Local term in the anomaly-induced action of Weyl quantum gravity

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The finite local conformally noninvariant R^2 term emerges in the one-loop effective action of the model of quantum gravity based on the Weyl-squared classical action. This term is related to the $\Box R$ contribution to the conformal anomaly, which in a wide class of regularization schemes is determined by the second Schwinger-DeWitt (or Gilkey-Seeley) coefficient of the heat kernel expansion for inverse propagators of the theory. The calculation of this term requires evaluating the contributions of the fourth-order derivative minimal and of the second-order nonminimal operators in the tensor and vector sectors of the theory, corresponding to metric, ghost, and gauge-fixing operators. To ensure the correctness of existing formulas, we derived (and confirmed) the result using a special technique of calculations, based on the heat-kernel representation of the Euclidean Green's function and the method of universal functional traces.

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I. INTRODUCTION

Conformal models play a very special role in modern quantum field theory and there is an extensive literature about various aspects of these theories. One of the important facts in even spacetime dimensions is that the local conformal symmetry is violated by the conformal (trace) anomaly $\langle T^\mu{}_\mu \rangle$, starting from the one-loop level [1,2]. Violation of local conformal symmetry comes in the form of local and nonlocal terms in the effective action, generating this anomaly. Breakdown of local Weyl invariance in classically conformally invariant theories with $\langle T^\mu{}_\mu \rangle = 0$ is the result of regularization and subtraction of UV divergences by local diffeomorphism invariant counterterms. In curved spacetime, when gravity plays the role of an external classical background, one-loop divergences of the classically conformal matter field theory has been proven to be universally Weyl invariant [3]. For Weyl-squared quantized conformal gravity, the same statement has a more involved status. Initially it has been derived with the use of the Fradkin-Vilkovisky conformization procedure [4] in [5], then confirmed in [6] and passed verification by direct calculations in [7].[1]

Despite conformal invariance of one-loop divergences, their subtraction from the regularized effective action entails

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[1]Modulo quantum anomalies, local gauge invariance of counterterms generically follows to all loops of perturbation expansion in the class of local background covariant gauges [8], but their application in the case of Weyl squared quantum gravity stumbles upon the problem of the search for such gauges satisfying the condition of background covariance for both diffeomorphism and conformal gauge transformations. Various approaches to this problem include [5], but can be circumvented by direct calculations confirming Weyl invariance of one-loop divergences in the conformally noncovariant background gauges as was done in [6,7,9]. Gauge independence of this result is discussed below in Sec. III.

nonvanishing $\langle T^\mu{}_\mu\rangle$, the structure of this trace anomaly reflecting the structure of divergences. According to the general classification of the possible terms composing anomaly in the purely metric sector of the four dimensional theory (we restrict ourselves with this dimension only) [10,11], the expression for $\langle T^\mu{}_\mu\rangle$ consists of the following three types of terms: (i) conformally covariant square of the Weyl tensor $C^2_{\mu\nu\alpha\beta} = R^2_{\mu\nu\alpha\beta} - 2R^2_{\mu\nu} + \frac{1}{3}R^2$ (its densitized version $\sqrt{g}C^2_{\mu\nu\alpha\beta}$ being conformally invariant); (ii) topological invariant Gauss-Bonnet term $E_4 = R^2_{\mu\nu\alpha\beta} - 4R^2_{\mu\nu} + R^2$ (its densitized version $\sqrt{g}E_4$ being conformally invariant up to the addition of the total derivative term) and, finally, (iii) the total derivative $\Box R$-term.

All these structures, when they are densitized, i.e., multiplied by $\sqrt{g}$, represent the trace of the functional variational derivatives of certain metric functionals. The principal difference between them is that this functional is nonlocal for $\sqrt{g}C^2_{\mu\nu\alpha\beta}$ and $\sqrt{g}E_4$ and local only for the last total derivative contribution due to the relation

$$-\frac{2}{\sqrt{g}}g_{\mu\nu}\frac{\delta}{\delta g_{\mu\nu}}\int d^4x\sqrt{g}R^2 = 12\Box R. \qquad (1.1)$$

This relation shows that the $\Box R$-term in the quantum trace anomaly can be modified or even completely removed by adding a finite term $\int d^4x\sqrt{g}R^2$ to the classical action of gravity theory. This is legitimate for quantum theory of conformal matter in external gravitational field, where such a finite term belongs to the so-called vacuum part of the action (involving only classical background metric field). The vacuum action may not follow the symmetry of the quantum fields and does not affect the number of active degrees of freedom. Thus, the $\Box R$-term in the semiclassical anomaly is renormalization ambiguous (see e.g., [12], the detailed analysis of this issue in [13] and further developments for metric-scalar models in [14]) and this opens the way for fixing the coefficient of the R^2-term according to observations as is usually done in high-energy particle physics, e.g., related to the Starobinsky model of inflation [15,16] (see [17–21] for more recent developments concerning the anomaly-induced inflation and further references).

For quantized metric of Weyl invariant gravity theory, the situation is qualitatively different. Its classical action is

$$S_W = -\int d^4x\sqrt{g}\left\{\frac{1}{2\lambda}C^2 + \frac{1}{2\rho}E_4 + \frac{1}{2\xi}\Box R\right\}, \qquad (1.2)$$

where $C^2 = C^2_{\mu\nu\alpha\beta}$ is the square of the Weyl tensor, $E_4 = R^2_{\mu\nu\alpha\beta} - 4R^2_{\mu\nu} + R^2$ is the Gauss-Bonnet integrand and λ, ρ and ξ are coupling constants of the dynamical, topological and total derivative terms. The action satisfies the conformal Noether identity

$$-T^\mu{}_\mu = \frac{2}{\sqrt{g}}g_{\mu\nu}\frac{\delta S_W}{\delta g_{\mu\nu}} = 0. \qquad (1.3)$$

According to the existing tradition, $T^\mu{}_\mu$ is called the trace of the metric stress tensor. Correspondingly, the violation of the identity (1.3) with S_W replaced by the quantum effective action Γ is called the quantum trace anomaly.

The different status of the theory (1.2) from that of a conformal matter in external gravitational field is that the counterterms needed to cancel the $C^2_{\mu\nu\alpha\beta}$ and E_4 anomalies are nonlocal, and their nonlocality contradicts the concept and the rules of renormalization by local counterterms [22]. Now these metric functionals no longer belong to the vacuum (external field) sector of the theory and carry quantum degrees of freedom of the theory. In higher-order loops of semiclassical expansion the nonvanishing one-loop anomaly will irrecoverably destroy renormalizability of the theory and its Ward identities providing its unitarity, as this was originally stated in [23,24]. The $\Box R$ part of the anomaly will also make the theory inconsistent because the finite local R^2-counterterm needed for its cancellation is itself conformally not invariant. Therefore, consistency of the renormalization scheme would require introducing this term already at the classical level, which would mean the loss of local Weyl invariance from the very beginning of the quantization procedure.

Despite inconsistency of the Weyl theory (1.2) at the quantum level, there was much interest in this model considered in the series of papers [5–7,9,25] where the Weyl squared and Gauss-Bonnet terms of the trace anomaly were fully calculated while its $\Box R$ part was ignored. Lack of interest in this contribution might be explained by the fact that it is usually considered to be ambiguous and depending on the chosen regularization and renormalization scheme. For instance, in zeta function regularization [26] or in the covariant cutoff of the lower limit of the proper time integral [13], the $\Box R$-term enters the anomaly as the trace of the coincidence limit of the coefficient $\hat{a}_2$ of the Schwinger-DeWitt expansion, i.e., it is proportional to the corresponding term in the one-loop divergence. On the contrary, in the dimensional regularization, it is ambiguous and strongly depends on the details of analytic continuation into the complex plane of spacetime dimensionality [2,12,27].

On the other hand, the $\Box R$ might be important in various implications because its contribution to the finite nonzero $\langle T^\mu{}_\mu\rangle$ in view of Eq. (1.1) is responsible for a finite R^2-term in effective action, and this term represents the core of the Starobinsky model of inflation [15,16]. In addition to this, the knowledge of $\Box R$ in the anomaly allows one to pose the question of complete calculation of the surface terms in the one-loop divergences of the theory and their dependence on the boundary conditions at spacetime boundaries. As the trace anomaly in fact represents the spacetime integrand of the one-loop divergences—the local Schwinger-DeWitt coefficient a_2 (or Gilkey-Seeley E_4 coefficient) [28,29], its $\Box R$ part after integration gets washed out from the spacetime bulk $\mathcal{M}$ and reduces to the surface integral $\int_{\partial\mathcal{M}} d\Sigma^\mu \nabla_\mu R$ over the boundary $\Sigma = \partial\mathcal{M}$, whereas another part of this surface integral comes from the boundary part of the Gilkey-Seeley coefficient E_3. This issue was a subject of a very preliminary analyses [5] and never fully considered within Weyl invariant gravity theory. All this explains our interest in the $\Box R$ part of the trace anomaly in Weyl gravity theory.

Thus, the derivation of the term $\Box R$ in the one-loop trace anomaly is the main subject of the present work. As we will see the calculation of this term requires not only the use of known algorithms for the minimal fourth-order operator [5,9,30,31], but also the generalization of the algorithms for the nonminimal second-order vector operator [5,9,30,32].

Though the $\Box R$ term was ignored in previous calculations in conformal gravity [5–7,9,25], its calculation in generic curvature squared nonconformal theory was successfully accomplished [5]. We emphasize again that the status of this term in nonconformal case is different because the relevant R^2 term in the classical action is not forbidden by the requirement of local Weyl invariance and is subject to the renormalization and experimental adjustment in the subtraction point.

The paper is organized as follows. In Sec. II we explain the role of $\Box R$ term in quantum field theory on classical curved background and in conformal quantum gravity and emphasize the fundamental difference between the two cases. In the subsequent sections, we describe the calculation of the $\Box R$-type anomaly in the Weyl conformal gravity. Section III discusses the background field method in the conformal quantum gravity, presents the results for Hessian operators and introduces gauge fixing condition. On top of this, we repeat the proof of [33] of the gauge-fixing independence of the result for the one-loop divergences in this theory. Section IV gives the final forms of Faddeev-Popov ghosts and weight operator contributing to the one-loop divergences along with the tensor sector coming from quantum metric. In Sec. V we briefly summarize the algorithms for the heat-kernel-based algorithms for the minimal fourth-derivative operators acting in space of arbitrary fields [5,9,30,31] and for the nonminimal second-order vector operator [32]. Most of the calculational efforts of the present work consisted in the verification of this result by the method of universal functional traces of [30]. We report on this extensive calculation in Sec. VI and show that the mentioned algorithm of [32] is confirmed. Section VII reports on the final derivation of the anomaly $\Box R$ term by using these algorithms. In Sec. VIII we present the conclusions and final discussions of the conformal symmetry breakdown in Weyl quantum gravity by the local term. Throughout the paper we use Euclidean notations.

II. TRACE ANOMALY AND INDUCED ACTION OF GRAVITY

Let us briefly review the conformal anomaly and derivation of the anomaly-induced action [24,34]. In the discussion of this subject, we shall pay special attention to the ambiguities related to the choice of regularization and to the difference between conformal theories in external gravitational field and conformal quantum gravity.

The starting classical theory of the fields Φ_i and the metric, is conformal, i.e., its action $S_{\rm conf}$ satisfies the conformal Noether identity, that is a generalization of (1.3),

$$\frac{1}{\sqrt{g}}\left\{2g_{\mu\nu}\frac{\delta}{\delta g_{\mu\nu}} + \sum_i w_i \Phi_i \frac{\delta}{\delta\Phi_i}\right\} S = 0, \qquad (2.1)$$

were w_i is the conformal weight of the field Φ_i and S is the action of gravity and fields Φ_i. In the purely gravitational sector, the action has the form (1.2), and the form of conformal actions of scalars, fermions and vectors can be found, e.g., in [35] or elsewhere. In pure quantum gravity or on shell of matter fields, when $\delta S/\delta\Phi_i = 0$, the Noether identity (2.1) reduces to (1.3). In this sense, conformal quantum gravity and semiclassical theories are similar. Another common point is that, at the one-loop level, the identity (1.3) gets violated, i.e., acquires a nonvanishing mean value

$$\langle T^\mu_{\ \mu}\rangle = -\frac{2}{\sqrt{g}} g_{\mu\nu}\frac{\delta\Gamma}{\delta g_{\mu\nu}} \neq 0. \qquad (2.2)$$

In the case of quantum matter fields, the trace anomaly corresponds to the violation of conformal symmetry in the finite vacuum part of effective action. Depending on the regularization scheme, the expression for $\langle T^\mu_{\ \mu}\rangle$ is proportional to the local one-loop divergences modulo the $\Box R$ term which is the object of our prime interest. In quantum theory of conformal matter fields, these divergences and trace anomaly consist of the C^2, E_4, and $\Box R$ [3]. Thus within the existing classification of invariants [10,11]

$$\langle T^\mu_{\ \mu}\rangle = \omega C^2 + bE_4 + c\Box R, \qquad (2.3)$$

where ω, b, and c depend on the fields content of the model. Equation (2.2) can be used to find the part of the full effective action, which is responsible for the anomaly and also called induced effective action.

The anomaly can be integrated using the relations (1.1) and $\sqrt{g}(E_4 - \frac{2}{3}\Box R) = \sqrt{\bar g}(\bar E_4 - \frac{2}{3}\bar\Box\,\bar R + 4\bar\Delta_4\sigma)$, where the metrics $g_{\mu\nu}$ and $\bar g_{\mu\nu}$ are related by $g_{\mu\nu} = \bar g_{\mu\nu}e^{2\sigma}$ and

$$\Delta_4 = \Box^2 + 2R^{\mu\nu}\nabla_\mu\nabla_\nu - \frac{2}{3}R\Box + \frac{1}{3}(\nabla^\mu R)\nabla_\mu, \qquad (2.4)$$

is the fourth-order Hermitian conformal invariant operator [36,37] acting on a scalar field of zero conformal weight. The covariant solution of Eq. (1.1) has the form

$$\Gamma_{ind} = S_c(g_{\mu\nu}) + \frac{\omega}{4}\int_x\int_y C^2(x)G(x,y)\left(E_4 - \frac{2}{3}\Box R\right)_y + \frac{b}{8}\int_x\int_y \left(E_4 - \frac{2}{3}\Box R\right)_x G(x,y)\left(E_4 - \frac{2}{3}\Box R\right)_y - \frac{3c+2b}{36}\int_x R^2(x). \quad (2.5)$$

Here $S_c(g_{\mu\nu})$ is arbitrary conformally invariant functional of metric, serving as an integration constant for this solution. This term is important for short distance behavior of stress-tensor correlators [38] or in the model of initial conditions for inflationary cosmology driven by a conformal field theory [39,40], but for the purposes of our paper it is largely irrelevant (see, e.g., the discussion in [41]). The next two terms include the conformal Green function $G(x,y)$ of the operator Δ_4 and are free of ambiguities. On the other hand, the situation is more complex with the local term $\int\sqrt{g}R^2$, which is directly related to the $\Box R$-type anomaly owing to the relation (1.1).

In semiclassical theories, one can modify $\Box R$-term in the anomalous trace simply by adding the $\int\sqrt{g}R^2$ term to the classical action. This procedure can be also hidden in the details of dimensional or Pauli-Villars regularizations [13]. However the corresponding ambiguities are in fact equivalent to adding a classical $\int\sqrt{g}R^2$ term. In a semiclassical model, this vacuum term does not produce changes in the quantum theory because the metric is not quantized. However, things change if we add such a term in the theory of conformal quantum gravity, because this operation violates the classical conformal symmetry, increases the number of degrees of freedom and changes the quantum theory.

III. BACKGROUND FIELD METHOD AND GAUGE FIXING

For a one-loop calculation, we shall use the background field method, as it was done in the previous works in the same model, starting from [5]. The first step is to separate the metric field into background $g_{\mu\nu}$ and quantum $h_{\mu\nu}$ counterparts,

$$g_{\mu\nu} \to g'_{\mu\nu} = g_{\mu\nu} + h_{\mu\nu}. \quad (3.1)$$

The one-loop calculations include contribution of the Hessian for the fluctuation $h_{\mu\nu}$, so we need to expand the action up to second order in this field. One detail makes our calculation different from what was done before. We need only the $\Box R$-type term and, therefore, can restrict our attention to the linear in curvature tensor terms, since other contributions were calculated and verified in [5–7,9].

In what follows, we use standard condensed notations of DeWitt [29]. The unity operator and the covariant derivatives obeying the Leibnitz rule read as

$$\delta_{\mu\nu,\alpha\beta} = \frac{1}{2}(g_{\mu\alpha}g_{\nu\beta} + g_{\mu\beta}g_{\nu\alpha}), \quad \nabla_\mu A = (\nabla_\mu A) + A\nabla_\mu. \quad (3.2)$$

In the framework of usual Faddeev-Popov approach, we add the background-invariant gauge-fixing term fixing the diffeomorphism invariance for the quantum fields,

$$S_{gf} = \int d^4x\sqrt{g}\chi_\mu Y^{\mu\nu}\chi_\nu, \quad (3.3)$$

with the gauge condition χ_μ and the gauge fixing operator $Y^{\mu\nu}$ of the general form

$$\chi_\mu = \nabla_\alpha h^\alpha_\mu + \tau\nabla_\mu h, \qquad Y^{\mu\nu} = \gamma_1 g^{\mu\nu}\Box + \gamma_2\nabla^\mu\nabla^\nu, \quad (3.4)$$

where h is the trace, $h = g^{\mu\nu}h_{\mu\nu}$, while τ and $\gamma_{1,2}$ are three gauge-fixing parameters. For arbitrary choice of these parameters, the calculations become complicated. The total Hessian contributed by the sum of the action (1.2) and the gauge-fixing term (3.3) is a nonminimal fourth-order operator. On the other hand, there is a unique special choice of the gauge-fixing parameters [7,9,25].

$$\gamma_1 = \frac{1}{2}, \qquad \gamma_2 = -\frac{1}{6}, \qquad \tau = -\frac{1}{4}, \quad (3.5)$$

that reduces this Hessian to the minimal form. Then, the quadratic form of the action reads as

$$S^{(2)}_{\rm conf} = \frac{1}{4}\int d^4x\sqrt{g}h^{\mu\nu}H_{\mu\nu,\alpha\beta}h^{\alpha\beta}, \quad (3.6)$$

where

$$\hat{H} = H_{\mu\nu,\alpha\beta} = \hat{1}\Box^2 + \hat{V}^{\rho\sigma}\nabla_\rho\nabla_\sigma + \hat{N}^\rho\nabla_\rho + \hat{U}. \quad (3.7)$$

At this point, we encounter a special feature of conformal gravity, which has not only diffeomorphism invariance, but also conformal symmetry. For the gauge fixing of conformal symmetry we follow [5,9] and choose the degenerate (delta function type) gauge $\phi \equiv g^{\mu\nu}h_{\mu\nu} = 0$. This means that the quantum metric fluctuation in the path integral $h_{\mu\nu}$ becomes traceless and the unity matrix in the metric sector is a projector operator to the subspace of traceless tensors (the hat indicating the action of the tensor matrix objects in the space of tensors)

$$\hat{1} = \delta_{\mu\nu,\alpha\beta} - \frac{1}{4}g_{\mu\nu}g_{\alpha\beta}. \quad (3.8)$$

After some algebra, we get the following elements of the operator (3.7):

$$\hat{V}^{\rho\sigma} = -\frac{2}{3} R\delta_{\mu\nu,\alpha\beta} g^{\rho\sigma} + \frac{4}{3} R g_{\nu\beta}\delta_{\mu\alpha,}{}^{\rho\sigma} + \frac{4}{3} R_{\alpha\beta}\delta_{\mu\nu,}{}^{\rho\sigma} + \frac{4}{3} R_{\mu\nu}\delta_{\alpha\beta,}{}^{\rho\sigma} + 2R_{\mu\alpha}\delta_{\nu\beta,}{}^{\rho\sigma} - 4R^{\rho}_{\mu} g_{\nu\beta}\delta^{\sigma}_{\alpha} - 4R^{\rho}_{\alpha} g_{\nu\beta}\delta^{\sigma}_{\mu} + 4R_{\mu\alpha\nu\beta} g^{\rho\sigma} + 2\delta_{\mu\nu,\alpha\beta} R^{\rho\sigma}, \tag{3.9}$$

$$\hat{N}^{\lambda} = \frac{1}{3}\delta_{\mu\nu,\alpha\beta}(\nabla^{\lambda}R) - \frac{4}{3}(\nabla_{\mu}R_{\alpha\beta})\delta^{\lambda}_{\nu} - \frac{2}{3}(\nabla_{\alpha}R) g_{\nu\beta}\delta^{\lambda}_{\mu} - 2(\nabla_{\mu}R_{\nu\beta})\delta^{\lambda}_{\alpha} + 4(\nabla_{\alpha}R_{\mu\nu})\delta^{\lambda}_{\beta} + 4(\nabla_{\alpha}R_{\mu\beta})\delta^{\lambda}_{\nu} - 4(\nabla_{\alpha}R^{\lambda}_{\mu}) g_{\nu\beta} + 4(\nabla^{\lambda}R_{\mu\alpha\nu\beta}), \tag{3.10}$$

and

$$\hat{U} = -\frac{1}{3}\delta_{\mu\nu,\alpha\beta}(\Box R) - \frac{4}{3}(\nabla_{\mu}\nabla_{\alpha}R) g_{\nu\beta} + \frac{4}{3}(\nabla_{\mu}\nabla_{\nu}R_{\alpha\beta}) + 2(\Box R_{\mu\alpha}) g_{\nu\beta} + 2(\Box R_{\mu\alpha\nu\beta}). \tag{3.11}$$

Before starting the calculations, let us comment on the important question about the gauge-fixing dependence. Repeating the arguments of [33,35] (and those of [5] for a general higher-derivative quantum gravity) this issue may be stated as follows. As is well-known, the gauge and parametrization dependence of the one-loop effective action is proportional to classical equations of motion [42–45]. In view of locality of UV divergences [46,47] in the theory (1.2), the difference between its divergences $\Delta\Gamma^{(1)}_{\rm div}$ calculated in different background-covariantgauges [8], which necessarily has dimensionality 4 and is proportional to equations of motion, can only be of the following form [33]:

$$\Delta\Gamma^{(1)}_{\rm div} \propto \int d^4x g_{\mu\nu}\frac{\delta S_W}{\delta g_{\mu\nu}} \tag{3.12}$$

(other powers of $\delta S_W/\delta g_{\mu\nu}$ are obviously excluded for dimensional reasons). Therefore, in view of the conformal symmetry of the classical action (1.3), it is vanishing. Thus, in the conformal theory (1.2) in this class of gauges the one-loop divergences are parametrization and gauge-fixing independent. Hence, we can safely use the simplest choices of variables and gauge-fixing conditions.

IV. GAUGE GHOSTS AND GAUGE-FIXING OPERATOR

The general expression for the one-loop effective action in Euclidean notations reads [5] (see also [48,49] and the recent review in [50])

$$\Gamma = \frac{1}{2}{\rm Tr}\log\hat{H} - \frac{1}{2}{\rm Tr}\log\hat{Y} - {\rm Tr}\log\hat{M}, \tag{4.1}$$

where Tr denotes the functional trace, $\hat{H}$ is defined in (3.7), the gauge-fixing operator

$$\hat{Y} = Y^{\mu\nu} = \frac{1}{2}\left(g^{\mu\nu}\Box - \frac{1}{3}\nabla^{\mu}\nabla^{\nu}\right), \tag{4.2}$$

is defined in (3.4) and (3.5), and $\hat{M} = M^{\beta}{}_{\alpha}$ is the Hessian of the action of ghost fields responsible for diffeomorphism gauge transformations,

$$M_{\beta\alpha} = \frac{\delta\chi_{\beta}}{\delta h_{\mu\nu}}\mathcal{R}_{\mu\nu,\alpha}, \tag{4.3}$$

with the generator of diffeomorphism transformations $\mathcal{R}_{\mu\nu,\alpha} = -g_{\mu\alpha}\nabla_{\nu} - g_{\nu\alpha}\nabla_{\mu}$. This operator is built with the diffeomorphism gauge conditions in (3.4) and (3.5). Regarding the conformal gauge $h = 0$ we note that its ghost field does not contribute to the effective action (4.1) owing to the nondynamical (nonderivative) nature of its conformal ghost sector.[2] Thus, we get

$$\hat{M} = M^{\beta}{}_{\alpha} = -\left(\delta^{\beta}{}_{\alpha}\Box + \frac{1}{2}\nabla^{\beta}\nabla_{\alpha} + R^{\beta}{}_{\alpha}\right). \tag{4.4}$$

In both cases of operators (4.2) and (4.4) we meet the problem of calculating ${\rm Tr}\log\hat{F}$ for the nonminimal vector operator

$$\hat{F} \equiv F^{\mu}_{\nu} = \Box\delta^{\mu}_{\nu} - \lambda\nabla^{\mu}\nabla_{\nu} + P^{\mu}_{\nu} \tag{4.5}$$

with a generic value of the parameter λ and some potential term P^{μ}_{ν}.

[2]This is a nontrivial and very helpful corollary of the choice of the diffeomorphism gauge (3.4) and (3.5). In the full set of diffeomorphism and conformal gauge conditions (χ_{μ}, ϕ), $\phi \equiv g^{\mu\nu}h_{\mu\nu}$, the full Faddeev-Popov operator has a block matrix form with the off-diagonal element $(\delta\chi_{\alpha}/\delta h_{\mu\nu})\mathcal{R}_{\mu\nu}$, where $\mathcal{R}_{\mu\nu} = g_{\mu\nu}$ is the linearized generator of the conformal transformation of $h_{\mu\nu}$. But in the gauge (3.4) with the parameter $\tau = -1/4$ of (3.5) this block turns out to be vanishing, so that the total ghost determinant factorizes into the product of ${\rm Det}M_{\alpha\beta}$ and the determinant of its conformal-conformal block $(\delta\phi/\delta h_{\mu\nu})\mathcal{R}_{\mu\nu} = 4$. The latter one is ultralocal and does not contribute to the effective action.

V. THE ALGORITHMS FOR ANOMOLIES AND DIVERGENCES

One-loop UV divergences and anomalies in curved spacetime can be covariantly calculated by using the heat kernel of the corresponding wave operators of the theory. For the effective action built in terms of the functional determinants of the second-order operators $\hat{F}$ and the fourth-order operators $\hat{H}$, the heat kernels of these operators can be represented in the form of asymptotic expansions at small values of the proper time $s \to 0$,

$$e^{s\hat{F}}\delta(x,y)|_{y=x} = \frac{\sqrt{g(x)}}{(4\pi s)^{d/2}}\sum_{m=0}^{\infty} s^m \hat{a}_m^F(x), \qquad (5.1)$$

and

$$e^{s\hat{H}}\delta(x,y)|_{y=x} = \frac{\sqrt{g(x)}}{(4\pi s^{1/2})^{d/2}}\sum_{m=0}^{\infty} s^{m/2} \hat{a}_m^H(x), \qquad (5.2)$$

where d is the spacetime dimension, and $\hat{a}_m^{F,H}(x)$ are respectively the local Schwinger-DeWitt [29,30] coefficients of the operators $\hat{F}$ and $\hat{H}$ related to the Gilkey-Seeley coefficients [28] $\hat{E}_{2m}^{F,H}(x)$ by

$$\hat{a}_m^{F,H}(x) = (4\pi)^{d/2}\hat{E}_{2m}^{F,H}(x), \qquad (5.3)$$

These heat kernels generate by integration over the proper time parameter the contributions to the Euclidean effective action $\Gamma = \frac{1}{2}\mathrm{Tr}\ln F = \frac{1}{2}\int_0^\infty ds\, e^{-sF}/s$ coming from the operator $\hat{F}$ and correspondingly from $\hat{H}$. Their divergent parts read respectively as

$$\frac{1}{2}\mathrm{Tr}\log\hat{F}|_{\mathrm{div}} = -\frac{1}{\epsilon}\int d^4x\sqrt{g}\ \mathrm{tr}\ \hat{a}_2^F, \qquad (5.4)$$

$$\frac{1}{2}\mathrm{Tr}\log\hat{H}|_{\mathrm{div}} = -\frac{2}{\epsilon}\int d^4x\sqrt{g}\ \mathrm{tr}\ \hat{a}_2^H, \qquad (5.5)$$

where tr denotes the matrix trace with respect to the indices of the second Schwinger-DeWitt coefficient $\hat{a}_2$ and $\epsilon = (4\pi)^2(4-n) \to 0$ is the parameter of dimensional regularization with n denoting the regularized spacetime dimension. To avoid the ambiguities related to this regularization [12,13], one can use the covariant cutoff in the proper-time regularization with the dimensionless parameter $L \to \infty$ representing the ratio of the dimensional UV cutoff and the renormalization scale (see [30])

$$\frac{1}{\epsilon} = \frac{\log L^2}{32\pi^2}. \qquad (5.6)$$

Note the difference in coefficients in (5.4) and (5.5) associated with the fact that the operator $\hat{H}$ is quartic in derivatives (that is, it has the dimension four in units of mass and the corresponding conformal weight -4), whereas weight -2 operator $\hat{F}$ is quadratic in derivatives. This weight plays important role in the generation of trace anomaly, which can be most easily demonstrated within zeta-functional regularization as follows.

Consider a symmetric Weyl-covariant operator $\hat{H}$ of the conformal weight $-2w$ (correspondingly the weight of the field acted upon by $\hat{H}$ being w), which transforms under the infinitesimal conformal rescaling $\delta_\sigma g_{\mu\nu} = +2\sigma g_{\mu\nu}$ as $\delta_\sigma\hat{H} = -w(\sigma\hat{H} + \hat{H}\sigma)$. The conformal transform of its "effective action" $\Gamma = \frac{1}{2}\log\hat{H}$, expresses in zeta function regularization in terms of the particular value of the zeta function, $\zeta(z|x) = \mathrm{tr}(-\hat{H})^{-z}\delta(x,y)|_{y=x}$, at $z = 0$—the value which is given in 4-dimensional massless theories in terms of the local Schwinger-DeWitt (or Gilkey-Seeley) coefficient (5.3), $\zeta(0|x) = \mathrm{tr}\hat{E}_4^H(x)$. Namely,

$$\langle T_\mu^\mu\rangle \equiv -\delta_\sigma\Gamma = w\zeta(0|x) = w\,\mathrm{tr}\ \hat{E}_4^H(x). \qquad (5.7)$$

Thus, when the both operators $\hat{F}$ and $\hat{H}$ are individually Weyl invariant with respect to local conformal transformations, their functional determinants supply the total trace anomaly with the following contributions expressible in terms of the second Schwinger-DeWitt coefficient

$$\langle T_\mu^{\mu(F)}\rangle = \frac{\mathrm{tr}\ \hat{a}_2^F(x)}{16\pi^2}, \qquad \langle T_\mu^{\mu(H)}\rangle = 2\frac{\mathrm{tr}\ \hat{a}_2^H(x)}{16\pi^2}, \qquad (5.8)$$

in accordance with their conformal weights $-2w = -2$ and $-2w = -4$, respectively.

In Weyl gravity theory the fourth-order operator $\hat{H}$ given by (3.7) and nonminimal second-order operators $\hat{F} = (M_{\mu\nu}, Y^{\mu\nu})$ given by Eqs. (4.2) and (4.3) are not individually Weyl covariant, because the chosen diffeomorphism and conformal gauges are not background covariant with respect to the local conformal subgroup of the full set of gauge transformations. However, these operators are intertwined by Ward identities which provide the expression (3.12) for gauge conditions variation of the one-loop divergences, which in its turn occurs to be vanishing also off shell in view of the discussion above. Thus, the deviation of the actual conformal transformations from the expressions (5.8) above in fact cancels out in the total sum of contributions into effective action (4.1), which generate the overall trace anomaly

$$\langle T_\mu^\mu(x)\rangle = \frac{1}{16\pi^2}\Big(2\ \mathrm{tr}\ \hat{a}_2^H(x) - \mathrm{tr}\ \hat{a}_2^Y(x) - 2\ \mathrm{tr}\ \hat{a}_2^M(x)\Big). \qquad (5.9)$$

This expression thus turns out to be the integrand of the overall set of the one-loop divergences of the model,

$$\Gamma|_{\mathrm{div}} = -\frac{1}{\epsilon}\int d^4x\sqrt{g}\big(2\mathrm{tr}\hat{a}_2^H - \mathrm{tr}\hat{a}_2^Y - 2\mathrm{tr}\hat{a}_2^M\big). \qquad (5.10)$$

Note, however, that unless we include the surface terms at the boundary of spacetime, which are disregarded in this four-dimensional integral, the $\Box R$ contributions—the object of our prime interest—are completely washed out from the right-hand side of (5.10) because they do not contribute to the interior of spacetime domain. But they enter the right hand side of (5.9). They are ambiguous, as it was already remarked, in the dimensional regualarization, but in the zeta-functional regularization appear as uniquely defined ingredients of the relevant Schwinger-DeWitt (or Gilkey-Seeley) coefficients. Our goal is to calculate them here.

Let us start the calculation of various terms of (5.9) with the contribution of the minimal fourth-derivative operator (3.7). We shall ignore in what follows all the terms which are not of the desired $\Box R$ type, as they are not of our interest. The reader can easily find the corresponding expressions in [5,30] or in [31,32,51]. The expression for the $\Box R$-type terms for the fourth order minimal operator was derived in [31] using the heat-kernel method. The formula obtained in [31] is equivalent, for the operator (3.7), to

$$\mathrm{tr}\ \hat{a}_2^H(x) = \frac{1}{2}\mathrm{tr}\left\{\frac{\hat{1}}{15}(\Box R) + \frac{1}{9}\Box\hat{V} - \frac{5}{18}\nabla_\rho\nabla_\sigma\hat{V}^{\rho\sigma} + \frac{1}{2}\nabla_\lambda\hat{N}^\lambda - \hat{U}\right\}, \qquad (5.11)$$

It is worth mentioning that this expression passes the test of representing the fourth-order derivative operator as a product of two minimal second order ones [5] and calculating the divergent part of its functional determinant. The details of this verification sound as follows. For the basic second-order operator, the divergent part is [29]

$$-\frac{1}{2}\mathrm{Tr}\log(\widehat{\Box}+\hat{\Pi})|_{\mathrm{div}} = -\frac{1}{\epsilon}\int d^4x\sqrt{g}\,\mathrm{tr}\left\{\frac{\hat{1}}{30}\Box R + \frac{\hat{1}}{6}\Box\hat{\Pi}\right\}, \qquad (5.12)$$

Taking the particular form of the fourth-derivative operator

$$\hat{O} = (\widehat{\Box}+\hat{\Pi})^2 = \widehat{\Box}^2 + 2\hat{\Pi}\Box + 2(\nabla^\lambda\hat{\Pi})\nabla_\lambda + (\Box\hat{\Pi}),$$
$$\mathrm{Tr}\log\hat{O} = 2\mathrm{Tr}\log(\widehat{\Box}+\hat{\Pi}), \qquad (5.13)$$

we identify this with the special version of (3.7), when

$$\hat{V}^{\rho\sigma} = 2\hat{\Pi}g^{\rho\sigma}, \quad \hat{N}^\rho = 2(\nabla^\rho\hat{\Pi}), \quad \hat{U} = (\Box\hat{\Pi}). \qquad (5.14)$$

At this point, we can establish the form of the possible divergences of the $\Box R$-type. These divergences should be the total derivative expressions constructed from $\hat{V}^{\rho\sigma}$, $\hat{N}^\rho$ and $\hat{U}$. taking into account the dimension of these building blocks, we arrive at the expression for divergences which has four unknown coefficients $a_{1,\ldots,4}$, that are still to be defined,

$$\frac{1}{2}\mathrm{Tr}\log\hat{H}|_{\mathrm{div}} = -\frac{1}{\epsilon}\int d^4x\sqrt{g}\ \mathrm{tr}\left\{\frac{\hat{1}}{90}\Box R + a_1\nabla_\rho\nabla_\sigma\hat{V}^{\rho\sigma} + a_2\Box\hat{V} + a_3\nabla_\rho\hat{N}^\rho + a_4\hat{U}\right\}, \qquad (5.15)$$

From the nontotal derivative terms we know that $a_4 = -1$ (e.g., [5,30]). Other coefficients can be obtained using different doubling tricks [5]. In this case, we can identify $\nabla_\rho\nabla_\sigma\hat{V}^{\rho\sigma} = 2\Box\hat{\Pi}$ and $\Box\hat{V} = 8\Box\hat{\Pi}$. Using these relations and (5.13), we arrive at the equation

$$2a_1 + 8a_2 + 2a_3 + a_4 = \frac{1}{3}. \qquad (5.16)$$

Since Eq. (5.11) fits this condition, we conclude that it has passed this partial check.

Let us now consider the contribution of the nonminimal vector operator (4.5). The algorithms for the divergences in this case is known from [5] and [30]. However, in both cases the formula for divergences did not include the total derivative terms, such as $\Box R$, $\Box P$ and $\nabla_\mu\nabla_\mu P^{\mu\nu}$, where $P = P^{\mu\nu}g_{\mu\nu}$. The algorithm for these terms was obtained in [32]. The final result for the vector operator (4.5), $\hat{a}_2^F \equiv (a_2^F)^\mu_\nu$, $\mathrm{tr}\hat{a}_2^F = (a_2^F)^\mu_\mu$, reads

$$\mathrm{tr}\ \hat{a}_2^F = c_1\Box P + c_2\nabla^\alpha\nabla^\beta P_{\alpha\beta} + c_{11}\Box R, \qquad (5.17)$$

with the coefficients

$$c_1 = -\frac{8\lambda^2 - 21\lambda + 6}{36\lambda(1-\lambda)} + \frac{2\lambda - 1}{6\lambda^2}\log(1-\lambda),$$
$$c_2 = -\frac{13\lambda^2 + 6\lambda - 24}{36\lambda(1-\lambda)} + \frac{\lambda + 4}{6\lambda^2}\log(1-\lambda),$$
$$c_{11} = -\frac{133\lambda^2 - 168\lambda - 60}{360\lambda(1-\lambda)} - \frac{\lambda^2 - 5\lambda - 2}{12\lambda^2}\log(1-\lambda), \qquad (5.18)$$

where we keep enumeration of the coefficients adopted in [32]. To have an additional verification, in the next section we present an alternative derivation of the contribution of minimal-vector operator, by using a different approach.

VI. NEW DERIVATION FOR NONMINIMAL VECTOR OPERATOR

Consider the vector field operator (4.5)

$$F \equiv F^\mu_\nu = \Box\delta^\mu_\nu - \lambda\nabla^\mu\nabla_\nu + P^\mu_\nu. \qquad (6.1)$$

Coincidence limit of the heat kernel has Schwinger-DeWitt expansion given by [29,30]

$$e^{sF}\delta^\mu_\nu(x,y)|_{y=x} = \frac{1}{(4\pi s)^{d/2}}g^{1/2}\Big[a^\mu_{0\,\nu}(x,x) + a^\mu_{1\,\nu}(x,x)s + a^\mu_{2\,\nu}(x,x)s^2 + \cdots\Big], \qquad (6.2)$$

where the $a^{\mu}_{n,\nu}(x,x)$ are the coincidence limits of the two-point Schwinger-DeWitt coefficients $a^{\mu}_{n,\nu}(x,y)$ labeled above by one argument $a^{\mu}_{n,\nu}(x) \equiv a^{\mu}_{n,\nu}(x,x)$.

We intend to find total derivative terms in the second Schwinger-DeWitt coefficient $a^{\mu}_{2\,\nu}(x,x)$ which determine the corresponding one-loop divergences in the effective action. For the matrix trace of $a^{\mu}_{2\,\nu}(x,x)$ there are three such structures with some numerical coefficients a, b, and c,

$$a^{\mu}_{2\mu}(x,x) = a\Box R + b\Box P + c\nabla_{\mu}\nabla^{\nu}P^{\mu}_{\nu}, \quad P \equiv P^{\mu}_{\mu}. \tag{6.3}$$

These terms cannot be extracted from the integral quantity like $\mathrm{Tr}e^{sF}$ or $\mathrm{Tr}\log F$ because under integration over spacetime they get washed out and materialize as surface terms at the boundary which we do not control. Therefore, let us extract them from the local quantity. The simplest object is the coincidence limit of the Green's function

$$-\frac{1}{F}\delta^{\mu}_{\nu}(x,y)|_{y=x} = \int_0^{\infty} ds\, e^{sF}\,\delta^{\mu}_{\nu}(x,y)|_{y=x}. \tag{6.4}$$

Unfortunately, however, $a^{\mu}_{2\,\nu}(x,x)$ is contained in the UV finite part of this quantity which for massless operator is badly IR divergent within the local Schwinger-DeWitt expansion (6.2). Therefore we have to consider the massive Green's function

$$\begin{aligned}\frac{1}{m^2-F}\delta^{\mu}_{\nu}(x,y)|_{y=x} &= \int_0^{\infty} ds e^{sF-sm^2}\delta^{\mu}_{\nu}(x,y)|_{y=x}\\ &= \int_0^{\infty}\frac{ds e^{-sm^2}}{(4\pi s)^{d/2}}g^{1/2}\left[a^{\mu}_{0\,\nu}(x,x) + a^{\mu}_{1\,\nu}(x,x)s + a^{\mu}_{2\,\nu}(x,x)s^2 + \cdots\right]\\ &= \frac{1}{(4\pi)^{d/2}}g^{1/2}\left[\frac{\Gamma(1-\frac{d}{2})}{(m^2)^{1-\frac{d}{2}}}a^{\mu}_{0\,\nu}(x,x) + \frac{\Gamma(2-\frac{d}{2})}{(m^2)^{2-\frac{d}{2}}}a^{\mu}_{1\,\nu}(x,x) + \frac{1}{m^2}a^{\mu}_{2\,\nu}(x,x) + \cdots\right],\end{aligned} \tag{6.5}$$

for $d \to 4$. Then $a^{\mu}_{2\,\nu}(x,x)/16\pi^2$ is the coefficient of $1/m^2$ in the inverse mass expansion of a massive Green's function.

Let us calculate this Green's function by the method of universal functional traces of [30]. In the lowest order in curvatures, we have

$$\frac{1}{F-m^2} = K^{\mu}_{\alpha}\frac{\delta^{\alpha}_{\nu}}{(\Box - m^2)\left(\Box - \frac{m^2}{1-\lambda}\right)} + \cdots, \tag{6.6}$$

where

$$K \equiv K^{\mu}_{\nu} = \left(\Box - \frac{m^2}{1-\lambda}\right)\delta^{\mu}_{\nu} + \frac{\lambda}{1-\lambda}\nabla^{\mu}\nabla_{\nu}. \tag{6.7}$$

More precisely, we can derive the exact equality:

$$(F^{\mu}_{\alpha} - m^2\delta^{\mu}_{\alpha})K^{\alpha}_{\nu} = (\Box - m^2)\left(\Box - \frac{m^2}{1-\lambda}\right)\delta^{\mu}_{\nu} + M^{\mu}_{\nu}, \tag{6.8}$$

where the perturbation operator M^{μ}_{ν} equals

$$\begin{aligned}M^{\mu}_{\nu} = {}&\frac{\lambda}{1-\lambda}(R^{\mu}_{\alpha} + P^{\mu}_{\alpha})\nabla^{\alpha}\nabla_{\nu} + P^{\mu}_{\nu}\left(\Box - \frac{m^2}{1-\lambda}\right) - \lambda R_{\nu\alpha}\nabla^{\mu}\nabla^{\alpha}\\ &- \lambda(\nabla^{\mu}R_{\nu\alpha})\nabla^{\alpha} - \frac{\lambda}{2}(\nabla_{\nu}R)\nabla^{\mu} - \frac{\lambda}{2}(\nabla^{\mu}\nabla_{\nu}R).\end{aligned} \tag{6.9}$$

From (6.8), we find

$$\frac{1}{F-m^2} = K\frac{1}{F_1F_2 + M}, \tag{6.10}$$

where we have introduced the abbreviations

$$F_1 = \Box - m^2, \qquad F_2 = \Box - \frac{m^2}{1-\lambda}. \tag{6.11}$$

We expand (6.10) in the inverse powers of F_1F_2,

$$\frac{1}{F-m^2} = K\sum_{p=0}^{\infty}(-1)^p M_p\frac{1}{(F_1F_2)^{p+1}}. \tag{6.12}$$

We then find

$$M_0 = 1, \quad M_1 = M, \quad M_{p+1} = MM_p + \left[F_1F_2, M_p\right]. \tag{6.13}$$

For our purposes of finding linear $\nabla\nabla R$ and $\nabla\nabla P$ terms, we need the above operators up to $p = 3$ with

$$\begin{aligned}M_2 &= MM_1 + [F_1F_2, M_1] = M^2 + \left[F_1F_2, M\right]\\ &= [\Box, M](F_1 + F_2) + \left[\Box, [\Box, M]\right] + \cdots,\end{aligned} \tag{6.14}$$

where only $O(R)$ terms are kept. We also have

$$M_3 = MM_2 + [F_1F_2, M_2] = \Big[F_1F_2, M_2\Big] + \cdots = [\Box, [\Box, M]](F_1+F_2)^2 + \cdots, \tag{6.15}$$

where we keep only those terms linear in the curvature with two derivatives. Then the single and double commutators of $\Box$ with M are also needed:

$$\begin{aligned}[]
[\Box, M^\mu_\nu] = &\frac{2\lambda}{1-\lambda}\Big[\nabla_\beta(R^\mu_\alpha + P^\mu_\alpha)\Big]\nabla^\beta\nabla^\alpha\nabla_\nu + 2(\nabla^\alpha P^\mu_\nu)\nabla_\alpha F_2 \\
&- 2\lambda(\nabla_\beta R_{\nu\alpha})\nabla^\beta\nabla^\mu\nabla^\alpha + \frac{\lambda}{1-\lambda}\Big[\Box(R^\mu_\alpha + P^\mu_\alpha)\Big]\nabla^\alpha\nabla_\nu + (\Box P^\mu_\nu)F_2 \\
&- \lambda(\Box R_{\nu\alpha})\nabla^\mu\nabla^\alpha - 2\lambda(\nabla_\beta\nabla^\mu R_{\nu\alpha})\nabla^\beta\nabla^\alpha - \lambda(\nabla_\alpha\nabla_\nu R)\nabla^\alpha\nabla^\mu + \cdots,
\end{aligned} \tag{6.16}$$

$$\begin{aligned}
\Big[\Box, [\Box, M^\mu_\nu]\Big] = &\frac{4\lambda}{1-\lambda}\left[\nabla_\gamma\nabla_\beta(R^\mu_\alpha + P^\mu_\alpha)\right]\nabla^\gamma\nabla^\beta\nabla^\alpha\nabla_\nu \\
&+ 4(\nabla^\beta\nabla^\alpha P^\mu_\nu)\nabla_\beta\nabla_\alpha F_2 \\
&- 4\lambda(\nabla_\gamma\nabla_\beta R_{\nu\alpha})\nabla^\gamma\nabla^\beta\nabla^\mu\nabla^\alpha + \cdots,
\end{aligned} \tag{6.17}$$

up to terms $\mathcal{O}(R^2)$, $\mathcal{O}(RP)$ and higher derivatives of the curvatures.

A. M_0-term

Let us explicitly calculate the contribution of the first term in (6.12),

$$\begin{aligned}
K^\mu_\alpha \frac{\delta^\alpha_\nu}{F_1F_2}\delta(x,y)|_{y=x} = &\frac{\delta^\mu_\nu}{F_1}\delta(x,y)|_{y=x} \\
&+ \frac{\lambda}{1-\lambda}\nabla^\mu\nabla_\alpha\frac{\delta^\alpha_\nu}{F_1F_2}\delta(x,y)|_{y=x}
\end{aligned} \tag{6.18}$$

The $1/m^2$ term from the first term can be obtained from (6.5) and is given by

$$-g^{1/2}\frac{a^{\Box\mu}_{2\,\nu}(x,x)}{16\pi^2m^2}. \tag{6.19}$$

That from the second term in (6.18) is calculated as

$$\begin{aligned}
\frac{\lambda}{1-\lambda}\nabla^\mu\nabla_\alpha\frac{\delta^\alpha_\nu}{(\Box-m^2)(\Box-\frac{m^2}{1-\lambda})}\delta(x,y)|_{y=x} &= \frac{\lambda g^{1/2}}{1-\lambda}\nabla^\mu\nabla_\alpha\int_0^\infty ds\,dt\,\exp\left[(s+t)\Box - m^2\left(s+\frac{t}{1-\lambda}\right)\right]\delta(x,y)|_{y=x} \\
&= \frac{\lambda g^{1/2}}{1-\lambda}\int_0^\infty\frac{d\gamma\gamma e^{-m^2\frac{1-\alpha\lambda}{1-\lambda}}}{(4\pi\gamma)^{d/2}}\int_0^1 d\alpha\left(\nabla^\mu - \frac{\sigma^\mu}{2\gamma}\right)\left(\nabla_\alpha - \frac{\sigma_\alpha}{2\gamma}\right) \\
&\quad\times\Delta^{1/2}\left(a^{\Box\alpha}_{0\,\nu} + a^{\Box\alpha}_{1\,\nu}\gamma + a^{\Box\alpha}_{2\,\nu}\gamma^2 + \cdots\right)|_{y=x},
\end{aligned} \tag{6.20}$$

where we have used proper time representations for both massive Green's functions (s- and t-integrals) and made a change of integration variables $s,t\to\gamma,\alpha$, $s=\alpha\gamma$, $t=(1-\alpha)\gamma$, with $0\le s+t=\gamma<\infty$. We have also used the Schwinger-DeWitt expansion (A6) and pulled $\exp(-\sigma(x,y)/2\gamma)$ to the left through two covariant derivatives. Here $\sigma_\alpha = \nabla_\alpha\sigma(x,y)$ and similarly σ^μ.

Only two terms containing $\nabla\nabla R$ survive here after differentiation and taking the coincidence limits. These are where ∇^μ acts on σ_α times $a^\Box_2$ [they give $(\nabla^\mu\sigma_\alpha)a^{\Box\alpha}_{2\,\nu}|_{y=x} = a^{\Box\mu}_{2\,\nu}(x,x)$] and where both derivatives act on $a^\Box_1$. They have a needed dimensionality, and contain the needed $\nabla\nabla R$ terms. Taking in these two terms the integral over γ (which can be done directly in $d=4$, because these terms are UV finite) we get

$$\begin{aligned}
\frac{\lambda}{1-\lambda}\nabla^\mu\nabla_\alpha\frac{\delta^\alpha_\nu}{(\Box-m^2)(\Box-\frac{m^2}{1-\lambda})}\delta(x,y)|_{y=x} &= \lambda\frac{g^{1/2}}{16\pi^2m^2}\int_0^1\frac{d\alpha}{1-\alpha\lambda}\left[\nabla^\mu\nabla_\alpha a^{\Box\alpha}_{1\,\nu}(x,y) - \frac{1}{2}a^{\Box\mu}_{2\,\nu}(x,y)\right]\Bigg|_{y=x} + \cdots \\
&= -g^{1/2}\frac{\log(1-\lambda)}{16\pi^2m^2}\left[\nabla^\mu\nabla_\alpha a^{\Box\alpha}_{1\,\nu}(x,y) - \frac{1}{2}a^{\Box\mu}_{2\,\nu}(x,y)\right]\Bigg|_{y=x} + \cdots.
\end{aligned} \tag{6.21}$$

Using the coincidence limits (4.32) and (4.33) of [30] for the vector field Schwinger-DeWitt coefficients $a^{\Box\,\alpha}_{n\,\nu}(x,y)$, $n = 0, 1, \ldots$, of the operator $\Box\delta^\mu_\nu$:

$$\nabla^\mu\nabla_\alpha a^{\Box\,\alpha}_{1\,\nu}(x,y)|_{y=x} = -\frac{1}{15}\Box R^\mu_\nu + \frac{2}{15}\nabla^\mu\nabla_\nu R + \cdots, \tag{6.22}$$

$$a^{\Box\,\mu}_{2\,\nu}(x,x) = \frac{1}{30}\Box R\delta^\mu_\nu + \cdots, \tag{6.23}$$

one finally has

$$K^\mu_\alpha \frac{\delta^\alpha_\nu}{F_1F_2}\delta(x,y)|_{y=x} = \frac{g^{1/2}}{16\pi^2 m^2}\left\{\left[-\frac{1}{30}+\frac{1}{60}\log(1-\lambda)\right]\Box R\delta^\mu_\nu - \frac{2}{15}\log(1-\lambda)\nabla^\mu\nabla_\nu R + \frac{1}{15}\Box R^\mu_\nu\log(1-\lambda) + \cdots\right\}. \tag{6.24}$$

B. M_1-term

The rest of the terms can be calculated by using the formulas given in Appendix A. Using (6.7) and (6.9) and retaining only the terms with $\nabla^2 R$ and $\nabla^2 P$, we have, for the second term in (6.12),

$$\begin{aligned}-(KM)^\mu_\nu\frac{1}{F_1^2F_2^2} = g^{1/2}\Bigg\{&-\left[-\frac{\lambda}{2}\nabla^\mu\nabla_\nu R + \Box P^\mu_\nu + \frac{\lambda}{1-\lambda}\nabla^\mu\nabla_\alpha P^\alpha_\nu\right]\frac{1}{F_1^2F_2}\\ &-\left[\frac{\lambda}{1-\lambda}\Box(R^{\mu\alpha}+P^{\mu\alpha}) + \left(\frac{\lambda}{1-\lambda}\right)^2\nabla^\mu\nabla_\beta(R^{\beta\alpha}+P^{\beta\alpha})\right]\nabla_\alpha\nabla_\nu\frac{1}{F_1^2F_2^2}\\ &+\frac{1}{2}\frac{\lambda^2}{1-\lambda}(\nabla^\mu\nabla_\nu R)\Box\frac{1}{F_1^2F_2^2} + \frac{2\lambda}{1-\lambda}(\nabla_\alpha\nabla^\mu R_{\nu\beta})\nabla^\alpha\nabla^\beta\frac{1}{F_1^2F_2^2}\\ &+\frac{\lambda}{1-\lambda}\left[\nabla_\alpha\nabla_\nu R + \Box R_{\alpha\nu}\right]\nabla^\mu\nabla_\alpha\frac{1}{F_1^2F_2^2} + \cdots\Bigg\}.\end{aligned} \tag{6.25}$$

Using the results in Appendix A in (6.25), one gets the result for M_1-term,

$$\begin{aligned}-(KM)^\mu_\nu\frac{1}{F_1^2F_2^2} = g^{1/2}\frac{1-\lambda}{16\pi^2m^2}\Bigg\{&(\nabla^\mu\nabla_\nu R)\left[-\frac{\lambda}{2}I_{1,0} + \left(\frac{\gamma^2}{4}-\gamma\lambda-\gamma\right)I_{1,1}\right]\\ &+(\Box P^\mu_\nu)\left[I_{1,0}+\frac{\gamma}{2}I_{1,1}\right] + (\nabla^\mu\nabla_\alpha P^\alpha_\nu)\left[\gamma I_{1,0}+\frac{\gamma^2}{2}I_{1,1}\right]\Bigg\},\end{aligned} \tag{6.26}$$

where the integrals I_{n_1,n_2} here and below are defined in (A10) in Appendix A and

$$\gamma = \frac{\lambda}{1-\lambda}. \tag{6.27}$$

Note that $\Box R^\mu_\nu$-term does not appear in this order at all.

C. M_2-term

Using (A1), we have for the third term in (6.12),

$$\begin{aligned}(KM_2)^\mu_\nu &= K^\mu_\alpha[\Box, M^\alpha_\nu](F_1+F_2) + K^\mu_\alpha[\Box,[\Box,M^\alpha_\nu]] + \cdots\\ &= [\Box, M^\alpha_\nu]K^\mu_\alpha(F_1+F_2) + \left[K^\mu_\alpha,[\Box,M^\alpha_\nu]\right](F_1+F_2)\\ &\quad+\left[\Box,[\Box,M^\alpha_\nu]\right]K^\mu_\alpha + \cdots.\end{aligned} \tag{6.28}$$

The derivative operator is put to the right, and to the order we need, we find

$$\begin{aligned}(KM_2)^\mu_\nu = {}&[\Box, M^\mu_\nu]F_2(F_1+F_2)\\ &+\frac{\lambda}{1-\lambda}[\Box,M^\alpha_\nu]\nabla^\mu\nabla_\alpha(F_1+F_2)\\ &+\left[\Box,[\Box,M^\mu_\nu]\right](F_1+2F_2)\\ &+\frac{\lambda}{1-\lambda}\left[\Box,[\Box,M^\alpha_\nu]\right]\nabla^\mu\nabla_\alpha\\ &+\frac{\lambda}{1-\lambda}\left[\nabla^\mu\nabla_\alpha,[\Box,M^\alpha_\nu]\right](F_1+F_2)+\cdots.\end{aligned} \tag{6.29}$$

Using the results in Appendix A, we can calculate each term in (6.29). Collecting these terms, we find the M_2 term which reads

$$
\begin{aligned}
(KM_2)^\mu_\nu \frac{1}{F_1^3 F_2^3} = -\frac{g^{1/2}}{16\pi^2 m^2}\Bigg\{ & \frac{\lambda}{4}\Big[2\lambda I_{1,1} + \lambda I_{2,0} + (\gamma-4\lambda)I_{2,2} + (\gamma-3\lambda)I_{1,2} + (\gamma-\lambda)I_{2,1}\Big]\Box R^\mu_\nu \\
& + \frac{1}{4}\Big[2 + 4(1-\lambda)I_{1,0} + 4(2-\lambda)I_{1,1} + 2(4-3\lambda)I_{2,0} + \lambda(\gamma+2)I_{1,2} \\
& + \lambda(\gamma+6)I_{2,1} + \gamma\lambda I_{2,2}\Big]\Box P^\mu_\nu \\
& + \frac{\gamma\lambda}{8}(I_{1,2} + I_{2,1} + I_{2,2})\Big[\delta^\mu_\nu(2\Box R + \Box P + 2\nabla^\alpha\nabla^\beta P_{\alpha\beta}) + 2\nabla^\mu\nabla_\nu P\Big] \\
& + \frac{\lambda}{4}\Big[(3\gamma-4\lambda)(I_{2,1}+I_{2,2}) + (3\gamma-6\lambda)I_{1,2} - 2(1-\lambda)(I_{2,0}+2I_{1,1})\Big]\nabla^\mu\nabla_\nu R \\
& + \frac{\lambda}{4}\Big[(\gamma+8)I_{2,1} + (\gamma+4)I_{1,2} + 2\gamma I_{2,2}\Big]\nabla_\nu\nabla^\alpha P^\mu_\alpha \\
& + \frac{\lambda}{4}\Big[8I_{1,1} + 4I_{2,0} + 3\gamma I_{1,2} + (3\gamma+4)I_{2,1} + 2\gamma I_{2,2}\Big]\nabla^\mu\nabla^\alpha P_{\alpha\nu}\Bigg\}. \qquad (6.30)
\end{aligned}
$$

D. M_3-term

We find that the M_3-term is

$$
\begin{aligned}
-(KM_3)^\mu_\nu = -\Big[& 4\gamma\nabla_\gamma\nabla_\beta(R^\mu_\alpha + P^\mu_\alpha)\nabla^\gamma\nabla^\beta\nabla^\alpha\nabla_\nu F_2 + 4(\nabla^\beta\nabla^\alpha P^\mu_\nu)\nabla_\beta\nabla_\alpha F_2^2 \\
& - 4\lambda(\nabla_\gamma\nabla_\beta R_{\nu\alpha})\nabla^\gamma\nabla^\beta\nabla^\mu\nabla^\alpha F_2 + 4\gamma^2\nabla_\gamma\nabla_\beta(R^\rho_\alpha + P^\rho_\alpha)\nabla^\mu\nabla_\rho\nabla^\gamma\nabla^\beta\nabla^\alpha\nabla_\nu \\
& + 4\gamma(\nabla^\beta\nabla^\alpha P^\rho_\nu)\nabla^\mu\nabla_\rho\nabla_\beta\nabla_\alpha F_2 - 4\gamma\lambda(\nabla_\gamma\nabla_\beta R_{\alpha\nu})\Box\nabla^\mu\nabla^\gamma\nabla^\beta\nabla^\alpha\Big](F_1+F_2)^2. \qquad (6.31)
\end{aligned}
$$

By use of the results in Appendix A, this gives

$$
\begin{aligned}
-(KM_3)^\mu_\nu \frac{1}{F_1^4 F_2^4} = \frac{g^{1/2}}{16\pi^2 m^2}\Bigg[& (\Box R^\mu_\nu)\lambda\left(\frac{-3I_{1,2} + 3I_{2,1} + I_{3,0}}{6} + \gamma\frac{I_{1,3} + 3I_{2,2} + I_{3,1}}{6} - \frac{I_{0,3}}{6}\right) \\
& + (\Box P^\mu_\nu)\left(\frac{1}{3} + 2(1-\lambda)(I_{1,1} + I_{2,0}) + \lambda\frac{3I_{1,2} + 6I_{2,1} + I_{3,0}}{3} + \gamma\lambda\frac{I_{1,3} + 3I_{2,2} + I_{3,1}}{6}\right) \\
& + (\nabla^\mu\nabla^\alpha P_{\nu\alpha} + \nabla_\nu\nabla^\alpha P^\mu_\alpha)\lambda\left(\gamma\frac{I_{1,3} + 3I_{2,2} + I_{3,1}}{3} + \frac{3I_{1,2} + 6I_{2,1} + I_{3,0}}{3}\right) \\
& + (\nabla^\mu\nabla_\nu R)\lambda\left(\frac{-3I_{1,2} + 3I_{2,1} + I_{3,0} - I_{0,3}}{6} + \gamma\frac{I_{1,3} + 3I_{2,2} + I_{3,1}}{2}\right) \\
& + \gamma\lambda(\delta^\mu_\nu\{2\Box R + \Box P + 2\nabla^\alpha\nabla^\beta P_{\alpha\beta}\} + 2\nabla^\mu\nabla_\nu P)\frac{I_{1,3} + 3I_{2,2} + I_{3,1}}{12}\Bigg]. \qquad (6.32)
\end{aligned}
$$

E. The total result

Collecting all the results up to the M_3 terms and substituting the result of the integrals from Appendix A, we get

$$
\begin{aligned}
-\mathrm{tr}\hat{G}(x,x) \sim \frac{g^{1/2}}{360(4\pi)^2 m^2(1-\lambda)\lambda^2}\Big[& \lambda\Big\{(60 + 168\lambda - 133\lambda^2)\Box R \\
& - 10(6 - 21\lambda + 8\lambda^2)(\Box P) - 10(-24 + 6\lambda + 13\lambda^2)(\nabla^\alpha\nabla^\beta P_{\alpha\beta})\Big\} \\
& + 30(1-\lambda)\Big\{(2 + 5\lambda - \lambda^2)(\Box R) + 2(2\lambda - 1)(\Box P) \\
& + 2(4+\lambda)(\nabla^\alpha\nabla^\beta P_{\alpha\beta})\Big\}\log(1-\lambda)\Big]. \qquad (6.33)
\end{aligned}
$$

On the other hand, the result in [32] is

$$\mathrm{tr}\,\hat{E}_4(x) = \frac{1}{(4\pi)^2} a^{\mu}_{2,\mu}(x,x) = \frac{1}{(4\pi)^2}(c_1\Box P + c_2\nabla^\alpha\nabla^\beta P_{\alpha\beta} + c_{11}\Box R), \qquad (6.34)$$

with the coefficients given in (5.18). Thus, this result agrees with Eq. (6.33).

VII. FINAL RESULT FOR THE ANOMALOUS $\Box R$ TERM

Let us now apply the algorithm for the minimal fourth-order operator (5.11) and for the nonminimal vector operator (5.17), to derive the one-loop total-derivative divergence in the theory (1.2).

We start from the tensor sector. The intermediate expressions for the elements of the general formula (5.11), with the elements (3.9)–(3.11), can be found in Appendix B. Summing up these expressions, we arrive at

$$\mathrm{tr}\,\hat{a}_2^H = \frac{1}{2}\mathrm{tr}\left\{\frac{74}{135}\delta_{\mu\nu,\alpha\beta}\Box R + \frac{8}{27}g_{\mu\nu}\Box R_{\alpha\beta} - \frac{8}{3}g_{\nu\beta}\Box R_{\mu\alpha} + \frac{2}{3}\Box R_{\mu\alpha\nu\beta} - \frac{20}{27}\nabla_\mu\nabla_\nu R_{\alpha\beta} + \frac{4}{9}\nabla_\mu\nabla_\alpha R_{\nu\beta} + \frac{20}{27}g_{\nu\beta}\nabla_\mu\nabla_\alpha R\right\}. \qquad (7.1)$$

The rule of taking trace should take into account that the unite matrix is the projector to the traceless states, such that, e.g.,

$$\mathrm{tr}\delta_{\mu\nu,\alpha\beta} = \left(\delta^{\mu\nu,\alpha\beta} - \frac{1}{4}g^{\mu\nu}g^{\alpha\beta}\right)\delta_{\mu\nu,\alpha\beta} = 9. \qquad (7.2)$$

In this way, after small algebra, we get

$$\mathrm{tr}\,\hat{a}_2^H = \frac{13}{135}\Box R. \qquad (7.3)$$

For the nonminimal vector operators, we have to apply the results (5.17) and (5.18) to the operators (4.2) and (4.4). In the first case, $P^{\mu\nu} = 0$ and $\lambda = 1/3$. After a small algebra, we get

$$\mathrm{tr}\,\hat{a}_2^Y = \left\{\frac{911}{720} - \frac{8}{3}\log\left(\frac{3}{2}\right)\right\}\Box R. \qquad (7.4)$$

For the ghost operator $\hat{M}$, we have $P^{\mu\nu} = R^{\mu\nu}$ and $\lambda = -1/2$. The calculations give, in this case,

$$\mathrm{tr}\,\hat{a}_2^M = \left\{\frac{247}{540} - \frac{5}{12}\log\left(\frac{3}{2}\right)\right\}\Box R. \qquad (7.5)$$

Substituting these results in (5.9) we get

$$\langle T^\mu_\mu(x)\rangle|_{\text{total derivative}} = \frac{1}{16\pi^2}\left\{\frac{7}{2}\log\left(\frac{3}{2}\right) - \frac{159}{80}\right\}\Box R, \qquad (7.6)$$

where we explicitly indicated that this is a total derivative part of the full trace anomaly. Other terms in the anomaly can be recovered from the integrand of the one-loop divergences in Weyl gravity model, which can be found in [6] or [7,9]. As we have mentioned above, the result quoted in (7.6) cannot be modified by adding a finite classical R^2 term, and the last enters into the anomaly-induced action only in the form defined by the anomaly and the relation (1.1).

At this point, we can formulate the complete version of Eq. (2.2) for the effective action of gravity induced by conformal anomaly in Weyl-squared quantum conformal gravity. This equation has the form

$$\langle T^\mu_\mu(x)\rangle = -\frac{2}{\sqrt{g}}g_{\mu\nu}\frac{\delta\Gamma_{\text{renorm}}}{\delta g_{\mu\nu}} = \omega C^2 + bE_4 + c\Box R, \qquad (7.7)$$

where we stressed the fact that the equation is for the finite renormalized part Γ_{renorm} of the effective action with the divergences are subtracted in the process of renormalization. According to [6,7,9] and (7.6) the coefficients are

$$\omega = \frac{1}{(4\pi)^2}\frac{199}{15}, \quad b = -\frac{1}{(4\pi)^2}\frac{87}{20}, \quad c = \frac{1}{(4\pi)^2}\left[\frac{7}{2}\log\left(\frac{3}{2}\right) - \frac{159}{80}\right]. \qquad (7.8)$$

The solution of Eq. (7.7) has the general form (2.5) and, as usual, it includes a conformal invariant functional of the metric S_c, playing the role of an integration constant for the equation. This functional is not controlled by the trace anomaly.

VIII. CONCLUSIONS

We have presented original calculation of the $\Box R$ term in the one-loop trace anomaly of Weyl gravity model with the action (1.2). The mapping of this anomaly to one-loop divergences of the theory runs within the framework of the zeta-function regularization or the regularization by the covariant proper time cutoff in the heat kernel. Functional integration of the $\Box R$ anomaly term yields a finite local R^2 term in the one-loop effective action. Unlike the semi-classical theory, this term cannot be "removed" by adding a "finite counterterm." This situation demonstrates that the local conformal invariance gets violated at the one-loop level. This violation is related not only to the nonlocal terms shown in (2.5), but also to the local R^2 term. Starting from the second loop, one has to take this term into account, modify propagator by adding the dynamical scalar mode, modify vertices, etc. This confirms that the local conformal symmetry cannot be exact at the quantum level and, moreover, its violation beyond one-loop approximation cannot be controlled [23] unless some mechanisms like supersymmetry are used for the cancellation of anomalies [24].

Despite this set of intrinsic inconsistencies and regularization ambiguities of the above type, the calculation of anomalous $\Box R$ terms still makes sense for the sake of the potential analysis of the boundary terms in one-loop divergences and in view of cosmological implications of the related R^2 terms in the Starobinsky model of modified gravity theory. Let us also note that since in dimensional and Pauli-Villars regularizations there are ambiguities on the way from divergences to the trace anomaly, it would be certainly interesting to make derivation within nonlocal covariant curvature expansion of [52,53]. In particular, it is worth deriving the form factors of the R^2 term for the minimal fourth-order and nonminimal second-order vector operators. However, this challenging calculation is beyond the scope of the present work and we postpone it for future.

Let us mention two important aspects of the conformal anomaly. First, while the local conformal symmetry is violated by both local and nonlocal terms, the global symmetry still holds in the anomaly-induced action (2.5). This shows that the destinies of these two symmetries at the quantum level are very much different. Unlike the anomaly which was discussed above, the violation of the global symmetry requires the presence of a dimensional parameter, which may emerge either from an interaction with massive fields or from the phase transition and dimensional transmutation, as discussed in [54,55].

The only way to use local conformal symmetry in quantum theory is by assuming the corresponding hierarchy, as it was discussed in [7]. One can start with the theory that has both C^2 and R^2 terms, but the coefficient of the last is very small, such that its contributions in loops get strongly suppressed. There is a chance that this hierarchy may hold at higher loops. This scheme enables one to preserve the advantages of conformal theory, including the compact and useful form of anomaly-induced action.

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APPENDIX A: DETAILS OF THE CALCULATIONS

In this appendix, we summarize formulae necessary for the evaluation of the Green's functions performed in Sec. VI.

M_2 and M_3 take the following skeleton form in the needed approximation

$$M_2 = [\Box, M](F_1 + F_2) + \big[\Box, [\Box, M]\big] + \cdots, \qquad \text{(A1)}$$

$$M_3 = \big[\Box, [\Box, M]\big](F_1 + F_2)^2 + \cdots, \qquad \text{(A2)}$$

and the expansion (6.12) starts with the following four terms which contribute to needed $\nabla\nabla R$ and $\nabla\nabla P$ structures

$$\frac{1}{F - m^2} = K\frac{1}{F_1 F_2} - KM\frac{1}{F_1^2 F_2^2} + KM_2\frac{1}{F_1^3 F_2^3} - KM_3\frac{1}{F_1^4 F_2^4} + \cdots. \qquad \text{(A3)}$$

Note that the coefficient functions in (A3) are differential operators K, KM, KM_2, KM_3 which contain the mass parameter only in combinations F_1, F_2 given by (6.11) or their powers. Therefore, the final answer is a linear combination of the following massive universal traces

$$\nabla_{\alpha_1}\ldots\nabla_{\alpha_{2n}}\frac{1}{F_1^{n_1}F_2^{n_2}}\delta(x, y)\Big|_{y=x}, \qquad \text{(A4)}$$

where $n_1 + n_2 - n = 3$ and the restriction on n, n_1 and n_2 follows from their dimensionality $\sim 1/m^2$. To evaluate this, we need the proper time representation in terms of the heat kernel

$$\frac{1}{(\Box - m_1^2)^{n_1}(\Box - m_2^2)^{n_2}}\delta(x, y) = \frac{(-1)^{n_1+n_2}}{\Gamma(n_1)\Gamma(n_2)}\int_0^\infty ds_1 ds_2 s_1^{n_1-1}s_2^{n_2-1}e^{(s_1+s_2)\Box-(s_1m_1^2+s_2m_2^2)}\delta(x, y)$$
$$= \frac{(-1)^{n_1+n_2}}{\Gamma(n_1)\Gamma(n_2)}\int_0^\infty dt t^{n_1+n_2-1}\int_0^1 d\alpha\alpha^{n_1-1}(1-\alpha)^{n_2-1}e^{-tm^2(\alpha)}e^{t\Box}\delta(x, y), \qquad \text{(A5)}$$

where we have made the change of variables $s_1 = \alpha t, s_2 = (1-\alpha)t$ and $m^2(\alpha) \equiv m_1^2\alpha + m_2^2(1-\alpha)$. Substitution of the Schwinger-DeWitt expansion for the $\Box$-operator [29]

$$e^{s\Box}\delta^\mu_\nu(x,x') = \frac{\Delta^{1/2}(x,x')}{(4\pi s)^{d/2}} g^{1/2}(x') e^{-\frac{\sigma(x,x')}{2s}}\Big(a^{\Box\,\mu}_{0\,\nu}(x,x') + a^{\Box\,\mu}_{1\,\nu}(x,x')s + a^{\Box\,\mu}_{2\,\nu}(x,x')s^2 + \cdots\Big), \tag{A6}$$

then gives the final result with a given accuracy in curvatures. Here $\sigma(x,x')$ is a geodesic interval given by one half of the square of the distance along the geodesic between x and x', $\Delta(x,x') = -g(x)^{-1/2}\det(-\sigma_{,\mu\nu'})g(x')^{-1/2}$, and $a^{\Box\,\mu}_{n\,\nu}(x,x')$ are the Schwinger-DeWitt coefficients for the $\Box$-operator.

We need to keep only the first term to obtain

$$\begin{aligned} g^{-1/2}\nabla_{\alpha_1}\ldots\nabla_{\alpha_{2n}}\frac{1}{F_1^{n_1}F_2^{n_2}}\delta(x,y)\big|_{y=x} &= \frac{(-1)^{n_1+n_2}}{\Gamma(n_1)\Gamma(n_2)}\int_0^\infty \frac{dt t^{n_1+n_2-1}}{(4\pi t)^2}\int_0^1 d\alpha\alpha^{n_1-1}(1-\alpha)^{n_2-1} \\ &\quad\times e^{-tm^2(\alpha)}\nabla_{\alpha_1}\ldots\nabla_{\alpha_{2n}}e^{-\sigma(x,y)/2t}\big|_{y=x} + \cdots \\ &= -\frac{1}{2^n(n_1-1)!(n_2-1)!}\frac{1-\lambda}{16\pi^2m^2}\int_0^1 d\alpha\frac{\alpha^{n_1-1}(1-\alpha)^{n_2-1}}{1-\alpha\lambda}g^{(n)}_{\alpha_1\ldots\alpha_{2n}} + \cdots, \end{aligned} \tag{A7}$$

where we have taken into account that $(-1)^{n+n_1+n_2} = -1$ in view of the above restriction, ellipses denote terms other than $(\Box R)$ and $(\Box P)$ structures, and we have used the totally symmetric tensor built of the metric [30]

$$\begin{aligned} &\nabla_{\alpha_1}\ldots\nabla_{\alpha_{2n}}e^{-\sigma(x,y)/2t}\big|_{y=x} + \cdots \\ &= (\nabla_{\alpha_1} - \sigma_{\alpha_1}/2t)\ldots(\nabla_{\alpha_{2n}} - \sigma_{\alpha_{2n}}/2t)1\big|_{y=x} + \cdots \\ &= \left(-\frac{1}{2t}\right)^n g^{(n)}_{\alpha_1\ldots\alpha_{2n}} + \cdots, \end{aligned} \tag{A8}$$

$$g^{(1)}_{\alpha\beta} = g_{\alpha\beta}, \quad g^{(2)}_{\alpha\beta\mu\nu} = g_{\alpha\beta}g_{\mu\nu} + g_{\alpha\mu}g_{\beta\nu} + g_{\alpha\nu}g_{\beta\mu}, \ldots. \tag{A9}$$

It proves useful to introduce the notation for the α-integrals,

$$I_{n_1,n_2}(\lambda) = \int_0^1 d\alpha\frac{\alpha^{n_1}(1-\alpha)^{n_2}}{1-\alpha\lambda}. \tag{A10}$$

This integral can be easily evaluated for integer n_1, n_2 by using *Mathematica* or other software.

Next, it follows from (A7) that for all n_1, n_2 and the number of derivatives $2n = 2(n_1+n_2-3)$, the needed massive universal functional traces read

$$\begin{aligned} &\nabla_{\alpha_1}\ldots\nabla_{\alpha_{2n}}\frac{1}{F_1^{n_1}F_2^{n_2}}\delta(x,y)\big|_{y=x} \\ &= g^{1/2}\frac{1-\lambda}{16\pi^2m^2}\frac{(-1)^{n+n_1+n_2}I_{n_1-1,n_2-1}(\lambda)}{2^n(n_1-1)!(n_2-1)!}g^{(n)}_{\alpha_1\ldots\alpha_{2n}} + \cdots \\ &= -g^{1/2}\frac{1-\lambda}{16\pi^2m^2}\frac{I_{n_1-1,n_2-1}(\lambda)}{2^n(n_1-1)!(n_2-1)!}g^{(n)}_{\alpha_1\ldots\alpha_{2n}} + \cdots \end{aligned} \tag{A11}$$

We also need

$$\nabla_{\alpha_1}\ldots\nabla_{\alpha_{2n}}\frac{1}{F_1^{n_1}}\delta(x,y)\big|_{y=x}. \tag{A12}$$

First recall that

$$\frac{1}{F_1^{n_1}} = \frac{(-1)^{n_1}}{\Gamma(n_1)}\int_0^\infty ds s^{n_1-1}e^{sF_1}. \tag{A13}$$

So we get

$$\begin{aligned} &\nabla_{\alpha_1}\ldots\nabla_{\alpha_{2n}}\frac{1}{F_1^{n_1}}\delta(x,y)\big|_{y=x} \\ &= g^{1/2}\frac{(-1)^{n_1}}{\Gamma(n_1)}\int_0^\infty ds\, s^{n_1-1}\nabla_{\alpha_1}\ldots\nabla_{\alpha_{2n}}e^{sF_1}\delta(x,y)\big|_{y=x} \\ &= g^{1/2}\frac{(-1)^{n_1}}{\Gamma(n_1)}\int_0^\infty\frac{ds s^{n_1-1}}{(4\pi s)^2}\nabla_{\alpha_1}\ldots\nabla_{\alpha_{2n}}e^{-sm^2}e^{-\sigma(x,y)/2s}\big|_{y=x} \\ &= g^{1/2}\frac{(-1)^{n_1}}{16\pi^2\Gamma(n_1)}\left(-\frac{1}{2}\right)^n\int_0^\infty ds e^{-sm^2}g^{(n)}_{\alpha_1\ldots\alpha_{2n}} + \cdots \\ &= -\frac{g^{1/2}}{16\pi^2m^2}\frac{1}{2^n(n_1-1)!}g^{(n)}_{\alpha_1\ldots\alpha_{2n}} + \cdots, \end{aligned} \tag{A14}$$

where we have chosen $n = n_1 - 3$.

APPENDIX B: TENSOR CONTRIBUTION IN $\Box R$-TYPE DIVERGENCES

Let us list the particular results necessary to evaluate the general formula (5.11) in (7.1) for the tensor sector. These are derived from (3.9) and (3.10).

$$(\nabla_\rho \nabla_\sigma \hat{V}^{\rho\sigma})_{\mu\nu,\alpha\beta} = \frac{1}{3}\delta_{\mu\nu,\alpha\beta}(\Box R) - \frac{8}{3}(\nabla_\mu \nabla_\alpha R)g_{\nu\beta} + \frac{4}{3}(\nabla_\mu \nabla_\nu R_{\alpha\beta}) + \frac{4}{3}(\nabla_\alpha \nabla_\beta R_{\mu\nu}) + 2(\nabla_\nu \nabla_\beta R_{\mu\alpha}) + 4(\Box R_{\mu\alpha\nu\beta}), \tag{B1}$$

$$(\Box \hat{V})_{\mu\nu,\alpha\beta} = \frac{2}{3}\delta_{\mu\nu,\alpha\beta}(\Box R) + \frac{4}{3}g_{\mu\nu}(\Box R_{\alpha\beta}) + \frac{4}{3}g_{\alpha\beta}(\Box R_{\mu\nu}) - 6(\Box R_{\mu\alpha})g_{\nu\beta} + 16(\Box R_{\mu\alpha\nu\beta}), \tag{B2}$$

$$\begin{aligned}(\nabla_\lambda \hat{N}^\lambda)_{\mu\nu,\alpha\beta} = {} & \frac{1}{3}\delta_{\mu\nu,\alpha\beta}(\Box R) - \frac{4}{3}(\nabla_\mu \nabla_\nu R_{\alpha\beta}) - \frac{8}{3}(\nabla_\mu \nabla_\alpha R)g_{\nu\beta} - 2(\nabla_\mu \nabla_\alpha R_{\nu\beta}) \\ & + 4(\nabla_\alpha \nabla_\beta R_{\mu\nu}) + 4(\nabla_\nu \nabla_\beta R_{\mu\alpha}) + 4(\Box R_{\mu\alpha\nu\beta}),\end{aligned} \tag{B3}$$

where symmetrization $\mu \leftrightarrow \nu$, $\alpha \leftrightarrow \beta$ and $(\mu,\nu) \leftrightarrow (\alpha,\beta)$ should be understood.

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