Interaction of symmetric higher-spin gauge fields

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(Received 31 July 2023; accepted 2 October 2023; published 30 October 2023)

We show that the recently proposed equations for holomorphic sector of higher-spin theory in $d = 4$, also known as chiral, can be naturally extended to describe interacting symmetric higher-spin gauge fields in any dimension. This is achieved with the aid of Vasiliev's off shell higher-spin algebra. The latter contains ideal associated to traces that has to be factored out in order to set the equations on shell. To identify the ideal in interactions we observe the global $sp(2)$ that underlies it to all orders. The $sp(2)$ field dependent generators are found in closed form and appear to be remarkably simple. The traceful higher-spin vertices are analyzed against locality and shown to be all-order space-time spin-local in the gauge sector, as well as spin-local in the Weyl sector. The vertices are found manifestly in the form of curious integrals over hypersimplices. We also extend to any d the earlier observed in $d = 4$ higher-spin shift symmetry known to be tightly related to spin-locality.

DOI: 10.1103/PhysRevD.108.086031

I. INTRODUCTION

The higher-spin (HS) interaction problem [1] dates back to the early papers [2–4]. Being notoriously complicated, it still remains only partially solved. In particular, a wealth of no go arguments [5–9] (see also [10,11] for a more recent account) seemingly preclude nonlinear HS propagation on the Minkowski space.[1] A green light was given by the seminal results of Fradkin and Vasiliev [14,15], who addressed the problem on the AdS background and showed that some of the HS cubic vertices carry inverse powers of cosmological constant. Ever since the field develops rapidly and has led to many interesting ideas and new results (see e.g., [16–36] and [1] for substantial, but still not yet a fully comprehensive bibliography). One of the central results in the field is the all-order HS generating equations first obtained in $d = 4$ in [17] and later on for symmetric bosonic fields in any d in [24].

The AdS/CFT view proposed by Klebanov and Polyakov [21] (see also [22,23] for closely related papers) that suggests the correspondence of Vasiliev's HS theory and the $O(N)$ vector model has added a potentially testable playground for the weak-weak type duality with HS fields in the bulk and free matter fields on the boundary. This line of thought has been triggered by Giombi and Yin as they directly computed the three-point correlation functions of the boundary theory using the Vasiliev equations [37] confirming the holographic expectations. The dual picture suggests that the bulk theory is supposedly healthy. There are many indications in favor of this statement including those at quantum level, see e.g., [38–40].

In case the conjecture of [21] is valid[2] it is natural reconstructing HS theory from e.g., a free boundary scalar. This way a partially gauge fixed HS cubic action was successfully found in [41] and shown to be a perfect match with the corresponding conformal field theory (CFT) independent results from the Vasiliev theory [42–44]. At quartic order, however, the reconstruction bumps into the locality problem [45,46] that basically leads to a lack of computational control over the Noether procedure and to a potentially unverifiable final result for the holographic vertex.

On the HS side the locality issue is present too, though in a different form. Namely, Vasiliev equations are formulated in a way which is invariant under field redefinitions, while the reconstruction of HS vertices that they do amounts to solution of a cohomological problem of fixing field representatives[3] in one way or another. Already in [37] the

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[1]See, however, [12,13] where the flat higher spins have been recently reconsidered from the holography perspective.

[2]An alternative option proposed in [33] is the boundary dual being a conformal HS theory rather than vector model.

[3]Interestingly, the L_∞ structures arising within the unfolded approach in HS interactions appear to be way more rigid than the usual Noether perturbations. The former survive even beyond locality limitations.

locality problem develops at the cubic order (quadratic at the level of equations). It has been partly circumvented by a smart analytic continuation in spins [37], but is still shown to cause troubles in the form of artificial infinities in [47]. This problem was understood as the cohomological one and solved at this order in [48] leading to a perfectly local vertex. At the same time this analysis has raised a concern regarding a systematic reconstruction of HS vertices from the Vasiliev equations in a way compatible with locality, whatever this means.

The locality problem has proven to be to be a highly complicated one and has rolled out the ongoing research mainly in $d = 4$ [49–60] with a number of important results[4] already obtained. These include the introduction of concepts of spin-locality and ultralocality, locality theorems of [49,55], the manifest form of local interaction vertices at the first few orders [50,54] as well as the important observation of the limiting star product in [51]. The latter establishes spin-locality of the holomorphic sector (also known as chiral) of the $4d$ theory to all orders [60], thus proving the locality conjecture of [49] in this sector. In addition, spin-locality itself seems likely to hinge upon presence of the so-called HS shift symmetry [57,60]; the physical origin of which remains to be understood. Its observation was inspired by the earlier result of [49] known as the structure lemma.

In proving locality of the holomorphic sector, the Vasiliev-type equations in $d = 4$ have been proposed [60]. Unlike the original Vasiliev equations, the former have no room for both the holomorphic and antiholomorphic sectors of HS interactions simultaneously. Nevertheless, their advantage is in that they result in all-order spin-local vertices of the holomorhic theory effortlessly.[5] More importantly, these equations can be straightforwardly generalized to describe the interaction of symmetric HS gauge fields in any dimension using the standard Vasiliev construction of the off shell HS algebra [24], as well as to the $3d$ spinorial HS systems.

The goal of this paper is to come up with a naturally spin-local HS setup at least within the orders where spin-locality is already known or expected. To this end we frame equations of [60] to (i) describe interacting bosonic HS fields in d dimensions and (ii) set the stage for further analysis of locality. Our interest in generating the equations of [60] is motivated by their somewhat unexpected feature to capture spin-local rather than nonlocal HS vertices in an unforced way. Yet, the structure that controls consistency of the proposed equations is quite puzzling on its own and extends beyond the realm of application to holomorphic sector of the $4d$ HS theory.

The first problem (i) appears to be quite tractable as it rests on the oscillator realization of HS algebra developed in [24]. This leads us to the generating equations for the off shell, equivalently, unconstrained unfolded equations of nonlinear HS fields. The wording "off shell" means that the equations actually describe no field dynamics rather provide with a set of consistency conditions similar to the Bianchi identities of general relativity.

We then study the off shell vertices. Interestingly, these turn out to be spin-local even beyond the cubic approximation. Specifically, we show that the so-called *canonical* choice of physical fields gives rise to all-order ultralocality of the gauge sector. As has been shown in [52], ultralocality is equivalent to the standard space-time spin locality. The Weyl sector of the theory, where, in particular, the scalar resides is shown to be all-order spin-local. The developed formalism brings HS interaction vertices in their manifest form to any given order. All these vertices exhibit a curious structure of integrals over n-dimensional hypersimplices with n being the order of perturbations.

Apart from locality, HS vertices feature the shift symmetry transformation properties. Shift symmetry acts most naturally in the Fourier space as a certain invariance of vertices under shifts in "momenta" and, as it has been shown recently in [57], goes hand in hand with locality being an intriguing subject for future investigation. In particular, as follows from analysis of [57,60], shift symmetry offers a class of field redefinitions that respects spin locality.

To set our system on shell a quotienting of the trace ideal of the off shell HS algebra is required. Generally, this problem can be as complicated as constructing HS interactions from scratch. Indeed, to factor out the ideal one has to know its exact deformation due to interactions. The trace ideal is known to be generated by a certain $sp(2)$ in the case of totally symmetric fields [24]. In [24] such $sp(2)$ was found to all orders. Its very existence rests on the properties of the so-called deformed oscillators that underlie the original Vasiliev equations. These generate the required $sp(2)$ via bilinears. The lack of the deformed oscillators in our case makes on shell reduction challenging. For a reason, which is not immediately obvious to us, the required $sp(2)$ is still there to all orders, while the corresponding generators appear to be remarkably simple as we find them to be linear in fields in a closed form. This is one of the most important results of the present work. Having manifested the structure of the trace ideal one can systematically analyze on shell interactions order-by-order, which we plan to carry out elsewhere.

The paper is organized as follows. In Sec. II following [24] we review the structure of HS unfolded equations i.e., the spectrum of fields, higher-spin algebra and its oscillator realization. Then, in Sec. II A we specify the recently

[4]Some of the developed approaches in this quest were successfully applied to HS theories in $d = 3$ [61].

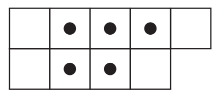
[5]In this regard it is worth mentioning papers [58,59] that propose spin-local holomorphic vertices too. The analysis relies on the homological perturbation theory that, however, renders uncertainties of the form $0 \cdot \infty$, thus compromising consistency. As the system of [60] has no issues with consistency, it can be used to offer validity checks of this proposal.

proposed in [60] HS generating equations for the case of the d-dimensional bosonic HS algebra and lay down their basic properties. Section III deals with unconstrained that is off shell analysis of HS vertices. We demonstrate the locality of the vertices, provide their manifest form as integrals over hypersimplices, as well as reveal the shift symmetry there. Section IV is all about bringing our generating equations on shell. Its central result is the manifest global $sp(2)$ that generates the trace ideal along with the explicitly given generators (4.36). We conclude in Sec. V. The paper is supplemented with three appendixes.

II. UNFOLDING OF HS INTERACTIONS

A powerful approach to HS problem is provided by the unfolded formulation [62] (see [63,64] for a recent development). The spectrum of fields suitable for description of HS symmetric fields in d dimensions consists of 1-forms and 0-forms. The 1-forms [24]

$$\omega^{a(s-1),\,b(n)} = \mathrm{d}x^{\mu}\omega_{\mu}^{a(s-1),\,b(n)} =$$



$$, \qquad 0 \le n \le s-1 \tag{2.1}$$

are given by the $o(d-1,1)$ two-row Young diagrams

$$\omega^{a(s-1),ab(n-1)} = 0, \qquad a, b = 0, \ldots d-1, \tag{2.2}$$

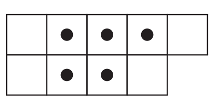
where as typical in HS literature we denote by one and the same letter symmetrization over a group of indices with Eq. (2.2) being the standard Young condition. At free level one imposes the irreducible traceless condition

$$\omega^{a(s-3)c}{}_{c,}{}^{b(n)} = 0. \tag{2.3}$$

The physical component which contains the usual Fronsdal spin-s field corresponds to $n = 0$, while $n > 0$ are auxiliary, i.e., they can be expressed in terms of derivatives of the physical field. Note that for each s there are finitely many ω's. Thus, introduced ω's play the role of HS potentials generalizing Maxwell potential $A = \omega$ for $s = 1$ and the Lorentz connection $\omega^{a,b}$ along with the frame field $\mathbf{e}^a = \omega^a$ of the Cartan gravity for $s = 2$.

The 0-forms

$$C^{a(m),\,b(s)} =$$



$$, \qquad m \ge s \tag{2.4}$$

having an arbitrary long first row are unbounded for fixed s. The $m = s$ component corresponds to the rectangular HS Weyl tensors generalizing Maxwell tensor $F^{a,b} = C^{a,b}$ for $s = 1$ and the Weyl tensor $C^{ab,cd}$ for $s = 2$. Components $m > s$ are auxiliary, being expressed roughly via derivatives of the primary fields with $m = s$. Similarly, C is subject to the Young constraint

$$C^{a(m),ab(s-1)} = 0 \tag{2.5}$$

along with the traceless condition

$$C^{a(m-2)c}{}_{c,}{}^{b(s)} = 0. \tag{2.6}$$

A convenient way to work with the two-row Young tensors was proposed in [24]. Accordingly, let us introduce the following commuting variables $\vec{\mathbf{y}}_\alpha \coloneqq \mathbf{y}^a_\alpha$ and y_α, where $\alpha = 1,\ 2$. The two-component indices can be associated with an $sp(2)$ algebra. To this end one introduces the canonical $sp(2)$ form $\epsilon_{\alpha\beta} = -\epsilon_{\beta\alpha}$ and its inverse $\epsilon^{\alpha\beta}\epsilon_{\alpha\gamma} = \delta_\gamma{}^\beta$ and defines the lowering and raising index convention, e.g.,

$$y^\alpha = \epsilon^{\alpha\beta} y_\beta, \qquad y_\alpha = y^\beta \epsilon_{\beta\alpha}, \tag{2.7}$$

then the Taylor coefficients of any analytic function $f(\vec{\mathbf{y}}, y)$ are supplemented with the following condition:

$$\left(\vec{\mathbf{y}}_\alpha \cdot \frac{\partial}{\partial \vec{\mathbf{y}}^\beta} + y_\alpha \frac{\partial}{\partial y^\beta} + \alpha \leftrightarrow \beta\right) f = 0 \tag{2.8}$$

which form the two-row Young diagrams with respect to Lorentz indices $a, b, \ldots$. In other words, they satisfy conditions like (2.2), but not the traceless constraint (2.3), which has to be imposed additionally in order to stay with $o(d-1,1)$ irreducible components. Equation (2.8) has the transparent meaning of an $sp(2)$ invariance. Indeed, by introducing the standard Moyal star product

$$(f * g)(y, \vec{\mathbf{y}}) = \int f(y+u, \vec{\mathbf{y}}+\vec{\mathbf{u}}) g(y+v, \vec{\mathbf{y}}+\vec{\mathbf{v}}) e^{iu_\alpha v^\alpha + i\vec{\mathbf{u}}_\alpha \vec{\mathbf{v}}^\alpha}, \tag{2.9}$$

where integration over u, v, $\vec{\mathbf{u}}$, and $\vec{\mathbf{v}}$ is assumed, it yields the following commutation relations

$$[\mathbf{y}^a_\alpha, \mathbf{y}^b_\beta]_* = 2i\eta^{ab}\epsilon_{\alpha\beta}, \qquad [y_\alpha, y_\beta]_* = 2i\epsilon_{\alpha\beta}, \qquad [\mathbf{y}^a_\alpha, y_\beta]_* = 0. \tag{2.10}$$

One then finds that (2.8) can be written down as

$$[t^0_{\alpha\beta}, f(y,\vec{\mathbf{y}})]_* = 0, \qquad t^0_{\alpha\beta} = \frac{1}{4i}(\mathbf{y}^a_\alpha \mathbf{y}_{a\beta} + y_\alpha y_\beta). \tag{2.11}$$

Generators $t^0_{\alpha\beta}$ form the so-called Howe dual algebra to $o(d-1,2)$ spanned by various $sp(2)$-invariant bilinears of $\vec{\mathbf{y}}$ and y

$$P^a = \mathbf{y}^a_\alpha y^\alpha, \qquad M^{ab} = \mathbf{y}^a_\alpha \mathbf{y}^{b\alpha}. \tag{2.12}$$

It is easy to observe $sp(2)$ commutation relations for the generators defined in (2.11)

$$[t^0_{\alpha\beta}, t^0_{\gamma\delta}]_* = \epsilon_{\alpha\gamma} t^0_{\beta\delta} + \epsilon_{\beta\gamma} t^0_{\alpha\delta} + \epsilon_{\alpha\delta} t^0_{\beta\gamma} + \epsilon_{\beta\delta} t^0_{\alpha\gamma}. \tag{2.13}$$

Having introduced the above formalism, the HS algebra can be constructed as follows [65]. Consider an associative algebra spanned by various formal power series in $\vec{\mathbf{y}}$ and y that has product (2.9). This algebra contains a subalgebra $\mathcal{S}$ generated by elements that fulfil $sp(2)$ singlet condition (2.11). Algebra $\mathcal{S}$ is not simple since it contains two-sided ideal I of the form

$$g = t^0_{\alpha\beta} * f^{\alpha\beta}(y,\vec{\mathbf{y}}) = f^{\alpha\beta}(y,\vec{\mathbf{y}}) * t^0_{\alpha\beta} \in I, \tag{2.14}$$

where g satisfies (2.11) which in practice means that $sp(2)$ indices of $f_{\alpha\beta}$ should be carried by y_α and $\vec{\mathbf{y}}_\alpha$. One can consider the quotient algebra $\mathcal{S}/I$ which is still an associative algebra and then build out of it a Lie algebra using the Lie bracket as a commutator. The latter algebra is the HS algebra in d dimensions.[6] Treating algebra $\mathcal{S}$ as a linear space one can strip its ideal off. This means the following decomposition for general $f \in \mathcal{S}$

$$f = \mathbf{f} + g, \quad g \in I, \tag{2.15}$$

where one has to specify any particular way of choosing the representative. For the $\mathbf{f} \in \mathcal{S}/I$ choice, its coefficients in Taylor expansion can be chosen to correspond to traceless Young diagrams [e.g., (2.3)].

Following the standard HS philosophy, consider now the 1-form ω and 0-form C taking values in the off shell HS algebra. These encode fields (2.1) and (2.4) in the Taylor-like expansion. The unfolded equations are

$$\mathrm{d}_x\omega + \omega * \omega = \Upsilon(\omega,\omega,C) + \Upsilon(\omega,\omega,C,C) + \ldots, \tag{2.16}$$

$$\mathrm{d}_x C + \omega * C - C * \pi(\omega) = \Upsilon(\omega,C,C) + \Upsilon(\omega,C,C,C) + \ldots, \tag{2.17}$$

where π is the reflection automorphism

$$\pi f(y,\vec{\mathbf{y}}) = f(-y,\vec{\mathbf{y}}) \tag{2.18}$$

and Υ's are interaction terms that describe the HS gauge-invariant off shell field identities. Abusing the terminology we call such vertices "off shell" as opposed to the on shell ones based on the factor algebra corresponding to dynamical HS interaction. To arrive at the on shell HS evolution one has to strip off the associated $sp(2)$ generated ideal. As mentioned, this problem can be really complicated due to the fact that Υ's on the right-hand side of (2.16) and (2.17) are not built out of field star products. Instead, the off shell HS algebra gets deformed which entails the deformation of the ideal. As a result the factorization cannot be carried out using the undeformed prescription (2.15). In other words, generators of the $sp(2)$ algebra defined in (2.11) become field dependent in the interactions.

A. Generating equations

To arrive at Eqs. (2.16) and (2.17) we follow [24,60]. Namely, using Vasiliev's trick we enhance the space spanned by $Y = (y, \vec{\mathbf{y}})$ by introducing the auxiliary two-component[7] z_α and embed the 1-form ω into this larger space

$$\omega \to W(z;Y) \coloneqq \omega(Y) + W_1(z;Y) + W_2(z;Y) + \ldots, \tag{2.19}$$

where W_n are yet to be defined perturbative in C corrections. We call embedding (2.19) *canonical* provided

$$W_n(0,Y) = 0, \quad \forall \;\; n \geq 1 \tag{2.20}$$

which means that the physical field representative $\omega = W(0,Y)$ is chosen to be fixed and is not subject to field redefinition in the interactions; this would not be the case if $W_k(0,Y) \neq 0$ for some integer k. To avoid any confusion at this point we want to stress here that constraint (2.20) just defines the way one can solve the generating system rather than forbids field redefinitions.[8]

We then also enhance the Moyal star product (2.9) to a large z-dependent product [60] (see also [51])

$$\begin{aligned}(f * g)(z;Y) = \int & f(z+u', y+u; \mathbf{y}) \\ & \star g(z-v, y+v+v'; \mathbf{y}) \\ & \times \exp(iu_\alpha v^\alpha + iu'_\alpha v'^\alpha),\end{aligned} \tag{2.21}$$

[6]This algebra was originally obtained in terms of the enveloping algebra in [66], where higher symmetries of the massless scalar were studied. The corresponding algebra was called conformal higher-spin algebra.

[7]Notice, however, that we do not double variables Y's unlike [24]. Our choice corresponds to a particular gauge fixing of Stückelberg fields along with the conventional choice for the compensator.

[8]Similarly, the so-called z-dominated constraints have been imposed (see e.g., [49]) to the first few orders in solving Vasiliev's generating system to obtain spin-local vertices.

where $\star$ is a part of star product (2.9) that acts on $\vec{\mathbf{y}}$ only

$$(f\star g)(\vec{\mathbf{y}}) = \int f(\vec{\mathbf{y}}+\vec{\mathbf{u}})g(\vec{\mathbf{y}}+\vec{\mathbf{v}})\exp(i\vec{\mathbf{u}}_\alpha\vec{\mathbf{v}}^\alpha). \quad (2.22)$$

Notice, the (z, y) ordering in (2.21) differs from the original Vasiliev one [24]. For z-independent functions (2.21) is identical to (2.9). Moreover, (2.21) coincides with that of Vasiliev if either f or g is z-independent. The large star product mixes z and y leaving transformation with respect to $\vec{\mathbf{y}}$ unaffected. A distinguishing property of (2.21) is

$$[z_\alpha, z_\beta]_* = 0. \quad (2.23)$$

More generally, one comes along with the following action

$$y* = y + i\frac{\partial}{\partial y} - i\frac{\partial}{\partial z}, \qquad z* = z + i\frac{\partial}{\partial y}, \quad (2.24)$$

$$*y = y - i\frac{\partial}{\partial y} - i\frac{\partial}{\partial z}, \qquad *z = z + i\frac{\partial}{\partial y}, \quad (2.25)$$

$$\vec{\mathbf{y}}* = \vec{\mathbf{y}} + i\frac{\partial}{\partial\vec{\mathbf{y}}}, \qquad *\vec{\mathbf{y}} = \vec{\mathbf{y}} - i\frac{\partial}{\partial\vec{\mathbf{y}}}. \quad (2.26)$$

The important element of the star product (2.21) is a "δ-function"

$$\delta \coloneqq e^{iz_\alpha y^\alpha}, \qquad f(z)*\delta = \delta * f(z) = \delta\cdot f(0). \quad (2.27)$$

Its square $\delta*\delta = 0\cdot\infty$ is ill-defined that along with (2.27) emphasizes an analogy with δ-function.[9]

Generating equations that reconstruct the right-hand sides of (2.16) and (2.17) order-by-order has the form [60]

$$\mathrm{d}_x W + W*W = 0, \quad (2.28)$$

$$\mathrm{d}_z W + \{W,\Lambda\}_* + \mathrm{d}_x\Lambda = 0, \quad (2.29)$$

$$\mathrm{d}_x C + (W(z';y,\vec{\mathbf{y}})*C - C*W(z';-y,\vec{\mathbf{y}}))\Big|_{z'=-y} = 0, \quad (2.30)$$

where $C = C(y,\vec{\mathbf{y}}|x)$ is z-independent and Λ is the following $\mathrm{d}z$-form

$$\Lambda = \mathrm{d}z^\alpha z_\alpha\int_0^1 \mathrm{d}\tau\,\tau C(-\tau z,\vec{\mathbf{y}})e^{i\tau z_\alpha y^\alpha}, \quad (2.31)$$

which satisfies

$$\mathrm{d}_z\Lambda = C*\gamma, \qquad \gamma = \frac{1}{2}e^{izy}\mathrm{d}z^\beta\mathrm{d}z_\beta \qquad \mathrm{d}_z \coloneqq \mathrm{d}z^\alpha\frac{\partial}{\partial z^\alpha}, \quad (2.32)$$

where we use the following shorthand notation for index contraction

$$\xi_\alpha\eta^\alpha \coloneqq \xi\eta = -\eta\xi. \quad (2.33)$$

Note that Eq. (2.30), unlike (2.28) and (2.29), contains no z and, therefore the star product acts in Eq. (2.9) by ignoring dummy argument z' of W which is set to $z' = -y$ at the end.

1. Consistency

Unlike the original Vasiliev equations, consistency of system (2.28)–(2.30) is far from being obvious. It rests on the properties of functional class that admits evolution on (2.28)–(2.30) and a particular *projective identity* (2.36). All necessary details are laid in [60]. Here we briefly sketch the basic steps:

(i) First, apart from formal consistency it is important to examine whether or not star products of various master fields in (2.28)–(2.30) exist at all. For example, $\Lambda*\Lambda$ can be checked to be ill-defined. To answer that question one searches for a class of functions that respects operations on the generating equations. It turns out that if system (2.28)–(2.30) has perturbative solutions, then they belong to dz-graded class $\mathbf{C}^r$, where r is the rank of dz-differential form (see Appendix A),

$$\mathbf{C}^{r_1}*\mathbf{C}^{r_2}\to\mathbf{C}^{r_1+r_2}, \qquad r_1+r_2<2, \quad (2.34)$$

$$\mathrm{d}_z\mathbf{C}^r\to\mathbf{C}^{r+1}, \qquad \mathrm{d}_x\mathbf{C}^r\to\mathbf{C}^r. \quad (2.35)$$

Correspondingly, $W\in\mathbf{C}^0$, $\Lambda\in\mathbf{C}^1$, and $C\in\mathbf{C}^0$. In [60] it was proven that functions (A3) indeed respect requirements (2.34) and (2.35) and result in well-defined star products provided $r_1+r_2\neq 2$. The following products $\mathbf{C}^0*\mathbf{C}^2$, $\mathbf{C}^2*\mathbf{C}^0$, and $\mathbf{C}^1*\mathbf{C}^1$ are sick unless $\mathbf{C}^0$ is z-independent. Luckily, there are no ill-defined products in (2.28)–(2.30). It was also shown that arbitrary z-independent functions of grading $r=0$ do belong to $\mathbf{C}^0$, while $\delta\in\mathbf{C}^2$ (2.27).

(ii) Now that we have identified proper functions (A3), one proves the following projecting identity [60]. Suppose $\phi(z,y;\vec{\mathbf{y}})\in\mathbf{C}^0$ is a dx 0-form,[10] then

$$\mathrm{d}_z(\phi*\Lambda) = (\phi(z',y;\vec{\mathbf{y}})*C)\Big|_{z'=-y}*\gamma,$$
$$\mathrm{d}_z(\Lambda*\phi) = (C*\phi(z',-y;\vec{\mathbf{y}}))\Big|_{z'=-y}*\gamma. \quad (2.36)$$

[9] Let us stress that the square of $\varkappa = \exp izy$ is perfectly well-defined in the Vasiliev ordering, $\varkappa*\varkappa = 1$, [24].

[10] For space-time differential forms of higher ranks one should bear in mind $\{\mathrm{d}x, \mathrm{d}z_\alpha\} = 0$ that may result in overall signs in (2.36).

In proving (2.36) one needs (A3) and (2.31). Identity (2.36) was coined *projective* in [60], since it somehow projected out the dependence on z of ϕ into δ-element (2.27) stored in γ. We note also, that while it is tempting to use the Leibniz rule and Eq. (2.32) in proving (2.36), this would lead to uncertainty because both terms in decomposition $\mathrm{d}_z(\mathbf{C}^0 * \mathbf{C}^1) = \mathrm{d}_z\mathbf{C}^0 * \mathbf{C}^1 + \mathbf{C}^0 * \mathrm{d}_z\mathbf{C}^1$ do not exist separately.[11]

(iii) Armed with the class (2.34) and (2.35) and identity (2.36) all is set to guarantee consistency of (2.28)–(2.30). Equation (2.28) is clearly consistent under $\mathrm{d}_x^2 = 0$. Hitting (2.28) with d_x and substituting $\mathrm{d}_x W$ from (2.28) back one faces no ill-defined objects either. Now, applying d_z to (2.28) and using (2.29) we again find neither new constraints nor ill-defined expressions. Less trivial is to check consistency of (2.29) and (2.30). Applying d_z to (2.29) gives the following constraint

$$\mathrm{d}_z\{W, \Lambda\}_* = \mathrm{d}_x C * \gamma. \tag{2.37}$$

The problem here is that one can not use Leibniz rule on the left of (2.37) because product of $\mathbf{C}^{r_1} * \mathbf{C}^{r_2}$ with $r_1 + r_2 = 2$ is ill-defined as noted earlier. Another point is since $\mathrm{d}_x C$ is z-independent by definition, the z-dependence of the left-hand side of (2.37) must be of the very specific form $F(Y) * \gamma$. At this stage identity (2.36) helps. Using it one has

$$\left(C * W(z'; -y, \vec{\mathbf{y}}) - W(z'; y, \vec{\mathbf{y}}) * C\right)\Big|_{z'=-y} * \gamma = \mathrm{d}_x C * \gamma, \tag{2.38}$$

which holds in view of (2.30). Finally, one has to check that (2.30) is consistent with $\mathrm{d}_x^2 = 0$. This is indeed so, because (2.30) comes out as a consequence of (2.29) in the form of (2.37), which is equivalent to (2.30). The former is manifestly consistent as can be seen upon applying d_x to it and using again (2.28) and (2.29). Let us also note in this regard that (2.30), therefore, is not an independent equation.

2. Gauge symmetries

Equations (2.28)–(2.30) are invariant under the following local gauge transformations parametrized by $\epsilon \in \mathbf{C}^0$

$$\delta_\epsilon W = \mathrm{d}_x \epsilon + [W, \epsilon]_*, \tag{2.39}$$

$$\delta_\epsilon \Lambda = \mathrm{d}_z \epsilon + [\Lambda, \epsilon]_*, \tag{2.40}$$

$$\delta_\epsilon C = (\epsilon(z', y) * C - C * \epsilon(z', -y))_{z'=-y}. \tag{2.41}$$

The above transformations leave (2.28)–(2.30) and (2.32) invariant, but not yet (2.31). In order to keep (2.31) intact one has to require

$$\delta_\epsilon \Lambda(C) = \Lambda(\delta_\epsilon C), \tag{2.42}$$

which is equivalent to

$$\mathrm{d}_z \epsilon + [\Lambda(C), \epsilon]_* - \Lambda(\delta_\epsilon C) = 0, \tag{2.43}$$

where the last term in (2.43) is understood as (2.31) with C being replaced by (2.41). Equation (2.43) is consistent as can be seen upon acting with d_z and using (2.36) and, therefore, can be solved order-by-order using the standard contracting homotopy. Functions $\epsilon \in \mathbf{C}^0$ that satisfy (2.43) determine true local gauge symmetries of HS generating Eqs. (2.28)–(2.30). The corresponding solution space we label by $\mathbf{C}_c^0$, where subscript C stands for a z-independent element that Λ depends on. It is easy to see that these functions form the Lie algebra. Indeed, on one hand from (2.40) one finds

$$(\delta_\epsilon \delta_\eta - \delta_\eta \delta_\epsilon)\Lambda = \mathrm{d}_z[\epsilon, \eta]_* + [\Lambda, [\epsilon, \eta]_*]_*. \tag{2.44}$$

On the other hand, from (2.42) it follows $\delta_\eta \delta_\epsilon \Lambda = \Lambda(\delta_\eta \delta_\epsilon C)$ and so

$$(\delta_\epsilon \delta_\eta - \delta_\eta \delta_\epsilon)\Lambda = \Lambda((\delta_\epsilon \delta_\eta - \delta_\eta \delta_\epsilon)C), \tag{2.45}$$

where we have also taken into account that Λ is linear in C

$$\Lambda(C_1 + C_2) = \Lambda(C_1) + \Lambda(C_2). \tag{2.46}$$

Those parameters $\epsilon^{gl} \in \mathbf{C}_c^0$ that leave fields W, Λ, and C invariant generate global symmetries

$$\delta W = \delta \Lambda = \delta C = 0. \tag{2.47}$$

In this case the corresponding global symmetry parameter satisfies

$$\mathrm{d}_x \epsilon^{gl} + [W, \epsilon^{gl}]_* = 0, \tag{2.48}$$

$$\mathrm{d}_z \epsilon^{gl} + [\Lambda, \epsilon^{gl}]_* = 0. \tag{2.49}$$

Note that $\delta_{\epsilon^{gl}} C = 0$ comes about as the integrability consequence of (2.49). It also entails validity of (2.43).

3. Projectively twisted-adjoint module

The role of projective identity (2.36) is crucial for consistency (2.28)–(2.30). One of its remarkable properties is it generates a kind of a twisted-adjoint representation

[11]It is worth to mention that the failure of Leibniz rule offers some freedom in definition of d_z in sector of $\mathrm{d}z$ one-forms in a way that may differ from (2.32). Unfortunate choice might lead to inconsistent vertices, however the one used in this paper and in [60] is consistent. More details are in [67].

of the large algebra that acts on its module spanned by z-independent functions from the off shell HS algebra. Since the module is z-independent, we call it *projective*. Specifically, for a z-independent element $C(y, \vec{\mathbf{y}})$ and any $\epsilon \in \mathbf{C}^0_c$ let us define the following action

$$\rho_\epsilon(C) \coloneqq (\epsilon(z', y) * C - C * \epsilon(z', -y))_{z'=-y}. \tag{2.50}$$

Recall, that star-product in (2.50) is insensitive to z', which stands idle there and then is set to $-y$. One can show that ρ forms a representation of the large Lie algebra, as for two such $\epsilon_{1,2}$ we have

$$(\rho_{\epsilon_1}\rho_{\epsilon_2} - \rho_{\epsilon_2}\rho_{\epsilon_1})C = \rho_{[\epsilon_1,\epsilon_2]_*}C. \tag{2.51}$$

The proof of this fact follows from (2.44). A brief comment in regard of (2.51) is in order. We can not make a stronger statement than (2.51). Looking at the large algebra as at the associative one, we could have expected the following map

$$\tau_\epsilon(C) \coloneqq (\epsilon(z', y) * C)_{z'=-y} \tag{2.52}$$

to be its representation. However, this is not the case as it is not difficult to show that

$$\tau_{\epsilon_1}\tau_{\epsilon_2} \neq \tau_{\epsilon_1 * \epsilon_2} \tag{2.53}$$

for z-dependent $\epsilon_{1,2} \in \mathbf{C}^0$.

Notice, (2.51) guarantees that local gauge symmetries act properly on the Weyl module C to form an algebra, i.e., $\delta_{[\epsilon_1}\delta_{\epsilon_2]} = \delta_{[\epsilon_1,\epsilon_2]_*}$. It is important also to stress that parameter $\epsilon(z, Y|x)$ is *not* an arbitrary analytic function of z and Y but belongs to a much smaller subset of analytic functions from $\mathbf{C}^0$. Still, the latter class is big enough to capture HS dynamics as it contains all analytic z-independent functions.

III. OFF SHELL: VERTICES, LOCALITY AND SHIFT SYMMETRY

Equations (2.28)–(2.30) generate unconstrained, which is off shell, HS vertices of (2.16) and (2.17). The procedure is pretty standard for the Vasiliev-like systems. In our case it boils down to a resolution for z-dependence of field W using (2.29) and substitution of the result into (2.28) and (2.30). Equation (2.30) being manifestly z-independent reproduces (2.17). While z-independence of (2.28) that brings (2.16) is not manifest, it is guaranteed by the fact that application of d_z to the left-hand side of (2.28) is zero provided (2.29) imposed. This makes the analysis particularly simple as substitution of any W that satisfies (2.29) into (2.28) can be carried out at an arbitrary convenient value of z, e.g., at $z = 0$. Below we provide all necessary details. The important conclusion here is the off shell vertices appear to be properly local at any interaction order. Namely, we will show that all $\Upsilon(\omega, \omega, C\ldots C)$ are space-time spin-local, while all $\Upsilon(\omega, C\ldots C)$ are spin-local.

Another observation, which is tightly related to locality is the shift symmetry of the obtained vertices. It has been already observed in [60] as a symmetry of the holomorphic on shell HS vertices in four dimensions. Its effect has been analyzed on general grounds within the Vasiliev theory in $d = 4$, [57] with a conclusion that if present, it leads to HS spin-locality under some mild extra assumptions.

A. Contracting homotopy

First order in z partial differential equation (2.29) determines z-dependence of the perturbative corrections $W_n(z; Y)$ from (2.19). It brings a freedom in z-independent functions at each perturbation order. This freedom is naturally interpreted as field redefinition of the physical field $\omega(Y)$ by higher order in C terms

$$\omega \to \omega + F(\omega, C\ldots C). \tag{3.1}$$

Field ω makes its appearance at zeroth order. For reasons that will become significant upon on shell reduction of HS equations, we stick to the canonical form (2.20) of perturbative corrections W_n implying that we set

$$F(\omega, C\ldots C) = 0. \tag{3.2}$$

In other words, we fix field ω at the very beginning assuming no further field redefinitions. This choice lives up to the standard contracting homotopy as will be clear soon.

Consider the following equation for dz-zero form $f(z)$

$$\mathrm{d}_z f(z) = g(z) \coloneqq \mathrm{d}z^\alpha g_\alpha(z), \tag{3.3}$$

which is consistent provided $\mathrm{d}_z g(z) = 0$. In this case it can be solved up to a constant as

$$f(z) = \Delta g(z) \coloneqq z^\alpha \int_0^1 \mathrm{d}\tau\, g_\alpha(\tau z). \tag{3.4}$$

Operator Δ is referred to as the contracting homotopy operator. Note, that no part of $g_\alpha(z)$ of the form $z_\alpha \phi(z)$ contributes to the solution for f, since $\Delta(\mathrm{d}z^\alpha z_\alpha \phi(z)) \equiv 0$ as $z^\alpha z_\alpha = 0$. Now, substituting decomposition (2.19) into (2.29) one finds

$$\mathrm{d}_z W_n = -\{W_{n-1}, \Lambda\}_* - (\mathrm{d}_x \Lambda)_n, \qquad W_0 \coloneqq \omega(Y|x). \tag{3.5}$$

The solution satisfying (2.20) can be then written down as

$$W_n = -\Delta\{W_{n-1}, \Lambda\}_*, \tag{3.6}$$

where $(\mathrm{d}_x \Lambda)_n$, being proportional to z_α, (2.31) vanishes upon application of Δ. Starting from physical ω, Eq. (3.6) generates any order $O(C^n)$ corrections $W_n(z;Y)$ such, that $W_n(0,Y)=0$ in accordance with the canonical embedding (2.20). The corresponding HS vertices from (2.16) and (2.17) acquire the following form:

$$\Upsilon(\omega,\omega,\underbrace{C\ldots C}_{n}) = -\left(\sum_{j+k=n} W_j * W_k\right)\Big|_{z=0}, \tag{3.7}$$

$$\Upsilon(\omega,\underbrace{C\ldots C}_{n}) = -(W_{n-1}(z';y,\vec{\mathbf{y}}) * C - C * W_{n-1}(z';-y,\vec{\mathbf{y}}))\Big|_{z'=-y}, \tag{3.8}$$

where in (3.7) we set $z=0$ for convenience as the vertex is z-independent anyway. Since $W_n(0;Y)=0$ for $n>0$, this choice leads to no contribution from $\mathrm{d}_x W_n$.

B. Vertices

Star product (2.21) is well-suited for exponentials. It is then convenient to use the Taylor representation with respect to variable y of our fields

$$C(y,\vec{\mathbf{y}}) \equiv e^{-i\mathrm{p}^\alpha y_\alpha} C(y',\vec{\mathbf{y}})|_{y'=0}, \qquad \mathrm{p}_\alpha = -i\frac{\partial}{\partial y'^\alpha}, \tag{3.9}$$

$$\omega(y,\vec{\mathbf{y}}) \equiv e^{-i\mathrm{t}^\alpha y_\alpha} \omega(y',\vec{\mathbf{y}})|_{y'=0}, \qquad \mathrm{t}_\alpha = -i\frac{\partial}{\partial y'^\alpha}, \tag{3.10}$$

where p acts on C, while t on ω. As the HS vertices, Υ, from (2.16) and (2.17) are strings of several C's the number of which depends on the order of perturbations, we endow p with index p_k which points at which C it acts upon as seen from left. Similarly, in (2.16) where two ω's present, we distinguish between t_1 and t_2. The only star product that one faces in extracting vertices is the one of the Gaussian exponentials. Notice also, that $\star$-product with respect to $\vec{\mathbf{y}}$'s stays undeformed in interaction that makes it dummy in practical calculation. Therefore, the vertices can be conveniently written down in the generating form using functions $\Phi^{[\delta_1,\delta_2]}(y;\mathrm{t}_{1,2},\mathrm{p}_1,\ldots,\mathrm{p}_n)$ that reproduce various order n vertex contributions via

$$\Upsilon = \sum_{\delta_2>\delta_1} \Phi^{[\delta_1,\delta_2]}(y;\mathrm{t}_1,\mathrm{t}_2,\mathrm{p}_i) \times (C\star\ldots\star\omega\star\ldots\star\omega\star\ldots\star C)(y'_I,\vec{\mathbf{y}})|_{y'_I=0}, \tag{3.11}$$

where $\delta_{1,2}$ labels positions of two ω's in the string of (3.11). Vertices from (2.17) look similar

$$\Upsilon = \sum_\delta \Phi^{[\delta]}(y;\mathrm{t},\mathrm{p}_i)(C\star\ldots\star\omega\star\ldots\star C)(y'_I,\vec{\mathbf{y}})|_{y'_I=0}. \tag{3.12}$$

So, the HS vertices are unambiguously determined via functions Φ, which we will call the off shell or unconstrained vertices abusing the terminology. We will also ignore the part on which operators t and p act upon, that is we adopt the following shorthand notation

$$\omega \to e^{-i\mathrm{t}^\alpha y_\alpha}, \tag{3.13}$$

$$C \to e^{-i\mathrm{p}^\alpha y_\alpha}, \tag{3.14}$$

$$\Lambda \to \mathrm{d}z^\alpha \int_0^1 \mathrm{d}\tau\, \tau z_\alpha e^{i\tau z_\beta (y+\mathrm{p})^\beta}. \tag{3.15}$$

Using (3.6) it is not difficult to arrive at manifested expressions for Φ at any order. However, since these vertices are mere the generalized Bianchi constraints, one has to strip off the traceful ideal properly to set them on shell. This problem will be addressed elsewhere, while here we provide with examples of all order off shell vertices, as well as spell out some properties that in fact do not rely on their explicit form.

Using that $W_0=\omega(y,\vec{\mathbf{y}})$ at the lowest order one finds from (3.6), (3.13), and (3.15)

$$W_1 = W^{(1)}_{\omega C} + W^{(1)}_{C\omega}, \qquad W^{(1)}_{\omega C} = -\Delta(\omega * \Lambda),$$
$$W^{(1)}_{C\omega} = -\Delta(\Lambda * \omega), \tag{3.16}$$

$$W^{(1)}_{\omega C} = \mathrm{t}^\alpha z_\alpha \int \mathrm{d}^3_\triangle \tau e^{i\tau_1 z_\alpha (y+\mathrm{p}+\mathrm{t})^\alpha + i\tau_2 \mathrm{t}_\alpha y^\alpha + i(1-\tau_2)\mathrm{p}_\alpha t^\alpha}, \tag{3.17}$$

$$W^{(1)}_{C\omega} = -\mathrm{t}^\alpha z_\alpha \int \mathrm{d}^3_\triangle \tau e^{i\tau_1 z_\alpha (y+\mathrm{p}-\mathrm{t})^\alpha + i\tau_2 \mathrm{t}_\alpha y^\alpha + i(1-\tau_2)\mathrm{p}_\alpha \mathrm{t}^\alpha}, \tag{3.18}$$

where we have used the following shorthand notation for τ-integrals

$$\int \mathrm{d}^3_\triangle \tau \coloneqq \int \mathrm{d}\tau_1 \mathrm{d}\tau_2 \mathrm{d}\tau_3 \theta(\tau_1)\theta(\tau_2)\theta(\tau_3)\delta(1-\tau_1-\tau_2-\tau_3). \tag{3.19}$$

This simplex integration arises upon suitable integration variable change along the lines of [48,50]. The vertex is then obtained via (3.7)

$$\Upsilon(\omega,\omega,C) = -(\omega * W_1 + W_1 * \omega)_{z=0} \tag{3.20}$$

with the final result readily accessible

$$\Phi^{[1,2]} = \mathrm{t}_1\mathrm{t}_2 \int \mathrm{d}^3_\triangle \tau e^{i(1-\tau_1)\mathrm{t}_{1\alpha}y^\alpha + i\tau_2 \mathrm{t}_{2\alpha}y^\alpha - i(\tau_1 t_1 + (1-\tau_2)\mathrm{t}_2)_\alpha \mathrm{p}^\alpha + i(1-\tau_3)\mathrm{t}_{2\alpha}\mathrm{t}_1^\alpha}, \tag{3.21}$$

$$\Phi^{[1,3]} = \mathrm{t}_2\mathrm{t}_1 \int \mathrm{d}^3_\triangle \tau e^{i(1-\tau_1)\mathrm{t}_{2\alpha}y^\alpha + i\tau_2 \mathrm{t}_{1\alpha}y^\alpha - i(\tau_1 \mathrm{t}_2 + (1-\tau_2)\mathrm{t}_1)_\alpha \mathrm{p}^\alpha + i(\tau_2-\tau_1)\mathrm{t}_{2\alpha}\mathrm{t}_1^\alpha} + \mathrm{t}_2\mathrm{t}_1 \int \mathrm{d}^3_\triangle \tau e^{i(1-\tau_1)\mathrm{t}_{1\alpha}y^\alpha + i\tau_2 \mathrm{t}_{2\alpha}y^\alpha - i(\tau_1 \mathrm{t}_1 + (1-\tau_2)\mathrm{t}_2)_\alpha \mathrm{p}^\alpha + i(\tau_2-\tau_1)\mathrm{t}_{2\alpha}\mathrm{t}_1^\alpha}, \tag{3.22}$$

$$\Phi^{[2,3]} = \mathrm{t}_1\mathrm{t}_2 \int \mathrm{d}^3_\triangle \tau e^{i(1-\tau_1)\mathrm{t}_{2\alpha}y^\alpha + i\tau_2 \mathrm{t}_{1\alpha}y^\alpha - i(\tau_1 \mathrm{t}_2 + (1-\tau_2)\mathrm{t}_1)_\alpha \mathrm{p}^\alpha + i(1-\tau_3)\mathrm{t}_{2\alpha}\mathrm{t}_1^\alpha}, \tag{3.23}$$

where we also use the notation (4.21). Vertices $\Upsilon(\omega, C, C)$ can be found from (3.8) as easily

$$\Upsilon(\omega, C, C) = -(W_1(z'; y, \vec{\mathbf{y}}) * C - C * W_1(z'; -y, \vec{\mathbf{y}}))\Big|_{z'=-y} \tag{3.24}$$

with the final result being

$$\Phi^{[1]} = -\mathrm{t}y \int \mathrm{d}^3_\triangle \tau e^{i(\tau_2 \mathrm{p}_2 + (1-\tau_2)\mathrm{p}_1)_\alpha \mathrm{t}^\alpha + i((1-\tau_1)\mathrm{p}_2 + \tau_1 \mathrm{p}_1 + (1-\tau_3)\mathrm{t})_\alpha y^\alpha}, \tag{3.25}$$

$$\Phi^{[2]} = \mathrm{t}y \int \mathrm{d}^3_\triangle \tau e^{i(\tau_2 \mathrm{p}_2 + (1-\tau_2)\mathrm{p}_1)_\alpha \mathrm{t}^\alpha + i((1-\tau_1)\mathrm{p}_2 + \tau_1 \mathrm{p}_1 - (\tau_1-\tau_2)\mathrm{t})_\alpha y^\alpha} + \mathrm{t}y \int \mathrm{d}^3_\triangle \tau e^{i(\tau_2 \mathrm{p}_1 + (1-\tau_2)\mathrm{p}_2)_\alpha \mathrm{t}^\alpha + i((1-\tau_1)\mathrm{p}_1 + \tau_1 \mathrm{p}_2 + (\tau_1-\tau_2)\mathrm{t})_\alpha y^\alpha}, \tag{3.26}$$

$$\Phi^{[3]} = -\mathrm{t}y \int \mathrm{d}^3_\triangle \tau e^{i(\tau_2 \mathrm{p}_1 + (1-\tau_2)\mathrm{p}_2)_\alpha \mathrm{t}^\alpha + i((1-\tau_1)\mathrm{p}_1 + \tau_1 \mathrm{p}_2 - (1-\tau_3)\mathrm{t})_\alpha y^\alpha}. \tag{3.27}$$

These simple examples above illustrate a few general phenomena typical of all orders. Namely: (i) the exponential part of the vertices never has contractions $\mathrm{p}_i \cdot \mathrm{p}_j$. Moreover, there is no a single contraction $y \cdot \mathrm{p}_i$ within $\Phi^{[\delta_1, \delta_2]}$; and (ii) integration over τ's goes along a simplex. At higher orders the integration domain includes a peculiar structure of two hypersimplices. For example, it is not difficult to extract the following any order vertex explicitly

$$\Phi^{[1]}_{n+1} = -(W^{(n)}_{\omega C \ldots C}(z'; Y) * C)_{z'=-y} = -W^{(n)}_{\omega C \ldots C}(-y; y - \mathrm{p}_{n+1}, \vec{\mathbf{y}}) e^{-iy\mathrm{p}_{n+1}}. \tag{3.28}$$

To do so we need W to the nth order. It can be conveniently found using that $W \in \mathbf{C}^0$ that makes representation (A4) particularly useful. The concise form follows from the iterative equation

$$W^{(n)} = -\Delta(W^{(n-1)} * \Lambda), \tag{3.29}$$

which gives us the final result (see Appendix B)

$$W^{(n)} = (z\mathrm{t})^n \int_0^1 \mathrm{d}\tau\, \tau^{n-1}(1-\tau) \int_{\mathcal{D}} e^{-i(1-\tau)r_n y\mathrm{t} + i\tau z(y+B_n) - i\tau r_n B_n \mathrm{t} + ic_n}, \tag{3.30}$$

where

$$B_n = \sum_{i=1}^n \lambda_i \mathrm{p}_i + \left(1 + \sum_{i<j}^n (\lambda_i \nu_j - \nu_i \lambda_j)\right)\mathrm{t}, \tag{3.31}$$

$$c_n = \sum_{i=1}^n \nu_i \mathrm{p}_i \mathrm{t}, \tag{3.32}$$

$$r_n = 1 - \sum_{i=1}^n \nu_i, \tag{3.33}$$

with the integration domain $\mathcal{D}$ being the product of two hypersimplices

$$\mathcal{D} = \triangle_n \times \triangle_n^*, \tag{3.34}$$

where

$$\triangle_n = \left\{0 \le \lambda_i \le 1; \sum_{i=1}^n \lambda_i = 1\right\}, \quad \triangle_n^* = \left\{0 \le \nu_i \le 1; \sum_{i=1}^n \nu_i \le 1\right\}. \tag{3.35}$$

Note that the integration over λ_i goes along faces of hypersimplex $\triangle_n$, while the integration over ν_i goes across its volume. Note also for $n = 1$ the integration over λ_1 trivializes in a sense that $\lambda_1 \equiv 1$ as follows from (3.35). The appearance of integrals over hypersimplices might not be too surprising. A suggested analogy is the structure of cubic coupling constants for helicities h_i obtained in [8]

$$C_{h_1,h_2,h_3} = \frac{1}{\Gamma(h_1+h_2+h_3)} \tag{3.36}$$

while the typical simplex integral features the gamma function in the denominator as well,

$$\int_{\sum_i \nu_i \le 1} \nu_1^{s_1-1} \dots \nu_n^{s_n-1} = \frac{\Gamma(s_1)\dots\Gamma(s_n)}{\Gamma(1+s_1+\dots+s_n)}. \tag{3.37}$$

Another comment is since the result (3.30) belongs to class $\mathbf{C}^0$, (A4), it can be conveniently rewritten in the factorized form with respect to operation (A10)

$$W^{(n)} = \int_0^1 \mathrm{d}\tau\, \tau^{n-1}(1-\tau) \int_{\mathcal{D}} e^{-ir_n y \mathrm{t} + ic_n} \circledast (\mathrm{t}z)^n e^{i\tau z(y+B_n)}. \tag{3.38}$$

Vertex Φ_{n+1} is reproduced via (3.28). It carries the structure of the hypersimplex integration domain just found. Simplicity of the final result is noteworthy. All vertices including those from (3.11) have similar form based on the unique configuration space (3.34). For example, from (3.7) one obtains

$$\Phi_n^{[1,2]} = -W^{(n)}(-\mathrm{t}_1, y+\mathrm{t}_1, \vec{\mathbf{y}}) e^{-iy\mathrm{t}_1} \tag{3.39}$$

with $W^{(n)}(z, y, \vec{\mathbf{y}}; \mathrm{t}_2, \mathrm{p}_i)$ from (3.30). It would be interesting to elaborate more on the geometric properties of the obtained integrals.

The resulting vertices feature a particular simple transformation under shifts $\mathrm{p}_i \to \mathrm{p}_i + a$, where a is an arbitrary spinor. Specifically, one observes using (3.33) and (3.35) that

$$B_n(\mathrm{p}_i + a, \mathrm{t}) = B_n(\mathrm{p}_i, \mathrm{t}) + a, \tag{3.40}$$

$$c_n(\mathrm{p}_i + a, \mathrm{t}) = c_n(\mathrm{p}_i + a, \mathrm{t}) + (1 - r_n) a\mathrm{t}. \tag{3.41}$$

These relations underlie the so-called shift symmetry of the interaction vertices, which we discuss in the next section. Technically, most of the properties discussed above are literally coincide with the analysis from [60], where reader can find more details.

C. Locality

1. Spin-locality

The problem of locality of the HS interactions in d dimensions boils down to the analysis of the number of Lorentz index contractions[12] of various C's within vertices $\Upsilon(\omega, C\dots C)$ and $\Upsilon(\omega, \omega, C\dots C)$. This number can be arbitrarily large even for fixed spins. Indeed, as the first row in (2.4) is unbounded, there can be infinitely many contractions of the form, for example

$$\sum_m g(m, s_1, s_2) C^{a(m)\dots, b(s_1)} C_{a(m)\dots,}{}^{b(s_2)}. \tag{3.42}$$

Whenever this happens the vertex is said to be spin nonlocal. An infinite length of the first row of (2.4) is supported by infinitely many $y^\alpha \vec{\mathbf{y}}_\alpha$ contracted with it. Therefore, nonlocality can be rephrased using the language of Φ from (3.11) and (3.12) as a nonpolynomial dependence on various contractions $\mathrm{p}_i \cdot \mathrm{p}_j$. So, the vertex is said to be spin-local if Φ contains no nonpolynomial contractions $\mathrm{p} \cdot \mathrm{p}$ [49,52]. Let us note, that dependence on $\mathrm{t} \cdot \mathrm{t}$ and $\mathrm{t} \cdot \mathrm{p}$ is irrelevant for spin-locality. This is due to the fact that 1-forms ω belong to a finite-dimensional module of the HS algebra for fixed spin (2.1) as opposed to C, which is infinite dimensional (2.4).

2. Spin ultralocality

Another important concept introduced in [50] is spin ultralocality. Suppose Φ from (3.11) depends on contractions $y \cdot \mathrm{p}_i$. If such a dependence is at most polynomial, then the corresponding vertex is called spin ultralocal. The meaning of this concept is roughly the following. Imagine (3.11) is spin-local but not ultralocal. This implies that the number of various contractions is finite for fixed spins but grows with spin sufficiently fast. As explained in [60], in order for vertices (2.17) to be spin-local (3.11) should be spin-ultralocal, which means that $\Phi^{[\delta_1,\delta_2]}$ must contain no nonpolynomial $y \cdot \mathrm{p}$ contractions. These, if present, imply that while vertex is still spin local, the depth of contractions of first rows between various C's grows with spins. This ruins spin-locality of (3.12) at the next order in perturbation via the integrability condition [60]. In addition, it was shown in [52] in $d = 4$ that if a vertex is spin-ultralocal, then it is space-time spin-local in the usual sense.

To summarize, (non)locality of the vertex constrained to a fixed set of spins s_i depends on function Φ. To be spin-nonlocal, Φ must contain at least one nonpolynomial contractions $\mathrm{p}_k \cdot \mathrm{p}_l$. This is equivalent to having infinitely many contractions with respect to first rows of two C's. Notice, that the presence of star product $\star$ in (3.11) and (3.12) that acts on variable $\vec{\mathbf{y}}$ yields no such contractions and therefore can not affect locality. The important property of Eqs. (2.28)–(2.30) is that the off shell vertices they generate in (2.16) and (2.17) are naturally spin-ultralocal and spin-local, respectively. Therefore, the problem of the HS locality on shell resolves into the algebraic one of a proper factorization of the trace ideal. We do not pursue on shell (non)locality here leaving this analysis for the future.

Spin-locality of traceful Υ's follows literally from the analysis of [60]. The only difference is one has to replace

[12] Recall, the rectangular Young diagram $C^{a(s),b(s)}$ corresponds to a spin s Weyl tensor, which is a physical field, while $C^{a(s+n),b(s)}$ is its descendant made of the derivatives for $n > 0$. The number of these derivatives grows with n.

$\bar{*}$ with $\star$ leaving the rest as is. For details we refer the reader to Sec. V in Ref. [60]. Here we state the final result. Namely, $\Phi^{[\delta_1,\delta_2]}$ contains neither $\mathrm{p}\cdot y$, nor $\mathrm{p}\cdot\mathrm{p}$ contractions which implies that Υ's from (2.16) are ultralocal and therefore, according to [52], are space-time spin-local at any order. Analogously, it follows that $\Phi^{[\delta]}$ carry no $\mathrm{p}\cdot\mathrm{p}$ contractions while vertices from (2.17) are spin-local off shell.

D. Shift symmetry

Remarkable property of vertices (3.11) and (3.12) is particularly simple transformation laws under a shift of parameters p_i and oscillator y. Specifically, as was shown in [60] and as is clear from (3.40) and (3.41) the following transformations take place

$$\Phi^{[\delta_1,\delta_2]}(y-a;\mathrm{t}_1,\mathrm{t}_2,\mathrm{p}_i+a)=e^{i(\mathrm{t}_1+\mathrm{t}_2)^\alpha a_\alpha}\Phi^{[\delta_1,\delta_2]}(y;\mathrm{t}_1,\mathrm{t}_2,\mathrm{p}_i), \tag{3.43}$$

$$\Phi^{[\delta]}(y;\mathrm{t},\mathrm{p}_i+b)=e^{i(\mathrm{t}+y)^\alpha b_\alpha}\Phi^{[\delta]}(y;\mathrm{t},\mathrm{p}_i), \tag{3.44}$$

where a and b are arbitrary spinor parameters. In particular, the above shift transformations leave vertices unaffected for $a_\alpha=\mu(\mathrm{t}_1+\mathrm{t}_2)_\alpha$ and $b_\alpha=\nu(\mathrm{t}+y)_\alpha$ with arbitrary numbers μ and ν. The proof is based on the transformations (3.40) and (3.41) in the perturbative expansion. A more detailed derivation can be found in [60].

The symmetry transformations (3.43) and (3.44) have been already observed in four dimensions and dubbed *shift symmetry*. Hence, shift symmetry has a straightforward generalization to any d. This observation is closely related to the earlier analysis of [49], where it was shown how certain parameter shifts of the so-called shifted homotopies reduce the degree of nonlocality of HS vertices. While we use no shifted homotopies in our studies, the idea to look at what shifts in (3.43) and (3.44) might lead to was to a large extent inspired by the result [49]. Since (3.11) and (3.12) represent the Fourier transformed HS vertices, the symmetry naturally acts in the Fourier space. However, its physical interpretation remains unclear. The recent analysis of [57] has revealed that spin-locality is intimately related to shift symmetry. Given that, it would be interesting to see whether or not shift symmetry in d dimensions is on the on shell factorization.

IV. ON SHELL REDUCTION

Equations (2.28)–(2.30) with (2.31) are shown to be consistent. They generate unfolded Eqs. (2.16) and (2.17). As has been mentioned, the latter describe HS field dynamics not until a proper factorization condition over the traceful ideal is imposed. Recall, that at the free level the ideal is generated by the Howe dual $sp(2)$, (2.14), provided the HS fields ω and C are $sp(2)$ singlets

$$[t^0_{\alpha\beta},C]_*=[t^0_{\alpha\beta},\omega]_*=0, \tag{4.1}$$

Equation (4.1) guarantees the spectrum to consist of the two-row Young diagrams (2.1) and (2.4), which, however, are not traceless yet. To make them traceless one strips off the ideal (2.14). This simply implies a subtraction of traceful components of Young diagrams in practice.

Now, nonlinear equations (2.16) and (2.17) that we can denote $A_1=0$ for 1-forms and $A_0=0$ for 0-forms, correspondingly, span two-row Young diagrams as well and, therefore, one should have

$$[A_1,t^0_{\alpha\beta}]_*=[A_0,t^0_{\alpha\beta}]_*=0. \tag{4.2}$$

A seemingly natural idea is to set equations $A_{0,1}$ on shell by assuming the fields ω and C are properly traceless and then by dropping terms associated with traceful contributions of the form $t^0_{\alpha\beta}*A^{\alpha\beta}=A^{\alpha\beta}*t^0_{\alpha\beta}$, where $A^{\alpha\beta}$ are chosen in such a way, so as to make e.g., Eq. (2.16) totally traceless. The procedure just prescribed however does not lead to consistent interaction, because in adding terms from the ideal one has to make sure that the resulting equations remain consistent up to terms from the same ideal. For the ideal in question this is not granted in interactions since the off shell HS algebra along with its ideal get deformed. In other words, the corresponding Howe dual $sp(2)$ generators (2.11) receive field-dependent corrections

$$t_{\alpha\beta}=t^0_{\alpha\beta}+O(C). \tag{4.3}$$

So, the problem that one would like to address is whether there still exists an algebra spanned by (possibly) field-dependent generators that satisfy commutation relations (2.13)

$$[t_{\alpha\beta},t_{\gamma\delta}]_*=\epsilon_{\alpha\gamma}t_{\beta\delta}+\epsilon_{\beta\gamma}t_{\alpha\delta}+\epsilon_{\alpha\delta}t_{\beta\gamma}+\epsilon_{\beta\delta}t_{\alpha\gamma}, \tag{4.4}$$

and that allows for a consistent truncation of the associated trace ideal at the level of the nonlinear equations, (2.16) and (2.17).

A. Global $sp(2)$ and the quotienting

The described problem can be solved systematically if it is noted that the singlet condition (2.11) can be viewed as a requirement of existence of the global symmetry $sp(2)$ at the level of the off shell equations. While deformed, it should still be there in interactions to guarantee proper degrees of freedom. Once the generating system (2.28)–(2.30) reproduces (2.16) and (2.17), we require the $sp(2)$ to be its global symmetry,

$$\delta_t W=\delta_t\Lambda=\delta_t C=0. \tag{4.5}$$

Using (2.39)–(2.41) one finds the corresponding generators $t_{\alpha\beta}$ satisfy

$$\mathrm{d}_x t_{\alpha\beta} + [W, t_{\alpha\beta}]_* = 0, \tag{4.6}$$

$$\mathrm{d}_z t_{\alpha\beta} + [\Lambda, t_{\alpha\beta}]_* = 0, \tag{4.7}$$

$$(t_{\alpha\beta}(z'; y, \vec{\mathbf{y}}) * C - C * t_{\alpha\beta}(z'; -y, \vec{\mathbf{y}}))|_{z'=-y} = 0, \tag{4.8}$$

where one should also remember (4.4). Recall, (4.8) is a consequence of (4.7). Notice, that neither W nor Λ are $sp(2)$ singlets in a sense of (2.11). However, the equations are. This can be shown using (4.6) and (4.7) provided $t_{\alpha\beta} \in \mathbf{C}^0$

$$[\mathrm{d}_x W + W * W, t_{\alpha\beta}]_* = 0, \tag{4.9}$$

$$[\mathrm{d}_z W + \{W, \Lambda\}_* + \mathrm{d}_x \Lambda, t_{\alpha\beta}]_* = 0, \tag{4.10}$$

where $[A, B]_* \coloneqq A * B - B * A$. Following [24] one wishes to drop terms from the ideal at the level of the generating equations to set it on shell. This is equivalent to adding an arbitrary ideal contribution to the generating system (2.28)–(2.30) so, that its consistency gets ruined in its ideal part only. If this is the case, then the factorization results in a consistent on shell system from the factor space. To be a bit more specific, consider Eq. (2.16), for example, which we have denoted by A_1 already. Being an $sp(2)$ singlet, thanks to (4.9), it is an element of the large HS off shell algebra

$$[A_1, t_{\alpha\beta}]_* = 0. \tag{4.11}$$

Decomposing

$$A_1 = \mathbf{A} + A^{\mathbf{id}}, \qquad A^{\mathbf{id}} = A^{\alpha\beta} * t_{\alpha\beta} = t_{\alpha\beta} * A^{\alpha\beta}, \tag{4.12}$$

where $\mathbf{A}$ is any particular representative of the HS algebra (factor algebra), while $A^{\mathbf{id}}$ is an element of the ideal. Both should commute with $t_{\alpha\beta}$

$$[\mathbf{A}, t_{\alpha\beta}]_* = [A^{\mathbf{id}}, t_{\alpha\beta}]_* = 0. \tag{4.13}$$

In that case the equation of motion $A_1 = 0$ entails

$$\mathbf{A} = 0, \tag{4.14}$$

which is equivalent to dropping off the part associated with the ideal from equations. In practice, once the z-dependence of W is resolved via (2.29), Eq. (2.16) becomes z-independent. The natural representative for physical field ω is z-independent too. Hence, one has to require the ideal part to be z-independent

$$\mathrm{d}_z (A^{\alpha\beta} * t_{\alpha\beta}) = 0. \tag{4.15}$$

Let us check now if the addition of $A^{\mathbf{id}}$ of the form (4.12) to (2.16) is indeed consistent. To this end consider

$$\mathrm{d}_x W + W * W + A^{\alpha\beta} * t_{\alpha\beta} \simeq 0, \tag{4.16}$$

where $\simeq$ means equality up to terms that belong to the ideal. Hitting with d_x on (4.16) gives

$$t_{\alpha\beta} * \mathrm{D}_x A^{\alpha\beta} \simeq 0, \qquad \mathrm{D}_x = \mathrm{d}_x + [W, \bullet], \tag{4.17}$$

which do belong to the ideal, since $\mathrm{D}_x t_{\alpha\beta} = 0$ and $t_{\alpha\beta} * A^{\alpha\beta} = A^{\alpha\beta} * t_{\alpha\beta}$ and, therefore, $t_{\alpha\beta} * \mathrm{D}_x A^{\alpha\beta} = \mathrm{D}_x A^{\alpha\beta} * t_{\alpha\beta}$ is a two-sided ideal.

A similar analysis should be carried out for the rest of the Eqs. (2.28)–(2.30). We do not do it here as we plan to consider it in detail elsewhere.

B. Explicit form of $sp(2)$ generators

Let us collect all the conditions for $t_{\alpha\beta}$ together

$$[t_{\alpha\beta}, t_{\gamma\delta}]_* = \epsilon_{\alpha\gamma} t_{\beta\delta} + \epsilon_{\beta\gamma} t_{\alpha\delta} + \epsilon_{\alpha\delta} t_{\beta\gamma} + \epsilon_{\beta\delta} t_{\alpha\gamma}, \tag{4.18}$$

$$\mathrm{d}_x t_{\alpha\beta} + [W, t_{\alpha\beta}]_* = 0, \tag{4.19}$$

$$\mathrm{d}_z t_{\alpha\beta} + [\Lambda, t_{\alpha\beta}]_* = 0. \tag{4.20}$$

While the differential equations (4.19) and (4.20) are consistent and must have solutions, *a priori* it is not guaranteed that the $sp(2)$ condition (4.18) can be satisfied. There is an elegant explanation on as to why the proper $sp(2)$ exists within the approach of [24]. Its origin is the algebra of the deformed oscillators that underlies Vasiliev's master equations. It generates the required $sp(2)$ in the covariant way in terms of the field analogous to Λ via its quadratic star-product combinations. Our approach is different in some aspects. In particular, the star product operation (2.21) leads to an ill-defined product $\mathbf{C}^1 * \mathbf{C}^1$ in contrast with the Vasiliev case. Since $\Lambda \in \mathbf{C}^1$, there is no hope we can define quadratic combinations out of it. Nevertheless, if Eqs. (2.28)–(2.30) describe a certain deformation of the free off shell HS unfolded equations, the global symmetry of which is $sp(2)$ spanned by (2.13), then one should expect a relevant deformation in interactions. Baring this option in mind one can try to look at solutions of (4.18)–(4.20) in perturbations in C (4.3).

At order zero it is easy to see that t^0 of (2.11) solves all the above conditions. Indeed, (4.18) is fulfilled by its definition (2.13). Equation (4.19) is satisfied because t^0 is space-time constant, while $W^0 = \omega(Y)$ at this order is an $sp(2)$ singlet, (4.1). Equation (4.20) is satisfied since t^0 is z-independent, while the second term in (4.20) is of the order $O(C)$.

1. First order

To proceed to the next order, let us resolve the undeformed singlet condition (2.11) explicitly. To this end we use convention (2.33) and introduce the following shorthand notation

$$\eta_{ab}\mathbf{y}^a_\alpha \mathbf{y}^b_\beta \coloneqq \vec{\mathbf{y}}_\alpha \cdot \vec{\mathbf{y}}_\beta, \tag{4.21}$$

$$\mathbf{y}^a_\alpha \mathbf{y}^{\alpha b} \coloneqq \vec{\mathbf{y}} \otimes \vec{\mathbf{y}}. \tag{4.22}$$

It is straightforward to check using (2.8), that functions of the form

$$C(y,\vec{\mathbf{y}}|x) \coloneqq C(\vec{\mathbf{y}}y;\vec{\mathbf{y}}\otimes\vec{\mathbf{y}}|x) \tag{4.23}$$

are manifest t^0-singlets. At order $O(C)$ Eq. (4.19) boils down to

$$\mathrm{d}_z t^1_{\alpha\beta} = [t^0_{\alpha\beta},\Lambda]_*. \tag{4.24}$$

Its consistency (4.8) is just the free $sp(2)$-singlet condition

$$[t^0_{\alpha\beta},C]_* = 0, \tag{4.25}$$

which is solved by (4.23). Therefore, solution of (4.24) exists and we can write it down using the standard conventional homotopy. Substituting (2.31) into (4.24) and using

$$[t^0_{\alpha\beta},\bullet]_* = \vec{\mathbf{y}}_\alpha\cdot\frac{\partial}{\partial\vec{\mathbf{y}}^\beta} + y_\alpha\frac{\partial}{\partial y^\beta} - i\frac{\partial}{\partial y^\alpha}\frac{\partial}{\partial z^\beta} + (\alpha\leftrightarrow\beta) \tag{4.26}$$

gives (upon changing the integration variable)

$$t^1_{\alpha\beta} = -z_\alpha z_\beta\int_0^1 \mathrm{d}\tau\,\tau(1-\tau)C(-\tau\vec{\mathbf{y}}z;\vec{\mathbf{y}}\otimes\vec{\mathbf{y}})e^{i\tau zy}. \tag{4.27}$$

A possible freedom in z-independent function of this solution is fixed to zero by (4.18). Indeed, suppose $t^1_{\alpha\beta}$ ruins $sp(2)$ commutation relations (4.18) up to $O(C)$. Using that antisymmetric pair of spinor indices is proportional to the $sp(2)$ epsilon, we have at this order

$$[t_{\alpha\alpha},t_{\beta\beta}]_*|_{O(C)} = [t^0_{\alpha\alpha},t^1_{\beta\beta}]_* + [t^1_{\alpha\alpha},t^0_{\beta\beta}]_* = 2\epsilon_{\alpha\beta}S_{\alpha\beta}, \tag{4.28}$$

where $S_{\alpha\beta}=S_{\beta\alpha}$ and we suppose $S_{\alpha\beta}\neq t^1_{\alpha\beta}$. From (4.20) one concludes, however, that

$$\mathrm{d}_z S_{\alpha\beta} = [t^0_{\alpha\beta},\Lambda]_*. \tag{4.29}$$

Therefore, $S_{\alpha\beta} = t^1_{\alpha\beta} + \sigma_{\alpha\beta}(y,\vec{\mathbf{y}})$, where σ is some arbitrary function. Setting $z=0$ in (4.28) one finds that its left-hand side vanishes, while the σ-contribution on the right-hand side survives; thus, $\sigma_{\alpha\beta}=0$. Notice also that $t^1\in\mathbf{C}^0$ as required.

2. *Second order*

Let us now inspect (4.20) at order $O(C^2)$

$$\mathrm{d}_z t^2_{\alpha\beta} = [t^1_{\alpha\beta},\Lambda]_*. \tag{4.30}$$

Consistency $\mathrm{d}_z^2=0$ by virtue of (2.36) constrains

$$\left.\left(t^1_{\alpha\beta}(z',y)*C - C*t^1_{\alpha\beta}(z',-y)\right)\right|_{z=-y} = 0. \tag{4.31}$$

We can check whether (4.31) is fulfilled by direct computation that gives

$$\left.(t^1_{\alpha\beta}(z,y)*C)\right|_{z=-y} = -y_\alpha y_\beta\int_0^1\mathrm{d}\tau\,\tau(1-\tau)C(\tau\vec{\mathbf{y}}y;\vec{\mathbf{y}}\otimes\vec{\mathbf{y}})\star C((1-\tau)\vec{\mathbf{y}}y;\vec{\mathbf{y}}\otimes\vec{\mathbf{y}}), \tag{4.32}$$

$$\left.(C*t^1_{\alpha\beta}(z,-y))\right|_{z=-y} = -y_\alpha y_\beta\int_0^1\mathrm{d}\tau\,\tau(1-\tau)\times C((1-\tau)\vec{\mathbf{y}}y;\vec{\mathbf{y}}\otimes\vec{\mathbf{y}})\star C(\tau\vec{\mathbf{y}}y;\vec{\mathbf{y}}\otimes\vec{\mathbf{y}}), \tag{4.33}$$

where $\star$ is a part of the Moyal product (2.9) associated with $\vec{\mathbf{y}}$ variables (2.22). Note that the prefactor in (4.32) and (4.33) is symmetric under $\tau\to1-\tau$ and therefore the two terms above are the same as consistency (4.31) holds. Less expected is that the whole right-hand side of (4.30) vanishes for t^1 from (4.27)

$$[t^1_{\alpha\beta},\Lambda]_* = 0 \tag{4.34}$$

and, therefore, t^2 is z-independent. The cancellation in (4.30) can be observed by straightforward star-product calculation using the singlet ansatz (4.23) along with symmetry $\tau\to1-\tau$ in the measure of (4.27). Moreover, similar analysis results in (see Appendix C)

$$[t^1_{\alpha\beta},t^1_{\gamma\delta}]_* = 0. \tag{4.35}$$

This suggests higher order in C corrections to $t_{\alpha\beta}$ are absent as the final result that satisfies both (4.18) and (4.20) is at most linear in C

$$t_{\alpha\beta} = \frac{1}{4i}(\vec{\mathbf{y}}_\alpha\cdot\vec{\mathbf{y}}_\beta + y_\alpha y_\beta) - z_\alpha z_\beta\int_0^1\mathrm{d}\tau\,\tau(1-\tau)\times C(-\tau\vec{\mathbf{y}}z;\vec{\mathbf{y}}\otimes\vec{\mathbf{y}})e^{i\tau zy}. \tag{4.36}$$

There are no higher order in C corrections as (4.36) is all-order exact.

3. *Covariant constancy*

Solution (4.36) is shown to satisfy all the required conditions except for (4.19). Note, there is no freedom left in (4.36). Given the freedom in solutions of (2.29) for W, Eq. (4.19) can be satisfied for some particular choice of W. We now prove that $t_{\alpha\beta}$ from (4.36) satisfies the remaining covariant constancy condition (4.19) provided field W is fixed canonically (2.20).

Suppose now Eq. (4.19) is not valid and therefore,

$$\mathrm{d}_x t_{\alpha\beta} + [W, t_{\alpha\beta}]_* = R_{\alpha\beta} \neq 0. \tag{4.37}$$

Using (4.20) one finds the following consistency constraint for R

$$\mathrm{d}_z R_{\alpha\beta} + [\Lambda, R_{\alpha\beta}]_* = 0. \tag{4.38}$$

Let us analyze (4.38) in perturbations. At zeroth order we know that $R^0_{\alpha\beta} = 0$ and, therefore, it follows from (4.38) that

$$R^1_{\alpha\beta} = R^1_{\alpha\beta}(y, \vec{\mathbf{y}}) \tag{4.39}$$

is z-independent. Hence, R^1 can be found from (4.37) by setting $z = 0$

$$R^1_{\alpha\beta} = ([\omega, t^1_{\alpha\beta}]_* + [W_1, t^0_{\alpha\beta}]_*)|_{z=0}. \tag{4.40}$$

Let us show that

$$([\omega, t^1_{\alpha\beta}]_*)_{z=0} = ([t^0_{\alpha\beta}, W_1]_*)_{z=0} \Rightarrow R^1_{\alpha\beta} = 0. \tag{4.41}$$

To this end we use (4.26)

$$[t^0_{\alpha\beta}, W_1]_*|_{z=0} = -\frac{i}{2}\left(\frac{\partial}{\partial y^\alpha}\frac{\partial}{\partial z^\beta} + \frac{\partial}{\partial y^\beta}\frac{\partial}{\partial z^\alpha}\right) W_1|_{z=0}, \tag{4.42}$$

where $y\frac{\partial}{\partial y}$ in t^0 does not contribute at $z = 0$ since $W_1(z = 0; Y) = 0$. Then, from the z-evolution equation (2.29) we have

$$\frac{\partial}{\partial z^\alpha} W_1|_{z=0} = (\omega * \Lambda_\alpha - \Lambda_\alpha * \omega)_{z=0}, \tag{4.43}$$

where one also takes into account that $\mathrm{d}_x\Lambda = 0$ at $z = 0$. Note, W_1 contains two possible orderings $\omega \star C$ and $C \star \omega$. If (4.41) is valid, it should be so for each ordering separately. Let us check ordering $\omega \star C$ first. It is convenient to use the Taylor form of ω (3.13)

$$\omega(y, \vec{\mathbf{y}}) = e^{-iy\mathrm{t}}\omega(y', \vec{\mathbf{y}})|_{y'=0}. \tag{4.44}$$

Straightforward $*$ calculation yields

$$\frac{\partial}{\partial y^\alpha}(\omega * \Lambda_\beta)_{z=0} = -i\mathrm{t}_\alpha \mathrm{t}_\beta \int_0^1 \mathrm{d}\tau\, \tau(1-\tau) e^{-i(1-\tau)y\mathrm{t}}\omega(y', \vec{\mathbf{y}}) \star C(-\tau\mathrm{t}, \vec{\mathbf{y}})|_{y'=0}. \tag{4.45}$$

At the same time,

$$(\omega * t^1_{\alpha\beta})_{z=0} = -\mathrm{t}_\alpha \mathrm{t}_\beta \int_0^1 \mathrm{d}\tau\, \tau(1-\tau) e^{-i(1-\tau)y\mathrm{t}}\omega(y', \vec{\mathbf{y}}) \star C(-\tau\mathrm{t}, \vec{\mathbf{y}})|_{y'=0} \tag{4.46}$$

which precisely matches (4.41) for ordering $\omega \star C$. Similarly, the required cancellation takes place for ordering $C \star \omega$.

At second order we find from (4.38) that once $R^1_{\alpha\beta} = 0$, then $R^2_{\alpha\beta}$ is z-independent and therefore, from (4.38) the latter can be found as

$$R^2_{\alpha\beta} = ([W_1, t^1_{\alpha\beta}]_* + [W_2, t^0_{\alpha\beta}]_*)_{z=0}. \tag{4.47}$$

The canonical W_2 is of $O(z^2)$ and, hence, once t^0 contains no more than first derivative over z, the second term on the right-hand side of (4.47) vanishes. Furthermore, before setting $z = 0$ each term from commutator $[W_1, t^1_{\alpha\beta}]_*$ can be shown to be of $O(z)$ by direct calculation. So, the first term on the right of (4.47) vanishes at $z = 0$ just as well. One then concludes that $R^2_{\alpha\beta} = 0$. The above consideration extends analogously to higher orders and thanks to W_n is at least $O(z^2)$ for $n \geq 2$, provided the canonical choice of field ω is made. This proves (4.19) and provides the final simple form of all-order generators (4.36).

V. OUTLOOK AND DISCUSSION

We have shown that the Vasiliev type generating equations of [60] can be framed to describe interacting symmetric HS gauge fields in any dimension. Based on the off shell HS algebra they provide a set of HS compatibility constraints. The dynamical evolution originates from factorization over the trace ideal governed by the Howe dual $sp(2)$ that reduces the system down to its mass shell. Since the off shell HS algebra gets deformed in interaction, the factorization procedure becomes highly nontrivial and *a priori* unknown. To solve this problem we explicitly find field-dependent generators of $sp(2)$ commuting with the generating equations. These allow one to set equations on shell by stripping the traceful contributions off. The remarkable feature of the generators obtained is a very simple linear in field form (4.36), which is one of the main results of the current research. The cancellation of higher-order field corrections was not obvious and came as a surprise. This is not typical of the original Vasiliev case as field dependence of $sp(2)$ is not bounded by linear terms in general. A concise form of the generators found makes on shell analysis feasible, while the HS locality issue in d dimensions amenable.

The main objective of using generating equations of the proposed type is all-order spin-locality of the (off shell) HS vertices manifestly available via canonical choice of field variables (2.20). The respective vertices were shown to be space-time spin-local in (2.16) and spin-local in (2.17). To arrive at this result one uses the standard contracting homotopy resolution operator all the way in perturbations without any further field redefinitions, thus implementing the canonical embedding. The manifest form of off shell HS vertices have been obtained at any order in a neat form of integrals over hypersimplices, the dimension of which grows with the order of perturbation theory.

While we have not carried out the on shell reduction in this paper, an important remark in this regard is in order. The canonical choice (2.20), being irrelevant to consistency of the generating equations, plays an essential role in factorization of the trace ideal. Namely, in manifestly deriving $sp(2)$ generators (4.36), it was the canonical embedding that guaranteed the proposed ansatz to satisfy all necessary constraints.

Let us comment now on the differences between generating Eqs. (2.28)–(2.30) and the Vasiliev ones from [24]. The minor difference is we do not double variables Y's at nonlinear level, rather add a two-component z_α which is sufficient to grasp at nonlinear level. We do not take advantage of the AdS covariant setting of the original equations [24] either as we prefer to stay in the Lorentz frame. The major departure is the choice of the z-commuting large algebra (2.21) which significantly shifts the whole setting.[13] As explained in detail in [60] this choice does not quite live up to the original Vasiliev construction causing early infinities in interactions. This is one of the reasons why Vasiliev's Z does not commute.[14] Nevertheless, it is precisely the star product (2.21) that effectively manifests while evaluating HS vertices [51] constrained by locality using the original Vasiliev framework. This apparent contradiction is resolved by postulating in [60] equations of the form (2.28)–(2.30). As compared to the original Vasiliev ones, they lack the 0-form module B, while HS vertices of (2.17) are generated by (2.30). Consistency rests on the remarkable, yet so far poorly understood projective identity (2.36). This identity is strongly tied to a particular solution (2.31) of (2.32). On the contrary, there is no formal solution selection for the analog of field Λ within the Vasiliev equations.

Another important difference is the lack of quadratic contribution $\Lambda * \Lambda$ in (2.29) in contrast with the similar Vasiliev case, where not only it is present due to consistency, it plays a fundamental role. This term is responsible for the local Lorentz symmetry of the spinorial $d = 4$ equations [17], that would be not otherwise guaranteed. It as well plays a key role in the on shell projection of [24]. In both cases the deformed oscillator algebra of the aforementioned quadratic contribution does the job. This missing ingredient challenges the global $sp(2)$, which is necessary to make sense of our equations on shell. Nevertheless, the global $sp(2)$ does exist in our case, although we find its presence not obvious beforehand. It is also worth recalling that the corresponding generators are of remarkably simple form for the canonical definition of the physical field (2.20). The latter choice then again is in consonance with spin-locality.

Despite the differences we believe Eqs. (2.28)–(2.30) should follow from Vasiliev's ones perhaps nontrivially. It would be interesting to understand the link between the two systems. It is not unlikely that the former results from the latter upon star-product contraction along the lines of Sec. VI from [51] and a proper renormalization of the Vasiliev vacuum state. Constrained by associativity, however, the limiting procedure makes its implementation beyond order C^2 challenging.[15] One reason to expect Eqs. (2.28)–(2.30) may appear as a result of a certain contraction is the structure of functional class (A3), which is a subclass of the one from [52] designed to reconcile locality. However, in our case the system has no room for a class bigger than (A3) at least in perturbations that indicates its reductive origin.

In conclusion let us point out some interesting problems for the future investigation:

(i) Computation of on shell HS vertices at orders C^2 and C^3 and beyond along with examination of spin-locality. The developed formalism seems quite suitable for that, as on the one hand HS vertices are shown to be space-time local to any order in C at the off shell level; thus one is left with a careful analysis of the factorization condition. On the other hand, the factorization procedure looks encouraging in view of remarkably simple $sp(2)$ generators (4.36). At this stage it is not clear whether one should proceed along the lines of the quasiprojector technique [65,71–73] (see also [74] for the somewhat degenerate lower-dimensional case) in attacking this problem or some new tools should be developed. Indeed, the action of quasiprojectors might not be necessarily compatible with locality.

(ii) An intriguing HS shift symmetry that manifests itself within the spin-local setting has not received an adequate explanation so far (a technical reason is the presence of hypersimplex in the integration domain). In particular, it is interesting to check if it stands the on shell factorization. Given the results of [57], where spin-locality was shown to follow from the shift symmetry assumption, it seems likely to define a class of proper on shell field representatives from factor space.

[13] An observation of the limiting star product that leads to (2.21) in [51] was stimulated by the results of [68], where a relation of specific homotopy shifts to star-product ordering was pointed out (see also [69,70] for further discussion on orderings). Equation (2.21) plays a pivotal role in the analysis of [60].

[14] In [71] it was stressed that Z's should be noncommuting for a yet another reason.

[15] At order C^2 equivalence of the two systems can be established using a peculiar identity (6.29) of [51].

(iii) An immediate application of Eqs. (2.28)–(2.30) could be the $3d$ HS interactions of Prokushkin and Vasiliev [75], which are on shell due to their spinorial setup. It would be interesting to revisit locality issue for this theory.

ACKNOWLEDGMENTS

We are grateful to Kostya Alkalaev for useful discussions. We also acknowledge Mikhail Povarnin for his careful reading of the manuscript and pointing out a few typos in formulas in Sec. III. V. E. acknowledges the financial support from the Foundation for the Advancement of Theoretical Physics and Mathematics "BASIS".

APPENDIX A: CLASSES OF FUNCTIONS

The important step is to identify a class of functions $\mathbf{C}^r$ that lives on all operations entering (2.28)–(2.30). As the degree of dz- is a grading, the class is graded correspondingly. We define $r = 0, 1, 2$ to be a label prescribed to zero-, one-, and two-forms, respectively. Accordingly, we require

$$\mathbf{C}^{r_1} * \mathbf{C}^{r_2} \in \mathbf{C}^{r_1+r_2}, \tag{A1}$$

$$\mathrm{d}_z \mathbf{C}^r = \mathbf{C}^{r+1}. \tag{A2}$$

Conditions (A1) and (A2) have been analyzed in [60] with the following explicit result: $\mathbf{C}^r$ consists of the following functions

$$\phi(z, y; \vec{\mathbf{y}}, \mathrm{d}z) = \int_0^1 \mathrm{d}\tau \frac{1-\tau}{\tau} \int \frac{\mathrm{d}u \mathrm{d}v}{(2\pi)^2} f \times \left(\tau z + v, (1-\tau)(y+u); \vec{\mathbf{y}}, \frac{\tau}{1-\tau}\mathrm{d}z\right) \times e^{i\tau z_\alpha y^\alpha + i u_\alpha v^\alpha}, \tag{A3}$$

where $f(z, y; \vec{\mathbf{y}}, \mathrm{d}z)$ is such that the integration is well defined and is otherwise arbitrary analytic function. A convenient way to visualize the above class is to use the source parametrization which gives ϕ in different dz-sectors

$$1: \int_0^1 \mathrm{d}\tau \frac{1-\tau}{\tau} e^{i\tau z_\alpha y^\alpha + i(1-\tau)A^\alpha y_\alpha + i\tau B^\alpha z_\alpha - i\tau A^\alpha B_\alpha}, \tag{A4}$$

$$\mathrm{d}z^\alpha: \int_0^1 \mathrm{d}\tau e^{i\tau z_\alpha y^\alpha + i(1-\tau)A^\alpha y_\alpha + i\tau B^\alpha z_\alpha + i(1-\tau)A^\alpha B_\alpha}, \tag{A5}$$

$$\mathrm{d}z^\alpha \mathrm{d}z_\alpha: \int_0^1 \mathrm{d}\tau \frac{\tau}{1-\tau} e^{i\tau z_\alpha y^\alpha + i(1-\tau)A^\alpha y_\alpha + i\tau B^\alpha z_\alpha + i(1-\tau)A^\alpha B_\alpha}. \tag{A6}$$

Differentiating with respect to sources A and B and then setting them to zero one reproduces various elements of $\mathbf{C}^r$. Freedom in coefficients contains arbitrary functions of $\vec{\mathbf{y}}$. Note, that in order τ-integration to be well-defined, (A4) and (A6) should be at least linear in A.

The important properties of functions from $\mathbf{C}^r$ are as follows:

(i) While not immediately obvious, it has been shown in [60] that functions (A3) exhaust all possible perturbative solutions of (2.28)–(2.30). In particular, z-independent zero-forms do belong to $\mathbf{C}^0$. Therefore, field redefinitions $\omega \to \omega + F(\omega, C\ldots C)$ are not constrained by (A3). It is also important that $\Delta\mathbf{C}^1 \in \mathbf{C}^0$, where Δ is the homotopy operator given in (3.4).

(ii) There is an invariance (under proper rescaling) of functions from $\mathbf{C}^r$ with respect to the following star-product reordering operator

$$O_\beta \phi(z, y; \vec{\mathbf{y}}) = \int \frac{\mathrm{d}^2 u \mathrm{d}^2 v}{(2\pi)^2} \phi(z+v, y+\beta u; \vec{\mathbf{y}}) \times \exp(i u_\alpha v^\alpha). \tag{A7}$$

It can be shown that

$$O_\beta \phi(z, y; \vec{\mathbf{y}}) = \phi\left(\frac{z}{1-\beta}, y; \vec{\mathbf{y}}\right), \quad \forall\ \phi \in \mathbf{C}^r, \tag{A8}$$

where β is an arbitrary number. It plays a role of a reordering parameter of the original Vasiliev star-product [51]. The limiting star product (2.21) emerges from the Vasiliev one in the limit $\beta \to -\infty$. Equation (A7) implies that (A3) is a fixed point of the reordering operator (upon a proper rescaling).

(iii) Lastly, product $\mathbf{C}^{r_1} * \mathbf{C}^{r_2}$ is well-defined provided $r_1 + r_2 < 2$ or one of the functions (from $\mathbf{C}^0$) is z-independent, while another one belongs to $\mathbf{C}^2$.

There is a useful factorization formula for (A4)–(A6) convenient in practice ([57,60]). Namely,

$$e^{i\tau z(y+B) + i(1-\tau)yA - i\tau BA} = e^{iyA} \circledast e^{i\tau(z+y)}, \tag{A9}$$

where

$$f(y) \circledast g(z, y) = \int f(y+u) g(z-v, y) e^{iuv}. \tag{A10}$$

Note that $f(y)$ is z-independent.

APPENDIX B: HIGHER ORDERS OFF SHELL

Here we derive Eq. (3.30). To do so we use the following generating formula. For

$$f = \int_0^1 \mathrm{d}\tau \frac{1-\tau}{\tau} e^{iyA} \circledast e^{i\tau z(y+B)} \tag{B1}$$

one has

$$f * \Lambda = \mathrm{d}z^\alpha \int_{[0,1]^2} \mathrm{d}\tau \mathrm{d}\sigma \frac{\sigma}{1-\sigma} e^{iyA} \circledast \tau z_\alpha e^{i\tau z(y+\sigma(\mathrm{p}-A)+(1-\sigma)B)}, \tag{B2}$$

where Λ is given in (3.15), while (B1) should be understood as generating in terms of sources A and B. In order to be well-defined there must be no contribution at $B = 0$. In other words, f should be at least linear in B.

As we consider mostly left ordering $\omega C \ldots C$, we have at order $n+1$

$$W_{n+1} = -\Delta(W_n * \Lambda), \tag{B3}$$

where Δ is the standard contracting homotopy defined in (3.4) that suggests the following ansatz

$$W_n = \int \mu_n e^{-ir_n y\mathrm{t} + ic_n} \circledast (z\mathrm{t})^n \tau^{n-1}(1-\tau) e^{i\tau z(y+B_n)}. \tag{B4}$$

Using (B2) and the identity

$$\int_{[0,1]^2} \mathrm{d}\tau_1 \mathrm{d}\tau_2 f(\tau_1\tau_2, \tau_1) = \int_{[0,1]^2} \mathrm{d}\tau \mathrm{d}\rho \frac{1-\tau}{1-(1-\tau)\rho} \times f(\tau, 1-(1-\tau)\rho) \tag{B5}$$

one finds from (B3)

$$W_{n+1} = \int \mu_{n+1} e^{-ir_{n+1} y\mathrm{t} + ic_{n+1}} \circledast (z\mathrm{t})^{n+1} \tau^n (1-\tau) e^{i\tau z(y+B_{n+1})}, \tag{B6}$$

where

$$\mu_{n+1} = \mu_n r_n \sigma_{n+1}(1-\sigma_{n+1})^{n-1}, \tag{B7}$$

$$B_{n+1} = \sigma_{n+1}(\mathrm{p}_{n+1} + r_n \mathrm{t}) + (1-\sigma_{n+1})B_n, \tag{B8}$$

$$c_{n+1} = c_n + r_n(1-\rho_{n+1})(B_{n+1}\mathrm{t}), \tag{B9}$$

$$r_{n+1} = \rho_{n+1} r_n, \tag{B10}$$

where by σ_{n+1} and ρ_{n+1} we denote integration variables σ and ρ from (B5) that arise at the stage $n+1$. The initial values are

$$\mu_1 = 1, \qquad \sigma_1 = 1, \qquad r_1 = \rho_1,$$
$$c_1 = (1-\rho_1)(\mathrm{p}_1\mathrm{t}), \qquad B_1 = \mathrm{p}_1 + \mathrm{t}. \tag{B11}$$

The integration goes along σ_i $(i \geq 2)$, ρ_i $(i \geq 1)$, and τ in the range [0, 1]. The solution of (B8)–(B10) is given by (3.31) and (3.32) that can be proven by induction. Indeed, assuming (3.31) and (3.32) hold at order n and plugging these expressions explicitly in (B8) and (B9) we have

$$B_{n+1} = \sigma_{n+1}(\mathrm{p}_{n+1} + \rho_1 \ldots \rho_n \mathrm{t}) + (1-\sigma_{n+1}) \times \left(\sum_{i=1}^n \lambda_i \mathrm{p}_i + \left(1 + \sum_{i<j}^n (\lambda_i \nu_j - \nu_i \lambda_j)\right)\mathrm{t} \right), \tag{B12}$$

$$c_{n+1} = \sum_{i=1}^n \nu_i(\mathrm{p}_i\mathrm{t}) + \rho_1 \ldots \rho_n (1-\rho_{n+1}) \times \left(\sigma_{n+1}\mathrm{p}_{n+1} + (1-\sigma_{n+1}) \sum_{i=1}^n \lambda_i \mathrm{p}_i \right)\mathrm{t}, \tag{B13}$$

where

$$1 - \rho_1 \ldots \rho_n = \sum_{i=1}^n \nu_i, \qquad \sum_{i=1}^n \lambda_i = 1 \tag{B14}$$

holds by inductive assumption. Introducing new variables

$$\lambda'_{n+1} = \sigma_{n+1}, \qquad \lambda'_i = (1-\sigma_{n+1})\lambda_i \Rightarrow \sum_{i'} \lambda'_{i'} = 1, \tag{B15}$$

$$\nu'_i = \nu_i + \rho_1 \ldots \rho_n (1-\rho_{n+1})\lambda'_i,$$
$$\nu'_{n+1} = \rho_1 \ldots \rho_n (1-\rho_{n+1})\lambda'_{n+1}, \tag{B16}$$

$$1 - \rho_1 \ldots \rho_{n+1} = \sum_{i'} \nu'_{i'}, \qquad 1 - \rho_1 \ldots \rho_n = \sum_i \nu_i, \tag{B17}$$

where index i' ranges $n+1$ values and observing that

$$\lambda'_i \nu_j - \nu_i \lambda'_j = \lambda'_i \nu'_j - \nu'_i \lambda'_j \tag{B18}$$

one easily finds

$$B_{n+1} = \sum_{i'} \lambda'_{i'} \mathrm{p}_{i'} + \left(1 + \sum_{i'<j'} (\lambda'_{i'}\nu'_{j'} - \nu'_{i'}\lambda'_{j'})\right)\mathrm{t}, \tag{B19}$$

$$c_{n+1} = \sum_{i'} \nu'_{i'}(\mathrm{p}_{i'}\mathrm{t}), \tag{B20}$$

$$r_{n+1} = 1 - \sum_{i'} \nu'_{i'}, \tag{B21}$$

thus proving (3.31)–(3.33).

1. Jacobian

Let us show now that (λ, ν) variables imply $\mu_{n+1} \times J = 1$, where J is the Jacobian of $(\sigma, \rho) \to (\lambda, \nu)$ variable change. For $n = 1$ one notes that $\mu_1 = 1$. For $n \geq 1$ we can show $\mu_n = 1$. The proof goes inductively. Assuming $\mu_n = 1$, we then have from (B7)

$$\mu_{n+1} = \rho_1 \ldots \rho_n \sigma_{n+1}(1-\sigma_{n+1})^{n-1}. \tag{B22}$$

Variable change (B15) and (B16) yields its further multiplication by the Jacobian arising from integration over the following δ-functions

$$J = \delta(\lambda_i' - (1-\sigma_{n+1})\lambda_i)\delta(\nu_{n+1}' - \rho_1\ldots\rho_n(1-\rho_{n+1})\sigma_{n+1}) \\ \times \delta(\nu_i' - \nu_i - \rho_1\ldots\rho_n(1-\rho_{n+1})\lambda_i')\delta\left(1-\sum_i \lambda_i\right)\delta\left(1-\rho_1\ldots\rho_n - \sum_i \nu_i\right), \tag{B23}$$

with respect to λ_i and ν_i. By integrating out delta-functions we see that they exactly cancel out (B22)

$$\mu_{n+1} \times J = 1. \tag{B24}$$

2. Integration domain

The configuration space is easily identified by induction. Assuming it is given by (3.35) at order n, being the case at $n=1$ (somewhat trivially, since $\lambda_1 = 1$) and $n=2$ (regularly), one sees from (B15) that the λ-hypersimplex extends to the order $n+1$, while from (B16) and (B17) the ν-hypersimplex holds at $n+1$ too.

APPENDIX C: $sp(2)$ PROOF

Let us check that (4.36) indeed generates $sp(2)$ commutation relations (4.18). We have

$$t_{\alpha\beta} = t^0_{\alpha\beta} + t^1_{\alpha\beta}, \tag{C1}$$

where t^0 is given in (2.11), while t^1 is defined in (4.27). In order to see that (4.18) is fulfilled we are left to check that

$$[t^1_{\alpha\beta}, t^1_{\gamma\delta}]_* = 0. \tag{C2}$$

Specific dependence on $\vec{\mathbf{y}}$ in C is not going to be important in what follows, so we just set $C = C(y, \vec{\mathbf{y}})$ and, correspondingly,

$$t^1_{\alpha\beta} = -z_\alpha z_\beta \int_0^1 \mathrm{d}\tau\, \tau(1-\tau) C(-\tau z; \vec{\mathbf{y}}) e^{i\tau z y}. \tag{C3}$$

We proceed with the Taylor representation of field C, (3.14)

$$C(y, \vec{\mathbf{y}}) \equiv e^{-iy\mathrm{p}} C(y', \vec{\mathbf{y}})|_{y'=0}, \qquad \mathrm{p}_\alpha = -i\frac{\partial}{\partial y'^\alpha}. \tag{C4}$$

Using (2.21) it is easy to derive the following useful product

$$z_\alpha z_\beta \int_0^1 \mathrm{d}\tau_1 \tau_1(1-\tau_1) e^{i\tau_1 z(y+\mathrm{p}_1)} * z_\gamma z_\delta \\ \times \int_0^1 \mathrm{d}\tau_2 \tau_2(1-\tau_2) e^{i\tau_1 z(y+\mathrm{p}_2)} \\ = z_\alpha z_\beta z_\gamma z_\delta \int_{[0,1]^2} \mathrm{d}\tau \mathrm{d}\sigma \sigma(1-\sigma)\tau^3(1-\tau) e^{i\tau z(y+\sigma \mathrm{p}_1 + (1-\sigma)\mathrm{p}_2)}. \tag{C5}$$

Having it we obtain

$$t^1_{\alpha\beta} * t^1_{\gamma\delta} = z_\alpha z_\beta z_\gamma z_\delta \int_{[0,1]^2} \mathrm{d}\tau\, \mathrm{d}\sigma\, \sigma(1-\sigma)\tau^3(1-\tau) \\ \times e^{i\tau z y} C(-\tau\sigma z, \vec{\mathbf{y}}) \star C(-\tau(1-\sigma)z, \vec{\mathbf{y}}), \tag{C6}$$

where $\star$ acts on $\vec{\mathbf{y}}$ via (2.22). Since measure in (C6) is invariant under $\sigma \to 1-\sigma$ one trivially obtains (C2). This completes the proof of (4.18).

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