}\right] + \left[-\frac{64\ln^3(2)\pi(z^2-2)^2}{9(z-2)z^2(z+2)}\right.$$

$$- \frac{128\pi^2\frac{1}{(4-z^2)^{3/2}}\ln^2(z)Q_6}{27z^3} - \frac{16\ln(2)\pi^3Q_{30}}{27(z-2)z^4(z+2)} + \frac{16\ln^2(2)\pi Q_{41}}{27(z-2)z^2(z+2)} + \frac{32\ln(2)\pi\ln(z)Q_{41}}{9(z-2)z^2(z+2)}$$

$$+ \frac{16\pi\ln^2(z)Q_{46}}{27(z-2)z^2(z+2)} - \frac{\pi^2\frac{1}{(4-z^2)^{3/2}}\ln(z)Q_{79}}{81z^5} - \frac{224\pi(z^2-2)(12-62z^2+31z^4)\zeta_3}{27(z-2)z^4(z+2)}$$

$$- \frac{64\ln(2)^2\pi(z^2-2)^2\ln(z)}{(z-2)z^2(z+2)} - \frac{8\pi^3(z^2-2)(-32-278z^2+139z^4)\ln(z)}{27(z-2)z^4(z+2)}$$

$$- \frac{64\ln(2)\pi^2(14-5z^4+z^6)\frac{1}{(4-z^2)^{3/2}}\ln(z)}{9z^3} - \frac{192\ln(2)\pi(z^2-2)^2\ln^2(z)}{(z-2)z^2(z+2)}$$

$$- \frac{10432\pi(z^2-2)^2\ln^3(z)}{81(z-2)z^2(z+2)}\Bigg] G(\sqrt{4-\tau}\sqrt{\tau};z^2) + \Bigg[\frac{2\pi^2\ln(z)Q_{20}}{9(z-2)z^4(z+2)}$$

$$+ \frac{16\pi\frac{1}{(4-z^2)^{3/2}}\ln^2(z)Q_{21}}{27z^3} + \frac{24\ln(2)\pi^3\frac{1}{\sqrt{4-z^2}}}{z^3} - \frac{16\ln^2(2)\pi(4-22z^2+11z^4)\frac{1}{\sqrt{4-z^2}}}{27z^3}$$

$$- \frac{32\ln(2)\pi(4-22z^2+11z^4)\frac{1}{\sqrt{4-z^2}}\ln(z)}{9z^3}\Bigg] G(\sqrt{4-\tau}\sqrt{\tau};z^2)^2 + \Bigg[\frac{64\ln^2(2)\pi(z^2-2)^2}{27(z-2)z^2(z+2)}$$

$$+ \frac{128\ln(2)\pi(z^2-2)^2\ln(z)}{9(z-2)z^2(z+2)} - \frac{64\pi^2(-26+18z^2-7z^4+z^6)\frac{1}{(4-z^2)^{3/2}}\ln(z)}{27z^3}$$

$$+ \frac{320\pi(z^2-2)^2\ln^2(z)}{27(z-2)z^2(z+2)}\Bigg] G\left(\frac{1}{4-\tau},\sqrt{4-\tau}\sqrt{\tau};z^2\right) + \Bigg[\frac{32\ln^2(2)\pi(z^2-2)^2}{3(z-2)z^2(z+2)}$$

$$+ \frac{16\pi^2\frac{1}{(4-z^2)^{3/2}}\ln(z)Q_{17}}{9z^3} + \frac{64\ln(2)\pi(z^2-2)^2\ln(z)}{(z-2)z^2(z+2)} + \frac{2848\pi(z^2-2)^2\ln^2(z)}{27(z-2)z^2(z+2)}\Bigg]$$

$$\times G\left(\frac{1}{\tau},\sqrt{4-\tau}\sqrt{\tau};z^2\right) + \Bigg[\frac{32\ln^2(2)\pi(z^2-2)^2}{3(z-2)z^2(z+2)} + \frac{64\ln(2)\pi(z^2-2)^2\ln(z)}{(z-2)z^2(z+2)}$$

$$+ \frac{96\pi(z^2-2)^2\ln^2(z)}{(z-2)z^2(z+2)}\Bigg] G\left(\sqrt{4-\tau}\sqrt{\tau},\frac{1}{4-\tau};z^2\right) - \frac{\pi^2\kappa_2\frac{1}{\sqrt{4-z^2}}}{12z^3} - \frac{\kappa_9\frac{1}{\sqrt{4-z^2}}}{2z^3}$$

$$+ \frac{32}{9}\ln^3(2)\pi z(z^2-2)\sqrt{4-z^2} - \frac{\pi^3(-2404+1441z^2)\frac{\zeta_3}{(4-z^2)^{3/2}}}{45z^3} + \frac{2166080\frac{\zeta_3}{\sqrt{4-z^2}}}{243\pi z^3}$$

$$- \frac{5376\mathrm{Li}_4(\frac{1}{2})\frac{\zeta_3}{\sqrt{4-z^2}}}{\pi z^3} - \frac{2034368\ln(2)\frac{\zeta_3}{\sqrt{4-z^2}}}{243\pi z^3} - \frac{17024\ln^2(2)\frac{\zeta_3}{\sqrt{4-z^2}}}{81\pi z^3} + \frac{7168\ln^3(2)\frac{\zeta_3}{\sqrt{4-z^2}}}{9\pi z^3}$$

$$- \frac{224\ln^4(2)\frac{\zeta_3}{\sqrt{4-z^2}}}{\pi z^3} + \frac{64256\ln(2)\pi\frac{\zeta_3}{\sqrt{4-z^2}}}{27z^3} - \frac{736\ln^2(2)\pi\frac{\zeta_3}{\sqrt{4-z^2}}}{z^3} + \frac{17\kappa_1\frac{\zeta_3}{\sqrt{4-z^2}}}{24z^3} - \frac{\kappa_3\frac{\zeta_3}{\sqrt{4-z^2}}}{3z^3}$$

$$- \frac{49112\frac{\zeta_3^2}{\sqrt{4-z^2}}}{27\pi z^3} + \frac{40\pi(8+z^2)\frac{\zeta_5}{(4-z^2)^{3/2}}}{z^3} - \frac{32\ln(2)\pi^2(24-6z^2-3z^4+z^6)\ln(z)}{9z^2}$$

$$- \frac{640\pi^4(z^2-2)^2\ln(z)}{27(z-2)z^2(z+2)} + 32\ln^2(2)\pi z(z^2-2)\sqrt{4-z^2}\ln(z)$$

$$+ \frac{496}{9}\ln(2)\pi z(4-z^2)^{3/2}\ln(z) + \frac{4\pi^2\frac{1}{\sqrt{4-z^2}}G(\sqrt{4-\tau}\sqrt{\tau};z^2)^3\ln(z)}{3z^3}$$

$$+ 96\ln(2)\pi z(z^2-2)\sqrt{4-z^2}\ln^2(z) + \pi^2\Bigg[-\frac{1024}{27} + \frac{4096}{81z^2} - \frac{1024z^2}{81(z-2)(z+2)}\Bigg]\ln^3(z)$$

$$+ \frac{16}{3}\ln^2(2)\pi z(z^2-2)\sqrt{4-z^2}\ln\left(1-\frac{z^2}{4}\right) + 32\ln(2)\pi z(z^2-2)\sqrt{4-z^2}\ln(z)$$

$$\times \ln\left(1-\frac{z^2}{4}\right) + 48\pi z(z^2-2)\sqrt{4-z^2}\ln^2(z)\ln\left(1-\frac{z^2}{4}\right) + \tilde{F}_{A,1}^{(0),\mathrm{rest}}(z), \tag{D2}$$

$$F_{A,2,1}^{(0)}(z) + \zeta_2 F_{A,2,2}^{(0)}(z) + \zeta_3 F_{A,2,3}^{(0)}(z)$$

$$= \frac{512\pi^2 \ln^2(z) Q_4}{9(z-2)^2 z^2 (z+2)^2} - \frac{16\pi \frac{1}{(4-z^2)^{3/2}} \ln^2(z) Q_{58}}{27 z^3} + \frac{2\pi^2 \ln(z) Q_{70}}{81(z-2)^2 z^2 (z+2)^2}$$

$$- \frac{8\ln(2)\pi^3 \frac{1}{(4-z^2)^{5/2}} Q_{75}}{27 z^5} + \text{Li}_4\left(\frac{1}{2}\right)\pi\left[-\frac{1024(192 - 140 z^2 + 29 z^4) \frac{1}{(4-z^2)^{5/2}}}{9 z^3} \right.$$

$$\left. - \frac{2048 \frac{1}{(4-z^2)^{3/2}} \ln(1 - \frac{z^2}{4})}{z^3} \right] + \ln^4(2)\pi\left[-\frac{128(192 - 140 z^2 + 29 z^4) \frac{1}{(4-z^2)^{5/2}}}{27 z^3} \right.$$

$$\left. - \frac{256 \frac{1}{(4-z^2)^{3/2}} \ln(1 - \frac{z^2}{4})}{3 z^3} \right] + \pi^5\left[\frac{16(6288 - 4393 z^2 + 805 z^4) \frac{1}{(4-z^2)^{5/2}}}{405 z^3} \right.$$

$$\left. - \frac{32 \frac{1}{(4-z^2)^{3/2}} \ln(1 - \frac{z^2}{4})}{45 z^3} \right] + \ln^2(2)\pi^3\left[\frac{128(720 - 445 z^2 + 67 z^4) \frac{1}{(4-z^2)^{5/2}}}{9 z^3} \right.$$

$$\left. + \frac{256 \frac{1}{(4-z^2)^{3/2}} \ln(1 - \frac{z^2}{4})}{3 z^3} \right] + \left[\frac{128\ln^2(2)\pi(92 - 84 z^2 + 17 z^4)}{27(z-2)^2 z^2 (z+2)^2} - \frac{32\ln(2)\pi^3 Q_{15}}{9(z-2)^2 z^4 (z+2)^2} \right.$$

$$+ \frac{64\pi \ln^2(z) Q_{26}}{27(z-2)^2 z^2 (z+2)^2} + \frac{4\pi^2 \frac{1}{(4-z^2)^{5/2}} \ln(z) Q_{67}}{27 z^5} - \frac{3584\pi(z^2 - 2)\zeta_3}{9(z-2)^2 z^4 (z+2)^2}$$

$$+ \frac{256\ln(2)\pi(92 - 84 z^2 + 17 z^4)\ln(z)}{9(z-2)^2 z^2 (z+2)^2} + \frac{256\ln(2)\pi^2(z^2 - 7)(z^2 - 2) \frac{1}{(4-z^2)^{5/2}} \ln(z)}{9 z^3}$$

$$\left. + \frac{1024\pi^3(z^2 - 2)\ln(z)}{27(z-2)^2 z^4 (z+2)^2} + \frac{512\pi^2(1130 - 425 z^2 + 36 z^4)\sqrt{4 - z^2}\ln^2(z)}{27(z-2)^3 z^3 (z+2)^3} \right] G(\sqrt{4 - \tau}\sqrt{\tau}; z^2)$$

$$+ \left[\frac{256\ln^2(2)\pi \frac{1}{(4-z^2)^{3/2}}}{27 z^3} - \frac{96\ln(2)\pi^3 \frac{1}{(4-z^2)^{3/2}}}{z^3} - \frac{8\pi^2(-256 - 16 z^2 + 9 z^6)\ln(z)}{9(z-2)^2 z^4 (z+2)^2} \right.$$

$$\left. + \frac{512\ln(2)\pi \frac{1}{(4-z^2)^{3/2}} \ln(z)}{9 z^3} + \frac{64\pi(3340 - 1270 z^2 + 111 z^4)\sqrt{4 - z^2}\ln^2(z)}{27(z-2)^3 z^3 (z+2)^3} \right]$$

$$\times G(\sqrt{4 - \tau}\sqrt{\tau}; z^2)^2 + \frac{128}{27}\ln^2(2)\pi z(-23 + 8 z^2)\frac{1}{\sqrt{4 - z^2}} + \frac{\pi^2 \kappa_2 \frac{1}{(4-z^2)^{3/2}}}{3 z^3}$$

$$+ \frac{2\kappa_9 \frac{1}{(4-z^2)^{3/2}}}{z^3} + \frac{1}{\pi}\left[-\frac{135380}{243} z^3 \frac{1}{(4 - z^2)^{3/2}} - \frac{135380(4 - 2z + z^2)(4 + 2z + z^2) \frac{1}{\sqrt{4-z^2}}}{243 z^3} \right]$$

$$\times \zeta_3 + \frac{4\pi^3(-2404 + 3121 z^2) \frac{\zeta_3}{(4-z^2)^{5/2}}}{45 z^3} + \frac{21504\text{Li}_4(\frac{1}{2}) \frac{\zeta_3}{(4-z^2)^{3/2}}}{\pi z^3} + \frac{8137472\ln(2) \frac{\zeta_3}{(4-z^2)^{3/2}}}{243\pi z^3}$$

$$+ \frac{68096\ln^2(2) \frac{\zeta_3}{(4-z^2)^{3/2}}}{81\pi z^3} - \frac{28672\ln^3(2) \frac{\zeta_3}{(4-z^2)^{3/2}}}{9\pi z^3} + \frac{896\ln^4(2) \frac{\zeta_3}{(4-z^2)^{3/2}}}{\pi z^3}$$

$$- \frac{257024\ln(2)\pi \frac{\zeta_3}{(4-z^2)^{3/2}}}{27 z^3} + \frac{128\pi(2891 - 5994 z^2 + 999 z^4) \frac{\zeta_3}{(4-z^2)^{3/2}}}{243 z^3} + \frac{4\kappa_3 \frac{\zeta_3}{(4-z^2)^{3/2}}}{3 z^3}$$

$$+ \frac{2944\ln^2(2)\pi \frac{\zeta_3}{(4-z^2)^{3/2}}}{z^3} - \frac{17\kappa_1 \frac{\zeta_3}{(4-z^2)^{3/2}}}{6z^3} + \frac{196448\frac{\zeta_3^2}{(4-z^2)^{3/2}}}{27\pi z^3} - \frac{160\pi(8+7z^2)\frac{\zeta_5}{(4-z^2)^{5/2}}}{z^3}$$

$$- \frac{128\ln(2)\pi^2(-8+z^2)(z^2-2)\ln(z)}{9(z-2)z^2(z+2)} - \frac{512\ln(2)\pi^3(866-535z^2+80z^4)\frac{\ln(z)}{(4-z^2)^{5/2}}}{27z^3}$$

$$+ \frac{64\pi^3(z^2-2)\frac{1}{\sqrt{4-z^2}}\ln(z)}{27z^3} + \frac{256}{9}\ln(2)\pi z(-23+8z^2)\frac{1}{\sqrt{4-z^2}}\ln(z)$$

$$- \frac{16\pi^2\frac{1}{(4-z^2)^{3/2}}G(\sqrt{4-\tau}\sqrt{\tau};z^2)^3\ln(z)}{3z^3} - \frac{256\pi^2(26-9z^2+z^4)\frac{1}{(4-z^2)^{5/2}}\ln(z)}{27z^3}$$

$$\times G\left(\frac{1}{4-\tau}, \sqrt{4-\tau}\sqrt{\tau};z^2\right) + \frac{64\pi^2(4808-1880z^2+171z^4)\frac{1}{(4-z^2)^{5/2}}\ln(z)}{9z^3}$$

$$\times G\left(\frac{1}{\tau}, \sqrt{4-\tau}\sqrt{\tau};z^2\right) + \frac{128\ln(2)\pi^3(860-550z^2+83z^4)\frac{1}{(4-z^2)^{5/2}}\ln(1-\frac{z^2}{4})}{27z^3}$$

$$- \frac{512\pi(z^2-2)\frac{1}{\sqrt{4-z^2}}\ln^3(z)}{9z^3} + \tilde{F}_{A,2}^{(0),\text{rest}}(z), \tag{D3}$$

$$F_{S,1}^{(0)}(z) + \zeta_2 F_{S,2}^{(0)}(z) + \zeta_3 F_{S,3}^{(0)}(z)$$

$$= \frac{4}{27}\ln^2(2)\pi z\sqrt{4-z^2}Q_1 + \frac{8}{9}\ln(2)\pi z\sqrt{4-z^2}\ln(z)Q_1 - \frac{64\pi^2\ln^2(z)Q_{38}}{27(z-2)z^2(z+2)}$$

$$- \frac{4\pi^3\frac{1}{\sqrt{4-z^2}}\ln(z)Q_{49}}{27z^3} - \frac{32\pi\frac{1}{\sqrt{4-z^2}}\ln^3(z)Q_{50}}{81z^3} - \frac{16\pi\frac{\zeta_3}{\sqrt{4-z^2}}Q_{53}}{27z^3} - \frac{4\ln(2)\pi^3\frac{1}{\sqrt{4-z^2}}Q_{61}}{81z^5}$$

$$+ \frac{4\pi\frac{1}{\sqrt{4-z^2}}\ln^2(z)Q_{71}}{27z^3} - \frac{\pi^2\ln(z)Q_{77}}{243(z-2)z^2(z+2)} + \left[-\frac{64\ln^3(2)\pi(z^2-2)^2}{9(z-2)z^2(z+2)}\right.$$

$$- \frac{8\ln(2)\pi^3(1172-1118z^2+239z^4)}{27(z-2)z^2(z+2)} + \frac{16\ln^2(2)\pi Q_{39}}{27(z-2)z^2(z+2)} + \frac{32\ln(2)\pi\ln(z)Q_{39}}{9(z-2)z^2(z+2)}$$

$$+ \frac{16\pi\ln^2(z)Q_{44}}{27(z-2)z^2(z+2)} + \frac{4\pi^2\frac{1}{\sqrt{4-z^2}}\ln(z)Q_{59}}{81z^5} - \frac{6944\pi(z^2-2)^2\zeta_3}{27(z-2)z^2(z+2)}$$

$$- \frac{64\ln^2(2)\pi(z^2-2)^2\ln(z)}{(z-2)z^2(z+2)} - \frac{1112\pi^3(z^2-2)^2\ln(z)}{27(z-2)z^2(z+2)} - \frac{192\ln(2)\pi(z^2-2)^2\ln^2(z)}{(z-2)z^2(z+2)}$$

$$+ \frac{64\ln(2)\pi^2(z^2-3)(1+z^2)\frac{1}{\sqrt{4-z^2}}\ln(z)}{9z^3} + \frac{128\pi^2(101-34z^2+9z^4)\frac{1}{\sqrt{4-z^2}}\ln^2(z)}{27z^3}$$

$$- \frac{10432\pi(z^2-2)^2\ln^3(z)}{81(z-2)z^2(z+2)}\right]G(\sqrt{4-\tau}\sqrt{\tau};z^2) + \left[-\frac{176\ln^2(2)\pi(z^2-2)\frac{1}{\sqrt{4-z^2}}}{27z}\right.$$

$$+ \frac{4\pi^2(68-53z^2+11z^4)\ln(z)}{3(z-2)z^2(z+2)} - \frac{16\pi(-318-86z^2+67z^4)\frac{1}{\sqrt{4-z^2}}\ln^2(z)}{27z^3}$$

$$- \frac{352\ln(2)\pi(z^2-2)\frac{1}{\sqrt{4-z^2}}\ln(z)}{9z}\right]G(\sqrt{4-\tau}\sqrt{\tau};z^2)^2 + \left[\frac{64\ln^2(2)\pi(z^2-2)^2}{27(z-2)z^2(z+2)}\right.$$

$$
+ \frac{128\ln(2)\pi(z^2-2)^2\ln(z)}{9(z-2)z^2(z+2)} + \frac{64\pi^2(5-2z^2+z^4)\frac{1}{\sqrt{4-z^2}}\ln(z)}{27z^3} + \frac{320\pi(z^2-2)^2\ln^2(z)}{27(z-2)z^2(z+2)} \Bigg]
$$

$$
\times G\left(\frac{1}{4-\tau}, \sqrt{4-\tau}\sqrt{\tau}; z^2\right) + \Bigg[\frac{32\ln^2(2)\pi(z^2-2)^2}{3(z-2)z^2(z+2)} + \frac{64\ln(2)\pi(z^2-2)^2\ln(z)}{(z-2)z^2(z+2)}
$$

$$
- \frac{16\pi^2(476-181z^2+45z^4)\frac{1}{\sqrt{4-z^2}}\ln(z)}{9z^3} + \frac{2848\pi(z^2-2)^2\ln^2(z)}{27(z-2)z^2(z+2)} \Bigg] G\left(\frac{1}{\tau}, \sqrt{4-\tau}\sqrt{\tau}; z^2\right)
$$

$$
+ \Bigg[\frac{32\ln^2(2)\pi(z^2-2)^2}{3(z-2)z^2(z+2)} + \frac{64\ln(2)\pi(z^2-2)^2\ln(z)}{(z-2)z^2(z+2)} + \frac{96\pi(z^2-2)^2\ln^2(z)}{(z-2)z^2(z+2)} \Bigg]
$$

$$
\times G\left(\sqrt{4-\tau}\sqrt{\tau}, \frac{1}{4-\tau}; z^2\right) - \frac{256\mathrm{Li}_4(\frac{1}{2})\pi(36-43z^2+8z^4)\frac{1}{\sqrt{4-z^2}}}{27z^3}
$$

$$
- \frac{32\ln^4(2)\pi(36-43z^2+8z^4)\frac{1}{\sqrt{4-z^2}}}{81z^3} - \frac{32\ln^2(2)\pi^3(1350-659z^2+100z^4)\frac{1}{\sqrt{4-z^2}}}{81z^3}
$$

$$
- \frac{4\pi^5(1818-2711z^2+1612z^4)\frac{1}{\sqrt{4-z^2}}}{1215z^3} + \frac{32}{9}\ln^3(2)\pi z(z^2-2)\sqrt{4-z^2} - \frac{16\pi^3\frac{\zeta_3}{\sqrt{4-z^2}}}{z^3}
$$

$$
- \frac{80\pi\frac{\zeta_5}{\sqrt{4-z^2}}}{z^3} - \frac{640\pi^4(z^2-2)^2\ln(z)}{27(z-2)z^2(z+2)} - \frac{32\ln(2)\pi^2(2+z^2)(12-6z^2+z^4)\ln(z)}{9z^2}
$$

$$
+ \frac{64\ln(2)\pi^3(142-120z^2+33z^4)\frac{1}{\sqrt{4-z^2}}\ln(z)}{9z^3} + 32\ln^2(2)\pi z(z^2-2)\sqrt{4-z^2}\ln(z)
$$

$$
+ 96\ln(2)\pi z(z^2-2)\sqrt{4-z^2}\ln^2(z) - \frac{32\ln(2)\pi^3(74-41z^2+7z^4)\frac{1}{\sqrt{4-z^2}}\ln(1-\frac{z^2}{4})}{9z^3}
$$

$$
- \frac{4096\pi^2(z^2-2)^2\ln^3(z)}{81(z-2)z^2(z+2)} + \frac{16}{3}\ln^2(2)\pi z(z^2-2)\sqrt{4-z^2}\ln\left(1-\frac{z^2}{4}\right)
$$

$$
+ 32\ln(2)\pi z(z^2-2)\sqrt{4-z^2}\ln(z)\ln\left(1-\frac{z^2}{4}\right) + 48\pi z(z^2-2)\sqrt{4-z^2}\ln^2(z)
$$

$$
\times \ln\left(1-\frac{z^2}{4}\right) + \tilde{F}_S^{(0),\mathrm{rest}}(z), \tag{D4}
$$

$$
F_{P,1}^{(0)}(z) + \zeta_2 F_{P,2}^{(0)}(z) + \zeta_3 F_{P,3}^{(0)}(z)
$$

$$
= -\frac{64\pi^2\ln^2(z)Q_7}{27(z-2)(z+2)} - \frac{4\pi^3\frac{1}{\sqrt{4-z^2}}\ln(z)Q_{28}}{27z} - \frac{32\pi\frac{1}{\sqrt{4-z^2}}\ln^3(z)Q_{31}}{81z} - \frac{16\pi\frac{\zeta_3}{\sqrt{4-z^2}}Q_{32}}{27z}
$$

$$
- \frac{4\ln(2)\pi^3\frac{1}{(4-z^2)^{3/2}}Q_{66}}{81z^5} - \frac{4\pi(z-2)(z+2)\frac{1}{(4-z^2)^{3/2}}\ln^2(z)Q_{72}}{27z^3} - \frac{\pi^2\ln(z)Q_{78}}{243(z-2)z^2(z+2)}
$$

$$
+ \mathrm{Li}_4\left(\frac{1}{2}\right)\pi\left[-\frac{256}{9}z^3\frac{1}{(4-z^2)^{3/2}} - \frac{256(-31+11z^2)\frac{1}{\sqrt{4-z^2}}}{27z} \right] + \Bigg[-\frac{64\ln^3(2)\pi(z^2-2)^2}{9(z-2)z^2(z+2)}
$$

$$
- \frac{8\ln(2)\pi^3(740-1010z^2+239z^4)}{27(z-2)z^2(z+2)} + \frac{16\ln^2(2)\pi Q_{40}}{27(z-2)z^2(z+2)} + \frac{32\ln(2)\pi\ln(z)Q_{40}}{9(z-2)z^2(z+2)}
$$

$$+\frac{16\pi\ln^2(z)Q_{45}}{27(z-2)z^2(z+2)}-\frac{4\pi^2\frac{1}{(4-z^2)^{3/2}}\ln(z)Q_{76}}{81z^5}-\frac{6944\pi(z^2-2)^2\zeta_3}{27(z-2)z^2(z+2)}$$

$$-\frac{64\ln^2(2)\pi(z^2-2)^2\ln(z)}{(z-2)z^2(z+2)}-\frac{64\ln(2)\pi^2(z^2-3)^2\frac{1}{(4-z^2)^{3/2}}\ln(z)}{9z}-\frac{1112\pi^3(z^2-2)^2\ln(z)}{27(z-2)z^2(z+2)}$$

$$-\frac{192\ln(2)\pi(z^2-2)^2\ln^2(z)}{(z-2)z^2(z+2)}-\frac{128\pi^2(137-70z^2+9z^4)\frac{1}{(4-z^2)^{3/2}}\ln^2(z)}{27z}$$

$$-\frac{10432\pi(z^2-2)^2\ln^3(z)}{81(z-2)z^2(z+2)}\Bigg]G(\sqrt{4-\tau}\sqrt{\tau};z^2)+\Bigg[-\frac{176\ln^2(2)\pi(z^2-2)\frac{1}{\sqrt{4-z^2}}}{27z}$$

$$+\frac{4\pi^2(44-47z^2+11z^4)\ln(z)}{3(z-2)z^2(z+2)}+\frac{16\pi(338-354z^2+67z^4)\frac{1}{(4-z^2)^{3/2}}\ln^2(z)}{27z}$$

$$-\frac{352\ln(2)\pi(z^2-2)\frac{1}{\sqrt{4-z^2}}\ln(z)}{9z}\Bigg]G^2(\sqrt{4-\tau}\sqrt{\tau};z^2)+\Bigg[\frac{64\ln^2(2)\pi(z^2-2)^2}{27(z-2)z^2(z+2)}$$

$$+\frac{128\ln(2)\pi(z^2-2)^2\ln(z)}{9(z-2)z^2(z+2)}-\frac{64\pi^2(z^2-3)^2\frac{1}{(4-z^2)^{3/2}}\ln(z)}{27z}+\frac{320\pi(z^2-2)^2\ln^2(z)}{27(z-2)z^2(z+2)}\Bigg]$$

$$\times G\left(\frac{1}{4-\tau},\sqrt{4-\tau}\sqrt{\tau};z^2\right)+\Bigg[\frac{32\ln^2(2)\pi(z^2-2)^2}{3(z-2)z^2(z+2)}+\frac{64\ln(2)\pi(z^2-2)^2\ln(z)}{(z-2)z^2(z+2)}$$

$$+\frac{16\pi^2(656-343z^2+45z^4)\frac{1}{(4-z^2)^{3/2}}\ln(z)}{9z}+\frac{2848\pi(z^2-2)^2\ln^2(z)}{27(z-2)z^2(z+2)}\Bigg]$$

$$\times G\left(\frac{1}{\tau},\sqrt{4-\tau}\sqrt{\tau};z^2\right)+\Bigg[\frac{32\ln^2(2)\pi(z^2-2)^2}{3(z-2)z^2(z+2)}+\frac{64\ln(2)\pi(z^2-2)^2\ln(z)}{(z-2)z^2(z+2)}$$

$$+\frac{96\pi(z^2-2)^2\ln^2(z)}{(z-2)z^2(z+2)}\Bigg]G\left(\sqrt{4-\tau}\sqrt{\tau},\frac{1}{4-\tau};z^2\right)$$

$$-\frac{32\ln^4(2)\pi(-124+75z^2-8z^4)\frac{1}{(4-z^2)^{3/2}}}{81z}+\frac{32\ln^2(2)\pi^3(1358-735z^2+100z^4)\frac{1}{(4-z^2)^{3/2}}}{81z}$$

$$+\frac{4\pi^5(11642-9159z^2+1612z^4)\frac{1}{(4-z^2)^{3/2}}}{1215z}+\frac{32}{9}\ln^3(2)\pi z(z^2-2)\sqrt{4-z^2}$$

$$-\frac{4}{27}\ln^2(2)\pi z(z^2-2)(28-44z^2+11z^4)\sqrt{4-z^2}+\frac{112\pi^3\frac{\zeta_3}{(4-z^2)^{3/2}}}{3z}-\frac{240\pi\frac{\zeta_5}{(4-z^2)^{3/2}}}{z}$$

$$-\frac{32\ln(2)\pi^2(8+4z^2-4z^4+z^6)\ln(z)}{9z^2}-\frac{64\ln(2)\pi^3(1010-648z^2+99z^4)\frac{1}{(4-z^2)^{3/2}}\ln(z)}{27z}$$

$$-\frac{640\pi^4(z^2-2)^2\ln(z)}{27(z-2)z^2(z+2)}+32\ln^2(2)\pi z(z^2-2)\sqrt{4-z^2}\ln(z)$$

$$-\frac{8}{9}\ln(2)\pi z(z^2-2)(28-44z^2+11z^4)\sqrt{4-z^2}\ln(z)+96\ln(2)\pi z(z^2-2)\sqrt{4-z^2}\ln^2(z)$$

$$-\frac{4096\pi^2(z^2-2)^2\ln^3(z)}{81(z-2)z^2(z+2)}+\frac{32\ln(2)\pi^3(274-153z^2+21z^4)\frac{1}{(4-z^2)^{3/2}}\ln(1-\frac{z^2}{4})}{27z}$$

$$+ \frac{16}{3}\ln^2(2)\pi z(z^2 - 2)\sqrt{4 - z^2}\ln\left(1 - \frac{z^2}{4}\right) + 32\ln(2)\pi z(z^2 - 2)\sqrt{4 - z^2}\ln(z)\ln\left(1 - \frac{z^2}{4}\right)$$

$$+ 48\pi z(z^2 - 2)\sqrt{4 - z^2}\ln^2(z)\ln\left(1 - \frac{z^2}{4}\right) + \tilde{F}_P^{(0),\text{rest}}(z). \tag{D5}$$

The polynomials Q_i are given by

$$Q_1 = -11z^6 + 66z^4 - 116z^2 + 312, \tag{D6}$$

$$Q_2 = 2z^6 + 3z^4 - 74z^2 + 144, \tag{D7}$$

$$Q_3 = 3z^6 - 3444z^4 + 10544z^2 - 2496, \tag{D8}$$

$$Q_4 = 6z^6 - 69z^4 + 196z^2 + 32, \tag{D9}$$

$$Q_5 = 9z^6 - 479z^4 + 3528z^2 - 7032, \tag{D10}$$

$$Q_6 = 9z^6 - 106z^4 + 562z^2 - 1130, \tag{D11}$$

$$Q_7 = 9z^6 - 88z^4 + 292z^2 - 272, \tag{D12}$$

$$Q_8 = 9z^6 - 70z^4 + 562z^2 - 1758, \tag{D13}$$

$$Q_9 = 11z^6 - 66z^4 + 118z^2 + 8, \tag{D14}$$

$$Q_{10} = 11z^6 - 66z^4 + 180z^2 - 240, \tag{D15}$$

$$Q_{11} = 11z^6 - 22z^4 + 4z^2 - 48, \tag{D16}$$

$$Q_{12} = 21z^6 - 236z^4 + 824z^2 - 860, \tag{D17}$$

$$Q_{13} = 21z^6 - 202z^4 + 300z^2 + 156, \tag{D18}$$

$$Q_{14} = 27z^6 - 187z^4 + 274z^2 + 156, \tag{D19}$$

$$Q_{15} = 27z^6 - 144z^4 + 80z^2 + 128, \tag{D20}$$

$$Q_{16} = 31z^6 - 62z^4 + 12z^2 - 144, \tag{D21}$$

$$Q_{17} = 45z^6 - 514z^4 + 2536z^2 - 4808, \tag{D22}$$

$$Q_{18} = 45z^6 - 352z^4 + 2356z^2 - 7032, \tag{D23}$$

$$Q_{19} = 52z^6 - 505z^4 + 1188z^2 - 32, \tag{D24}$$

$$Q_{20} = 57z^6 - 282z^4 + 280z^2 + 256, \tag{D25}$$

$$Q_{21} = 67z^6 - 243z^4 - 932z^2 + 3340, \tag{D26}$$

$$Q_{22} = 67z^6 + 19z^4 - 1636z^2 + 4836, \tag{D27}$$

$$Q_{23} = 99z^6 - 808z^4 + 2080z^2 - 1732, \tag{D28}$$

$$Q_{24} = 99z^6 - 524z^4 + 672z^2 - 156, \tag{D29}$$

$$Q_{25} = 100z^6 - 1338z^4 + 5363z^2 - 6480, \tag{D30}$$

$$Q_{26} = 111z^6 - 1230z^4 + 3448z^2 + 536, \tag{D31}$$

$$Q_{27} = 135z^6 + 540z^4 - 40096z^2 + 129672, \tag{D32}$$

$$Q_{28} = 139z^6 - 834z^4 + 200z^2 + 1832, \tag{D33}$$

$$Q_{29} = 139z^6 - 278z^4 - 32z^2 + 384, \tag{D34}$$

$$Q_{30} = 160z^6 - 721z^4 + 490z^2 + 192, \tag{D35}$$

$$Q_{31} = 163z^6 - 978z^4 + 692z^2 + 1152, \tag{D36}$$

$$Q_{32} = 217z^6 - 1302z^4 + 1145z^2 + 738, \tag{D37}$$

$$Q_{33} = 309z^6 - 2914z^4 + 6744z^2 - 296, \tag{D38}$$

$$Q_{34} = 1612z^6 - 11574z^4 + 24821z^2 - 18864, \tag{D39}$$

$$Q_{35} = 3z^8 - 51z^6 + 156z^4 + 64z^2 - 128, \tag{D40}$$

$$Q_{36} = 3z^8 - 21z^6 + 36z^4 - 128z^2 + 256, \tag{D41}$$

$$Q_{37} = 9z^8 - 124z^6 + 706z^4 - 1448z^2 - 192, \tag{D42}$$

$$Q_{38} = 9z^8 - 88z^6 + 392z^4 - 544z^2 - 256, \tag{D43}$$

$$Q_{39} = 11z^8 - 88z^6 + 196z^4 - 200z^2 + 288, \tag{D44}$$

$$Q_{40} = 11z^8 - 88z^6 + 196z^4 - 72z^2 + 32, \tag{D45}$$

$$Q_{41} = 11z^8 - 88z^6 + 230z^4 - 240z^2 + 216, \tag{D46}$$

$$Q_{42} = 11z^8 - 66z^6 + 118z^4 + 232z^2 - 288, \tag{D47}$$

$$Q_{43} = 21z^8 + 6z^6 - 296z^4 + 1792z^2 - 3072, \tag{D48}$$

$$Q_{44} = 67z^8 - 488z^6 + 484z^4 + 992z^2 + 1216, \tag{D49}$$

$$Q_{45} = 67z^8 - 488z^6 + 796z^4 + 208z^2 + 384, \tag{D50}$$

$$Q_{46} = 67z^8 - 377z^6 - 434z^4 + 3656z^2 + 920, \tag{D51}$$

$$Q_{47} = 139z^8 - 834z^6 + 196z^4 + 1856z^2 - 32, \tag{D52}$$

$$Q_{48} = 139z^8 - 834z^6 + 196z^4 + 2368z^2 - 1056, \tag{D53}$$

$$Q_{49} = 139z^8 - 834z^6 + 200z^4 + 1848z^2 - 32, \tag{D54}$$

$$Q_{50} = 163z^8 - 978z^6 + 692z^4 + 1008z^2 + 288, \tag{D55}$$

$$Q_{51} = 163z^8 - 978z^6 + 728z^4 + 936z^2 + 288, \tag{D56}$$

$$Q_{52} = 214z^8 - 1585z^6 + 3082z^4 + 1344z^2 - 2304, \tag{D57}$$

$$Q_{53} = 217z^8 - 1302z^6 + 1145z^4 - 150z^2 + 1776, \tag{D58}$$

$$Q_{54} = 1953z^8 - 11718z^6 + 12303z^4 - 5346z^2 + 5782, \tag{D59}$$

$$Q_{55} = 9z^{10} - 124z^8 + 986z^6 - 4976z^4 + 9568z^2 - 256, \tag{D60}$$

$$Q_{56} = 11z^{10} - 88z^8 + 230z^6 - 152z^4 - 8z^2 + 288, \tag{D61}$$

$$Q_{57} = 67z^{10} - 383z^8 - 1202z^6 + 13712z^4 - 26888z^2 + 1184, \tag{D62}$$

$$Q_{58} = 111z^{10} - 1190z^8 + 2952z^6 + 2784z^4 + 4912z^2 + 3520, \tag{D63}$$

$$Q_{59} = 297z^{10} - 2025z^8 + 2792z^6 - 754z^4 + 12112z^2 + 10240, \tag{D64}$$

$$Q_{60} = 309z^{10} - 4622z^8 + 22854z^6 - 38000z^4 - 816z^2 + 15808, \tag{D65}$$

$$Q_{61} = 717z^{10} - 4788z^8 + 10572z^6 - 4640z^4 - 13584z^2 + 18496, \tag{D66}$$

$$Q_{62} = 783z^{10} - 4860z^8 + 2132z^6 - 832z^4 + 315328z^2 - 1149440, \tag{D67}$$

$$Q_{63} = 972z^{10} - 20412z^8 + 78927z^6 + 27852z^4 - 206992z^2 + 499680, \tag{D68}$$

$$Q_{64} = 5859z^{10} - 35154z^8 + 36909z^6 - 52326z^4 + 8306z^2 + 367272, \tag{D69}$$

$$Q_{65} = 6003z^{10} - 50112z^8 + 123660z^6 + 25152z^4 - 329080z^2 + 477024, \tag{D70}$$

$$Q_{66} = -717z^{12} + 7332z^{10} - 26484z^8 + 35984z^6 + 2160z^4 - 20160z^2 - 15104, \tag{D71}$$

$$Q_{67} = -27z^{12} + 876z^8 - 4872z^6 + 61696z^4 - 180480z^2 - 65536, \tag{D72}$$

$$Q_{68} = 27z^{12} - 297z^{10} + 756z^8 + 3660z^6 - 59760z^4 + 331520z^2 - 667648, \tag{D73}$$

$$Q_{69} = 27z^{12} - 297z^{10} + 972z^8 - 724z^6 - 20808z^4 + 176384z^2 - 380416, \tag{D74}$$

$$Q_{70} = 27z^{12} + 108z^{10} - 2052z^8 + 4512z^6 - 76608z^4 + 277504z^2 + 212992, \tag{D75}$$

$$Q_{71} = 67z^{12} - 622z^{10} + 1530z^8 - 1768z^6 + 3468z^4 + 12696z^2 + 2368, \tag{D76}$$

$$Q_{72} = 67z^{12} - 622z^{10} + 1842z^8 - 2064z^6 - 1396z^4 + 3000z^2 - 2304, \tag{D77}$$

$$Q_{73} = 67z^{12} - 517z^{10} - 470z^8 + 15864z^6 - 45684z^4 + 1928z^2 + 13504, \tag{D78}$$

$$Q_{74} = 67z^{12} - 511z^{10} + 652z^8 + 888z^6 + 1388z^4 + 7912z^2 + 1216, \tag{D79}$$

$$Q_{75} = 81z^{12} - 864z^{10} + 2804z^8 - 3136z^6 - 13848z^4 + 87456z^2 - 59392, \tag{D80}$$

$$Q_{76} = 297z^{12} - 3051z^{10} + 9272z^8 - 6222z^6 - 6088z^4 + 5184z^2 + 8192, \tag{D81}$$

$$Q_{77} = 891z^{12} - 11421z^{10} + 41232z^8 + 10668z^6 - 263024z^4 + 321536z^2 - 180224, \tag{D82}$$

$$Q_{78} = 891z^{12} - 10935z^{10} + 35400z^8 + 37092z^6 - 341072z^4 + 447488z^2 + 65536, \tag{D83}$$

$$Q_{79} = 1107z^{12} - 12204z^{10} + 39716z^8 - 39504z^6 + 160736z^4 - 520704z^2 - 163840, \tag{D84}$$

$$Q_{80} = 1377z^{12} - 17658z^{10} + 56220z^8 + 64968z^6 + 129440z^4 - 3691520z^2 + 8142848, \tag{D85}$$

$$Q_{81} = 1677z^{12} - 17256z^{10} + 61380z^8 - 81376z^6 - 45864z^4 + 302688z^2 - 147968, \tag{D86}$$

$$Q_{82} = 1701z^{12} - 22194z^{10} + 76956z^8 + 60648z^6 - 452320z^4 + 62464z^2 - 507904. \tag{D87}$$

APPENDIX E: NUMERICAL VALUES OF THE NEW CONSTANTS

Here we present the 14 new constants needed to express the coefficients of the expansion around $\hat{s} = 4$ with 60 digits,

$$\tilde{\kappa}_1 = -1264.9432224278092329957750523362172006762421108620998 6111296, \tag{E1}$$

$$\tilde{\kappa}_2 = -26176.46676087246839492168201111273297550514989318646 72207674, \tag{E2}$$

$$\tilde{\kappa}_3 = -2729.299217750581120003422590692510669156974355218788 29616461, \tag{E3}$$

$$\tilde{\kappa}_4 = 55185.66704306030293623174582182803894284296377596593 05766923, \tag{E4}$$

$$\tilde{\kappa}_5 = -231417.5433206241973350291333548322775137629566421686 70916934, \tag{E5}$$

$$\tilde{\kappa}_6 = 27058.06741559393924027338507376749420361763992697107 32266681, \tag{E6}$$

$$\tilde{\kappa}_7 = 37228.13930962831923213195694841360357480287239267809 36227023, \tag{E7}$$

$$\tilde{\kappa}_8 = -13339.44689938064109552942856630030958544703021198711 83700172, \tag{E8}$$

$$\tilde{\kappa}_9 = 36376.06778256936901207780608324931235857880864258819 20483389, \tag{E9}$$

$$\tilde{\kappa}_{10} = 44168.36701540207489178045289699246400549155107288085 20969412, \tag{E10}$$

$$\tilde{\kappa}_{11} = 216837.1191056016042984235154720743505272680683085353 84925274, \tag{E11}$$

$$\tilde{\kappa}_{12} = -5730.871558438944817192640393442256646366043806054619 96009706, \tag{E12}$$

$$\tilde{\kappa}_{13} = -135665.0668062562684803898005593667922857694812717318 24723568, \tag{E13}$$

$$\tilde{\kappa}_{14} = 25026.21943170395282185918025145123899691692098021431 92666245. \tag{E14}$$

In the ancillary files, we provide these constants with 1800 digits of precision.

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