

Geometric bounds on the 1-form gauge sector

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We classify the allowed structures of the discrete 1-form gauge sector in six-dimensional supergravity theories realized as F-theory compactifications. This provides upper bounds on the 1-form gauge factors \mathbb{Z}_m and in particular demands each cyclic factor obey $m \leq 6$. Our bounds correspond to the universal geometric constraints on the torsion subgroup of the Mordell-Weil group of elliptic Calabi-Yau threefolds. For any F-theory vacua with at least one tensor multiplet, we derive the constraints from the \mathbb{P}^1 fibration structure of the base twofold and identify their physical origin in terms of the worldsheet symmetry of the associated effective heterotic string. The bounds are also extended to the F-theory vacua with no tensor multiplets via a specific deformation of the theory followed by a small instanton transition, along which the 1-form gauge sector is not reduced. We envision that our geometric bounds can be promoted to a swampland constraint on any six-dimensional gravitational theories with minimal supersymmetry and also extend them to four-dimensional F-theory vacua.

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I. INTRODUCTION AND SUMMARY

The notion of symmetries has been significantly generalized in recent years. In particular, ordinary symmetries acting on local operators have been generalized to p -form symmetries with $p \geq 1$, under which nonlocal operators extended along p spatial dimensions are charged [1]. As various interesting field-theoretic constructions involve gauging of symmetries, it is of utmost importance to understand the gauging of generalized symmetries to reveal the full symmetry structure of a theory.

Perhaps the simplest such generalization is the 1-form gauge symmetries. A prototypical example arises from the discrete 1-form symmetry,

$$\Gamma \subseteq Z(\hat{G}), \quad (1)$$

of a simply-connected non-Abelian¹ 0-form gauge group \hat{G} , where $Z(\hat{G})$ denotes the center of \hat{G} . Gauging of this 1-form symmetry Γ is responsible for the global structure of the 0-form gauge sector, resulting in the nonsimply-connected quotient,

$$G = \hat{G}/\Gamma, \quad (2)$$

for the actual gauge group.

If the gauge theory in scrutiny is coupled to gravity, such gauging of the 1-form symmetry Γ , or equivalently, taking the quotient (2) by Γ of the 0-form gauge sector, is not an option but a must unless the symmetry is broken. This follows from the absence of global symmetries [2,3], which is one of the conjectured properties that any consistent quantum theories of gravity are subject to. In the *swampland program* initiated in [4], many general criteria of this kind have been proposed as universal features of quantum gravity, dubbed *swampland constraints*, leading to invaluable inspirations as well as explicit guiding principles for a variety of recent research activities in high energy physics.

¹In this paper we focus on the part of the center symmetry originating purely from the non-Abelian 0-form gauge sector if \hat{G} acquires $U(1)$ factors. In the presence of the latter, the quotient action Γ may receive additional contributions from mixing of the center $U(1)$ s, which we hope to address in future work.

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The aim of this paper is to propose general bounds on the discrete 1-form gauge symmetry Γ of the form (2) in six-dimensional gravitational theories with minimal supersymmetry, going beyond the necessity for its gauging. Mostly focusing for concreteness on F-theory vacua, we will constrain the structure of the 1-form gauge sector in 6d $\mathcal{N} = (1, 0)$ supergravity theories as

$$\Gamma = \mathbb{Z}_n \times \mathbb{Z}_m, \quad (3)$$

where the allowed values of (n, m) with $n \leq m$ are restricted as

$$\begin{array}{c|ccc} n & 1 & 2 & 3 \\ \hline m & 1, \dots, 6 & 2, 4 & 3 \end{array}. \quad (4)$$

This in particular imposes a definite upper bound,

$$m \leq 6, \quad (5)$$

on every cyclic factor \mathbb{Z}_m .

As will be reviewed in Sec. II, similar constraints were studied in [5] for the 8d $\mathcal{N} = 1$ supergravity theories with 0-form gauge sector of rank 18, where a few more cases with the following values of (n, m) are also allowed:

$$\begin{array}{c|ccc} n & 1 & 2 & 4 \\ \hline m & 7, 8 & 6 & 4 \end{array}, \quad (6)$$

in addition to those in (4). This restriction can be understood in F-theory as an immediate consequence of the classification results on the elliptic K3 surfaces, whose field-theoretic derivation can also be given from the anomaly consideration alone.² It is thus intriguing to note that the constraints persist, even in an enhanced form, in lower dimensional theories with less supersymmetry. Also from the geometric point of view, the set of elliptic Calabi-Yau threefolds available for 6d F-theory vacua is much wilder than that of the elliptic K3 surfaces for 8d vacua, and hence, the extension of the geometric constraints on surfaces to those on threefolds naively looks surprising.

In this regard, the connection between the 6d and the 8d vacua will be clarified in Sec. III, where we will constrain the 1-form gauge sector of 6d $\mathcal{N} = (1, 0)$ supergravity theories in the framework of geometric F-theory compactifications to six dimensions. We will find that the generic \mathbb{P}^1 fibration structure of the base twofold will play a pivotal role. The only exceptional base lacking such a fibration is \mathbb{P}^2 , which leads to F-theory vacua with $T = 0$ tensor multiplets. However, we will be able to deform those

²The one-to-one correspondence between the elliptic K3 surfaces and the fully consistent supergravity theories can be further sharpened by considering the worldvolume theories of probe 3-branes [6].

models and to make a transition to the models with $T > 0$ tensor multiplets, without reducing the 1-form gauge sector along the way, thereby validating the derivation of the generic constraints even to the theories without tensor multiplets.

In principle, it still remains a logical possibility that the geometric bounds derived as such may not serve as a general swampland constraint. However, the validity of our proposal is bolstered as follows. First, in Sec. IV we will provide a rationale behind the bound (5) from the dual heterotic perspective. Our main message is that the Γ quotient (2) must act nontrivially on the *perturbative* 0-form gauge sector of the dual heterotic string, whether or not it affects the nonperturbative sector.³ One can already infer from this perturbative nature of Γ that the definite upper bound of 6 arises for each 1-form factor, which follows from the simple group theoretical fact that \mathbb{Z}_6 is the maximal cyclic subgroup of E_8 .⁴ Such a perturbative nature of the 1-form gauge sector is reminiscent of that of the $U(1)$ 0-form gauge sector [8]; we will emphasize this resemblance in Sec. IV. Furthermore, inspired by how the F-theoretic geometry and the heterotic physics constrain the 1-form gauge sector, we will be naturally led in Sec. V to envision a concrete route for extending the bounds to general effective theories via decompactification to eight dimensions.

In Sec. V we will in fact provide a number of applications and extensions of our findings. Upon addressing the aforementioned possibility of enhancing our string-theoretic bounds to swampland ones, we will connect the putative 6d F-theory vacua violating the bounds to certain infinite distance limits in the moduli space, which will also be argued to result in decompactification. Another natural extension to be discussed concerns the discrete 0-form gauge sector. We will argue that a cyclic 0-form gauge factor with order beyond 6 cannot arise in 6d F-theory models with $T \geq 1$ tensor multiplets. We will then also conjecture that no matter states in those models can have a $U(1)$ charge larger than 6. Moreover, our derivation of the six-dimensional bounds will also be extended to the *geometric* part of the 1-form gauge sector in 4d F-theory models, with a systematic analysis of the flux-induced sector left for future work.

II. BACKGROUND AND REVIEW

A most versatile tool to date for classifying supergravity theories in $D = 12 - 2d$ is F-theory compactifications on elliptically fibered Calabi-Yau d -folds. In $D \geq 6$

³The nonperturbative sector will also be affected in general by the quotient; see Ref. [7].

⁴To be precise, this intuition is valid under the assumption that the perturbative sector in six dimension is embeddable into the worldsheet symmetry of the 10d heterotic string, as will be emphasized later.

dimensions, i.e., with $d \leq 3$, this allows us to define the effective theories in terms of algebraic properties of the fibration alone, which can always be brought into a Weierstrass form,

$$y^2 = x^3 + fxz^4 + gz^6. \quad (7)$$

Here, z, x, y are the coordinates of the weighted projective space $\mathbb{P}_{[1:2:3]}^2$ and (f, g) a pair of sections,

$$f \in H^0(B_{d-1}, \bar{K}_{B_{d-1}}^{\otimes 4}), \quad g \in H^0(B_{d-1}, \bar{K}_{B_{d-1}}^{\otimes 6}), \quad (8)$$

with $\bar{K}_{B_{d-1}}$ denoting the anticanonical bundle of the F-theory base B_{d-1} .

As is well known, the $[p, q]$ 7-brane stacks are located in the codimension-one loci in the base where the discriminant $\Delta := 4f^3 + 27g^2$ vanishes, and the worldvolume gauge algebra \mathfrak{g}_i carried by each brane stack is read off from the vanishing orders of f, g and Δ via the classification of singular fibers by Kodaira and Néron. On the other hand, the refined information as to which gauge group G_i arises from each stack is encoded in the torsional sections that generate the finite subgroup $\text{MW}(Y_d)_{\text{Tors}}$ of the Mordell-Weil group. In particular, it is known that the global structure of the total gauge group,

$$G = \prod_i G_i, \quad (9)$$

is determined as [9,10]

$$\pi_1(G) = \text{MW}(Y_d)_{\text{Tors}}, \quad (10)$$

which embeds into the total center of the simply-connected cover \hat{G} ,

$$Z(\hat{G}) = \prod_i \mathbb{Z}_{n_i}, \quad (11)$$

via the embedding factors

$$k_i \in \mathbb{Z}_{n_i}, \quad (12)$$

Geometrically, the allowed torsional structures, as well as the generic forms of the associated Weierstrass modes, have been systematically studied in [9], leading to the realization of

$$\text{MW}(Y_d)_{\text{Tors}} = \mathbb{Z}_n \times \mathbb{Z}_m, \quad (13)$$

with the possibilities given in (4). Furthermore, it is known [11] that there are additional outliers with the values of (n, m) given in (6), which only apply to the cases with $d = 2$, i.e., to K3 surfaces.

One of the goals of this paper is to derive these geometric constraints for Calabi-Yau threefolds in an explicit and constructive manner, and to understand them physically from the perspective of the dual heterotic string. In particular we will see, under a certain simplifying assumption, that the global gauge group structure must always embed into the perturbative $E_8 \times E_8$ [or $SO(32)$] current of the heterotic worldsheet, from which the constraints immediately follow.

This is to be contrasted with the following independent, but less constructive, route to the geometric bounds [11,12]: In type IIB string theory the axiodilaton τ in general takes any values in the upper half-plane \mathcal{H} modulo $\text{SL}(2, \mathbb{Z})$. Upon adding the point at infinity, the moduli space of τ is represented by the modular curve,

$$X(1) = \mathcal{H}^*/\text{SL}(2, \mathbb{Z}). \quad (14)$$

The full $\text{SL}(2, \mathbb{Z})$ group, however, does not preserve the $\mathbb{Z}_n \times \mathbb{Z}_m$ torsion points but only the congruence subgroup⁵ $\Gamma(n) \cap \Gamma_1(m) \subset \text{SL}(2, \mathbb{Z})$ does. The associated moduli space of τ , i.e., its respective compactified modular curve, is given as

$$X(n, m) = \mathcal{H}^*/\Gamma(n) \cap \Gamma_1(m). \quad (17)$$

Those curves admit an intricate structure of cusps, of which τ values correspond to the singular fibers of Kodaira type I_s , in perfect match with those found in the generic Weierstrass models [9]. This structure is not unlike that of the gauging of an $\text{SL}(2, \mathbb{Z})$ subgroup, i.e., taking an orbifold which leads to additional fixed points. Those fixed points are loci where D7 brane stacks reside as required by the discrete 1-form gauge symmetry [12]. For the torsion with large orders (n, m) , the modular curve $X(n, m)$ is sufficient to fix the full gauge group of the 8d F-theory vacuum.

While it was not claimed in [9] that (4) is the full list, the proof of its completeness has later been provided in [11]. In this paper, we will find another, more constructive route to this classification result, manifesting the connection in geometry between elliptic K3 surfaces and elliptic Calabi-Yau threefolds.

⁵The relevant congruence subgroups are defined as

$$\Gamma_1(n) = \left\{ \gamma \in \text{SL}(2, \mathbb{Z}) : \gamma \equiv \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \pmod{n} \right\}, \quad (15)$$

$$\Gamma(n) = \left\{ \gamma \in \text{SL}(2, \mathbb{Z}) : \gamma \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod{n} \right\}. \quad (16)$$

A. 8d EFTs

8d $\mathcal{N} = 1$ supergravity theories with $\text{rk}(G) = 18$ have been characterized via F-theory in terms of elliptic K3 surfaces Y_2 over the base onefold $B_1 = \mathbb{P}^1$. In particular, the inequivalent fiber structures of Y_2 , including the torsion subgroups of the Mordell-Weil groups, have been classified [13]. Moreover, as pointed out in [14], $\text{MW}_{\text{Tors}}(Y_2)$ is equally well understood in terms of the nonprimitive embedding of the ADE singularities into the $\Lambda_{2,18}$ lattice of the elliptic K3 surfaces.

All in all, the torsion subgroup, and hence, the 1-form gauge sector, takes the form

$$\Gamma = \mathbb{Z}_n \times \mathbb{Z}_m, \quad (18)$$

with the allowed values of (n, m) given by (4) and (6). As emphasized in [11], the outliers (6) do not apply to higher-dimensional fibrations $Y_{d>2}$, and the K3 surfaces realizing them are necessarily rigid, i.e., *extremal*, as the associated F-theory compactifications turn out to have the non-Abelian groups of ranks $\text{rk}(G) = 18$.

In the meantime, as pointed out in [5] via a purely field-theoretic consideration, the 1-form symmetry might in general lead to a global anomaly (see also [15]). Absence of such an anomaly for the 1-form gauge sector thus imposes field-theoretic constraints on the embedding factors k_i (12) in terms of the cyclic orders n_i of the center (11), leading in turn to constraints on the 1-form gauge sector Γ . In particular, for each of the extremal outliers (6), the 0-form gauge sector, including its global structure, turns out to be uniquely determined by such constraints: e.g., for the 1-form gauge sector $\Gamma = \mathbb{Z}_7$ with $(n, m) = (1, 7)$ in (6), the unique consistent 0-form gauge group is $G = SU(7)^3/\mathbb{Z}_7$, where the global structure is realized by the embedding factors,

$$(k_1, k_2, k_3) = (1, 2, 3), \quad (19)$$

into the total center $Z(\hat{G}) = \mathbb{Z}_7^3$.

Given such a concrete classification of consistent gauge groups, a natural question arises as to what would go wrong e.g. if Γ acquired a cyclic factor \mathbb{Z}_m with $m \geq 9$. One way to address this is to invoke the orbifold picture of the fundamental domain of the axiodilaton discussed before. The modular curve $X(1, m)$ for $m \geq 9$ introduces cusps corresponding to the singular fibers of Kodaira type I_{s_i} , which must thus be included in the geometry, leading to gauge groups with

$$\text{rk}(G) = \sum_i (s_i - 1) \geq 24, \quad (20)$$

where the inequality saturates for $m = 9$. However, this exceeds the geometric upper bound of 18, which follows

from the Tate-Shioda-Wazir theorem [16–18], and hence, $m \geq 9$ cannot be realized in 8d F-theory vacua.

B. 6d EFTs

F-theory has proven to be an invaluable geometric tool⁶ also for systematically exploring 6d $\mathcal{N} = (1, 0)$ supergravity theories in terms of elliptic Calabi-Yau threefolds Y_3 over a base twofold B_2 . Compactness of the base B_2 is crucial as its finite volume relates to the Planck mass,

$$M_{\text{Pl}}^4 \propto \text{Vol}(B_2), \quad (21)$$

keeping the effective theory gravitational. Moreover, the gravitational and the gauge anomalies are beautifully reflected in the topological intersection structure of the base [19]. To begin with, the anomaly coefficient of each gauge algebra factor \mathfrak{g}_i is identified with the respective curve class b_i of the discriminantal component, with gauge coupling given as

$$g_i^{-2} \propto \text{Vol}(b_i). \quad (22)$$

Furthermore, the anomaly cancellation via the Green-Schwarz-Sagnotti-West mechanism [20,21] follows from the geometry of the elliptic fibration Y_3 : for instance, the matter spectrum of the fields charged under $\mathfrak{g}_i = \mathfrak{su}(N)$ carried by the curve of class b_i with self-intersection n and genus g is fixed purely in terms of these topological data as⁷

$$[(8 - N)n + 16(1 - g)]\mathbf{F} + (n + 1 - g)\mathbf{\Lambda} + g\mathbf{Adj}, \quad (23)$$

whose contribution to the anomalies can be shown to cancel with the intersection pairings between the anomaly coefficients.

As it turns out, not every six-dimensional supergravity theory fulfilling the anomaly criteria has an F-theoretic realization. This can be easily inferred from the fact that the set of distinct elliptic Calabi-Yau threefolds is finite [23,24], while infinite classes of such apparently consistent supergravity theories arise (see, e.g., [25]). It has recently been found [26,27], however, that some of those infinite classes can be ruled out by the consistency requirement on certain Bogomol'nyi-Prasad-Sommerfield (BPS) strings, whose existence is addressed by one of the swampland constraints known as the completeness hypothesis (see Ref. [3] for a recent account).

In much the same way, the analog of the field-theoretic 8d global anomalies by itself does not lead to as strong constraints on the 1-form gauge sector as the geometric

⁶In this paper we focus on the geometric, i.e., unfrozen, phase of F-theory.

⁷Symmetric representations can be obtained through singular divisors [22]: an adjoint representation there is anomaly equivalent to a symmetric and an antisymmetric representation.

ones (4). The latter was first addressed in [11] in terms of the torsion subgroup of the Mordell-Weil group of Y_3 via modular curves. The precise argument there was based on the structure of the modular curves $X(n, m)$: the latter indicated that the generic Weierstrass models with $\text{MW}(Y_3)_{\text{Tors}} = \mathbb{Z}_n \times \mathbb{Z}_m$ involve collisions of singularities which effectively result in nonminimal codimension-one fibers with vanishing orders,

$$(\text{ord}(f), \text{ord}(g)) \geq (4, 6), \quad (24)$$

and hence, should be ruled out. In the ensuing section, a constructive derivation will be given of the same geometric constraints. Furthermore, we will also provide in Sec. IV a heterotic worldsheet perspective on those geometric constraints, thereby interpreting them physically, still in a string-theoretic context. In fact, as will be envisioned in Sec. V, it is our belief that the purely effective-field-theoretic constraints will eventually enhance, possibly to the geometric ones, once again by applying appropriate swampland constraints if necessary.

III. GEOMETRIC BOUNDS FROM F-THEORY

Let us consider 6d $\mathcal{N} = (1, 0)$ supergravity theories realized as F-theory compactification on an elliptic Calabi-Yau threefold,

$$\pi: Y_3 \rightarrow B_2. \quad (25)$$

Upon requiring the absence of nonminimal fibers for Y_3 to admit a Calabi-Yau resolution, the base twofolds B_2 allowed by F-theory are restricted [23,24] to be either⁸

$$B_2 = \mathbb{P}^2 \quad (26)$$

or

$$B_2 = \text{Bl}^k(\mathbb{F}_a), \quad (27)$$

a k -fold blowup of Hirzebruch surface \mathbb{F}_a . The F-theory vacua of these two types, respectively, have $T = 0$ and $T = 1 + k \geq 1$ tensor multiplets. In this section we will derive geometric bounds on the 1-form gauge sector, starting from the theories of the latter type with $T \geq 1$.

A. F-theory vacua with $T \geq 1$

We start by noting that every base twofold B_2 of the form (27) admits a rational fibration,

$$\rho: B_2 \rightarrow \mathbb{P}^1, \quad (28)$$

⁸While the Enriques surfaces provide yet another viable base topology, we do not discuss them as they only lead to a trivial fibration.

whose generic fiber f is a rational curve.⁹ Restricting the fibration (25) to this rational fiber, we then obtain an elliptic surface $Y_2 := \pi^{-1}(B_1)$ for $B_1 = f \simeq \mathbb{P}^1$,

$$p: Y_2 \rightarrow B_1. \quad (29)$$

The adjunction formula can then be exploited as follows to show that Y_2 is a K3 surface:

$$\begin{aligned} c_1(\bar{K}_{Y_2}) &= c_1(\bar{K}_{Y_3}|_{Y_2}) - c_1(N_{Y_2/Y_3}) \\ &= -P^*c_1(N_{B_1/B_2}) \\ &= 0, \end{aligned} \quad (30)$$

where we have used in the second line that Y_3 has a trivial first Chern class and that the fibration (29) is a restriction of (25), and in the third line that B_1 is a fiber of (28).

Furthermore, it follows by construction that the Mordell-Weil groups of Y_3 and Y_2 satisfy the obvious inclusion relation,

$$\text{MW}(Y_3) \subseteq \text{MW}(Y_2). \quad (31)$$

This can be seen by observing that a given section $\sigma \in \text{MW}(Y_3)$ restricts to $s \in \text{MW}(Y_2)$ under the embedding of the two fibrations (25) and (29) as

$$\begin{array}{ccc} \sigma: & B_2 & \dashrightarrow & Y_3 \\ & \uparrow & & \uparrow \\ s: & B_1 & \longrightarrow & Y_2 \end{array} \quad (32)$$

where the restricted section s is holomorphic even when σ is not since $B_1 = f$ is a generic fiber of (28), hence evading potential nongeneric points in B_2 where the section σ wraps a fibral curve.

From the fact that B_1 is a generic fiber of B_2 , it immediately follows that such a section restriction is one-to-one and, importantly, upon restriction a torsional section remains torsional with the same order. In other words, the inclusion (31) of the Mordell-Weil groups leads to

$$\text{MW}(Y_3)_{\text{Tors}} \subseteq \text{MW}(Y_2)_{\text{Tors}}, \quad (33)$$

so that the torsional subgroup $\text{MW}(Y_3)_{\text{Tors}}$ of $\text{MW}(Y_3)$ must take the form

$$\text{MW}(Y_3)_{\text{Tors}} = \mathbb{Z}_{n'} \times \mathbb{Z}_{m'} \subseteq \mathbb{Z}_n \times \mathbb{Z}_m, \quad (34)$$

where (n, m) take values from (4) and (6) that are realizable for an elliptic K3 surface. We therefore conclude that the

⁹The base B_2 is a Hirzebruch surface if $k = 0$ in (27), in which case even the *general* fiber f is rational.

bounds on the 1-form gauge sector of 8d supergravity theories persist in 6d F-theory vacua with $T \geq 1$.

However, all 1-form gauge symmetries viable in eight dimensions are not necessarily realizable also in six dimensions. Indeed the 6d bounds turn out to be stronger due to the outliers (6) that are known not to be realized as the structure of the Mordell-Weil torsion of Y_3 [11]. In particular, \mathbb{Z}_7 and \mathbb{Z}_8 are not a viable 1-form gauge sector, which we will also support by switching to the dual heterotic frame in Sec. IV.¹⁰

One may also attempt to rule out those outliers by combining field-theoretic and geometric constraints, independently of the rationale based on modular curves. In the following we will illustrate how this works with the example \mathbb{Z}_7 ; only the key ideas will be presented here, with the full details relegated to Appendix A.

As reviewed in Sec. II A, the anomaly consideration leaves

$$G = SU(7)^3/\mathbb{Z}_7 \quad (35)$$

as a viable gauge group configuration in 8d supergravity theories. This still survives the analogous constraints in the dimensionally reduced 6d theories.¹¹ However, additional constraints arise in six dimensions: In order for the 1-form symmetry $\Gamma = \mathbb{Z}_7$ to be gauged, the matter spectrum should lack certain representations that transform under the Γ action. To be concrete, the fundamental, the anti-symmetric and the adjoint representations under each $SU(N)$ factor naturally appear in the spectrum as (23) and, respectively, have the center charges,

$$q_{\mathbb{Z}_N}(\mathbf{R}_r) = \frac{1}{N}, \frac{2}{N}, 0 \pmod{1}. \quad (36)$$

One thus requires that the following Γ charge of the representation $\mathbf{R} = (\mathbf{R}_{r_i})_{i=1}^3$ under G ,

$$\sum_i q_{\mathbb{Z}_7}(\mathbf{R}_{r_i}) \cdot k_i, \quad (37)$$

¹⁰In view of the explicit description for the fibration Y_3 , as explained in Sec. II A, its elliptic K3 fiber with the torsional orders (6) is extremal and hence rigid, rendering the elliptic fibration within the K3 almost trivial in that τ is constant. One might naively think that extremal K3 surfaces can only be fibered over T^2 . It is known [28], however, that nontrivial smooth fibrations with $\mathfrak{su}(3)$ holonomy are realizable in certain rigid threefolds, where the K3 fibers may become reducible over points in the \mathbb{P}^1 base.

¹¹Unlike in eight dimensions, field-theoretic consideration in six dimensions does not by itself guarantee that G of the form (35) is the only allowed 0-form gauge symmetry with a \mathbb{Z}_7 quotient. However, thanks to the geometric constraints [13] for the generic K3 fiber Y_2 , we know that the \mathbb{Z}_7 quotient leads precisely to (35) for the perturbative 0-form gauge sector even in six dimensions. It thus suffices to rule this unique choice out to show that \mathbb{Z}_7 is not a viable 6d 1-form gauge factor.

should vanish modulo 1, if matter states in \mathbf{R} are to be present in the spectrum at all. Here, the embedding factors k_i are fixed as (19) either via the same anomaly consideration as in eight dimensions [5] or via the geometric classification results on the K3 surface fiber Y_2 [13]. Note that r_i can take the values 0 through 3, where the trivial representation $\mathbf{R}_0 = 1$ with vanishing center charge is included to systematically express the Γ charge of \mathbf{R} . The vanishing of (37) turns out to restrict the topology of the curve classes b_i severely; one can in particular show that the genus g_i of each 7-brane locus is strictly positive,

$$g_i \geq 1, \quad (38)$$

which in the end leads to the violation of the Calabi-Yau condition, in that the residual discriminant Δ' turns out to have its class,

$$[\Delta'] = [\Delta] - 7(b_1 + b_2 + b_3) = \bar{K}_{B_2}^{\otimes 12} - 7(b_1 + b_2 + b_3), \quad (39)$$

lie outside of the Mori cone (see Appendix A for the proof). We therefore conclude that \mathbb{Z}_7 is ruled out as a 1-form gauge sector.

As it turns out, we can use the arguments along precisely the same line to also rule out \mathbb{Z}_8 , thereby establishing in particular the definite upper bound of 6 for every cyclic 1-form gauge factor as in (5).

The geometric classification results on K3 surfaces [13] also fix the embedding factors for the two double-factor outliers in (6), eventually ruling the latter out in a similar fashion. See Appendix B for the details.

B. F-theory vacua with $T = 0$

What remains to be analyzed is the geometry of the F-theory background Y_3 with base $B_2 = \mathbb{P}^2$, for which no tensor multiplets arise. In case the effective supergravity theories contained a strongly coupled conformal matter sector [29], one could blow up the geometry at the location of the conformal matter in the base as

$$\begin{array}{ccc} \pi: & Y_3 & \rightarrow & B_2 \\ & \uparrow & & \uparrow \\ \hat{\pi}: & \hat{Y}_3 & \rightarrow & \hat{B}_2 \end{array} \quad (40)$$

which brings in a tensor multiplet to the theory. The Mordell-Weil torsion $\text{MW}(\hat{Y}_3)_{\text{Tors}}$ of the blown-up geometry \hat{Y}_3 would then be subject to the same constraints as those derived earlier in Sec. III A. Crucially, the Mordell-Weil group is a birational invariant and hence the pair of birationally equivalent fibrations Y_3 and \hat{Y}_3 in (40) satisfy

$$\text{MW}(\hat{Y}_3) \simeq \text{MW}(Y_3). \quad (41)$$

In particular, we have

$$\text{MW}(\hat{Y}_3)_{\text{Tors}} \simeq \text{MW}(Y_3)_{\text{Tors}}, \quad (42)$$

indicating that the 1-form gauge sector of the effective theories with no tensor multiplets should also be subject to the constraints that apply to the theories with tensor multiplets, as long as conformal matter is present.

In the rest of this section we will thus focus on the theories without conformal matter. Let us suppose that the fibration (25) has a nontrivial Mordell-Weil torsion $\text{MW}(Y_3)_{\text{Tors}}$ with an m -torsional generator. The generic Weierstrass models for such a torsional geometry have the sections $f \in H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(12))$ and $g \in H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(18))$ naturally constrained to take a particular polynomial form in terms of a collection of sections,

$$\begin{aligned} a_{i,\alpha_i} &\in H^0(B_2, \bar{K}_{B_2}^{\otimes i}) \\ &= H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(3i)), \quad i = 1, 2, \dots, 6, \end{aligned} \quad (43)$$

as worked out in [9] for a list of torsion subgroups.¹² Here, the second index α_i for each i runs from 1 to $P_i \in \mathbb{Z}_{\geq 0}$ with P_i representing the number of degree- $3i$ sections, which can be 0, 1 or beyond. As an illustration, the ansatz derived for $\text{MW}(Y_3)_{\text{Tors}} = \mathbb{Z}_2$ takes the following form:

$$\begin{aligned} f &= a_{4,1} - \frac{1}{3}a_{2,1}^2, \\ g &= \frac{1}{27}a_{2,1}(2a_{2,1}^2 - 9a_{4,1}), \\ \Delta &= a_{4,1}^2(4a_{4,1} - a_{2,1}^2), \end{aligned} \quad (44)$$

where the discriminant Δ follows from the expressions for f and g . In our notation, the values of P_i in this example are

$$P_i = \begin{cases} 1 & \text{if } i = 2 \text{ or } 4 \\ 0 & \text{otherwise,} \end{cases} \quad (45)$$

and the generic Weierstrass models (44) are described by a total of two sections $a_{2,1}$ and $a_{4,1}$ of the line bundles $\bar{K}_{B_2}^{\otimes 2}$ and $\bar{K}_{B_2}^{\otimes 4}$, respectively.

Let us denote by $P(\geq 1)$ the total number of sections (43) that appear in the polynomial ansatz for f and g , i.e.,

$$P := \sum_{i=1}^6 P_i. \quad (46)$$

If $P \geq 2$, we can split the set of a_{i,α_i} into a pair of complementary proper subsets and tune the sections by demanding common linear factors, e.g., $z_{I=0,1}$, raised to appropriate powers as

$$a_{i,\alpha_i} = z_I^i b_{i,\alpha_i} \quad \text{for } I = 0, 1, \quad (47)$$

depending on which of the two subsets they belong to. Here, $z_{I=0,1}$ are part of the homogeneous coordinates $[z_0 : z_1 : z_2]$ of \mathbb{P}^2 and $b_{i,\alpha_i} \in H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2i))$ are generic sections.¹³ It then follows that the Weierstrass sections f and g have vanishing orders

$$(\text{ord}(f), \text{ord}(g)) = (4, 6) \quad (48)$$

at the point $[0:0:1]$, while along each of the two individual loci $z_{I=0,1} = 0$ at least one of the two orders is reduced. For instance, in the above example (44), we have $P = 2$ so that the tuning (47) goes as

$$a_{2,1} = z_0^2 b_{2,1}, \quad (49)$$

$$a_{4,1} = z_1^4 b_{4,1}, \quad (50)$$

which in turn leads to

$$f = z_1^4 b_{4,1} - \frac{1}{3}z_0^4 b_{2,1}^2, \quad (51)$$

$$g = \frac{1}{27}z_0^2(2z_0^4 b_{2,1}^2 - 9z_1^4 b_{4,1}), \quad (52)$$

exhibiting the codimension-two nonminimality at $[0:0:1]$ as expected by (48).

If $P = 1$ on the other hand, f and g are expressible as a power of a unique section,

$$a_{i_0} \in H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(3i_0)), \quad (53)$$

where i_0 divides $\text{gcd}(4, 6) = 2$, implying that $i_0 = 1$ or 2 . In the former case we have

$$f \propto a_1^4 \quad \text{and} \quad g \propto a_1^6, \quad (54)$$

leading to codimension-one nonminimal fibers along the cubic locus $a_1 = 0$, which we may thus discard. In the latter case we do not have such nonminimal singularities as

$$f \propto a_2^2 \quad \text{and} \quad g \propto a_2^3. \quad (55)$$

We may then tune the section a_2 to take the form

$$a_2 = z_0 z_1 b_2, \quad b_2 \in H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(4)), \quad (56)$$

¹²Similar constraints arise even when $\text{MW}(Y_3)_{\text{Tors}}$ has two generators, as also exemplified in the same reference.

¹³Note that the generic choice of b_{i,α_i} in (47) could in principle result in a partial Higgsing of the non-Abelian 0-form symmetries in case the sections a_{i,α_i} had not been generic to begin with before the tuning. However, this does not affect our discussions about the bounds on the 1-form gauge sector.

so that the theory develops a conformal matter sector at the point $[0:0:1]$.

Therefore, whether $P \geq 2$ or $P = 1$, the geometry can be tuned within the given m -torsional ansatz to develop an isolated codimension-two nonminimal fiber, allowing us to blow it up in the same fashion as in (40). The resulting effective theory thus acquires a tensor multiplet while the 1-form gauge sector remains intact under the blowup operation at least. Even though the prerequisite tuning operation might possibly have enhanced the 1-form gauge sector, the upper bounds obtained for theories with $T \geq 1$ would still apply to those with $T = 0$.

IV. HETEROTIC VIEWS ON THE BOUNDS

The F-theoretic bounds obtained in the previous section hinge on the special geometric role played by the rational fiber f , which is a 0-curve in B_2 .¹⁴ From the perspective of the effective theory as well, such a rational fiber f in B_2 plays a distinguished physical role since the D3 brane wrapped on it leads to an effective heterotic string. In fact such a realization of the heterotic string in F-theory has already played an important role in studying various swampland constraints such as the weak gravity conjectures [30–33] and the swampland distance conjecture [33–35], and also in constraining the $U(1)$ 0-form gauge sector [8]. In this section we will provide a clean physical intuition for the derived geometric bounds on the 1-form gauge sector in view of this heterotic string.

The gauge sector of the 6d supergravity theory then naturally splits into the perturbative sector that is “visible” to the heterotic string, hence leading to worldsheet global symmetries, and the nonperturbative sector that is invisible on the string worldsheet. Geometrically, the former arises from the 7-brane loci that intersect with f and the latter from the loci that do not, i.e., they are either a fibral or an exceptional \mathbb{P}^1 in B_2 . The inclusion relation (33) of the Mordell-Weil torsion then implies the effective theory cannot have (a proper subset of) the center 1-form symmetry of such a nonperturbative sector gauged by itself. In other words, every center gauging must involve the perturbative sector.

The perturbative nature of the 1-form gauge symmetries is reminiscent of what is known about the $U(1)$ 0-form gauge symmetries. It was addressed in [8] that the latter should embed into the current algebra of the perturbative heterotic string, based on which the number of $U(1)$ gauge factors was bounded. In geometric terms, this led to the bold proposal for a definite upper bound of 16 for $\text{rank}(\text{MW}(Y_3))$, which concerns the free part of the $\text{MW}(Y_3)$. The current proposal about the 1-form gauge symmetries on the other hand concerns the torsion part of

$\text{MW}(Y_3)$ and demands in particular that each cyclic factor of $\text{MW}(Y_3)_{\text{Tors}}$ be a finite cyclic subgroup of $E_8 \times E_8$ or $SO(32)$. Interestingly, the maximal such cyclic subgroup occurs for E_8 and is known to be \mathbb{Z}_6 (see, e.g., [36]), agreeing with the constraints derived in III.

At this point one may wonder how \mathbb{Z}_7 and \mathbb{Z}_8 could be realized as a 1-form gauge factor in 8d supergravity theories even if they are not a subgroup of E_8 . This naively seems to contradict the developed heterotic perspective. However, a simple explanation comes from the fact that the internal space for the 8d heterotic string theory is simply a two-torus T^2 . The latter leads to a pair of Kaluza-Klein $U(1)$ s that can in principle enhance to a non-Abelian gauge group in a nongeometric background and also mix with the 10d-induced perturbative sector of maximal rank 16. The perturbative 0-form gauge sector may thus enhance in eight dimensions to a rank-18 non-Abelian group.

Motivated by this eight-dimensional phenomenon, we must thus revisit the potential intricate enhancement of the 10d-induced perturbative sector to the 6d $\mathcal{N} = (1, 0)$ supergravity theories. While imposing only the minimal supersymmetry sets the heterotic internal space as a K3 surface, there may still arise a pair of Kaluza-Klein $U(1)$ s, e.g., for a singular geometry. Nevertheless, the indirect observation in [11] via modular curves that \mathbb{Z}_7 and \mathbb{Z}_8 cannot serve as a Mordell-Weil torsion informs us that such an enhanced 0-form sector cannot be quotiented by those cyclic groups. We are currently not aware of a simple reasoning based on heterotic worldsheet symmetry that can rule out $\mathbb{Z}_{\geq 7}$ even when the mixing of the 10d-induced perturbative sector mixes with Kaluza-Klein $U(1)$ s. In Sec. V, however, we will envision the possibility of having the 6d bulk supergravity theory decompactify to an 8d one at an appropriate limiting regime in the moduli space, where the field-theoretic consideration in [5] would eventually rule those high-order cyclic 1-form sector.

Going back to the simple situation where no such intricate mixing arises in the perturbative sector, the derived bound of 6 for each cyclic 1-form gauge factor is also consistent with the worldsheet current generated by the small instanton transitions. We exploited the latter in Sec. III B to address the geometric bounds for F-theory vacua with $T = 0$. The codimension-two nonminimal singularities with vanishing orders (48) are well known to correspond to the tensionless E-string, which exhibits an E_8 flavor symmetry. Depending on the way this string couples to the bulk gravitational theory, parts of the flavor symmetry are typically gauged. This gauging naturally constrains the representations, $U(1)$ charges and the dimension of the residual Higgs branch (see Ref. [37]). Thus, when performing such an E-string transition, we expect the string to couple to the global 1-form gauge group Γ as well. However, such a (partial) gauging is only possible if Γ is a subgroup of E_8 , and hence $|\Gamma|$ cannot exceed 6 in the same vein as the heterotic case.

¹⁴Recall that for $B_2 = \mathbb{P}^2$, we have even enforced a codimension-two nonminimal fiber to be able to blow B_2 up so that a 0-curve f may be present in the blown-up base.

V. DISCUSSION AND APPLICATIONS

In this paper, the general upper bounds on the 1-form gauge sector of 8d supergravity theories have proven to apply also to F-theory compactifications to six dimensions. By focusing on the 6d F-theory vacua conserving precisely eight real supercharges, we have in fact ruled out some of the high-order symmetries that are realizable in 8d supergravity, thereby proposing the following reduced set of viable 1-form gauge symmetries for 6d F-theory vacua:

$$\mathbb{Z}_m \quad \text{for } 2 \leq m \leq 6 \quad (57)$$

$$\mathbb{Z}_n \times \mathbb{Z}_m \quad \text{for } (n, m) = (2, 2), (2, 4), (3, 3). \quad (58)$$

The geometric derivation of this result has hinged upon the distinguished role played by the ubiquitous rational fibration within the base twofold B_2 other than \mathbb{P}^2 . As a key step we have identified the elliptic K3 surface Y_2 with the following property:

$$\text{MW}(Y_3)_{\text{Tors}} \subseteq \text{MW}(Y_2)_{\text{Tors}}, \quad (59)$$

by restricting the elliptic fibration Y_3 over B_2 to the generic rational fiber of B_2 . For F-theory vacua with $T \geq 1$ tensor multiplets, definite upper bounds on the 6d 1-form gauge sector have thus been established as the known geometric bounds on $\text{MW}(Y_2)_{\text{Tors}}$. Of the allowed set of Mordell-Weil torsions for K3 surfaces, the following four,

$$\mathbb{Z}_m \quad \text{for } m = 7, 8, \quad (60)$$

$$\mathbb{Z}_n \times \mathbb{Z}_m \quad \text{for } (n, m) = (2, 6), (4, 4), \quad (61)$$

have been argued not to be viable for $\text{MW}(Y_3)_{\text{Tors}}$ by also requiring absence of field-theoretic anomaly, while it is still possible to fiber the relevant rigid K3 surfaces nontrivially over \mathbb{P}^1 .

With the 1-form gauge sector of the 6d F-theory vacua identified as $\text{MW}(Y_3)_{\text{Tors}}$, this leads to the candidates (57) and (58) for 1-form gauge symmetries. On the other hand, for F-theory on elliptic Y_3 over $B_2 = \mathbb{P}^2$, it has been shown that the theory can be deformed to develop a conformal matter without reducing the 1-form gauge sector; since the tensor transition keeps the latter intact, the same upper bounds apply also to F-theory vacua without tensor multiplets.

While the bounds (57) and (58) originate from the geometric constraints on the internal space of F-theory, we have also provided a worldsheet interpretation of this result. That a central role is played by the rational fiber within B_2 strongly suggests that the physical object governing the geometric bounds must be the effective heterotic string; the latter can be identified, e.g., as in [8,30–33], as a solitonic string arising from the $D3$ brane wrapped on the rational fiber. Indeed, we have been able to

rederive the upper bound 6 for the cyclic orders in (57) and (58) from the fact that \mathbb{Z}_6 is the maximal cyclic Abelian subgroup of E_8 , at least when the perturbative sector of the six-dimensional bulk theory does not acquire any intricate mixing with the Kaluza-Klein $U(1)$ s. In such a simple situation, the validity of this heterotic interpretation has been addressed by showing that every discrete quotient action on the 0-form gauge sector necessarily invokes the perturbative sector. Interestingly, such a desired perturbative nature of the 1-form gauge sector has immediately followed from the inclusion relation (59) derived in F-theory. We have also observed as an aside that the upper bound of 6 is also consistent with the E-string transitions, where the relevant Γ quotient needs to embed into the E_8 flavor group.

To some extent the advocated perturbative nature of the 1-form gauge sector nicely parallels that of the $U(1)$ 0-form gauge sector of 6d F-theory vacua with $T \geq 1$. It was proven in [8] that every $U(1)$ gauge factor, if present in the 0-form gauge sector of such theories, must embed into the perturbative current algebra of the effective heterotic string. Interestingly, the geometric origins of the $U(1)$ 0-form and the 1-form gauge sectors are, respectively, the free and the torsional sectors of the Mordell-Weil group. The heterotic interpretation of the Abelian 0-form gauge sector led in [8] to the geometric conjecture that

$$\text{rank}(\text{MW}(Y_3)) \leq 16, \quad (62)$$

for an elliptic Calabi-Yau threefold. In fact, already from the purely geometric perspective, the inclusion (59) leads to a milder bound of 18 since any elliptic K3 surface Y_2 obeys

$$\text{rank}(\text{MW}(Y_2)) \leq 18, \quad (63)$$

which follows from the Tate-Shioda-Wazir theorem, so that

$$\text{rank}(\text{MW}(Y_3)) \leq \text{rank}(\text{MW}(Y_2)) \leq 18. \quad (64)$$

There is a chance that the bound (62) might possibly be violated since the rank-16 current algebra of the 10d heterotic string can in principle enhance in a compactified theory, for instance, by mixing with a pair of Kaluza-Klein $U(1)$ s. Whether the milder upper bound (64) could indeed be realized in an explicit 6d F-theory vacuum is yet to be clarified.

While both the 1-form and the $U(1)$ 0-form gauge sectors are perturbatively realized from the heterotic perspective, the addressed extension of the constraints (57) and (58) on the former to F-theory models with $T = 0$ does not have a natural counterpart on the latter. Those $T = 0$ models can always be deformed to develop a conformal matter sector, which then lead to $T \geq 1$ models through a tensor transition. It was thus crucial that the deformation of our choice does not reduce the 1-form

gauge sector, in arguing that the upper bounds on the 1-form sector of the latter models also bound that of the former. However, without generic ansatz available for $U(1)$ models in F-theory for a given positive rank, it is not clear how to find a systematic way to deform the models in a similar fashion, not reducing the $U(1)$ 0-form sector along the way. The rank bound (64) thus remains conjectural for $T = 0$ models.

Let us end with several extensions and applications of the current work, organized by the dimension D of the spacetime in which the physics phenomena of interest take place.

A. Decompactification ($D=8$)

1. Beyond the geometric bounds

Our geometric bounds on the 1-form gauge sector in 6d F-theory have been built upon the analogous 8d bounds. However, there is a notable difference between the 6d and the 8d constraints: the latter can be derived by the absence of gauge anomalies, which provides a purely effective field theoretic perspective. One may then naturally attempt to also promote the geometric bounds in 6d to a general swampland constraint. The geometric arguments in this paper suggest that one could achieve this by decompactifying the starting 6d $\mathcal{N} = (1, 0)$ supergravity theory to an 8d supergravity endowed with the defect sector of a particular kind. Such a decompactification, if viewed as an F-theoretic deformation, corresponds to taking an infinite-distance limit in the Kähler moduli space, where the base \mathbb{P}^1 of (28) expands while the fibral \mathbb{P}^1 remains finite. One could thus analyze the obvious counterpart limit in supergravity to show that the resulting theory indeed decompactifies to begin with. Since this supergravity limit itself lies at infinite distance, it is natural to expect decompactification to occur, in line with the emergent string conjecture [38]. Importantly, the 1-form gauge sector needs to be in good control along the way, as is the case for the prescribed 8d decompactification in F-theory. We leave it to future work to develop a general effective-theoretic picture for the desired, controlled decompactification.

2. Beyond the minimal fibers

While the aforementioned decompactification would occur in F-theory at infinite distance in the Kähler moduli space, F-theory also provides a qualitatively distinct route to decompactification via complex structure limits, as initially analyzed for 8d F-theory vacua in [34,35] and generalized to 6d vacua in [39]. In these infinite-distance complex structure limits, several 7-brane stacks each carrying a 0-form Lie-type gauge algebra collide to enhance the symmetry to an emergent affine algebra. Geometrically, such an affine enhancement of the 0-form gauge algebra is realized by the nonminimal singularities in codimension-one fibers of the elliptic Calabi-Yau manifold.

In view of this, it is intriguing that the geometric bounds (4) and (6) for elliptic Calabi-Yau manifolds were understood in [11] as the orders of the cyclic groups beyond which codimension-two nonminimal singularities at infinite distance arise. In other words, demanding that the 1-form gauge sector violate the bounds enforces the brane moduli to sit at infinite distance, resulting in a decompactification [39]. We leave a thorough investigation of this intriguing connection with infinite distance limits to future work.

B. Mirror ($D=6$)

1. Bounds on discrete 0-form gauge factors

An elliptically fibered Calabi-Yau threefold Y_3 with an n -torsional section is conjectured¹⁵ to be paired with a genus-one fibered mirror dual threefold Z_3 with an n section. In view of the torsional bounds (5), it is therefore conjectured that the n -section geometry Z_3 should also obey

$$n \leq 6. \quad (65)$$

Explicit geometric constructions known to date of the multisection geometry are consistent with the bound (65). The highest n that has so far been achieved is 5, which is realized via non-Abelian gauged linear sigma models [45] and attempts are made to construct explicit examples with a 6-section.

While the bound (65) for multisection geometry arises from the mirror conjecture, which is purely geometric and construction dependent in nature, the heterotic interpretation in Sec. IV leads to an interesting string-theoretic intuition as follows. We first recall that compactifications of F-theory on Y_3 and Z_3 result in effective theories with \mathbb{Z}_n 1-form and 0-form gauge symmetries, respectively. In particular, the constraint (65) is understood as the upper bound for the discrete 0-form gauge symmetry \mathbb{Z}_n . From the heterotic perspective, one can naturally obtain the latter by embedding¹⁶ into E_8 a gauge bundle with structure group G of the form

$$G = \hat{G}/\mathbb{Z}_n \subset E_8, \quad (66)$$

¹⁵The mirror conjecture for fibrations is based on the mirror duality established for the elliptic curves and their generic structures. Strong evidence of the conjecture arises from explicit toric constructions [40,41] and more generally in the context of gauged linear sigma models [42]. Also see Refs. [43,44] for the heterotic/F-theory duality perspective.

¹⁶To be most general, we should consider embeddings of a gauge bundle into the full 10d heterotic current algebra potentially mixed with a pair of Kaluza-Klein $U(1)$ s and rearranged afterwards. Here, we assume embeddings into E_8 only for simplicity of presentation; the physical intuition behind the mirror conjecture works in full generality.

where \hat{G} is a simply-connected Lie group (or a product of such factors). This results in the 0-form gauge symmetry with a discrete \mathbb{Z}_n factor,

$$H = \mathbb{Z}_n \times \hat{H} \subset E_8, \quad (67)$$

where \hat{H} is the commutant of \hat{G} in E_8 . To turn the tables around, one can consider a gauge bundle with structure group \hat{H} endowed with an \mathbb{Z}_n -holonomy instanton to obtain G as the effective gauge group, so that a discrete \mathbb{Z}_n 1-form symmetry is realized instead.

The heterotic string thus provides us with the physical intuition behind the geometric mirror conjecture, pairing an elliptic Calabi-Yau threefold Y_3 with a genus-one fibered Z_3 . In particular, we conclude from this dual heterotic perspective that the upper bound 6 must apply not only to the 1-form but also to the 0-form factors, conforming with the geometric conjecture (65).

Strictly speaking, however, the physical argument above is clear only for F-theory models with $T \geq 1$. It would be worth clarifying if this heterotic perspective extends to the models with $T = 0$.

2. Bounds on $U(1)$ charges

We emphasize that the aforementioned heterotic intuition behind the mirror conjecture relies heavily on the perturbative nature of the 1-form gauge sector as addressed in Sec. IV, as well as that of the discrete 0-form gauge sector. The latter follows from the observation in [8] that $U(1)$ 0-form gauge factors of 6d F-theory vacua should lie entirely in the perturbative sector. Presuming that a \mathbb{Z}_n 0-form gauge factor arises from the Higgsing of $U(1)$, the former should also be perturbative in nature.

In view of this Higgsing phenomenon, we are naturally lead to the closely related conjecture that the $U(1)$ charges q of the matter states must be subject to

$$q \leq 6, \quad (68)$$

for any F-theory models with $T \geq 1$. This conjecture is particularly interesting in view of recent attempts of classifying possible $U(1)$ charges in 6d F-theory (see e.g. [46–51]).

While the conjectured charge bound (68) does not necessarily apply to the models on $B_2 = \mathbb{P}^2$, it is particularly interesting to note that models on Hirzebruch surfaces were proposed in [50] to have matter states with a $U(1)$ charge as large as 21. The apparent conflict between the conjectured bound (68) and the proposed models with such a large $U(1)$ charge would in principle be resolved by carefully analyzing the specific Higgsing processes assumed. On the one hand, particular chains of Higgsings are required in [50], both for non-Abelian and for (intermediate) multiple- $U(1)$ gauge theories; if any of them fail to be realized in F-theory the proposed models

will not exist. On the other hand, our bound (68) assumes that any matter state with a $U(1)$ charge q could acquire a vacuum expectation value (VEV) to break $U(1)$ to a \mathbb{Z}_q 0-form gauge factor. It will be interesting to clarify which Higgsing assumptions are invalid and also find the rationale behind such an obstruction, from the perspectives of both the geometry and the effective theory.¹⁷

C. Compactification ($D=4$)

As already emphasized, the key to our geometric derivation of the bounds (4) is the presence of a \mathbb{P}^1 fibration within the F-theory base B_2 , with the exception of $B_2 = \mathbb{P}^2$, which, nevertheless, can be blown up without reducing the 1-form gauge sector. In order to extend our results to 4d F-theory models, we must argue that B_3 also admits a \mathbb{P}^1 fibration. Interestingly, a birational characterization of the F-theory base threefolds has recently been accomplished in terms of the so-called Fano towers of fibrations [53,54]. This suggests that the following two exceptions to a \mathbb{P}^1 fibration should be analyzed:

- (1) $B_3 = \mathbb{P}^3$.
- (2) $B_3 =$ a \mathbb{P}^2 fibration over \mathbb{P}^1 .

In the former case, the same tuning used for $B_2 = \mathbb{P}^2$ in Sec. III B enables us to develop codimension-two non-minimal fibers along a line, which, once blown up, will turn $B_3 = \mathbb{P}^3$ into a \mathbb{P}^2 fibration over \mathbb{P}^1 , which falls into the latter type. Similarly, the base B_3 of this latter type can be tuned within the torsional ansatz in a similar fashion, this time by choosing two toric coordinates of the \mathbb{P}^2 fiber as the analog of the linear factors in (47). The common zero locus of these two toric divisors is a section of the \mathbb{P}^2 fibration. Blowing up the threefold B_3 along this section results in \hat{B}_3 , which is a fibration of the Hirzebruch surfaces \mathbb{F}_1 , which may as well be interpreted as a \mathbb{P}^1 fibration over yet another Hirzebruch surface. This assures that the torsional constraints (4) persist in the elliptic Calabi-Yau fourfolds $Y_4 \rightarrow B_3$, thereby establishing the same bounds also on the 1-form gauge sector of 4d $\mathcal{N} = 1$ F-theory vacua.

Note, however, that this only constrains the 1-form gauge sector realized by the geometry alone. Unlike in 6d F-theory models, the 0-form gauge group of a 7-brane stack may be broken by gauge fluxes, which may in principle serve as an independent source of a 1-form gauge symmetry even for the nonperturbative sector, to which our heterotic argument based on E_8 embedding does not apply.

¹⁷For instance, for the $U(1)$ Higgsing to be realizable in the effective theory, it is necessary that more than one Higgs field exists and acquires a VEV, so that D-flatness is satisfied. From this effective theory consideration alone, our conjectured bound (65) on discrete 0-form gauge factor does not rule out any potential models in which precisely one massless field is charged under $U(1)$ with a large charge $q > 6$. It is not clear, however, if such models can exist at all to begin with (see, e.g., [52]).

Relatedly, it has recently been proposed [55] that flux breaking of a non-Abelian group may possibly lead to the Cartan $U(1)$ s, a particular linear combination of which may admit a charged matter with charge as large as 657, far beyond the torsional bound of 6. Unless the matter fields with such a large $U(1)$ charge $q \gg 6$ face a fundamental obstruction in acquiring a VEV, the $U(1)$ 0-form symmetry of the theory can be Higgsed to \mathbb{Z}_q . In view of the mirror conjecture, this may serve as an indication for the existence of a 4d F-theory model with a \mathbb{Z}_q 1-form gauge factor, where q violates the torsional bound.

It would be interesting to extend our constraints on the perturbative 1-form gauge symmetry to such a nonperturbative sector.

D. Decomposition ($D=2$)

Another interesting direction worth pursuing further arises from the worldsheet theories of the BPS noncritical strings that couple to bulk tensor fields. As was briefly mentioned in Sec. II, many of the seemingly consistent configurations for the 0-form gauge sector can be successfully ruled out [26] by requiring the unitarity of the 2d superconformal field theories (SCFTs), to which the worldsheet theories of such BPS strings flow in the infrared. However, this approach has not been strong enough to rule out all configurations known so far, which are believed to eventually be ruled out. Also, the consistency of the exceptional configurations such as $G = SU(7)^3/\mathbb{Z}_7$, which arise from the choices of the 1-form gauge sector with (n, m) from (6), remain agnostic from the unitarity consideration of such string probes alone.

On the other hand, one might expect that the 1-form gauge sector descends to the 2d SCFTs as a global symmetry, as is the case for the 0-form sector [56]. It is therefore worth questioning if any other world-sheet-theoretic notions, e.g., that of decomposition [57], which addresses the equivalence between two-dimensional quantum field theories with 1-form global symmetries and disjoint unions of other theories, might possibly lead us to novel consistency constraints on the 6d bulk supergravity theories. It would be very interesting to clarify if and how the string worldsheet theories reflect the nonsimply-connected nature of the 1-form global symmetry in a nontrivial fashion.

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APPENDIX A: INCONSISTENCY OF \mathbb{Z}_7 AND \mathbb{Z}_8 1-FORM GAUGE FACTORS

In this Appendix we provide the detailed explanation on the inconsistency in geometry arising from demanding the 1-form gauge sector to take the form

$$\Gamma = \mathbb{Z}_7, \quad (\text{A1})$$

in an F-theory compactification to six dimensions. Let us first recall from Sec. III A that the choice (A1) made for the 1-form gauge sector uniquely fixes the 0-form gauge group G as

$$G = SU(7)^3/\mathbb{Z}_7, \quad (\text{A2})$$

which follows from the geometric classification of elliptic K3 surfaces [13]. Here, each simply-connected factor, $\hat{G}_i = SU(7)$ for $i = 1, 2, 3$, arises, in the language of the Calabi-Yau threefold Y_3 , from the singular fibers of Kodaira type I_7 along the curve in B_2 of class b_i . We denote the genus and self-intersection number of each such curve by g_i and n_i , respectively. As explained around (37), the embedding factors k_i are given as

$$(k_1, k_2, k_3) = (1, 2, 3), \quad (\text{A3})$$

so that the four representations $(\mathbf{R}_r)_{r=0}^4$ under each $SU(7)_i$ contribute to the Γ charge, $q_\Gamma(\mathbf{R})$, of $\mathbf{R} = (\mathbf{R}_r)_{r=1}^3$ as

$$\begin{aligned} q_{Z_{n_1}}(\mathbf{R}_r) \cdot k_1 &= 0, \frac{1}{7}, \frac{2}{7}, 0, \\ q_{Z_{n_2}}(\mathbf{R}_r) \cdot k_2 &= 0, \frac{2}{7}, \frac{4}{7}, 0, \\ q_{Z_{n_3}}(\mathbf{R}_r) \cdot k_3 &= 0, \frac{3}{7}, \frac{6}{7}, 0, \end{aligned} \quad (\text{A4})$$

if they occur in \mathbf{R} . Here, \mathbf{R}_r for $r = 0, 1, 2, 3$ are, respectively, $\mathbf{1}$, \mathbf{F} , $\mathbf{\Lambda}$ and \mathbf{Adj} with the multiplicities, $\mu_{r=1,2,3}$, of the charged ones given as (23).

Suppose now that a given representation \mathbf{R} appears in the spectrum. Then, its Γ charge $q_\Gamma(\mathbf{R})$ in (37) necessarily vanishes modulo 1. In view of (A4), we thus observe for a fixed i th simple factor that if \mathbf{R} transforms as \mathbf{R}_1 under G_i then it should transform as \mathbf{R}_2 under some other $G_{i'}$. In other words, if $r_i = 1$, then $r_{i'} = 2$ for some $i' \neq i$. Similarly, it also follows that if $r_i = 2$, then $r_{i'} = 1$ for some $i' \neq i$. This simple observation then leads to the following constraints on the total multiplicities μ_1 and μ_2 for the given i th sector:

$$\mu_1 = n_i + 16(1 - g_i) \stackrel{!}{=} 21m_1, \quad (\text{A5})$$

$$\mu_2 = n_i + 1 - g_i \stackrel{!}{=} 7m_2, \quad (\text{A6})$$

for $m_1, m_2 \in \mathbb{Z}_{\geq 0}$, where we have used (23) for the multiplicities. Upon subtracting (A6) from (A5), however, we obtain

$$15(1 - g_i) = 7(3m_1 - m_2), \quad (\text{A7})$$

indicating that

$$g_i \equiv 1 \pmod{7}, \quad (\text{A8})$$

and, hence, in particular that

$$g_i \geq 1. \quad (\text{A9})$$

Note that analogous calculations can be performed when the 1-form gauge sector takes the form

$$\Gamma = \mathbb{Z}_8, \quad (\text{A10})$$

which results in the same conclusion (A9) again via the divisibility argument. In this case, the 0-form gauge group is uniquely fixed via the K3 geometry [13] as

$$G = (SU(2) \times SU(4) \times SU(8)^2)/\mathbb{Z}_8, \quad (\text{A11})$$

and the embedding factors as

$$(k_1, k_2, k_3, k_4) = (1, 1, 1, 5). \quad (\text{A12})$$

Let us now proceed to show that (A9), which should work for any choice of i_0 , implies violation of the Calabi-Yau condition, in the sense that the class (39) of the residual discriminant Δ' is calculated not to be effective. This way, we will have concluded that \mathbb{Z}_7 cannot be realized as a 1-form gauge sector. In the rest of this Appendix, we thus prove this inconsistency by considering in turn the following three types of the base twofold:

$$B_2 = \mathbb{P}^2, \quad \mathbb{F}_a \quad \text{and} \quad \text{Bl}^k \mathbb{F}_a \quad \text{with} \quad k > 0. \quad (\text{A13})$$

1. Type 1: $B_2 = \mathbb{P}^2$

The genus constraint (A9) implies that the curve class b_i is not too small, i.e.,

$$b_i = l_i H, \quad \text{with} \quad l_i \geq 3, \quad (\text{A14})$$

where H is the hyperplane class. Since the i th discriminantal component supports a singular fibers of Kodaira type I_7 , the class $[\Delta']$ obeys

$$[\Delta'] = \bar{K}_{B_2}^{\otimes 12} - 7(b_1 + b_2 + b_3) \quad (\text{A15})$$

$$= (36 - 7(l_1 + l_2 + l_3))H \quad (\text{A16})$$

$$\leq -27H, \quad (\text{A17})$$

in contradiction with the required effectiveness of $[\Delta']$.

2. Type 2: $B_2 = \mathbb{F}_a$

Let us denote the (irreducible) curve class b_i as

$$b_i = s_i S + f_i F, \quad s_i, f_i \geq 0, \quad (\text{A18})$$

where S and F are the class of the exceptional section with self-intersection $-a$ and that of the fiber, respectively. We need to have the residual discriminant effective, whose class is computed as

$$[\Delta'] = \bar{K}_{B_2}^{\otimes 12} - 7(b_1 + b_2 + b_3) \quad (\text{A19})$$

$$= (24 - 7s)S + (24 + 12a - 7f)F, \quad (\text{A20})$$

where

$$s := s_1 + s_2 + s_3, \quad f := f_1 + f_2 + f_3. \quad (\text{A21})$$

Since S and F generate the Mori cone of B_2 , it follows in particular that

$$24 - 7s > 0, \quad (\text{A22})$$

and hence,

$$0 \leq s_i \leq 3, \quad \forall i. \quad (\text{A23})$$

In the meantime, one can calculate the genus g_i via the Hirzebruch-Riemann-Roch theorem as

$$2g_i - 2 = b_i \cdot (b_i - \bar{K}_{B_2}), \quad (\text{A24})$$

which leads to $g_i = 0$ for $s_i = 0$ and 1. The genus constraint (A9), together with (A23) thus leave only two possibilities left: $s_i = 2$ or 3. In particular, we thus have

$$s_i \geq 2, \quad \forall i, \quad (\text{A25})$$

which, however, contradicts (A22).

3. Type 3: $B_2 = \text{Bl}^k \mathbb{F}_a$ ($k > 0$)

We can immediately deduce the inconsistency of this most general case from that of the Hirzebruch case. To this end, let us consider the morphism,

$$\nu: B_2 \rightarrow \mathbb{F}_a, \quad (\text{A26})$$

which blows down the exceptional divisors E_l on B_2 . The anticanonical classes of B_2 and \mathbb{F}_a are naturally related by

$$\bar{K}_{B_2} = \nu^*(\bar{K}_{\mathbb{F}_a}) + \sum_l c_l E_l, \quad (\text{A27})$$

for the coefficients $c_l = -1$.

Let us now denote the curve class b_i as

$$b_i = s_i S + f_i F + \sum_l e_{i,l} E_l, \quad (\text{A28})$$

where by abuse of notation we do not distinguish the classes S and F of the section and the fiber on \mathbb{F}_a from their pullbacks under ν . Then, $[\Delta']$ is computed as

$$\begin{aligned} [\Delta'] &= \bar{K}_{B_2}^{\otimes 12} - 7(b_1 + b_2 + b_3) \\ &= (24 - 7s)S + (24 + 12a - 7f)F \end{aligned} \quad (\text{A29})$$

$$+ \sum_l (12c_l - 7e_l)E_l, \quad (\text{A30})$$

where $e_l := e_{1,l} + e_{2,l} + e_{3,l}$ and s and f are again defined as (A21). Effectiveness of $[\Delta']$ implies that of its push forward,

$$\nu_*([\Delta']) = (24 - 7s)S + (24 + 12a - 7f)F, \quad (\text{A31})$$

so that we are still subject to (A22) and, hence, also to (A23). Here, we have used $s_i \geq 0$, which follows from the requirement that $\nu_*(b_i) = s_i S + f_i F$ is effective on \mathbb{F}_a . Invoking the Hirzebruch-Riemann-Roch theorem (A24), we once again learn that $g_i = 0$ for $s_i = 0$ and 1, and hence that $s_i \geq 2$, contradicting (A22).

APPENDIX B: INCONSISTENCY OF $\mathbb{Z}_4 \times \mathbb{Z}_4$ AND $\mathbb{Z}_2 \times \mathbb{Z}_6$ 1-FORM GAUGE GROUPS

In this Appendix we show that the 1-form gauge sector of the form

$$\Gamma = \mathbb{Z}_4 \times \mathbb{Z}_4 \quad \text{or} \quad \mathbb{Z}_2 \times \mathbb{Z}_6 \quad (\text{B1})$$

leads to inconsistency in geometry. The arguments will largely parallel those in Appendix A where the $\Gamma = \mathbb{Z}_7$ case was analyzed in detail. As in the latter case, demanding the 1-form gauge sector to take either of the two cases (B1) highly constrains the geometry of K3 surfaces so that the 0-form gauge group G , as well as the embedding factors of the Γ action on its simply-connected cover \hat{G} , is uniquely fixed. Specifically, for $\Gamma = \mathbb{Z}_4 \times \mathbb{Z}_4$, the global gauge group is determined as

$$G = SU(4)^6 / (\mathbb{Z}_4 \times \mathbb{Z}_4) \quad (\text{B2})$$

with the embedding factors of the order-4 generators given by [58]

$$k^{(4,1)} = (0, 2, 1, 1, 1, 1), \quad k^{(4,2)} = (1, 1, 0, 2, 3, 1). \quad (\text{B3})$$

On the other hand, for $\Gamma = \mathbb{Z}_2 \times \mathbb{Z}_6$ we have the gauge group

$$G = (SU(2)^3 \times SU(6)^3) / (\mathbb{Z}_2 \times \mathbb{Z}_6), \quad (\text{B4})$$

where the \mathbb{Z}_2 and \mathbb{Z}_6 quotients act with the embedding factors,

$$\begin{aligned} k^{(2)} &= (0, 1, 1, 0, 3, 3), \\ k^{(6)} &= (1, 1, 0, 5, 5, 2). \end{aligned} \quad (\text{B5})$$

Similarly to the $\Gamma = \mathbb{Z}_7$ case in Appendix A, the embedding structures (B3) and (B5) for the double-factor Γ do not allow for Γ -neutral representations, except possibly for high-rank tensor representations, unless the adjoint representations are involved.¹⁸ Then, a contradiction arises once again as the six nonrational discriminantal components result in a noneffective residual discriminant Δ' . We will close this Appendix by sketching the latter.

First, for the base twofold $B_2 = \mathbb{P}^2$, the six simple factors of both gauge groups, (B2) and (B4), are associated with the curve classes,

$$b_i = l_i H, \quad i = 1, \dots, 6, \quad (\text{B6})$$

of the non-Abelian loci, each supporting singular fibers of Kodaira type I_{n_i} with $n_i = 2, 4$, or 6. Here, H is the hyperplane class and the individual degrees l_i must obey $l_i \geq 3$ so that the genus of each curve may be strictly positive. Therefore, for each 1-form gauge group Γ in (B1), the combined class,

$$b := \sum_{i=1}^6 n_i b_i, \quad (\text{B7})$$

of the collective locus of the I_{n_i} loci, weighted with the individual vanishing orders n_i , is subject to

$$\Gamma = \mathbb{Z}_4 \times \mathbb{Z}_4: \quad b = 4 \left(\sum_{i=1}^6 l_i \right) H \geq 72H, \quad (\text{B8})$$

$$\Gamma = \mathbb{Z}_2 \times \mathbb{Z}_6: \quad b = 2 \left(\sum_{i=1}^3 l_i \right) H + 6 \left(\sum_{i=4}^6 l_i \right) H \geq 72H. \quad (\text{B9})$$

The residual discriminant thus receives an upper bound,

$$[\Delta'] = \bar{K}_{\mathbb{P}^2}^{\otimes 12} - b \leq -36H, \quad (\text{B10})$$

which contradicts its effectiveness.

¹⁸In the $\Gamma = \mathbb{Z}_2 \times \mathbb{Z}_6$ case, given the embedding factors (B5), lower-rank tensors of $SU(6)^3$, e.g., $(\mathbf{6}, \mathbf{6}, \mathbf{6})$, can be used to form Γ -neutral representations. However, we believe that no brane configurations can realize such neutral representations at finite distance.

Essentially the same line of arguments can be applied to the base twofolds $B_2 = \mathbb{F}_a$ or their blowups; we will present the analysis of the former type, which straightforwardly generalizes to the latter as in Appendix A. Denoting the curve classes of the I_{n_i} loci as

$$b_i = s_i S + f_i F, \quad i = 1, \dots, 6, \quad (\text{B11})$$

in terms of the section and the fiber classes, S and F , of \mathbb{F}_a , we require that the sectional degrees s_i obey $s_i \geq 2$ so that each curve may have a strictly positive genus. Next we turn to the combined class,

$$b := \sum_{i=1}^6 n_i b_i, \quad (\text{B12})$$

of the collective locus, especially to its sectional degree,

$$s := \sum_{i=1}^6 n_i s_i. \quad (\text{B13})$$

For each 1-form gauge group Γ in (B1), the aforementioned constraints $s_i \geq 2$ on the individual degrees lead to

$$\Gamma = \mathbb{Z}_4 \times \mathbb{Z}_4: s = 4 \sum_{i=1}^6 s_i \geq 48, \quad (\text{B14})$$

$$\Gamma = \mathbb{Z}_2 \times \mathbb{Z}_6: s = 2 \sum_{i=1}^3 s_i + 6 \sum_{i=4}^6 s_i \geq 48. \quad (\text{B15})$$

Accordingly, the residual discriminant,

$$[\Delta'] = \bar{K}_{\mathbb{P}^2}^{\otimes 12} - b, \quad (\text{B16})$$

receives an upper bound on its sectional degree,

$$24 - s \leq -24, \quad (\text{B17})$$

contradicting the effectiveness of $[\Delta']$.

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