Three-point functions of conserved currents in 4D CFT: General formalism for arbitrary spins

Evgeny I. Buchbinder* and Benjamin J. Stone†
*Department of Physics M013, The University of Western Australia,
35 Stirling Highway, Crawley, Western Australia 6009, Australia*

(Received 10 August 2023; accepted 19 September 2023; published 11 October 2023)

We analyze the general structure of the three-point functions involving conserved higher-spin currents $J_s \coloneqq J_{\alpha(i)\dot{\alpha}(j)}$ belonging to any Lorentz representation in four-dimensional conformal field theory. Using the constraints of conformal symmetry and conservation equations, we computationally analyze the general structure of three-point functions $\langle J_{s_1} J'_{s_2} J''_{s_3} \rangle$ for arbitrary spins and propose a classification of the results. For bosonic vectorlike currents with $i = j$, it is known that the number of independent conserved structures is $2\ \min(s_i) + 1$. For the three-point functions of conserved currents with arbitrarily many dotted and undotted indices, we show that in many cases the number of structures deviates from $2\ \min(s_i) + 1$.

DOI: 10.1103/PhysRevD.108.086017

I. INTRODUCTION

In conformal field theory (CFT), it is known that the general structure of three-point functions of conserved currents is highly constrained by conformal symmetry. A systematic approach to study three-point functions of primary operators was introduced in [1,2] (see also Refs. [3–12] for earlier works), which presented an analysis of the general structure of three-point functions involving the energy-momentum tensor and conserved vector currents. The analysis of three-point functions of conserved higher-spin bosonic currents was later undertaken by Stanev [13–15] (see also [16,17]) in the four-dimensional case, and by Zhiboedov [18] in general dimensions.[1] In four dimensions (4D) it was shown that the number of independent structures in the three-point function of conserved bosonic vector-like currents $J_{\mu_1 \ldots \mu_s}$ increases linearly with the minimum spin. This is quite different to the results found in three dimensions (3D), where it has been shown by many authors [37–43] that there are only three possible independent conserved structures for currents of arbitrary integer/half-integer spins. The aim of this paper is to study three-point functions of conserved currents belonging to arbitrary Lorentz representations in 4D CFT. An approach to this problem was outlined in [16], however, it did not study correlation functions when the operators are all conserved currents.

In this paper we provide a complete classification of three-point functions of conserved currents $J_{\alpha(i)\dot{\alpha}(j)}$, with i, $j \geq 1$ in four-dimensional conformal field theory. Such currents satisfy the conservation equation

$$\partial^{\beta\dot{\beta}} J_{\beta\alpha(i-1)\dot{\beta}\,\dot{\alpha}(j-1)} = 0, \tag{1.1}$$

and possess scale dimension $\Delta_J = s + 2$, where the spin s is given by $s = \frac{1}{2}(i + j)$. To classify the possible three-point functions of currents $J_{\alpha(i)\dot{\alpha}(j)}$, we find it more convenient to parametrize them in terms of their spin, s, and an integer, $q = i - j$, as follows:

$$J_{(s,q)} \coloneqq J_{\alpha(s+\frac{q}{2})\dot{\alpha}(s-\frac{q}{2})}. \tag{1.2}$$

With this convention q is necessarily even/odd when s is integer/half-integer valued. Note that the Hermitian conjugate of $J_{(s,q)}$ is $J_{(s,-q)}$, hence, we introduce $\bar{J}_{(s,q)} \coloneqq J_{(s,-q)}$ and view q as being non-negative, taking values $q = 0, 1, \ldots, 2s - 2$. The case $q = 0$ corresponds to "standard" bosonic conserved currents $J_{(s,0)} \equiv J_{\alpha(s)\dot{\alpha}(s)}$. Likewise, for $q = 1$ we obtain pairs of (higher-spin) "supersymmetry-like" currents $J_{(s,1)} \equiv J_{\alpha(s+\frac{1}{2})\dot{\alpha}(s-\frac{1}{2})}$, $\bar{J}_{(s,1)} \equiv J_{(s,-1)} = J_{\alpha(s-\frac{1}{2})\dot{\alpha}(s+\frac{1}{2})}$, where s is necessarily half-integer valued. For example, by setting $s = \frac{3}{2}$ we obtain supersymmetry currents. In nonsupersymmetric settings, the currents with $i = j$ (i.e., $q = 0$) were constructed explicitly in terms of free fields in [44].

*evgeny.buchbinder@uwa.edu.au
†benjamin.stone@research.uwa.edu.au

[1]The study of correlation functions of conserved currents has also been extended to superconformal field theories in diverse dimensions [19–36].

For the three-point functions of conserved currents, there are essentially only two possibilities to consider as a result of the notation (1.2):

$$\langle J_{(s_1,q_1)}(x_1) J'_{(s_2,q_2)}(x_2) J''_{(s_3,q_3)}(x_3)\rangle, \quad \langle J_{(s_1,q_1)}(x_1) \bar{J}'_{(s_2,q_2)}(x_2) J''_{(s_3,q_3)}(x_3)\rangle. \tag{1.3}$$

Any other possible three-point functions are equivalent to these up to permutations of the points or complex conjugation. The main aim of this paper is to develop a general formalism to study the structure of the three-point correlation functions (1.3), where we assume only the constraints imposed by conformal symmetry and conservation equations. In doing so we essentially provide a complete classification of all possible conserved three-point functions in 4D CFT. The three-point functions of currents with $q = 0$, 1 have been studied in e.g. [13,16–18]. For bosonic conserved currents ($q_i = 0$), it is known that three-point functions of conserved currents with spins s_1, s_2, s_3 are fixed up to $2\ \min(s_1, s_2, s_3) + 1$ solutions in general. We show that the same result also holds for three-point functions involving conserved currents with $q = 1$. The three-point functions of currents $J_{(s,q)}$ with $q \geq 2$, however, are relatively unexplored in the literature. Conserved currents with $q \geq 2$ naturally arise in superconformal field theories in four-dimensions. As an example, consider a $\mathcal{N} = 2$ superconformal field theory possessing a conserved higher-spin supercurrent, $\mathcal{J}_{\alpha(s)\dot{\alpha}(s)}$ with $s \geq 1$, satisfying the following superfield conservation equation [45]:

$$D^{\beta}_{i} \mathcal{J}_{\beta\alpha(s-1)\dot{\alpha}(s)} = 0, \tag{1.4}$$

where D^{β}_{i} is the spinor covariant derivative in $\mathcal{N} = 2$ superspace, and $i = 1$, 2 is an iso-spinor index. The component structure of these supercurrents was elucidated in [46,47]. The $\mathcal{N} = 2$ supercurrent $\mathcal{J}_{\alpha(s)\dot{\alpha}(s)}$ can be decomposed into the following collection of independent conformal $\mathcal{N} = 1$ supercurrent multiplets (see [46,47] for more details):

$$\{\mathbf{J}_{\alpha(s)\dot{\alpha}(s)}, \mathbf{J}_{\alpha(s+1)\dot{\alpha}(s)}, \mathbf{J}_{\alpha(s+1)\dot{\alpha}(s+1)}\}. \tag{1.5}$$

These $\mathcal{N} = 1$ supercurrents, in turn, contain a multiplet of conserved component currents [48]. In particular, the $\mathcal{N} = 1$ supercurrent, $\mathbf{J}_{\alpha(s+1)\dot{\alpha}(s)}$,[2] contains a conserved component current, $S_{\alpha(s+2)\dot{\alpha}(s)}$, defined as follows:

$$S_{\alpha(s+2)\dot{\alpha}(s)} = D_{\alpha} \mathbf{J}_{\alpha(s+1)\dot{\alpha}(s)}|, \tag{1.6}$$

where implicit symmetrization among all α-indices is assumed. Hence, the $\mathcal{N} = 2$ higher-spin supercurrent $\mathcal{J}_{\alpha(s)\dot{\alpha}(s)}$ contains a conserved component current $S_{\alpha(s+2)\dot{\alpha}(s)}$, which corresponds to $q = 2$ in our convention above. The $\mathcal{N} = 2$ supercurrents have been constructed explicitly for the free hypermultiplet and vector multiplet in [46,47].

The formalism, which augments the approach of [1] with auxiliary spinors, is suitable for constructing three-point functions of (conserved) primary operators in any Lorentz representation. Our approach is exhaustive in the sense that we construct all possible structures for the three-point function consistent with the conformal properties of the fields. We then systematically extract the linearly independent structures and impose the constraints arising from conservation equations, reality conditions, properties under inversion, and symmetries under permutations of spacetime points. The calculations are automated for arbitrary spins, and as a result we obtain the three-point function in an explicit form which can be presented up to our computational limit, $s_i = 10$. However, this limit is sufficient to propose a general classification of the results.

We would like to point out that though the formalism developed in this paper is conceptually similar to the one developed for three-dimensional CFT in [43], there are two important differences. First, in three dimensions, three-point functions of conserved currents can have at most three independent structures (two parity-even and one parity-odd), whereas in four dimensions the number of conserved structures (generically) grows linearly with the minimum spin. Second, for three-point functions in 3D CFT an important role is played by the triangle inequalities

$$s_1 \leq s_2 + s_3, \qquad s_2 \leq s_1 + s_3, \qquad s_3 \leq s_1 + s_2. \tag{1.7}$$

For three-point functions involving conserved currents which are within the triangle inequalities, there are two parity-even solutions and one parity-odd solution. However, if any of the triangle inequalities are not satisfied then the parity-odd solution is incompatible with conservation equations [37–43]. This statement has been proven in the lightcone limit in [38,39] (see also [40] for results in momentum space). However, we found that in 4D CFT the triangle inequalities appear to have no significance.

The content of this paper is organized as follows. In Sec. II we review the properties of the conformal building blocks used to construct correlation functions of primary operators in four dimensions. We then develop the formalism to construct three-point functions of primary operators of the form $J_{\alpha(i)\dot{\alpha}(j)}$, where we demonstrate how to impose all constraints arising from conservation equations, reality conditions and point switch symmetries. In particular, we utilize an index-free auxiliary spinor formalism to construct a generating function for the three-point functions, and we outline the pertinent aspects of our computational approach.

[2]This supercurrent was constructed explicitly in terms of a free massless hypermultiplet in [49].

In Sec. III, we demonstrate our formalism by analyzing the structure of three-point functions involving conserved vector currents, "supersymmetry-like" currents and the energy-momentum tensor. We reproduce the known results previously found in [1,13,18]. We then expand our discussion to include three-point functions of higher-spin currents belonging to any Lorentz representation, and provide a classification of the results. For this the structure of the solutions is more easily identified by using the notation $J_{(s,q)}$, $\bar{J}_{(s,q)}$ for the currents as outlined above. In particular, we show that special attention is required for three-point functions of the form $\langle J_{(s_1,q)}\bar{J}'_{(s_2,q)}J''_{(s_3,0)}\rangle$ with $q \geq 2$. In this case the formula for the number of independent conserved structures is found to be quite nontrivial. The Appendix A is devoted to mathematical conventions and various useful identities. In Appendix B we provide some examples of the three-point functions $\langle J_{(s_1,q)}\bar{J}'_{(s_2,q)}J''_{(s_3,0)}\rangle$ for which the number of independent conserved structures differs from $2\,\min(s_1,s_2,s_3)+1$. Then, as a consistency check, in Appendix C we provide further examples of three-point functions involving scalars, spinors and conserved currents to compare against the results in [16].

II. CONFORMAL BUILDING BLOCKS

In this section we will review the group theoretic formalism used to compute correlation functions of primary operators in four dimensional conformal field theories. For a more detailed introduction to the formalism as applied to correlation functions of bosonic primary fields see [1]. Our 4D conventions and notation are outlined in Appendix A.

A. Two-point functions

Consider 4D Minkowski space $\mathbb{M}^{1,3}$, parametrized by coordinates x^m, where $m = 0, 1, 2, 3$ are Lorentz indices. For any two points, x_1, x_2, we construct the covariant two-point functions

$$x_{12m} = (x_1 - x_2)_m. \tag{2.1}$$

The two-point functions can be converted to spinor notation using the conventions outlined in Appendix A:

$$x_{12\alpha\dot{\alpha}} = (\sigma^m)_{\alpha\dot{\alpha}}x_{12m}, \qquad x_{12}^{\dot{\alpha}\alpha} = (\tilde{\sigma}^m)^{\dot{\alpha}\alpha}x_{12m},$$
$$x_{12}^2 = -\frac{1}{2}x_{12}^{\dot{\alpha}\alpha}x_{12\alpha\dot{\alpha}}. \tag{2.2}$$

In this form the two-point functions possess the following useful properties:

$$x_{12}^{\dot{\alpha}\alpha}x_{12\beta\dot{\alpha}} = -x_{12}^2\delta_\beta^\alpha, \qquad x_{12}^{\dot{\alpha}\alpha}x_{12\alpha\dot{\beta}} = -x_{12}^2\delta_{\dot{\beta}}^{\dot{\alpha}}. \tag{2.3}$$

Hence, we find

$$(x_{12}^{-1})^{\dot{\alpha}\alpha} = -\frac{x_{12}^{\dot{\alpha}\alpha}}{x_{12}^2}. \tag{2.4}$$

We also introduce the normalized two-point functions, denoted by $\hat{x}_{12}$,

$$\hat{x}_{12\alpha\dot{\alpha}} = \frac{x_{12\alpha\dot{\alpha}}}{(x_{12}^2)^{1/2}}, \qquad \hat{x}_{12}^{\dot{\alpha}\alpha}\hat{x}_{12\beta\dot{\alpha}} = -\delta_\beta^\alpha. \tag{2.5}$$

From here we can now construct an operator analogous to the conformal inversion tensor acting on the space of symmetric traceless tensors of arbitrary rank. Given a two-point function, x, we define the operator

$$\mathcal{I}_{\alpha(k)\dot{\alpha}(k)}(x) = \hat{x}_{(\alpha_1(\dot{\alpha}_1}\ldots\hat{x}_{\alpha_k)\dot{\alpha}_k)}, \tag{2.6}$$

along with its inverse

$$\bar{\mathcal{I}}^{\dot{\alpha}(k)\alpha(k)}(x) = \hat{x}^{(\dot{\alpha}_1(\alpha_1}\ldots\hat{x}^{\dot{\alpha}_k)\alpha_k)}. \tag{2.7}$$

The spinor indices may be raised and lowered using the standard conventions as follows:

$$\mathcal{I}^{\alpha(k)}{}_{\dot{\alpha}(k)}(x) = \varepsilon^{\alpha_1\gamma_1}\ldots\varepsilon^{\alpha_k\gamma_k}\mathcal{I}_{\gamma(k)\dot{\alpha}(k)}(x), \tag{2.8a}$$

$$\bar{\mathcal{I}}_{\dot{\alpha}(k)}{}^{\alpha(k)}(x) = \varepsilon_{\dot{\alpha}_1\dot{\gamma}_1}\ldots\varepsilon_{\dot{\alpha}_k\dot{\gamma}_k}\bar{\mathcal{I}}^{\dot{\gamma}(k)\alpha(k)}(x). \tag{2.8b}$$

Now due to the property

$$\mathcal{I}_{\alpha(k)\dot{\alpha}(k)}(-x) = (-1)^k\mathcal{I}_{\alpha(k)\dot{\alpha}(k)}(x), \tag{2.9}$$

we have the following useful relations:

$$\mathcal{I}_{\alpha(k)\dot{\alpha}(k)}(x_{12})\bar{\mathcal{I}}^{\dot{\alpha}(k)\beta(k)}(x_{21}) = \delta_{(\alpha_1}^{(\beta_1}\ldots\delta_{\alpha_k)}^{\beta_k)}, \tag{2.10a}$$

$$\bar{\mathcal{I}}^{\dot{\beta}(k)\alpha(k)}(x_{12})\mathcal{I}_{\alpha(k)\dot{\alpha}(k)}(x_{21}) = \delta_{(\dot{\alpha}_1}^{(\dot{\beta}_1}\ldots\delta_{\dot{\alpha}_k)}^{\dot{\beta}_k)}. \tag{2.10b}$$

The objects (2.6) and (2.7) prove to be essential in the construction of correlation functions involving primary operators of arbitrary spins. Indeed, the vector representation of the inversion tensor may be recovered in terms of the spinor two-point functions as follows:

$$I_{mn}(x) = -\frac{1}{2}\mathrm{Tr}(\tilde{\sigma}_m\hat{x}\tilde{\sigma}_n\hat{x}). \tag{2.11}$$

Now let $\Phi_\mathcal{A}$ be a primary field with dimension Δ, where $\mathcal{A}$ denotes a collection of Lorentz spinor indices. The two-point correlation function of $\Phi_\mathcal{A}$ and its conjugate $\bar{\Phi}^{\bar{\mathcal{A}}}$ is fixed by conformal symmetry to the form

$$\langle\Phi_\mathcal{A}(x_1)\bar{\Phi}^{\bar{\mathcal{A}}}(x_2)\rangle = c\,\frac{\mathcal{I}_\mathcal{A}{}^{\bar{\mathcal{A}}}(x_{12})}{(x_{12}^2)^\Delta}, \tag{2.12}$$

where $\mathcal{I}$ is an appropriate representation of the inversion tensor and c is a constant complex parameter. The denominator of the two-point function is determined by the conformal dimension of $\Phi_{\mathcal{A}}$, which guarantees that the correlation function transforms with the appropriate weight under scale transformations.

B. Three-point functions

Given three distinct points in Minkowski space, x_i, with $i = 1, 2, 3$, we define conformally covariant three-point functions in terms of the two-point functions as in [1]

$$X_{ij} = \frac{x_{ik}}{x_{ik}^2} - \frac{x_{jk}}{x_{jk}^2}, \qquad X_{ji} = -X_{ij}, \qquad X_{ij}^2 = \frac{x_{ij}^2}{x_{ik}^2 x_{jk}^2}, \tag{2.13}$$

where (i, j, k) is a cyclic permutation of (1, 2, 3). For example, we have

$$X_{12}^m = \frac{x_{13}^m}{x_{13}^2} - \frac{x_{23}^m}{x_{23}^2}, \qquad X_{12}^2 = \frac{x_{12}^2}{x_{13}^2 x_{23}^2}. \tag{2.14}$$

There are several useful identities involving the two- and three-point functions and the conformal inversion tensor. For example we have the useful algebraic relations

$$I_m{}^a(x_{13}) I_{an}(x_{23}) = I_m{}^a(x_{12}) I_{an}(X_{13}), \qquad I_{mn}(x_{23}) X_{12}^n = \frac{x_{12}^2}{x_{13}^2} X_{13m}, \tag{2.15a}$$

$$I_m{}^a(x_{23}) I_{an}(x_{13}) = I_m{}^a(x_{21}) I_{an}(X_{32}), \qquad I_{mn}(x_{13}) X_{12}^n = \frac{x_{12}^2}{x_{23}^2} X_{32m}, \tag{2.15b}$$

and the differential identities

$$\partial_m^{(1)} X_{12n} = \frac{1}{x_{13}^2} I_{mn}(x_{13}), \qquad \partial_m^{(2)} X_{12n} = -\frac{1}{x_{23}^2} I_{mn}(x_{23}). \tag{2.16}$$

The three-point functions also may be represented in spinor notation as follows:

$$X_{ij,\alpha\dot{\alpha}} = (\sigma_m)_{\alpha\dot{\alpha}} X_{ij}^m, \qquad X_{ij,\alpha\dot{\alpha}} = (x_{ik}^{-1})_{\alpha\dot{\gamma}} x_{ij}^{\dot{\gamma}\gamma} (x_{jk}^{-1})_{\gamma\dot{\alpha}}. \tag{2.17}$$

These objects satisfy properties similar to the two-point functions (2.3). Indeed, it is convenient to define the normalized three-point functions, $\hat{X}_{ij}$, and the inverses, (X_{ij}^{-1})

$$\hat{X}_{ij,\alpha\dot{\alpha}} = \frac{X_{ij,\alpha\dot{\alpha}}}{(X_{ij}^2)^{1/2}}, \qquad (X_{ij}^{-1})^{\dot{\alpha}\alpha} = -\frac{X_{ij}^{\dot{\alpha}\alpha}}{X_{ij}^2}. \tag{2.18}$$

Now given an arbitrary three-point building block X, it is also useful to construct the following higher-spin operator:

$$\mathcal{I}_{\alpha(k)\dot{\alpha}(k)}(X) = \hat{X}_{(\alpha_1(\dot{\alpha}_1} \ldots \hat{X}_{\alpha_k)\dot{\alpha}_k)}, \tag{2.19}$$

along with its inverse

$$\bar{\mathcal{I}}^{\dot{\alpha}(k)\alpha(k)}(X) = \hat{X}^{(\dot{\alpha}_1(\alpha_1} \ldots \hat{X}^{\dot{\alpha}_k)\alpha_k)}. \tag{2.20}$$

These operators have properties similar to the two-point higher-spin inversion operators (2.6) and (2.7). There are also some useful algebraic identities relating the two- and three-point functions at various points, such as

$$\begin{aligned} \mathcal{I}_{\alpha\dot{\alpha}}(X_{12}) &= \mathcal{I}_{\alpha\dot{\gamma}}(x_{13}) \bar{\mathcal{I}}^{\dot{\gamma}\gamma}(x_{12}) \mathcal{I}_{\gamma\dot{\alpha}}(x_{23}), \\ \bar{\mathcal{I}}^{\dot{\alpha}\gamma}(x_{13}) \mathcal{I}_{\gamma\dot{\gamma}}(X_{12}) \bar{\mathcal{I}}^{\dot{\gamma}\alpha}(x_{13}) &= \bar{\mathcal{I}}^{\dot{\alpha}\alpha}(X_{32}). \end{aligned} \tag{2.21}$$

These identities are analogous to (2.15a) and (2.15b), and admit generalizations to higher-spins, for example

$$\bar{\mathcal{I}}^{\dot{\alpha}(k)\gamma(k)}(x_{13}) \mathcal{I}_{\gamma(k)\dot{\gamma}(k)}(X_{12}) \bar{\mathcal{I}}^{\dot{\gamma}(k)\alpha(k)}(x_{13}) = \bar{\mathcal{I}}^{\dot{\alpha}(k)\alpha(k)}(X_{32}). \tag{2.22}$$

In addition, similar to (2.16), there are also the following useful identities:

$$\begin{aligned} \partial^{(1)}_{\alpha\dot{\alpha}} X_{12}^{\dot{\sigma}\sigma} &= -\frac{2}{x_{13}^2} \mathcal{I}_\alpha{}^{\dot{\sigma}}(x_{13}) \bar{\mathcal{I}}_{\dot{\alpha}}{}^\sigma(x_{13}), \\ \partial^{(2)}_{\alpha\dot{\alpha}} X_{12}^{\dot{\sigma}\sigma} &= \frac{2}{x_{23}^2} \mathcal{I}_\alpha{}^{\dot{\sigma}}(x_{23}) \bar{\mathcal{I}}_{\dot{\alpha}}{}^\sigma(x_{23}). \end{aligned} \tag{2.23}$$

These identities allow us to account for the fact that correlation functions of primary fields can obey differential constraints which can arise due to conservation equations. Indeed, given a tensor field $\mathcal{T}_{\mathcal{A}}(X)$, there are the following differential identities which arise as a consequence of (2.23):

$$\partial_{(1)\alpha\dot{\alpha}} \mathcal{T}_{\mathcal{A}}(X_{12}) = \frac{1}{x_{13}^2} \mathcal{I}_\alpha{}^{\dot{\sigma}}(x_{13}) \bar{\mathcal{I}}_{\dot{\alpha}}{}^\sigma(x_{13}) \frac{\partial}{\partial X_{12}^{\dot{\sigma}\sigma}} \mathcal{T}_{\mathcal{A}}(X_{12}), \tag{2.24a}$$

$$\partial_{(2)\alpha\dot{\alpha}} \mathcal{T}_{\mathcal{A}}(X_{12}) = -\frac{1}{x_{23}^2} \mathcal{I}_\alpha{}^{\dot{\sigma}}(x_{23}) \bar{\mathcal{I}}_{\dot{\alpha}}{}^\sigma(x_{23}) \frac{\partial}{\partial X_{12}^{\dot{\sigma}\sigma}} \mathcal{T}_{\mathcal{A}}(X_{12}). \tag{2.24b}$$

Now let Φ, Ψ, Π be primary fields with scale dimensions Δ_1, Δ_2, and Δ_3 respectively. The three-point function may be constructed using the general ansatz

$$\langle \Phi_{\mathcal{A}_1}(x_1)\Psi_{\mathcal{A}_2}(x_2)\Pi_{\mathcal{A}_3}(x_3)\rangle = \frac{\mathcal{I}^{(1)}{}_{\mathcal{A}_1}{}^{\bar{\mathcal{A}}_1}(x_{13})\mathcal{I}^{(2)}{}_{\mathcal{A}_2}{}^{\bar{\mathcal{A}}_2}(x_{23})}{(x_{13}^2)^{\Delta_1}(x_{23}^2)^{\Delta_2}}\mathcal{H}_{\bar{\mathcal{A}}_1\bar{\mathcal{A}}_2\mathcal{A}_3}(X_{12}), \quad (2.25)$$

where the tensor $\mathcal{H}_{\bar{\mathcal{A}}_1\bar{\mathcal{A}}_2\mathcal{A}_3}$ encodes all information about the correlation function, and is constrained by the conformal symmetry as follows:

(i) Under scale transformations of Minkowski space $x^m \mapsto x'^m = \lambda^{-2}x^m$, the three-point building blocks transform as $X^m \mapsto X'^m = \lambda^2 X^m$. As a consequence, the correlation function transforms as

$$\langle \Phi_{\mathcal{A}_1}(x_1')\Psi_{\mathcal{A}_2}(x_2')\Pi_{\mathcal{A}_3}(x_3')\rangle = (\lambda^2)^{\Delta_1+\Delta_2+\Delta_3}\langle \Phi_{\mathcal{A}_1}(x_1)\Psi_{\mathcal{A}_2}(x_2)\Pi_{\mathcal{A}_3}(x_3)\rangle, \quad (2.26)$$

which implies that $\mathcal{H}$ obeys the scaling property

$$\mathcal{H}_{\bar{\mathcal{A}}_1\bar{\mathcal{A}}_2\mathcal{A}_3}(\lambda^2 X) = (\lambda^2)^{\Delta_3-\Delta_2-\Delta_1}\mathcal{H}_{\bar{\mathcal{A}}_1\bar{\mathcal{A}}_2\mathcal{A}_3}(X), \quad \forall\ \lambda \in \mathbb{R}\backslash\{0\}. \quad (2.27)$$

This guarantees that the correlation function transforms correctly under scale transformations.

(ii) If any of the fields Φ, Ψ, Π obey differential equations, such as conservation equations, then the tensor $\mathcal{H}$ is also constrained by differential equations. Such constraints may be derived with the aid of identities (2.24a) and (2.24b).

(iii) If any (or all) of the operators Φ, Ψ, Π coincide, the correlation function possesses symmetries under permutations of spacetime points, e.g.

$$\langle \Phi_{\mathcal{A}_1}(x_1)\Phi_{\mathcal{A}_1'}(x_2)\Pi_{\mathcal{A}_3}(x_3)\rangle = (-1)^{\epsilon(\Phi)}\langle \Phi_{\mathcal{A}_1'}(x_2)\Phi_{\mathcal{A}_1}(x_1)\Pi_{\mathcal{A}_3}(x_3)\rangle, \quad (2.28)$$

where $\epsilon(\Phi)$ is the Grassmann parity of Φ. As a consequence, the tensor $\mathcal{H}$ obeys constraints which will be referred to as "point-switch symmetries." A similar relation may also be derived for two fields which are related by complex conjugation.

The constraints above fix the functional form of $\mathcal{H}$ (and therefore the correlation function) up to finitely many independent parameters. Hence, using the general formula (2.29), the problem of computing three-point correlation functions is reduced to deriving the general structure of the tensor $\mathcal{H}$ subject to the above constraints.

1. *Comments on differential constraints*

For three-point functions of conserved currents, we must impose conservation on all three space-time points. For x_1 and x_2, this process is simple due to the identities (2.24a) and (2.24b), and the resulting conservation equations become equivalent to simple differential constraints on $\mathcal{H}$. However, conservation on x_3 is more challenging due to a lack of useful identities analogous to (2.24a) and (2.24b) for x_3. To correctly impose conservation on x_3, consider the correlation function $\langle \Phi_{\mathcal{A}_1}(x_1)\Psi_{\mathcal{A}_2}(x_2)\Pi_{\mathcal{A}_3}(x_3)\rangle$, with the ansatz

$$\langle \Phi_{\mathcal{A}_1}(x_1)\Psi_{\mathcal{A}_2}(x_2)\Pi_{\mathcal{A}_3}(x_3)\rangle = \frac{\mathcal{I}^{(1)}{}_{\mathcal{A}_1}{}^{\bar{\mathcal{A}}_1}(x_{13})\mathcal{I}^{(2)}{}_{\mathcal{A}_2}{}^{\bar{\mathcal{A}}_2}(x_{23})}{(x_{13}^2)^{\Delta_1}(x_{23}^2)^{\Delta_2}}\mathcal{H}_{\bar{\mathcal{A}}_1\bar{\mathcal{A}}_2\mathcal{A}_3}(X_{12}). \quad (2.29)$$

We now reformulate the ansatz with Π at the front

$$\langle \Pi_{\mathcal{A}_3}(x_3)\Psi_{\mathcal{A}_2}(x_2)\Phi_{\mathcal{A}_1}(x_1)\rangle = \frac{\mathcal{I}^{(3)}{}_{\mathcal{A}_3}{}^{\bar{\mathcal{A}}_3}(x_{31})\mathcal{I}^{(2)}{}_{\mathcal{A}_2}{}^{\bar{\mathcal{A}}_2}(x_{21})}{(x_{31}^2)^{\Delta_3}(x_{21}^2)^{\Delta_2}}\tilde{\mathcal{H}}_{\bar{\mathcal{A}}_3\bar{\mathcal{A}}_2\mathcal{A}_1}(X_{32}). \quad (2.30)$$

These two correlators are the same up to an overall sign due to Grassmann parity. Equating the two ansatz above yields the following relation:

$$\tilde{\mathcal{H}}_{\bar{\mathcal{A}}_3\bar{\mathcal{A}}_2\mathcal{A}_1}(X_{32}) = (x_{13}^2)^{\Delta_3-\Delta_1}\left(\frac{x_{21}^2}{x_{23}^2}\right)^{\Delta_2}\mathcal{I}^{(1)}{}_{\mathcal{A}_1}{}^{\bar{\mathcal{A}}_1}(x_{13}) \times \bar{\mathcal{I}}^{(2)}{}_{\bar{\mathcal{A}}_2}{}^{\mathcal{A}_2'}(x_{12})\mathcal{I}^{(2)}{}_{\mathcal{A}_2'}{}^{\bar{\mathcal{A}}_2'}(x_{23}) \times \bar{\mathcal{I}}^{(3)}{}_{\bar{\mathcal{A}}_3}{}^{\mathcal{A}_3}(x_{13})\mathcal{H}_{\bar{\mathcal{A}}_1\bar{\mathcal{A}}_2'\mathcal{A}_3}(X_{12}). \quad (2.31)$$

Now suppose $\mathcal{H}(X)$ (with indices suppressed) is composed of finitely many linearly independent tensor structures, $P_i(X)$, i.e. $\mathcal{H}(X) = \sum_i a_i P_i(X)$ where a_i are constant complex parameters. We define $\bar{\mathcal{H}}(X) = \sum_i \bar{a}_i \bar{P}_i(X)$, the conjugate of $\mathcal{H}$, and also $\mathcal{H}^c(X) = \sum_i a_i \bar{P}_i(X)$, which we denote as the complement of $\mathcal{H}$. As a consequence of (2.21), the following relation holds:

$$\mathcal{H}^c_{\mathcal{A}_1\mathcal{A}_2\bar{\mathcal{A}}_3}(X_{32}) = (x_{13}^2 X_{32}^2)^{\Delta_3-\Delta_2-\Delta_1}\mathcal{I}^{(1)}{}_{\mathcal{A}_1}{}^{\bar{\mathcal{A}}_1}(x_{13}) \times \mathcal{I}^{(2)}{}_{\mathcal{A}_2}{}^{\bar{\mathcal{A}}_2}(x_{13})\bar{\mathcal{I}}^{(3)}{}_{\bar{\mathcal{A}}_3}{}^{\mathcal{A}_3}(x_{13}) \times \mathcal{H}_{\bar{\mathcal{A}}_1\bar{\mathcal{A}}_2\mathcal{A}_3}(X_{12}). \quad (2.32)$$

After inverting this identity and substituting it directly into (2.31), we apply (2.21) to obtain an equation relating $\mathcal{H}^c$ and $\tilde{\mathcal{H}}$

$$\tilde{\mathcal{H}}_{\bar{\mathcal{A}}_3\bar{\mathcal{A}}_2\mathcal{A}_1}(X) = (X^2)^{\Delta_1-\Delta_3}\bar{\mathcal{I}}^{(2)}{}_{\bar{\mathcal{A}}_2}{}^{\mathcal{A}_2}(X)\mathcal{H}^c_{\mathcal{A}_1\mathcal{A}_2\bar{\mathcal{A}}_3}(X). \quad (2.33)$$

Conservation on x_3 may now be imposed by using (2.24a), with $x_1 \leftrightarrow x_3$. In principle, this procedure can be carried out for any configuration of the fields.

If we now consider the correlation function of three conserved primaries $J_{\alpha(i_1)\dot{\alpha}(j_1)}$, $J'_{\beta(i_2)\dot{\beta}(j_2)}$, $J''_{\gamma(i_3)\dot{\gamma}(j_3)}$, where $s_1 = \frac{1}{2}(i_1 + j_1)$, $s_2 = \frac{1}{2}(i_2 + j_2)$, $s_3 = \frac{1}{2}(i_3 + j_3)$, then the general ansatz is

$$\langle J_{\alpha(i_1)\dot{\alpha}(j_1)}(x_1) J'_{\beta(i_2)\dot{\beta}(j_2)}(x_2) J''_{\gamma(i_3)\dot{\gamma}(j_3)}(x_3)\rangle = \frac{1}{(x_{13}^2)^{\Delta_1}(x_{23}^2)^{\Delta_2}} \mathcal{I}_{\alpha(i_1)}{}^{\dot{\alpha}'(i_1)}(x_{13}) \bar{\mathcal{I}}_{\dot{\alpha}(j_1)}{}^{\alpha'(j_1)}(x_{13}) \times \mathcal{I}_{\beta(i_2)}{}^{\dot{\beta}'(i_2)}(x_{23}) \bar{\mathcal{I}}_{\dot{\beta}(j_2)}{}^{\beta'(j_2)}(x_{23}) \mathcal{H}_{\alpha'(j_1)\dot{\alpha}'(i_1)\beta'(j_2)\dot{\beta}'(i_2)\gamma(i_3)\dot{\gamma}(j_3)}(X_{12}), \tag{2.34}$$

where $\Delta_i = s_i + 2$. The constraints on $\mathcal{H}$ are then as follows:

(i) Homogeneity:

Recall that $\mathcal{H}$ is a homogeneous tensor field satisfying

$$\mathcal{H}_{\alpha(j_1)\dot{\alpha}(i_1)\beta(j_2)\dot{\beta}(i_2)\gamma(i_3)\dot{\gamma}(j_3)}(\lambda^2 X) = (\lambda^2)^{\Delta_3 - \Delta_2 - \Delta_1} \mathcal{H}_{\alpha(j_1)\dot{\alpha}(i_1)\beta(j_2)\dot{\beta}(i_2)\gamma(i_3)\dot{\gamma}(j_3)}(X). \tag{2.35}$$

It is often convenient to introduce $\hat{\mathcal{H}}_{\alpha(j_1)\dot{\alpha}(i_1)\beta(j_2)\dot{\beta}(i_2)\gamma(i_3)\dot{\gamma}(j_3)}(X)$, such that

$$\mathcal{H}_{\alpha(j_1)\dot{\alpha}(i_1)\beta(j_2)\dot{\beta}(i_2)\gamma(i_3)\dot{\gamma}(j_3)}(X) = X^{\Delta_3 - \Delta_2 - \Delta_1} \hat{\mathcal{H}}_{\alpha(j_1)\dot{\alpha}(i_1)\beta(j_2)\dot{\beta}(i_2)\gamma(i_3)\dot{\gamma}(j_3)}(X), \tag{2.36}$$

where $\hat{\mathcal{H}}_{\alpha(j_1)\dot{\alpha}(i_1)\beta(j_2)\dot{\beta}(i_2)\gamma(i_3)\dot{\gamma}(j_3)}(X)$ is homogeneous degree 0 in X, i.e.

$$\hat{\mathcal{H}}_{\alpha(j_1)\dot{\alpha}(i_1)\beta(j_2)\dot{\beta}(i_2)\gamma(i_3)\dot{\gamma}(j_3)}(\lambda^2 X) = \hat{\mathcal{H}}_{\alpha(j_1)\dot{\alpha}(i_1)\beta(j_2)\dot{\beta}(i_2)\gamma(i_3)\dot{\gamma}(j_3)}(X). \tag{2.37}$$

(ii) Differential constraints:

After application of the identities (2.24a) and (2.24b) we obtain the following constraints:

$$\text{Conservation at } x_1: \quad \partial_X^{\alpha\dot{\alpha}} \mathcal{H}_{\alpha\alpha(j_1-1)\dot{\alpha}\dot{\alpha}(i_1-1)\beta(j_2)\dot{\beta}(i_2)\gamma(i_3)\dot{\gamma}(j_3)}(X) = 0, \tag{2.38a}$$

$$\text{Conservation at } x_2: \quad \partial_X^{\beta\dot{\beta}} \mathcal{H}_{\alpha(j_1)\dot{\alpha}(i_1)\beta\beta(j_2-1)\dot{\beta}\dot{\beta}(i_2-1)\gamma(i_3)\dot{\gamma}(j_3)}(X) = 0, \tag{2.38b}$$

$$\text{Conservation at } x_3: \quad \partial_X^{\gamma\dot{\gamma}} \tilde{\mathcal{H}}_{\alpha(i_1)\dot{\alpha}(j_1)\beta(j_2)\dot{\beta}(i_2)\gamma\gamma(j_3-1)\dot{\gamma}\dot{\gamma}(i_3-1)}(X) = 0, \tag{2.38c}$$

where

$$\tilde{\mathcal{H}}_{\alpha(i_1)\dot{\alpha}(j_1)\beta(j_2)\dot{\beta}(i_2)\gamma(j_3)\dot{\gamma}(i_3)}(X) = (X^2)^{\Delta_1 - \Delta_3} \mathcal{I}_{\beta(j_2)}{}^{\dot{\beta}'(j_2)}(X) \bar{\mathcal{I}}_{\dot{\beta}(i_2)}{}^{\beta'(i_2)}(X) \times \mathcal{H}^c_{\alpha(i_1)\dot{\alpha}(j_1)\beta'(i_2)\dot{\beta}'(j_2)\gamma(j_3)\dot{\gamma}(i_3)}(X). \tag{2.39}$$

(iii) Point-switch symmetries:

If the fields J and J' coincide, then we obtain the following point-switch identity

$$\mathcal{H}_{\alpha(i_1)\dot{\alpha}(j_1)\beta(i_1)\dot{\beta}(j_1)\gamma(i_3)\dot{\gamma}(j_3)}(X) = (-1)^{\epsilon(J)} \mathcal{H}_{\alpha(i_1)\dot{\alpha}(j_1)\beta(i_1)\dot{\beta}(j_1)\gamma(i_3)\dot{\gamma}(j_3)}(-X), \tag{2.40}$$

where $\epsilon(J)$ is the Grassmann parity of J. Likewise, if the fields J and J'' coincide, then we obtain the constraint

$$\tilde{\mathcal{H}}_{\alpha(i_1)\dot{\alpha}(j_1)\beta(j_2)\dot{\beta}(i_2)\gamma(j_1)\dot{\gamma}(i_1)}(X) = (-1)^{\epsilon(J)} \mathcal{H}_{\alpha(j_1)\dot{\alpha}(i_1)\beta(j_2)\dot{\beta}(i_2)\gamma(i_1)\dot{\gamma}(j_1)}(-X). \tag{2.41}$$

(iv) Reality condition:

If the fields in the correlation function belong to the (s, s) representation, then the three-point function must satisfy the reality condition

$$\mathcal{H}_{\alpha(i_1)\dot{\alpha}(i_1)\beta(i_2)\dot{\beta}(i_2)\gamma(i_3)\dot{\gamma}(i_3)}(X) = \bar{\mathcal{H}}_{\alpha(i_1)\dot{\alpha}(i_1)\beta(i_2)\dot{\beta}(i_2)\gamma(i_3)\dot{\gamma}(i_3)}(X). \tag{2.42}$$

Similarly, if the fields at J, J' at x_1 and x_2 respectively possess the same spin and are conjugate to each other, i.e. $J' = \bar{J}$, we must impose a combined reality/point-switch condition using the following constraint

$$\mathcal{H}_{\alpha(i_1)\dot{\alpha}(j_1)\beta(j_1)\dot{\beta}(i_1)\gamma(i_3)\dot{\gamma}(j_3)}(X) = (-1)^{\epsilon(J)} \bar{\mathcal{H}}_{\beta(i_1)\dot{\beta}(j_1)\alpha(j_1)\dot{\alpha}(i_1)\gamma(i_3)\dot{\gamma}(j_3)}(-X), \tag{2.43}$$

where $\epsilon(J)$ is the Grassmann parity of J.

Working with the tensor formalism is quite messy and complicated in general, hence, to simplify the analysis we will utilize auxiliary spinors to carry out the computations.

2. Generating function formalism

Analogous to the approach of [43] we utilize auxiliary spinors to streamline the calculations. Consider a general spin-tensor $\mathcal{H}_{\mathcal{A}_1\mathcal{A}_2\mathcal{A}_3}(X)$, where $\mathcal{A}_1 = \{\alpha(i_1), \dot{\alpha}(j_1)\}$, $\mathcal{A}_2 = \{\beta(i_2), \dot{\beta}(j_2)\}$, $\mathcal{A}_3 = \{\gamma(i_3), \dot{\gamma}(j_3)\}$ represent sets of totally symmetric spinor indices associated with the fields at points x_1, x_2 and x_3 respectively. We introduce sets of commuting auxiliary spinors for each point; $U = \{u, \bar{u}\}$ at x_1, $V = \{v, \bar{v}\}$ at x_2, and $W = \{w, \bar{w}\}$ at x_3, where the spinors satisfy

$$u^2 = \varepsilon_{\alpha\beta} u^\alpha u^\beta = 0, \qquad \bar{u}^2 = \varepsilon_{\dot{\alpha}\dot{\beta}} \bar{u}^{\dot{\alpha}} \bar{u}^{\dot{\beta}} = 0,$$
$$v^2 = \bar{v}^2 = 0, \qquad w^2 = \bar{w}^2 = 0. \tag{2.44}$$

Now if we define the objects

$$\boldsymbol{U}^{\mathcal{A}_1} \equiv \boldsymbol{U}^{\alpha(i_1)\dot{\alpha}(j_1)} = u^{\alpha_1} \ldots u^{\alpha_{i_1}} \bar{u}^{\dot{\alpha}_1} \ldots \bar{u}^{\dot{\alpha}_{j_1}}, \tag{2.45a}$$

$$\boldsymbol{V}^{\mathcal{A}_2} \equiv \boldsymbol{V}^{\beta(i_2)\dot{\beta}(j_2)} = v^{\beta_1} \ldots v^{\beta_{i_2}} \bar{v}^{\dot{\beta}_1} \ldots \bar{v}^{\dot{\beta}_{j_2}}, \tag{2.45b}$$

$$\boldsymbol{W}^{\mathcal{A}_3} \equiv \boldsymbol{W}^{\gamma(i_3)\dot{\gamma}(j_3)} = w^{\gamma_1} \ldots w^{\gamma_{i_3}} \bar{w}^{\dot{\gamma}_1} \ldots \bar{w}^{\dot{\gamma}_{j_3}}, \tag{2.45c}$$

then the generating polynomial for $\mathcal{H}$ is constructed as follows:

$$\mathcal{H}(X; U, V, W) = \mathcal{H}_{\mathcal{A}_1\mathcal{A}_2\mathcal{A}_3}(X) \boldsymbol{U}^{\mathcal{A}_1} \boldsymbol{V}^{\mathcal{A}_2} \boldsymbol{W}^{\mathcal{A}_3}. \tag{2.46}$$

The tensor $\mathcal{H}$ can then be extracted from the polynomial by acting on it with the following partial derivative operators:

$$\frac{\partial}{\partial \boldsymbol{U}^{\mathcal{A}_1}} \equiv \frac{\partial}{\partial \boldsymbol{U}^{\alpha(i_1)\dot{\alpha}(j_1)}} = \frac{1}{i_1! j_1!} \frac{\partial}{\partial u^{\alpha_1}} \cdots \frac{\partial}{\partial u^{\alpha_{i_1}}} \frac{\partial}{\partial \bar{u}^{\dot{\alpha}_1}} \cdots \frac{\partial}{\partial \bar{u}^{\dot{\alpha}_{j_1}}}, \tag{2.47a}$$

$$\frac{\partial}{\partial \boldsymbol{V}^{\mathcal{A}_2}} \equiv \frac{\partial}{\partial \boldsymbol{V}^{\beta(i_2)\dot{\beta}(j_2)}} = \frac{1}{i_2! j_2!} \frac{\partial}{\partial v^{\beta_1}} \cdots \frac{\partial}{\partial v^{\beta_{i_2}}} \frac{\partial}{\partial \bar{v}^{\dot{\beta}_1}} \cdots \frac{\partial}{\partial \bar{v}^{\dot{\beta}_{j_2}}}, \tag{2.47b}$$

$$\frac{\partial}{\partial \boldsymbol{W}^{\mathcal{A}_3}} \equiv \frac{\partial}{\partial \boldsymbol{W}^{\gamma(i_3)\dot{\gamma}(j_3)}} = \frac{1}{i_3! j_3!} \frac{\partial}{\partial w^{\gamma_1}} \cdots \frac{\partial}{\partial w^{\gamma_{i_3}}} \frac{\partial}{\partial \bar{w}^{\dot{\gamma}_1}} \cdots \frac{\partial}{\partial \bar{w}^{\dot{\gamma}_{j_3}}}. \tag{2.47c}$$

The tensor $\mathcal{H}$ is then extracted from the polynomial as follows:

$$\mathcal{H}_{\mathcal{A}_1\mathcal{A}_2\mathcal{A}_3}(X) = \frac{\partial}{\partial \boldsymbol{U}^{\mathcal{A}_1}} \frac{\partial}{\partial \boldsymbol{V}^{\mathcal{A}_2}} \frac{\partial}{\partial \boldsymbol{W}^{\mathcal{A}_3}} \mathcal{H}(X; U, V, W). \tag{2.48}$$

The polynomial $\mathcal{H}$, (2.46), is now constructed out of scalar combinations of X, and the auxiliary spinors U, V, and W with the appropriate homogeneity. Such a polynomial can be constructed out of the following monomials:

$$P_1 = \varepsilon_{\alpha\beta} v^\alpha w^\beta, \qquad P_2 = \varepsilon_{\alpha\beta} w^\alpha u^\beta, \qquad P_3 = \varepsilon_{\alpha\beta} u^\alpha v^\beta, \tag{2.49a}$$

$$\bar{P}_1 = \varepsilon_{\dot{\alpha}\dot{\beta}} \bar{v}^{\dot{\alpha}} \bar{w}^{\dot{\beta}}, \qquad \bar{P}_2 = \varepsilon_{\dot{\alpha}\dot{\beta}} \bar{w}^{\dot{\alpha}} \bar{u}^{\dot{\beta}}, \qquad \bar{P}_3 = \varepsilon_{\dot{\alpha}\dot{\beta}} \bar{u}^{\dot{\alpha}} \bar{v}^{\dot{\beta}}, \tag{2.49b}$$

$$Q_1 = \hat{X}_{\alpha\dot{\alpha}} v^\alpha \bar{w}^{\dot{\alpha}}, \qquad Q_2 = \hat{X}_{\alpha\dot{\alpha}} w^\alpha \bar{u}^{\dot{\alpha}}, \qquad Q_3 = \hat{X}_{\alpha\dot{\alpha}} u^\alpha \bar{v}^{\dot{\alpha}}, \tag{2.49c}$$

$$\bar{Q}_1 = \hat{X}_{\alpha\dot{\alpha}} w^\alpha \bar{v}^{\dot{\alpha}}, \qquad \bar{Q}_2 = \hat{X}_{\alpha\dot{\alpha}} u^\alpha \bar{w}^{\dot{\alpha}}, \qquad \bar{Q}_3 = \hat{X}_{\alpha\dot{\alpha}} v^\alpha \bar{u}^{\dot{\alpha}}, \tag{2.49d}$$

$$Z_1 = \hat{X}_{\alpha\dot{\alpha}} u^\alpha \bar{u}^{\dot{\alpha}}, \qquad Z_2 = \hat{X}_{\alpha\dot{\alpha}} v^\alpha \bar{v}^{\dot{\alpha}}, \qquad Z_3 = \hat{X}_{\alpha\dot{\alpha}} w^\alpha \bar{w}^{\dot{\alpha}}. \tag{2.49e}$$

To construct linearly independent structures for a given three-point function, one must also take into account the following linear dependence relations between the monomials:

$$Z_2 Z_3 + P_1 \bar{P}_1 - Q_1 \bar{Q}_1 = 0, \tag{2.50a}$$

$$Z_1 Z_3 + P_2 \bar{P}_2 - Q_2 \bar{Q}_2 = 0, \tag{2.50b}$$

$$Z_1 Z_2 + P_3 \bar{P}_3 - Q_3 \bar{Q}_3 = 0, \tag{2.50c}$$

$$Z_1 P_1 + P_2 \bar{Q}_3 + P_3 Q_2 = 0, \qquad Z_1 \bar{P}_1 + \bar{P}_2 Q_3 + \bar{P}_3 \bar{Q}_2 = 0, \tag{2.51a}$$

$$Z_2 P_2 + P_3 \bar{Q}_1 + P_1 Q_3 = 0, \qquad Z_2 \bar{P}_2 + \bar{P}_3 Q_1 + \bar{P}_1 \bar{Q}_3 = 0, \tag{2.51b}$$

$$Z_3 P_3 + P_1 \bar{Q}_2 + P_2 Q_1 = 0, \qquad Z_3 \bar{P}_3 + \bar{P}_1 Q_2 + \bar{P}_2 \bar{Q}_1 = 0. \tag{2.51c}$$

$$Z_1 Q_1 + \bar{P}_2 P_3 - \bar{Q}_2 \bar{Q}_3 = 0, \qquad Z_1 \bar{Q}_1 + P_2 \bar{P}_3 + Q_2 Q_3 = 0, \tag{2.52a}$$

$$Z_2 Q_2 + \bar{P}_3 P_1 - \bar{Q}_3 \bar{Q}_1 = 0, \qquad Z_2 \bar{Q}_2 + P_3 \bar{P}_1 + Q_3 Q_1 = 0, \tag{2.52b}$$

$$Z_3 Q_3 + \bar{P}_1 P_2 - \bar{Q}_1 \bar{Q}_2 = 0, \qquad Z_3 \bar{Q}_3 + P_1 \bar{P}_2 + Q_1 Q_2 = 0. \tag{2.52c}$$

These allow elimination of the combinations $Z_i Z_j$, $Z_i P_i$, $Z_i \bar{P}_i$, $Z_i Q_i$, $Z_i \bar{Q}_i$. There are also the following relations involving triple products:

$$P_1\bar{P}_2\bar{P}_3 + \bar{P}_1 Q_2 \bar{Q}_3 + \bar{P}_2 \bar{Q}_3 \bar{Q}_1 + \bar{P}_3 Q_1 Q_2 = 0, \quad (2.53a)$$

$$P_2\bar{P}_3\bar{P}_1 + \bar{P}_2 Q_3 \bar{Q}_1 + \bar{P}_3 \bar{Q}_1 \bar{Q}_2 + \bar{P}_1 Q_2 Q_3 = 0, \quad (2.53b)$$

$$P_3\bar{P}_1\bar{P}_2 + \bar{P}_3 Q_1 \bar{Q}_2 + \bar{P}_1 \bar{Q}_2 \bar{Q}_3 + \bar{P}_2 Q_3 Q_1 = 0, \quad (2.53c)$$

$$\bar{P}_1 P_2 P_3 + P_1 \bar{Q}_2 Q_3 + P_2 Q_3 Q_1 + P_3 \bar{Q}_1 \bar{Q}_2 = 0, \quad (2.54a)$$

$$\bar{P}_2 P_3 P_1 + P_2 \bar{Q}_3 Q_1 + P_3 Q_1 Q_2 + P_1 \bar{Q}_2 \bar{Q}_3 = 0, \quad (2.54b)$$

$$\bar{P}_3 P_1 P_2 + P_3 \bar{Q}_1 Q_2 + P_1 Q_2 Q_3 + P_2 \bar{Q}_3 \bar{Q}_1 = 0, \quad (2.54c)$$

$$\bar{P}_1 P_2 \bar{Q}_3 - P_1 \bar{P}_2 Q_3 + \bar{Q}_1 \bar{Q}_2 \bar{Q}_3 - Q_1 Q_2 Q_3 = 0, \quad (2.55a)$$

$$\bar{P}_2 P_3 \bar{Q}_1 - P_2 \bar{P}_3 Q_1 + \bar{Q}_1 \bar{Q}_2 \bar{Q}_3 - Q_1 Q_2 Q_3 = 0, \quad (2.55b)$$

$$\bar{P}_3 P_1 \bar{Q}_2 - P_3 \bar{P}_1 Q_2 + \bar{Q}_1 \bar{Q}_2 \bar{Q}_3 - Q_1 Q_2 Q_3 = 0, \quad (2.55c)$$

which allow for elimination of the products $P_i\bar{P}_j\bar{P}_k$, $\bar{P}_i P_j P_k$, $\bar{P}_i P_j \bar{Q}_k$. These relations (which appear to be exhaustive) are sufficient to reduce any set of structures in a given three-point function to a linearly independent set.

The task now is to construct a complete list of possible (linearly independent) solutions for the polynomial $\mathcal{H}$ for a given set of spins. This process is simplified by introducing a generating function, $\mathcal{F}(X; U, V, W|\Gamma)$, defined as follows:

$$\mathcal{F}(X; U, V, W|\Gamma) = P_1^{k_1} P_2^{k_2} P_3^{k_3} \bar{P}_1^{\bar{k}_1} \bar{P}_2^{\bar{k}_2} \bar{P}_3^{\bar{k}_3} Q_1^{l_1} Q_2^{l_2} Q_3^{l_3} \bar{Q}_1^{\bar{l}_1} \bar{Q}_2^{\bar{l}_2} \times \bar{Q}_3^{\bar{l}_3} Z_1^{r_1} Z_2^{r_2} Z_3^{r_3}, \quad (2.56)$$

where the non-negative integers, $\Gamma = \bigcup_{i\in\{1,2,3\}}\{k_i, \bar{k}_i, l_i, \bar{l}_i, r_i\}$, are solutions to the following linear system:

$$k_2 + k_3 + r_1 + l_3 + \bar{l}_2 = i_1, \qquad \bar{k}_2 + \bar{k}_3 + r_1 + \bar{l}_3 + l_2 = j_1, \quad (2.57a)$$

$$k_1 + k_3 + r_2 + l_1 + \bar{l}_3 = i_2, \qquad \bar{k}_1 + \bar{k}_3 + r_2 + \bar{l}_1 + l_3 = j_2, \quad (2.57b)$$

$$k_1 + k_2 + r_3 + l_2 + \bar{l}_1 = i_3, \qquad \bar{k}_1 + \bar{k}_2 + r_3 + \bar{l}_2 + l_1 = j_3. \quad (2.57c)$$

Here i_1, i_2, i_3, j_1, j_2, j_3 are fixed integers corresponding to the spin representations of the fields in the three-point function. From here it is convenient to define

$$\Delta s = \frac{1}{2}(i_1 + i_2 + i_3 - j_1 - j_2 - j_3). \quad (2.58)$$

Using the system of equations (2.57), we obtain

$$\Delta s = k_1 + k_2 + k_3 - \bar{k}_1 - \bar{k}_2 - \bar{k}_3, \quad (2.59)$$

in addition to

$$k_1 + k_2 + k_3 \le \min(i_1 + i_2, i_1 + i_3, i_2 + i_3), \quad (2.60a)$$

$$\bar{k}_1 + \bar{k}_2 + \bar{k}_3 \le \min(j_1 + j_2, j_1 + j_3, j_2 + j_3). \quad (2.60b)$$

Hence, the conditions for a given three-point function to be nonvanishing are

$$-\min(j_1 + j_2, j_1 + j_3, j_2 + j_3) \le \Delta s \le \min(i_1 + i_2, i_1 + i_3, i_2 + i_3). \quad (2.61)$$

Indeed, this is the same condition found in [16]. Now given a finite number of solutions Γ_I, $I = 1, \ldots, N$ to (2.57) for a particular choice of i_1, i_2, i_3, j_1, j_2, j_3, the most general ansatz for the polynomial $\mathcal{H}$ in (2.46) is as follows:

$$\mathcal{H}(X; U, V, W) = X^{\Delta_3 - \Delta_2 - \Delta_1} \sum_{I=1}^{N} a_I \mathcal{F}(X; U, V, W|\Gamma_I), \quad (2.62)$$

where a_I are a set of complex constants. Hence, constructing the most general ansatz for the generating polynomial $\mathcal{H}$ is now equivalent to finding all non-negative integer solutions Γ_I of (2.57). Once this ansatz has been obtained, the linearly independent structures can be found by systematically applying the linear dependence relations (2.50)–(2.55).

Let us now recast the constraints on the three-point function into the auxiliary spinor formalism. Recalling that $s_1 = \frac{1}{2}(i_1 + j_1)$, $s_2 = \frac{1}{2}(i_2 + j_2)$, $s_3 = \frac{1}{2}(i_3 + j_3)$, first we define:

$$J_{s_1}(x_1; U) = J_{\alpha(i_1)\dot{\alpha}(j_1)}(x_1) \boldsymbol{U}^{\alpha(i_1)\dot{\alpha}(j_1)}, \quad (2.63a)$$

$$J'_{s_2}(x_2; V) = J'_{\alpha(i_2)\dot{\alpha}(j_2)}(x_2) \boldsymbol{V}^{\alpha(i_2)\dot{\alpha}(j_2)}, \quad (2.63b)$$

$$J''_{s_3}(x_3; W) = J''_{\gamma(i_3)\dot{\gamma}(j_3)}(x_3) \boldsymbol{W}^{\gamma(i_3)\dot{\gamma}(j_3)}, \quad (2.63c)$$

where, to simplify notation, we denote $J_{(s,q)} \equiv J_s$. The general ansatz can be converted easily into the auxiliary spinor formalism, and is of the form

$$\langle J_{s_1}(x_1; U) J'_{s_2}(x_2; V) J''_{s_3}(x_3; W)\rangle = \frac{\mathcal{I}^{(i_1,j_1)}(x_{13}; U, \tilde{U}) \mathcal{I}^{(i_2,j_2)}(x_{23}; V, \tilde{V})}{(x_{13}^2)^{\Delta_1} (x_{23}^2)^{\Delta_2}} \times \mathcal{H}(X_{12}; \tilde{U}, \tilde{V}, W), \quad (2.64)$$

where $\Delta_i = s_i + 2$. The generating polynomial, $\mathcal{H}(X; U, V, W)$, is defined as

$$\mathcal{H}(X;U,V,W)=\mathcal{H}_{\alpha(i_1)\dot{\alpha}(j_1)\beta(i_2)\dot{\beta}(j_2)\gamma(i_3)\dot{\gamma}(j_3)} \times(X)\boldsymbol{U}^{\alpha(i_1)\dot{\alpha}(j_1)}\boldsymbol{V}^{\beta(i_2)\dot{\beta}(j_2)}\boldsymbol{W}^{\gamma(i_3)\dot{\gamma}(j_3)}, \tag{2.65}$$

where

$$\mathcal{I}^{(i,j)}(x;U,\tilde{U})\equiv\mathcal{I}^{(i,j)}_x(U,\tilde{U}) =\boldsymbol{U}^{\alpha(i)\dot{\alpha}(j)}\mathcal{I}_{\alpha(i)}{}^{\dot{\alpha}'(i)}(x)\bar{\mathcal{I}}_{\dot{\alpha}(j)}{}^{\alpha'(j)}(x)\frac{\partial}{\partial\tilde{\boldsymbol{U}}^{\alpha'(j)\dot{\alpha}'(i)}}, \tag{2.66}$$

is the inversion operator acting on polynomials degree (i,j) in $(\tilde{\bar{u}},\tilde{u})$. It should also be noted that $\tilde{\boldsymbol{U}}$ has index structure conjugate to $\boldsymbol{U}$. Sometimes we will omit the indices (i,j) to streamline the notation. After converting the constraints summarized in the previous subsection into the auxiliary spinor formalism, we obtain:

(i) Homogeneity:

Recall that $\mathcal{H}$ is a homogeneous polynomial satisfying the following scaling property:

$$\mathcal{H}(\lambda^2X;U(i_1,j_1),V(i_2,j_2),W(i_3,j_3)) =(\lambda^2)^{\Delta_3-\Delta_2-\Delta_1}\mathcal{H}(X;U(i_1,j_1),V(i_2,j_2),W(i_3,j_3)), \tag{2.67}$$

where we have used the notation $U(i_1,j_1)$, $V(i_2,j_2)$, $W(i_3,j_3)$ to keep track of homogeneity in the auxiliary spinors $(u,\bar{u})$, $(v,\bar{v})$, and $(w,\bar{w})$. For compactness we will suppress the homogeneities of the auxiliary spinors in the results.

(ii) Differential constraints:

First, define the following three differential operators:

$$D_1=\partial_X^{\alpha\dot{\alpha}}\frac{\partial}{\partial u^\alpha}\frac{\partial}{\partial\bar{u}^{\dot{\alpha}}},\qquad D_2=\partial_X^{\alpha\dot{\alpha}}\frac{\partial}{\partial v^\alpha}\frac{\partial}{\partial\bar{v}^{\dot{\alpha}}},\qquad D_3=\partial_X^{\alpha\dot{\alpha}}\frac{\partial}{\partial w^\alpha}\frac{\partial}{\partial\bar{w}^{\dot{\alpha}}}. \tag{2.68}$$

Conservation on all three points may be imposed using the following constraints:

$$\text{Conservation at } x_1:\qquad D_1\mathcal{H}(X;U,V,W)=0, \tag{2.69a}$$

$$\text{Conservation at } x_2:\qquad D_2\mathcal{H}(X;U,V,W)=0, \tag{2.69b}$$

$$\text{Conservation at } x_3:\qquad D_3\tilde{\mathcal{H}}(X;U,V,W)=0, \tag{2.69c}$$

where, in the auxiliary spinor formalism, $\tilde{\mathcal{H}}$ is computed as follows:

$$\tilde{\mathcal{H}}(X;U,V,W)=(X^2)^{\Delta_1-\Delta_3}\mathcal{I}_X(V,\tilde{V})\mathcal{H}^c(X;U,\tilde{V},W). \tag{2.70}$$

Using the properties of the inversion tensor, it can be shown that this transformation is equivalent to the following replacement rules for the building blocks:

$$P_1\to Q_1,\qquad P_2\to-\bar{P}_2,\qquad P_3\to-\bar{Q}_3 \tag{2.71a}$$

$$\bar{P}_1\to\bar{Q}_1,\qquad \bar{P}_2\to-P_2,\qquad \bar{P}_3\to-Q_3 \tag{2.71b}$$

$$Q_1\to-P_1,\qquad Q_2\to\bar{Q}_2,\qquad Q_3\to\bar{P}_3 \tag{2.71c}$$

$$\bar{Q}_1\to-\bar{P}_1,\qquad \bar{Q}_2\to Q_2,\qquad \bar{Q}_3\to P_3 \tag{2.71d}$$

$$Z_1\to Z_1,\qquad Z_2\to-Z_2,\qquad Z_3\to Z_3. \tag{2.71e}$$

(iii) Point switch symmetries:

If the fields J and J' coincide (hence $i_1=i_2$, $j_1=j_2$), then we obtain the following point-switch constraint

$$\mathcal{H}(X;U,V,W)=(-1)^{\epsilon(J)}\mathcal{H}(-X;V,U,W), \tag{2.72}$$

where $\epsilon(J)$ is the Grassmann parity of J. Similarly, if the fields J and J'' coincide (hence $i_1=i_3$, $j_1=j_3$) then we obtain the constraint

$$\tilde{\mathcal{H}}(X;U,V,W)=(-1)^{\epsilon(J)}\mathcal{H}(-X;W,V,U). \tag{2.73}$$

(iv) Reality condition:

If the fields in the correlation function belong to the (s,s) representation, then the three-point function must satisfy the reality condition

$$\bar{\mathcal{H}}(X;U,V,W)=\mathcal{H}(X;U,V,W). \tag{2.74}$$

Similarly, if the fields at J, J' at x_1 and x_2 respectively possess the same spin and are conjugate to each other, i.e. $J'=\bar{J}$, we must impose a combined reality/point-switch condition using the following constraint

$$\mathcal{H}(X;U,V,W)=(-1)^{\epsilon(J)}\bar{\mathcal{H}}(-X;V,U,W), \tag{2.75}$$

where $\epsilon(J)$ is the Grassmann parity of J.

3. Inversion transformation

In general, whenever parity is a symmetry of a CFT, so too is invariance under inversions. An inversion transformation $\mathcal{I}$ maps fields in the (i,j) representation onto fields in the complex conjugate representation, (j, i).[3] Hence, inversions map correlation functions of fields onto correlation functions of their complex conjugate fields. In particular, if the fields in a given three-point function belong to the (s, s) representation then it is possible to construct linear combinations of structures for the three-point function which are eigenfunctions of the inversion operator. We denote these as parity-even and parity-odd solutions respectively. Indeed, given a tensor $\mathcal{H}_{\mathcal{A}_1\mathcal{A}_2\mathcal{A}_3}(X) = X^{\Delta_3-\Delta_2-\Delta_1}\hat{\mathcal{H}}_{\mathcal{A}_1\mathcal{A}_2\mathcal{A}_3}(X)$, the following inversion formula holds:

$$\hat{\mathcal{H}}^c_{\bar{\mathcal{A}}_1\bar{\mathcal{A}}_2\bar{\mathcal{A}}_3}(X) = \bar{\mathcal{I}}^{(1)}{}_{\bar{\mathcal{A}}_1}{}^{\mathcal{A}_1}(X)\bar{\mathcal{I}}^{(2)}{}_{\bar{\mathcal{A}}_2}{}^{\mathcal{A}_2}(X)\bar{\mathcal{I}}^{(3)}{}_{\bar{\mathcal{A}}_3}{}^{\mathcal{A}_3}(X)\hat{\mathcal{H}}_{\mathcal{A}_1\mathcal{A}_2\mathcal{A}_3}(-X). \tag{2.76}$$

Hence, we notice that under $\mathcal{I}$, $\mathcal{H}$ transforms into the complex conjugate representation. An analogous formula can be derived using the auxiliary spinor formalism. Given a polynomial $\mathcal{H}(X; U, V, W) = X^{\Delta_3-\Delta_2-\Delta_1}\hat{\mathcal{H}}(X; U, V, W)$, the following holds:

$$\hat{\mathcal{H}}^c(X; U, V, W) = \mathcal{I}_X(U, \tilde{U})\mathcal{I}_X(V, \tilde{V})\mathcal{I}_X(W, \tilde{W}) \times \hat{\mathcal{H}}(-X; \tilde{U}, \tilde{V}, \tilde{W}). \tag{2.77}$$

It is easy to understand this formula as the monomials (2.49) have simple transformation properties under $\mathcal{I}$:

$$P_i \xrightarrow{\mathcal{I}} -\bar{P}_i, \qquad Q_i \xrightarrow{\mathcal{I}} \bar{Q}_i, \qquad Z_i \xrightarrow{\mathcal{I}} Z_i, \tag{2.78}$$

with analogous rules applying for the building blocks $\bar{P}_i$, $\bar{Q}_i$. Since, for primary fields in the (s, s) representation the three-point maps onto itself under inversion, it is possible to classify the parity-even and parity-odd structures in $\mathcal{H}$ using (2.77). By letting $\hat{\mathcal{H}}(X) = \hat{\mathcal{H}}^{(+)}(X) + \hat{\mathcal{H}}^{(-)}(X)$, we have

$$\hat{\mathcal{H}}^{(\pm)}(X; U, V, W) = \pm\mathcal{I}_X(U, \tilde{U})\mathcal{I}_X(V, \tilde{V})\mathcal{I}_X(W, \tilde{W}) \times \hat{\mathcal{H}}^{(\pm)}(-X; \tilde{U}, \tilde{V}, \tilde{W}). \tag{2.79}$$

Structures satisfying the above property are defined as parity-even/odd for overall sign $+/-$. This is essentially the same approach used to classify parity-even and parity-odd three-point functions in 3D CFT [43], which proves to be equivalent to the classification based on the absence/presence of the Levi-Civita pseudotensor. However, it is crucial to note that in three dimensions the linearly independent basic monomial structures comprising $\mathcal{H}$ are naturally eigenfunctions of the inversion operator. The same is not necessarily true for three-point functions in four dimensions due to (2.78), as the monomials (2.49) now map onto their complex conjugates. Hence, we are required to take non-trivial linear combinations of the basic structures and use the linear dependence relations (2.50)–(2.55) to form eigenfunctions of the inversion operator. Our classification of parity-even/odd solutions obtained this way is in complete agreement with the results found in [13].

[3]For a more detailed discussion of parity transformations in 4D CFT, see [16].

III. THREE-POINT FUNCTIONS OF CONSERVED CURRENTS

In the next subsections we analyze the structure of three-point functions involving conserved currents in 4D CFT. We classify, using computational methods, all possible three-point functions involving the conserved currents $J_{(s,q)}$, $\bar{J}_{(s,q)}$ for $s_i \leq 10$. In particular, we determine the general structure and the number of independent solutions present in the three-point functions (1.3). As pointed out in the introduction, the number of independent conserved structures generically grows linearly with the minimum spin and the solution for the function $\mathcal{H}(X; U, V, W)$ quickly becomes too long and complicated even for relatively low spins. Thus, although our method allows us to find $\mathcal{H}(X; U, V, W)$ in a very explicit form for arbitrary spins (limited only by computer power), we find it practical to present the solutions when there is a small number of structures. Such examples involving low spins are discussed in Sec. III A. In Sec. III B we state the classification for arbitrary spins. Some additional examples are presented in Appendix B.

A. Conserved low-spin currents

We begin our analysis by considering correlation functions involving conserved low-spin currents such as the energy-momentum tensor, vector current, and "supersymmetry-like" currents in 4D CFT. Many of these results are known throughout the literature, but we derive them again here to demonstrate our approach.

1. Energy-momentum tensor and vector current correlators

The fundamental bosonic conserved currents in any conformal field theory are the conserved vector current, V_m, and the symmetric, traceless energy-momentum tensor, T_{mn}. The vector current has scale dimension $\Delta_V = 3$ and satisfies $\partial^m V_m = 0$, while the energy-momentum tensor has scale dimension $\Delta_T = 4$ and satisfies the conservation equation $\partial^m T_{mn} = 0$. Converting to spinor notation using the conventions outlined in Appendix A, we have:

$$V_{\alpha\dot{\alpha}}(x) = (\sigma^m)_{\alpha\dot{\alpha}}V_m(x),$$
$$T_{(\alpha_1\alpha_2)(\dot{\alpha}_1\dot{\alpha}_2)}(x) = (\sigma^m)_{(\alpha_1(\dot{\alpha}_1}(\gamma^n)_{\alpha_2)\dot{\alpha}_2)}T_{mn}(x). \tag{3.1}$$

These objects possess fundamental information associated with internal and spacetime symmetries, hence, their three-point functions are of great importance. The possible three-point functions involving the conserved vector current and the energy-momentum tensor are:

$$\langle V_{\alpha\dot{\alpha}}(x_1)V_{\beta\dot{\beta}}(x_2)V_{\gamma\dot{\gamma}}(x_3)\rangle, \quad \langle V_{\alpha\dot{\alpha}}(x_1)V_{\beta\dot{\beta}}(x_2)T_{\gamma(2)\dot{\gamma}(2)}(x_3)\rangle, \tag{3.2a}$$

$$\langle T_{\alpha(2)\dot{\alpha}(2)}(x_1)T_{\beta(2)\dot{\beta}(2)}(x_2)V_{\gamma\dot{\gamma}}(x_3)\rangle, \quad \langle T_{\alpha(2)\dot{\alpha}(2)}(x_1)T_{\beta(2)\dot{\beta}(2)}(x_2)T_{\gamma(2)\dot{\gamma}(2)}(x_3)\rangle. \tag{3.2b}$$

Let us first consider $\langle VVV\rangle$. By using the notation for the currents $J_{(s,q)}$, $\bar{J}_{(s,q)}$, this corresponds to the general three-point function $\langle J_{(1,0)}J'_{(1,0)}J''_{(1,0)}\rangle$.

Correlation function $\langle J_{(1,0)}J'_{(1,0)}J''_{(1,0)}\rangle$. The general ansatz for this correlation function, according to (2.34) is

$$\langle J_{\alpha\dot{\alpha}}(x_1)J'_{\beta\dot{\beta}}(x_2)J''_{\gamma\dot{\gamma}}(x_3)\rangle = \frac{\mathcal{I}_{\alpha}{}^{\dot{\alpha}'}(x_{13})\bar{\mathcal{I}}_{\dot{\alpha}}{}^{\alpha'}(x_{13})\mathcal{I}_{\beta}{}^{\dot{\beta}'}(x_{23})\bar{\mathcal{I}}_{\dot{\beta}}{}^{\beta'}(x_{23})}{(x_{13}^2)^3(x_{23}^2)^3}\mathcal{H}_{\alpha'\dot{\alpha}'\beta'\dot{\beta}'\gamma\dot{\gamma}}(X_{12}). \tag{3.3}$$

Using the formalism outlined in Sec. II B 2, all information about this correlation function is encoded in the following polynomial:

$$\mathcal{H}(X;U,V,W) = \mathcal{H}_{\alpha\dot{\alpha}\beta\dot{\beta}\gamma\dot{\gamma}}(X)\boldsymbol{U}^{\alpha\dot{\alpha}}\boldsymbol{V}^{\beta\dot{\beta}}\boldsymbol{W}^{\gamma\dot{\gamma}}. \tag{3.4}$$

Using *Mathematica* we solve (2.57) for the chosen spin representations of the currents and substitute each solution into the generating function (2.56). This provides us with the following list of (linearly dependent) polynomial structures:

$$\{Q_1Q_2Q_3, Z_1Z_2Z_3, P_3Q_2\bar{P}_1, P_1Z_1\bar{P}_1, P_1Q_3\bar{P}_2, P_2Z_2\bar{P}_2, \times P_2Q_1\bar{P}_3, P_3Z_3\bar{P}_3, Q_1Z_1\bar{Q}_1, P_3\bar{P}_2\bar{Q}_1, Q_2Z_2\bar{Q}_2, \times P_1\bar{P}_3\bar{Q}_2, Q_3Z_3\bar{Q}_3, P_2\bar{P}_1\bar{Q}_3, \bar{Q}_1\bar{Q}_2\bar{Q}_3\}. \tag{3.5}$$

Next, we systematically apply the linear dependence relations (2.50) to these lists, reducing them to the following sets of linearly independent structures:

$$\{Q_1Q_2Q_3, P_3Q_2\bar{P}_1, P_1Q_3\bar{P}_2, P_2Q_1\bar{P}_3, \bar{Q}_1\bar{Q}_2\bar{Q}_3\}. \tag{3.6}$$

Note that application of the linear-dependence relations eliminates all terms involving Z_i in this case. Since this correlation function is composed of fields in the (s,s) representation, the solutions for the three-point function may be split up into parity-even and parity-odd contributions. To do this we construct linear combinations for the polynomial $\hat{\mathcal{H}}(X;U,V,W)$ which are even/odd under inversion in accordance with (2.79):

$$A_1(\bar{Q}_1\bar{Q}_2\bar{Q}_3 + Q_1Q_2Q_3) + A_2(P_3Q_2\bar{P}_1 - \bar{Q}_1\bar{Q}_2\bar{Q}_3) + A_3(P_1Q_3\bar{P}_2 - \bar{Q}_1\bar{Q}_2\bar{Q}_3) + A_4(P_2Q_1\bar{P}_3 - \bar{Q}_1\bar{Q}_2\bar{Q}_3) + B_1(Q_1Q_2Q_3 - \bar{Q}_1\bar{Q}_2\bar{Q}_3). \tag{3.7}$$

We note here (and in all other examples) that the parity-even contributions possess the complex coefficients A_i, while the parity-odd solutions possess the complex coefficients B_i. It can be explicitly checked that these structures possess the appropriate transformation properties. Next, since the correlation function is overall real, we must impose the reality condition (2.74). As a result, we find that the parity-even coefficients A_i are purely real, i.e., $A_i = a_i$, while the parity-odd coefficients B_i are purely imaginary, i.e., $B_i = \mathrm{i}b_i$.

We must now impose the conservation of the currents. Following the procedure outlined in Sec. II B 2 we obtain a linear system in the coefficients a_i, b_i which can be easily solved computationally. We find the following solution for $\mathcal{H}(X;U,V,W)$ consistent with conservation on all three points:

$$\frac{a_1}{X^3}(Q_1Q_2Q_3 + 2P_1Q_3\bar{P}_2 - \bar{Q}_1\bar{Q}_2\bar{Q}_3) + \frac{a_2}{X^3}(P_3Q_2\bar{P}_1 - 3P_1Q_3\bar{P}_2 + P_2Q_1\bar{P}_3 + \bar{Q}_1\bar{Q}_2\bar{Q}_3) + \frac{\mathrm{i}b_1}{X^3}(Q_1Q_2Q_3 - \bar{Q}_1\bar{Q}_2\bar{Q}_3). \tag{3.8}$$

The only remaining constraints to impose are symmetries under permutations of spacetime points, which apply when the currents in the three-point function are identical, i.e. when $J = J' = J''$. After imposing (2.72) and (2.73), only the structure corresponding to the coefficient b_1 survives. However, the a_1, a_2 structures can exist if the currents are non-Abelian. This is consistent with the results of [1,2,13].[4]

The next example to consider is the mixed correlator $\langle VVT\rangle$. To study this case we may examine the correlation function $\langle J_{(1,0)}J'_{(1,0)}J''_{(2,0)}\rangle$.

Correlation function $\langle J_{(1,0)}J'_{(1,0)}J''_{(2,0)}\rangle$. Using the general formula, the ansatz for this three-point function is

$$\langle J_{\alpha\dot{\alpha}}(x_1)J'_{\beta\dot{\beta}}(x_2)J''_{\gamma(2)\dot{\gamma}(2)}(x_3)\rangle = \frac{\mathcal{I}_{\alpha}{}^{\dot{\alpha}'}(x_{13})\bar{\mathcal{I}}_{\dot{\alpha}}{}^{\alpha'}(x_{13})\mathcal{I}_{\beta}{}^{\dot{\beta}'}(x_{23})\bar{\mathcal{I}}_{\dot{\beta}}{}^{\beta'}(x_{23})}{(x_{13}^2)^3(x_{23}^2)^3} \times \mathcal{H}_{\alpha'\dot{\alpha}'\beta'\dot{\beta}'\gamma(2)\dot{\gamma}(2)}(X_{12}). \tag{3.9}$$

[4]The coefficient b_1 is related to the chiral anomaly of the CFT under consideration when it is coupled to a background vector field [2]. This anomaly exists in chiral theories which are not invariant under parity and, thus, admit a parity-odd contribution.

All information about this correlation function is encoded in the following polynomial:

$$\mathcal{H}(X;U,V,W) = \mathcal{H}_{\alpha\dot{\alpha}\beta\dot{\beta}\gamma(2)\dot{\gamma}(2)}(X)\boldsymbol{U}^{\alpha\dot{\alpha}}\boldsymbol{V}^{\beta\dot{\beta}}\boldsymbol{W}^{\gamma(2)\dot{\gamma}(2)}. \tag{3.10}$$

After solving (2.57), we find the following linearly dependent polynomial structures:

$$\{P_1P_2\bar{P}_1\bar{P}_2, P_2Q_1Q_2\bar{P}_1, P_3Q_2Z_3\bar{P}_1, P_3Z_3^2\bar{P}_3, Q_3Z_3^2\bar{Q}_3, Z_1Z_2Z_3^2, P_1Q_3Z_3\bar{P}_2, P_2Z_2Z_3\bar{P}_2, Q_1Q_2Q_3Z_3, P_2Q_1Z_3\bar{P}_3,$$
$$Q_1Z_1Z_3\bar{Q}_1, P_2Q_1\bar{P}_2\bar{Q}_1, P_3Z_3\bar{P}_2\bar{Q}_1, Q_2Z_2Z_3\bar{Q}_2, P_1Q_2\bar{P}_1\bar{Q}_2, P_1Z_3\bar{P}_3\bar{Q}_2, Q_1Q_2\bar{Q}_1\bar{Q}_2, P_1\bar{P}_2\bar{Q}_1\bar{Q}_2, P_1Z_1Z_3\bar{P}_1,$$
$$P_2Z_3\bar{P}_1\bar{Q}_3, Z_3\bar{Q}_1\bar{Q}_2\bar{Q}_3\}. \tag{3.11}$$

We now systematically apply the linear dependence relations (2.50)–(2.55) to obtain the linearly independent structures

$$\{P_2Q_1Q_2\bar{P}_1, P_1P_2\bar{P}_1\bar{P}_2, P_2Q_1\bar{P}_2\bar{Q}_1, P_1Q_2\bar{P}_1\bar{Q}_2, Q_1Q_2\bar{Q}_1\bar{Q}_2, P_1\bar{P}_2\bar{Q}_1\bar{Q}_2\}. \tag{3.12}$$

Next, we construct the following parity-even and parity-odd linear combinations which comprise the polynomial $\hat{\mathcal{H}}(X;U,V,W)$:

$$A_1(P_2Q_1Q_2\bar{P}_1 + P_1\bar{P}_2\bar{Q}_1\bar{Q}_2) + A_2P_1P_2\bar{P}_1\bar{P}_2 + A_3P_2Q_1\bar{P}_2\bar{Q}_1 + A_4P_1Q_2\bar{P}_1\bar{Q}_2 + A_5Q_1Q_2\bar{Q}_1\bar{Q}_2$$
$$+ B_1(P_2Q_1Q_2\bar{P}_1 - P_1\bar{P}_2\bar{Q}_1\bar{Q}_2). \tag{3.13}$$

We now impose conservation on all three points to obtain the final solution for $\mathcal{H}(X;U,V,W)$

$$\frac{a_1}{X^2}\left(P_2Q_1Q_2\bar{P}_1 + P_1Q_2\bar{P}_1\bar{Q}_2 + P_2Q_1\bar{P}_2\bar{Q}_1 + P_1\bar{P}_2\bar{Q}_1\bar{Q}_2 - \frac{2}{3}Q_1Q_2\bar{Q}_1\bar{Q}_2\right)$$
$$+\frac{a_2}{X^2}\left(-\frac{1}{2}P_2Q_1\bar{P}_2\bar{Q}_1 - \frac{1}{2}P_1Q_2\bar{P}_1\bar{Q}_2 + P_1P_2\bar{P}_1\bar{P}_2 + \frac{1}{3}Q_1Q_2\bar{Q}_1\bar{Q}_2\right) + \frac{\mathrm{i}b_1}{X^2}(P_2Q_1Q_2\bar{P}_1 - P_1\bar{P}_2\bar{Q}_1\bar{Q}_2). \tag{3.14}$$

In this case, only the parity-even structures (proportional to a_1 and a_2) survive after setting $J = J'$. Hence, this correlation function is fixed up to two independent parity-even structures with real coefficients.

The number of polynomial structures increases rapidly for increasing s_i, and for the three-point functions $\langle TTV\rangle$, $\langle TTT\rangle$ we will present only the linearly independent structures and the final results after imposing parity, reality, and conservation on all three points. For $\langle TTV\rangle$ we may consider the correlation function $\langle J_{(2,0)}J'_{(2,0)}J''_{(1,0)}\rangle$, which is constructed from the following list of linearly independent structures:

$$\{P_3Q_1Q_2Q_3\bar{P}_3, P_3^2Q_2\bar{P}_1\bar{P}_3, P_2P_3Q_1\bar{P}_3^2, Q_1Q_2Q_3^2\bar{Q}_3, P_1Q_3^2\bar{P}_2\bar{Q}_3, P_3Q_2Q_3\bar{P}_1\bar{Q}_3, P_2Q_1Q_3\bar{P}_3\bar{Q}_3,$$
$$P_3\bar{P}_3\bar{Q}_1\bar{Q}_2\bar{Q}_3, Q_3\bar{Q}_1\bar{Q}_2\bar{Q}_3^2\}. \tag{3.15}$$

We now construct linearly independent parity-even and parity-odd solutions consistent with (2.79). Then, after imposing all the constraints due to reality and conservation, we obtain the final solution for $\mathcal{H}(X;U,V,W)$:

$$\frac{a_1}{X^5}\left(3P_1Q_3^2\bar{P}_2\bar{Q}_3 + P_3Q_1Q_2Q_3\bar{P}_3 + 2P_3Q_2Q_3\bar{P}_1\bar{Q}_3 + 2P_2Q_1Q_3\bar{P}_3\bar{Q}_3 + P_3\bar{P}_3\bar{Q}_1\bar{Q}_2\bar{Q}_3 + \frac{7}{2}Q_1Q_2Q_3^2\bar{Q}_3 - \frac{7}{2}Q_3\bar{Q}_1\bar{Q}_2\bar{Q}_3^2\right)$$
$$+\frac{a_2}{X^5}\left(P_3^2Q_2\bar{P}_1\bar{P}_3 + P_2P_3Q_1\bar{P}_3^2 - 6P_3Q_2Q_3\bar{P}_1\bar{Q}_3 - 2P_3\bar{P}_3\bar{Q}_1\bar{Q}_2\bar{Q}_3 - 7P_1Q_3^2\bar{P}_2\bar{Q}_3 - 6P_2Q_1Q_3\bar{P}_3\bar{Q}_3\right.$$
$$\left.+\frac{17}{2}Q_3\bar{Q}_1\bar{Q}_2\bar{Q}_3^2 - \frac{21}{2}Q_1Q_2Q_3^2\bar{Q}_3\right) + \frac{\mathrm{i}b_1}{X^5}\left(P_3Q_1Q_2Q_3\bar{P}_3 - P_3\bar{P}_3\bar{Q}_1\bar{Q}_2\bar{Q}_3 - \frac{3}{2}Q_1Q_2Q_3^2\bar{Q}_3 + \frac{3}{2}Q_3\bar{Q}_1\bar{Q}_2\bar{Q}_3^2\right). \tag{3.16}$$

After setting $J = J'$ and imposing the required symmetries under the exchange of x_1 and x_2 we find that $b_1 = 0$, while a_1, a_2 remain unconstrained. Hence, the correlation function $\langle TTV\rangle$ is fixed up to two parity-even structures with real coefficients.

The final fundamental three-point function to study is $\langle TTT\rangle$, and for this we analyze the correlation function $\langle J_{(2,0)}J'_{(2,0)}J''_{(2,0)}\rangle$. In this case there are 15 linearly independent structures to consider:

$$\begin{aligned}&\{Q_1^2Q_2^2Q_3^2, P_3Q_1Q_2^2Q_3\bar{P}_1, P_3^2Q_2^2\bar{P}_1^2, P_1Q_1Q_2Q_3^2\bar{P}_2, P_1^2Q_3^2\bar{P}_2^2, P_2Q_1^2Q_2Q_3\bar{P}_3,\\ &P_3Q_1Q_2\bar{P}_3\bar{Q}_1\bar{Q}_2, P_2Q_1Q_3\bar{P}_2\bar{Q}_1\bar{Q}_3, P_1Q_2Q_3\bar{P}_1\bar{Q}_2\bar{Q}_3, Q_1Q_2Q_3\bar{Q}_1\bar{Q}_2\bar{Q}_3,\\ &P_2^2Q_1^2\bar{P}_3^2, P_3Q_2\bar{P}_1\bar{Q}_1\bar{Q}_2\bar{Q}_3, P_1Q_3\bar{P}_2\bar{Q}_1\bar{Q}_2\bar{Q}_3, P_2Q_1\bar{P}_3\bar{Q}_1\bar{Q}_2\bar{Q}_3, \bar{Q}_1^2\bar{Q}_2^2\bar{Q}_3^2\}.\end{aligned} \tag{3.17}$$

From these structures we construct linear combinations that are even/odd under parity, analogous to the previous examples. Then, after imposing reality and conservation on all three points we obtain the following solution for $\mathcal{H}(X;U,V,W)$:

$$\begin{aligned}&\frac{a_1}{X^4}(Q_1^2Q_2^2Q_3^2+2P_1^2Q_3^2\bar{P}_2^2-2Q_1Q_2Q_3\bar{Q}_1\bar{Q}_2\bar{Q}_3+2P_1Q_1Q_2Q_3^2\bar{P}_2-2P_1Q_3\bar{P}_2\bar{Q}_1\bar{Q}_2\bar{Q}_3+\bar{Q}_1^2\bar{Q}_2^2\bar{Q}_3^2)\\ &+\frac{a_2}{X^4}\bigg(P_2Q_1^2Q_2Q_3\bar{P}_3+P_3Q_1Q_2^2Q_3\bar{P}_1-\frac{17}{3}P_1Q_1Q_2Q_3^2\bar{P}_2+2P_2Q_1Q_3\bar{P}_2\bar{Q}_1\bar{Q}_3\\ &+3Q_1Q_2Q_3\bar{Q}_1\bar{Q}_2\bar{Q}_3+P_2Q_1\bar{P}_3\bar{Q}_1\bar{Q}_2\bar{Q}_3-\frac{20}{3}P_1^2Q_3^2\bar{P}_2^2-3\bar{Q}_1^2\bar{Q}_2^2\bar{Q}_3^2\\ &+2P_1Q_2Q_3\bar{P}_1\bar{Q}_2\bar{Q}_3+P_3Q_2\bar{P}_1\bar{Q}_1\bar{Q}_2\bar{Q}_3+\frac{23}{3}P_1Q_3\bar{P}_2\bar{Q}_1\bar{Q}_2\bar{Q}_3\bigg)\\ &+\frac{a_3}{X^4}\bigg(P_3^2Q_2^2\bar{P}_1^2+\frac{19}{3}P_1^2Q_3^2\bar{P}_2^2+\frac{16}{3}P_1Q_1Q_2Q_3^2\bar{P}_2-2Q_1Q_2Q_3\bar{Q}_1\bar{Q}_2\bar{Q}_3\\ &+P_2^2Q_1^2\bar{P}_3^2-2P_2Q_1Q_3\bar{P}_2\bar{Q}_1\bar{Q}_3-2P_1Q_2Q_3\bar{P}_1\bar{Q}_2\bar{Q}_3-\frac{22}{3}P_1Q_3\bar{P}_2\bar{Q}_1\bar{Q}_2\bar{Q}_3\\ &-3P_3Q_1Q_2\bar{P}_3\bar{Q}_1\bar{Q}_2-2P_3Q_2\bar{P}_1\bar{Q}_1\bar{Q}_2\bar{Q}_3-2P_2Q_1\bar{P}_3\bar{Q}_1\bar{Q}_2\bar{Q}_3+3\bar{Q}_1^2\bar{Q}_2^2\bar{Q}_3^2\bigg)\\ &+\frac{ib_1}{X^4}(Q_1^2Q_2^2Q_3^2+2P_1Q_1Q_2Q_3^2\bar{P}_2+\bar{Q}_1^2\bar{Q}_2^2\bar{Q}_3^2-2Q_1Q_2Q_3\bar{Q}_1\bar{Q}_2\bar{Q}_3-2P_1Q_3\bar{P}_2\bar{Q}_1\bar{Q}_2\bar{Q}_3)\\ &+\frac{ib_2}{X^4}(P_2Q_1^2Q_2Q_3\bar{P}_3+P_3Q_1Q_2^2Q_3\bar{P}_1-3P_1Q_1Q_2Q_3^2\bar{P}_2+Q_1Q_2Q_3\bar{Q}_1\bar{Q}_2\bar{Q}_3\\ &-P_2Q_1\bar{P}_3\bar{Q}_1\bar{Q}_2\bar{Q}_3-P_3Q_2\bar{P}_1\bar{Q}_1\bar{Q}_2\bar{Q}_3+3P_1Q_3\bar{P}_2\bar{Q}_1\bar{Q}_2\bar{Q}_3-\bar{Q}_1^2\bar{Q}_2^2\bar{Q}_3^2).\end{aligned} \tag{3.18}$$

In this case only three of the structures (corresponding to the real coefficients a_1, a_2, a_3) survive the point-switch symmetries upon exchange of x_1, x_2 and x_3. Hence, $\langle TTT\rangle$ is fixed up to three parity-even structures with real coefficients.

In all cases we note that the number of independent structures (prior to imposing exchange symmetries) is $2\,\min(s_1,s_2,s_3)+1$ in general, where $\min(s_1,s_2,s_3)+1$ are parity-even and $\min(s_1,s_2,s_3)$ are parity-odd. These results are in agreement with [1,13–15,18] in terms of the number of independent structures, however, our construction of the three-point function is quite different.

2. Spin-3/2 current correlators

In this section we will evaluate three-point functions involving conserved fermionic currents. The most important examples of fermionic conserved currents in 4D CFT are the supersymmetry currents, $Q_{m,\alpha}$, $\bar{Q}_{m,\dot{\alpha}}$, which appear in $\mathcal{N}$-extended superconformal field theories. Such fields are primary with dimension $\Delta_Q=\Delta_{\bar{Q}}=7/2$, and satisfy the conservation equations $\partial^m Q_{m,\alpha}=0$, $\partial^m\bar{Q}_{m,\dot{\alpha}}=0$. In spinor notation, we have:

$$Q_{\alpha\dot{\alpha},\beta}(x)=(\sigma^m)_{\alpha\dot{\alpha}}Q_{m,\beta}(x),\qquad \bar{Q}_{\alpha\dot{\alpha},\dot{\beta}}(x)=(\sigma^m)_{\alpha\dot{\alpha}}\bar{Q}_{m,\dot{\beta}}(x). \tag{3.19}$$

The correlation functions involving supersymmetry currents, vector currents, and the energy-momentum tensor are of fundamental importance. The four possible three-point functions involving Q, V and T which are of interest in $\mathcal{N}=1$ superconformal field theories are

$$\langle Q_{\alpha(2)\dot{\alpha}}(x_1)Q_{\beta(2)\dot{\beta}}(x_2)V_{\gamma\dot{\gamma}}(x_3)\rangle,\quad \langle Q_{\alpha(2)\dot{\alpha}}(x_1)Q_{\beta(2)\dot{\beta}}(x_2)T_{\gamma(2)\dot{\gamma}(2)}(x_3)\rangle, \tag{3.20a}$$

$$\langle Q_{\alpha(2)\dot{\alpha}}(x_1)\bar{Q}_{\beta\dot{\beta}(2)}(x_2)V_{\gamma\dot{\gamma}}(x_3)\rangle,\quad \langle Q_{\alpha(2)\dot{\alpha}}(x_1)\bar{Q}_{\beta(2)\dot{\beta}}(x_2)T_{\gamma(2)\dot{\gamma}(2)}(x_3)\rangle. \tag{3.20b}$$

These three-point functions were analyzed in [17] using a similar approach, but we present them again here for completeness and to demonstrate our general formalism. Note that in the subsequent analysis we assume only conformal symmetry, not supersymmetry.

We now present an explicit analysis of the general structure of correlation functions involving Q, $\bar{Q}$, V, and T that are compatible with the constraints of conformal symmetry and conservation equations. Using our conventions for the currents, we recall that $Q \equiv J_{(3/2,1)}$, $\bar{Q} \equiv \bar{J}_{(3/2,1)}$. Let us first consider $\langle QQV \rangle$, for which we may analyze the general structure of the correlation function $\langle J_{(3/2,1)} J'_{(3/2,1)} J''_{(1,0)} \rangle$.

Correlation function $\langle J_{(3/2,1)} J'_{(3/2,1)} J''_{(1,0)} \rangle$. Using the general formula, the ansatz for this three-point function:

$$\langle J_{\alpha(2)\dot{\alpha}}(x_1) J'_{\beta(2)\dot{\beta}}(x_2) J''_{\gamma\dot{\gamma}}(x_3) \rangle = \frac{\mathcal{I}_{\alpha(2)}{}^{\dot{\alpha}'(2)}(x_{13}) \bar{\mathcal{I}}_{\dot{\alpha}}{}^{\alpha'}(x_{13}) \mathcal{I}_{\beta(2)}{}^{\dot{\beta}'(2)}(x_{23}) \bar{\mathcal{I}}_{\dot{\beta}}{}^{\beta'}(x_{23})}{(x_{13}^2)^{7/2}(x_{23}^2)^{7/2}} \times \mathcal{H}_{\alpha'\dot{\alpha}'(2)\beta'\dot{\beta}'(2)\gamma\dot{\gamma}}(X_{12}). \tag{3.21}$$

Using the formalism outlined in Sec. II B 2, all information about this correlation function is encoded in the following polynomial:

$$\mathcal{H}(X; U, V, W) = \mathcal{H}_{\alpha\dot{\alpha}(2)\beta\dot{\beta}(2)\gamma\dot{\gamma}}(X) \boldsymbol{U}^{\alpha\dot{\alpha}(2)} \boldsymbol{V}^{\beta\dot{\beta}(2)} \boldsymbol{W}^{\gamma\dot{\gamma}}. \tag{3.22}$$

After solving (2.57), we find the following linearly dependent polynomial structures in the even and odd sectors respectively:

$$\{Q_2Z_1Z_2\bar{P}_1, Q_2Q_3Z_2\bar{P}_2, Q_1Q_2Q_3\bar{P}_3, Z_1Z_2Z_3\bar{P}_3, P_3Q_2\bar{P}_1\bar{P}_3, P_1Z_1\bar{P}_1\bar{P}_3, P_1Q_3\bar{P}_2\bar{P}_3, P_2Z_2\bar{P}_2\bar{P}_3, P_2Q_1\bar{P}_3^2, P_3Z_3\bar{P}_3^2, P_1\bar{P}_3^2\bar{Q}_2, Q_1Z_1\bar{P}_3\bar{Q}_1, P_3\bar{P}_2\bar{P}_3\bar{Q}_1, Q_2Z_2\bar{P}_3\bar{Q}_2, Z_1Z_2\bar{P}_2\bar{Q}_1, Q_2Q_3\bar{P}_1\bar{Q}_3, Q_3Z_3\bar{P}_3\bar{Q}_3, P_2\bar{P}_1\bar{P}_3\bar{Q}_3, Z_1\bar{P}_1\bar{Q}_1\bar{Q}_3, Q_3\bar{P}_2\bar{Q}_1\bar{Q}_3, \bar{P}_3\bar{Q}_1\bar{Q}_2\bar{Q}_3\}. \tag{3.23}$$

Next we systematically apply the linear dependence relations (2.50)–(2.55) and obtain the following linearly independent structures:

$$\{Q_1Q_2Q_3\bar{P}_3, P_3Q_2\bar{P}_1\bar{P}_3, P_2Q_1\bar{P}_3^2, Q_2Q_3\bar{P}_1\bar{Q}_3, Q_3\bar{P}_2\bar{Q}_1\bar{Q}_3, \bar{P}_3\bar{Q}_1\bar{Q}_2\bar{Q}_3\}. \tag{3.24}$$

We now impose conservation on all three points and find that the solution for $\mathcal{H}(X; U, V, W)$ is unique up to a complex coefficient, $A_1 = a_1 + \mathrm{i}\tilde{a}_1$:

$$\frac{A_1}{X^4}\left(Q_1Q_2Q_3\bar{P}_3 + \frac{5}{9}P_2Q_1\bar{P}_3^2 + \frac{5}{9}P_3Q_2\bar{P}_1\bar{P}_3 - \frac{1}{9}\bar{P}_3\bar{Q}_1\bar{Q}_2\bar{Q}_3 - \frac{2}{9}Q_2Q_3\bar{P}_1\bar{Q}_3 - \frac{2}{9}Q_3\bar{P}_2\bar{Q}_1\bar{Q}_3\right). \tag{3.25}$$

However, this three-point function is not compatible with the point-switch symmetry associated with setting $J = J'$. Therefore we conclude that the three-point function $\langle QQV \rangle$ must vanish in general.

Correlation function $\langle J_{(3/2,1)} \bar{J}'_{(3/2,1)} J''_{(1,0)} \rangle$. Using the general formula we obtain the following ansatz:

$$\langle J_{\alpha(2)\dot{\alpha}}(x_1) J'_{\beta\dot{\beta}(2)}(x_2) J''_{\gamma\dot{\gamma}}(x_3) \rangle = \frac{\mathcal{I}_{\alpha(2)}{}^{\dot{\alpha}'(2)}(x_{13}) \bar{\mathcal{I}}_{\dot{\alpha}}{}^{\alpha'}(x_{13}) \mathcal{I}_{\beta}{}^{\dot{\beta}'}(x_{23}) \bar{\mathcal{I}}_{\dot{\beta}(2)}{}^{\beta'(2)}(x_{23})}{(x_{13}^2)^{7/2}(x_{23}^2)^{7/2}} \times \mathcal{H}_{\alpha'\dot{\alpha}'(2)\beta'(2)\dot{\beta}'\gamma\dot{\gamma}}(X_{12}). \tag{3.26}$$

The tensor three-point function is encoded in the following polynomial:

$$\mathcal{H}(X; U, V, W) = \mathcal{H}_{\alpha\dot{\alpha}(2)\beta(2)\dot{\beta}\gamma\dot{\gamma}}(X) \boldsymbol{U}^{\alpha\dot{\alpha}(2)} \boldsymbol{V}^{\beta(2)\dot{\beta}} \boldsymbol{W}^{\gamma\dot{\gamma}}. \tag{3.27}$$

After solving (2.57), we find the following linearly dependent polynomial structures:

$$\{Q_1Q_2Z_1Z_2, P_3Q_2Z_2\bar{P}_2, P_1Z_1Z_2\bar{P}_2, P_3Q_1Q_2\bar{P}_3, P_1Q_1Z_1\bar{P}_3, P_1P_3\bar{P}_2\bar{P}_3, Q_1Q_2Q_3\bar{Q}_3, Z_1Z_2Z_3\bar{Q}_3, P_3Q_2\bar{P}_1\bar{Q}_3, P_1Z_1\bar{P}_1\bar{Q}_3, P_1Q_3\bar{P}_2\bar{Q}_3, P_2Z_2\bar{P}_2\bar{Q}_3, P_2Q_1\bar{P}_3\bar{Q}_3, P_3Z_3\bar{P}_3\bar{Q}_3, Q_1Z_1\bar{Q}_1\bar{Q}_3, P_3\bar{P}_2\bar{Q}_1\bar{Q}_3, Q_2Z_2\bar{Q}_2\bar{Q}_3, P_1\bar{P}_3\bar{Q}_2\bar{Q}_3, Q_3Z_3\bar{Q}_3^2, P_2\bar{P}_1\bar{Q}_3^2, \bar{Q}_1\bar{Q}_2\bar{Q}_3^2\}. \tag{3.28}$$

Next we systematically apply the linear dependence relations (2.50) to this list, which results in the following linearly independent structures:

$$\{P_3Q_1Q_2\bar{P}_3, Q_1Q_2Q_3\bar{Q}_3, P_3Q_2\bar{P}_1\bar{Q}_3, P_1Q_3\bar{P}_2\bar{Q}_3, P_2Q_1\bar{P}_3\bar{Q}_3, \bar{Q}_1\bar{Q}_2\bar{Q}_3^2\}. \tag{3.29}$$

We now construct the ansatz for this three-point function using the linearly independent structures above. After imposing conservation on all three points the final solution is

$$\frac{A_1}{X^4}\left(P_3Q_1Q_2\bar{P}_3+\frac{3}{2}P_1Q_3\bar{P}_2\bar{Q}_3-\frac{3}{4}\bar{Q}_1\bar{Q}_2\bar{Q}_3^2\right)+\frac{A_2}{X^4}(P_1Q_3\bar{P}_2\bar{Q}_3-\bar{Q}_1\bar{Q}_2\bar{Q}_3^2+Q_1Q_2Q_3\bar{Q}_3)$$
$$+\frac{A_3}{X^4}\left(P_3Q_2\bar{P}_1\bar{Q}_3-\frac{1}{2}P_1Q_3\bar{P}_2\bar{Q}_3+P_2Q_1\bar{P}_3\bar{Q}_3+\frac{3}{4}\bar{Q}_1\bar{Q}_2\bar{Q}_3^2\right). \tag{3.30}$$

Therefore we see that the correlation function $\langle J_{(3/2,1)}\bar{J}'_{(3/2,1)}J''_{(1,0)}\rangle$ and, hence, $\langle Q\bar{Q}V\rangle$, is fixed up to three independent complex coefficients. After imposing the combined point-switch/reality condition on Q and $\bar{Q}$, we find that the complex coefficients A_i must be purely imaginary, i.e., $A_i = \mathrm{i}\tilde{a}_i$. Hence, the correlation function $\langle Q\bar{Q}V\rangle$ is fixed up to three independent real parameters.

Next we determine the general structure of $\langle QQT\rangle$ and $\langle Q\bar{Q}T\rangle$, which are associated with the correlation functions $\langle J_{(3/2,1)}J'_{(3/2,1)}J''_{(2,0)}\rangle$, $\langle J_{(3/2,1)}\bar{J}'_{(3/2,1)}J''_{(2,0)}\rangle$ respectively using our general formalism. Since the number of structures grows rapidly with spin, we will simply present the final results after conservation. For $\langle J_{(3/2,1)}J'_{(3/2,1)}J''_{(2,0)}\rangle$ we obtain a single independent structure (up to a complex coefficient):

$$\frac{A_1}{X^3}\left(Q_1Q_3Q_2^2\bar{P}_1+\frac{7}{4}P_3Q_2^2\bar{P}_1^2+\frac{1}{2}P_1Q_3Q_2\bar{P}_1\bar{P}_2-\frac{5}{4}Q_1Q_3Q_2\bar{P}_2\bar{Q}_1-5Q_1Q_2\bar{P}_3\bar{Q}_1\bar{Q}_2-\frac{7}{2}Q_2\bar{P}_1\bar{Q}_1\bar{Q}_2\bar{Q}_3\right.$$
$$\left.+\frac{1}{2}P_1Q_3\bar{P}_2^2\bar{Q}_1+\frac{7}{4}P_2Q_1\bar{P}_2\bar{P}_3\bar{Q}_1-\frac{5}{4}\bar{P}_2\bar{Q}_1^2\bar{Q}_2\bar{Q}_3\right). \tag{3.31}$$

This solution is manifestly compatible with the point-switch symmetry resulting from setting $J = J'$, hence, $\langle QQT\rangle$ is unique up to a complex parameter. On the other hand, for $\langle J_{(3/2,1)}\bar{J}'_{(3/2,1)}J''_{(2,0)}\rangle$ we obtain four independent conserved structures proportional to complex coefficients

$$\frac{A_1}{X^3}\left(Q_1^2Q_3Q_2^2+\frac{6}{7}P_1Q_2\bar{P}_1\bar{Q}_2\bar{Q}_3+\frac{6}{7}P_2Q_1\bar{P}_2\bar{Q}_1\bar{Q}_3+\frac{6}{7}P_1\bar{P}_2\bar{Q}_1\bar{Q}_2\bar{Q}_3-\frac{10}{7}Q_1Q_2\bar{Q}_1\bar{Q}_2\bar{Q}_3\right)$$
$$+\frac{A_2}{X^3}(P_2Q_2Q_1^2\bar{P}_3+P_3Q_2^2Q_1\bar{P}_1-P_2Q_1\bar{P}_2\bar{Q}_1\bar{Q}_3-P_1Q_2\bar{P}_1\bar{Q}_2\bar{Q}_3-P_1\bar{P}_2\bar{Q}_1\bar{Q}_2\bar{Q}_3+Q_2Q_1\bar{Q}_1\bar{Q}_2\bar{Q}_3)$$
$$+\frac{A_3}{X^3}\left(P_1Q_1Q_2Q_3\bar{P}_2-\frac{3}{7}P_2Q_1\bar{P}_2\bar{Q}_1\bar{Q}_3-\frac{13}{14}P_1\bar{P}_2\bar{Q}_1\bar{Q}_2\bar{Q}_3-\frac{3}{7}P_1Q_2\bar{P}_1\bar{Q}_2\bar{Q}_3+\frac{3}{14}Q_1Q_2\bar{Q}_1\bar{Q}_2\bar{Q}_3\right)+\frac{A_4}{X^3}P_1^2Q_3\bar{P}_2^2. \tag{3.32}$$

After imposing the combined point-switch/reality condition, we find that the complex coefficients A_i must be purely real. Hence, the three-point function $\langle Q\bar{Q}T\rangle$ is fixed up to four independent real parameters. The results (3.25) and (3.30)–(3.32) are in agreement with those found in [17].

B. General structure of three-point functions for arbitrary spins

In four dimensions, three-point correlation functions of bosonic higher-spin conserved currents have been analyzed in the following publications [13,18] (see [20,24,34] for supersymmetric results). For three-point functions involving bosonic currents $J_{(s,0)} = J_{\alpha(s)\dot{\alpha}(s)}$, the general structure of the three-point function $\langle J_{(s_1,0)}J'_{(s_2,0)}J''_{(s_3,0)}\rangle$ was found to be fixed up to the following form [18,37,38]:

$$\langle J_{(s_1,0)}J'_{(s_2,0)}J''_{(s_3,0)}\rangle = \sum_{I=1}^{2\ \min(s_1,s_2,s_3)+1} a_I\langle J_{(s_1,0)}J'_{(s_2,0)}J''_{(s_3,0)}\rangle_I, \tag{3.33}$$

where a_I are real coefficients and $\langle J_{(s_1,0)}J'_{(s_2,0)}J''_{(s_3,0)}\rangle_I$ are linearly independent conserved structures.[5] Among these $2\ \min(s_1,s_2,s_3)+1$ structures, $\min(s_1,s_2,s_3)+1$ are parity-even while $\min(s_1,s_2,s_3)$ are parity odd. For correlation functions involving identical fields we must also impose point-switch symmetries. The following classification holds:

[5]Note that if the reality condition is not imposed, the three-point function is fixed up to $2\ \min(s_i)+1$ structures with complex coefficients.

(i) For three-point functions $\langle J_{(s,0)} J'_{(s,0)} J''_{(s,0)}\rangle$ there are $2s+1$ conserved structures, $s+1$ being parity even and s being parity odd. When the fields coincide, i.e. $J = J'$ the number of structures is reduced to the $s+1$ parity-even structures in the case when the spin s is even, or to the s parity-odd structures in the case when s is odd.

(ii) For three-point functions $\langle J_{(s_1,0)} J'_{(s_1,0)} J''_{(s_2,0)}\rangle$, there are $2\ \min(s_1, s_2) + 1$ conserved structures, $\min(s_1, s_2) + 1$ being parity even and $\min(s_1, s_2)$ being parity odd. For $J = J'$, the number of structures is reduced to the $\min(s_1, s_2) + 1$ parity-even structures in the case when the spin s_2 is even, or to the $\min(s_1, s_2)$ parity-odd structures in the case when s_2 is odd.

Note that the above classification is consistent with the results of [13], and we have explicitly reproduced them up to $s_i = 10$ in our computational approach.

Now let us discuss three-point functions involving currents with $q = 1$, which define "supersymmetrylike" fermionic higher-spin currents. The possible correlation functions that we can construct from these are $\langle J_{(s_1,1)} J'_{(s_2,1)} J''_{(s_3,0)}\rangle$ and $\langle J_{(s_1,1)} \bar{J}'_{(s_2,1)} J''_{(s_3,0)}\rangle$. Note that for $s_1 = s_2 = 3/2$ and $s_3 = 1, 2$ we obtain the familiar three-point functions (3.20). Based on our computational analysis we found that the three-point function $\langle J_{(s_1,1)} J'_{(s_2,1)} J''_{(s_3,0)}\rangle$ is fixed up to a unique structure after conservation in general. On the other hand, we found that three-point functions of the form $\langle J_{(s_1,1)} \bar{J}'_{(s_2,1)} J''_{(s_3,0)}\rangle$ are fixed up to $2\ \min(s_1, s_2, s_3) + 1$ independent conserved structures. It's important to note that for these three-point functions there is no notion of parity-even/odd structures.

We now dedicate the remainder of this section to classifying the number of independent structures in the general three-point functions

$$\langle J_{(s_1,q_1)} J'_{(s_2,q_2)} J''_{(s_3,q_3)}\rangle, \qquad \langle J_{(s_1,q_1)} \bar{J}'_{(s_2,q_2)} J''_{(s_3,q_3)}\rangle, \tag{3.34}$$

for arbitrary (s_i, q_i). We investigated the general structure of these three-point functions up to $s_i = 10$. Provided that the inequalities (2.61) are satisfied, we conjecture that the following classification holds in general:

(i) For three-point functions $\langle J_{(s_1,q_1)} J'_{(s_2,q_2)} J''_{(s_3,q_3)}\rangle$, $\langle J_{(s_1,q_1)} \bar{J}'_{(s_2,q_2)} J''_{(s_3,q_3)}\rangle$ with $q_1 \neq q_2 \neq q_3$, there is a unique solution in general. Similarly, the three-point function is also unique for the cases: (i) $q_1 = 0$, $q_2 \neq q_3$, and (ii) $q_1 = q_2 = 0$ with $q_3 \neq 0$.

(ii) For three-point functions $\langle J_{(s_1,q)} J'_{(s_2,q)} J''_{(s_3,0)}\rangle$ there is a unique solution up to a complex coefficient. However, for the case where $s_1 = s_2$ (fermionic or bosonic) and $J = J'$, the structure survives the resulting point-switch symmetry only when s_3 is an even integer.

(iii) For three-point functions $\langle J_{(s_1,q)} \bar{J}'_{(s_2,q)} J''_{(s_3,0)}\rangle$ we obtain quite a nontrivial result which we will now explain. The number of structures, $N(s_1, s_2, s_3; q)$, obeys the following formula:

$$N(s_1, s_2, s_3; q) = 2\ \min(s_1, s_2, s_3) + 1 - \max\left(\frac{q}{2} - |s_3 - \min(s_1, s_2)|, 0\right), \tag{3.35}$$

where s_1, s_2 are simultaneously integer/half-integer, for integer s_3. This formula can be arrived at using the following method. Let us fix s_1, s_2 and let $q \geq 2$. By varying s_3 and computing the resulting conserved three-point function, one can notice that if s_3 lies within the interval

$$\min(s_1, s_2) - \frac{q}{2} < s_3 < \min(s_1, s_2) + \frac{q}{2}, \tag{3.36}$$

then the number of structures is decreased from $2\ \min(s_1, s_2, s_3) + 1$ by

$$\delta N(s_1, s_2, s_3; q) = \frac{q}{2} - |s_3 - \min(s_1, s_2)|. \tag{3.37}$$

For s_3 outside the interval (3.36) there is always $2\ \min(s_1, s_2, s_3) + 1$ structures in general. It should also be noted that (3.35) is also valid for $q = 0$, 1 (by virtue of the max() function). In these cases the additional term does not contribute and we obtain $N(s_1, s_2, s_3; 0) = N(s_1, s_2, s_3; 1) = 2\ \min(s_1, s_2, s_3) + 1$.

As examples, below we tabulate the number of structures in the conserved three-point functions $\langle J_{(s_1,q)} \bar{J}'_{(s_2,q)} J''_{(s_3,0)}\rangle$ for some fixed s_1, s_2 while varying q and s_3. Let us recall that q is necessarily even/odd when s is integer/half-integer valued. In addition, since $J_{(s,q)} \coloneqq J_{\alpha(s+\frac{q}{2})\dot{\alpha}(s-\frac{q}{2})}$ it follows that the maximal allowed value of q in the above correlation function is $2\ \min(s_1, s_2) - 2$. Explicit solutions for particular cases are presented in Appendix B.

The highlighted values are within the interval (3.36) defined by s_1, s_2, and q, and we have used color to identify the pattern in the number of structures. Analogous tables can be constructed for any choice of s_1, s_2 and it is easy to see that the results are consistent with the general formula (3.35), which appears to hold for all such correlators within the bounds of our computational limitations ($s_i \leq 10$).

(iv) For three-point functions $\langle J_{(s_1,q)} \bar{J}'_{(s_1,q)} J''_{(s_2,0)}\rangle$ the number of structures adheres to the formula (3.35). However, for $J = J'$, we must impose the combined point-switch/reality condition. After imposing this constraint we find that the free complex parameters must be purely real/imaginary for s_2 even/odd.

The above classification appears to be complete, and we have not found any other permutations of fields/spins which give rise to new results.

ACKNOWLEDGMENTS

The authors are grateful to Sergei Kuzenko, Emmanouil Raptakis, and Daniel Hutchings for valuable discussions. The work of E. I. B. is supported in part by the Australian Research Council, Projects DP200101944 and DP230101629. The work of B. S. is supported by the *Bruce and Betty Green Postgraduate Research Scholarship* under the Australian Government Research Training Program.

APPENDIX A: 4D CONVENTIONS AND NOTATION

Our conventions closely follow that of [50]. For the Minkowski metric η_{mn} we use the "mostly plus" convention: $\eta_{mn} = \mathrm{diag}(-1,1,1,1)$. Spinor indices on spin-tensors are raised and lowered with the $\mathrm{SL}(2,\mathbb{C})$ invariant spinor metrics

$$\varepsilon_{\alpha\beta} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad \varepsilon^{\alpha\beta} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad \varepsilon_{\alpha\gamma}\varepsilon^{\gamma\beta} = \delta_\alpha{}^\beta, \tag{A1a}$$

$$\varepsilon_{\dot\alpha\dot\beta} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad \varepsilon^{\dot\alpha\dot\beta} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad \varepsilon_{\dot\alpha\dot\gamma}\varepsilon^{\dot\gamma\dot\beta} = \delta_{\dot\alpha}{}^{\dot\beta}. \tag{A1b}$$

Given the spinor fields ϕ_α, $\bar\phi_{\dot\alpha}$, the spinor indices $\alpha = 1, 2$, $\dot\alpha = \bar 1, \bar 2$ are raised and lowered according to the following rules:

$$\begin{aligned} \phi_\alpha &= \varepsilon_{\alpha\beta}\phi^\beta, & \phi^\alpha &= \varepsilon^{\alpha\beta}\phi_\beta, \\ \bar\phi_{\dot\alpha} &= \varepsilon_{\dot\alpha\dot\beta}\bar\phi^{\dot\beta}, & \bar\phi^{\dot\alpha} &= \varepsilon^{\dot\alpha\dot\beta}\bar\phi_{\dot\beta}. \end{aligned} \tag{A2}$$

It is also useful to introduce the complex 2×2 σ-matrices, defined as follows:

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \\ \sigma_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{A3}$$

The σ-matrices span the Lie group $\mathrm{SL}(2,\mathbb{C})$, the universal covering group of the Lorentz group $\mathrm{SO}(3,1)$. Now let $\sigma_m = (\sigma_0, \vec\sigma)$, we denote the components of σ_m as $(\sigma_m)_{\alpha\dot\alpha}$, and define:

$$(\tilde\sigma_m)^{\dot\alpha\alpha} \coloneqq \varepsilon^{\dot\alpha\dot\beta}\varepsilon^{\alpha\beta}(\sigma_m)_{\beta\dot\beta}. \tag{A4}$$

It can be shown that the σ-matrices possess the following useful properties:

$$(\sigma_m\tilde\sigma_n + \sigma_n\tilde\sigma_m)_\alpha{}^\beta = -2\eta_{mn}\delta_\alpha^\beta, \tag{A5a}$$

$$(\tilde\sigma_m\sigma_n + \tilde\sigma_n\sigma_m)^{\dot\alpha}{}_{\dot\beta} = -2\eta_{mn}\delta_{\dot\beta}^{\dot\alpha}, \tag{A5b}$$

$$\mathrm{Tr}(\sigma_m\tilde\sigma_n) = -2\eta_{mn}, \tag{A5c}$$

$$(\sigma^m)_{\alpha\dot\alpha}(\tilde\sigma_m)^{\dot\beta\beta} = -2\delta_\alpha^\beta\delta_{\dot\alpha}^{\dot\beta}. \tag{A5d}$$

The σ-matrices are then used to convert spacetime indices into spinor ones and vice versa according to the following rules:

$$X_{\alpha\dot\alpha} = (\sigma^m)_{\alpha\dot\alpha}X_m, \qquad X_m = -\frac{1}{2}(\tilde\sigma_m)^{\dot\alpha\alpha}X_{\alpha\dot\alpha}. \tag{A6}$$

For imposing conservation equations on three-point functions, one must act on the generating function (2.56) with the operators (2.68). For this, the following identities for the derivatives of the monomials Q_i, Z_i are useful:

$$\partial_X^{\alpha\dot\alpha}Q_1 = -\frac{1}{X}(2v^\alpha\bar w^{\dot\alpha} + \hat X^{\dot\alpha\alpha}Q_1), \tag{A7a}$$

$$\partial_X^{\alpha\dot\alpha}Q_2 = -\frac{1}{X}(2w^\alpha\bar u^{\dot\alpha} + \hat X^{\dot\alpha\alpha}Q_2), \tag{A7b}$$

$$\partial_X^{\alpha\dot\alpha}Q_3 = -\frac{1}{X}(2u^\alpha\bar v^{\dot\alpha} + \hat X^{\dot\alpha\alpha}Q_3), \tag{A7c}$$

$$\partial_X^{\alpha\dot\alpha}Z_1 = -\frac{1}{X}(2u^\alpha\bar u^{\dot\alpha} + \hat X^{\dot\alpha\alpha}Z_1), \tag{A8a}$$

$$\partial_X^{\alpha\dot\alpha}Z_2 = -\frac{1}{X}(2v^\alpha\bar v^{\dot\alpha} + \hat X^{\dot\alpha\alpha}Z_2), \tag{A8b}$$

$$\partial_X^{\alpha\dot\alpha}Z_3 = -\frac{1}{X}(2w^\alpha\bar w^{\dot\alpha} + \hat X^{\dot\alpha\alpha}Z_3). \tag{A8c}$$

Analogous identities for derivatives of $\bar Q_i$ may be obtained by complex conjugation.

APPENDIX B: EXAMPLES OF THREE-POINT FUNCTIONS $\langle J_{(s_1,q)}\bar J'_{(s_2,q)}J''_{(s_3,0)}\rangle$

In this appendix we provide some examples of three-point functions $\langle J_{(s_1,q)}\bar J'_{(s_2,q)}J''_{(s_3,0)}\rangle$. In particular, we compute two of the examples presented in Tables I and II to illustrate the decrease in the number of independent conserved structures for particular values of q. Due to

TABLE I. No. of structures in $\langle J_{(s_1,q)}\bar{J}'_{(s_2,q)}J''_{(s_3,0)}\rangle$ for $s_1=5$, $s_2=6$.

	s_3								
q	1	2	3	4	5	6	7	8	9
0	3	5	7	9	11	11	11	11	11
2	3	5	7	9	10	11	11	11	11
4	3	5	7	8	9	10	11	11	11
6	3	5	6	7	8	9	10	11	11
8	3	4	5	6	7	8	9	10	11

• $\delta N=1$; • $\delta N=2$; • $\delta N=3$; • $\delta N=4$.

TABLE II. No. of structures in $\langle J_{(s_1,q)}\bar{J}'_{(s_2,q)}J''_{(s_3,0)}\rangle$ for $s_1=9/2$, $s_2=11/2$.

	s_3							
q	1	2	3	4	5	6	7	8
1	3	5	7	9	10	10	10	10
3	3	5	7	8	9	10	10	10
5	3	5	6	7	8	9	10	10
7	3	4	5	6	7	8	9	10

• $\delta N=1$; • $\delta N=2$; • $\delta N=3$.

the large size of the solutions for increasing s_i, we only present the simplest cases.

First let us consider the three-point function $\langle J_{(s_1,q)}\bar{J}'_{(s_2,q)}J''_{(s_3,0)}\rangle$ with $s_1=5$, $s_2=6$, $q=8$ and $s_3=2$. Using our formalism, all information about this correlation function is encoded in the following polynomial:

$$\mathcal{H}(X;U,V,W)=\mathcal{H}_{\alpha(1)\dot{\alpha}(9)\beta(10)\dot{\beta}(2)\gamma(2)\dot{\gamma}(2)}(X)\boldsymbol{U}^{\alpha(1)\dot{\alpha}(9)}\boldsymbol{V}^{\beta(10)\dot{\beta}(2)}\boldsymbol{W}^{\gamma(2)\dot{\gamma}(2)}. \tag{B1}$$

There are 13 possible linearly independent structures that can be constructed in this case:

$$\{P_1P_3Q_1^2Q_2\bar{P}_3^2\bar{Q}_3^6, P_1Q_1^2Q_2Q_3\bar{P}_3\bar{Q}_3^7, P_3Q_1Q_2\bar{P}_1\bar{Q}_1\bar{Q}_3^8, P_3Q_1^2Q_2\bar{P}_3\bar{Q}_1\bar{Q}_3^7, P_1Q_1Q_2Q_3\bar{P}_1\bar{Q}_3^8, P_1P_3Q_1Q_2\bar{P}_1\bar{P}_3\bar{Q}_3^7,$$
$$P_1Q_1Q_3\bar{P}_2\bar{Q}_1\bar{Q}_3^8, P_2Q_1^2\bar{P}_3\bar{Q}_1\bar{Q}_3^8, P_1\bar{P}_1\bar{Q}_1\bar{Q}_2\bar{Q}_3^9, Q_1\bar{Q}_1^2\bar{Q}_2\bar{Q}_3^9, P_1P_3Q_2\bar{P}_1^2\bar{Q}_3^8, Q_1^2Q_2Q_3\bar{Q}_1\bar{Q}_3^8, P_1^2Q_3\bar{P}_1\bar{P}_2\bar{Q}_3^8\}. \tag{B2}$$

We now impose conservation on all three points. The following solution is obtained:

$$\frac{A_1}{X^{11}}\bigg(\frac{22}{189}Q_1\bar{Q}_1^2\bar{Q}_2\bar{Q}_3^9-\frac{11}{63}P_1\bar{P}_1\bar{Q}_1\bar{Q}_2\bar{Q}_3^9+\frac{11}{56}P_1^2Q_3\bar{P}_1\bar{P}_2\bar{Q}_3^8-\frac{121}{378}Q_1^2Q_2Q_3\bar{Q}_1\bar{Q}_3^8$$
$$-\frac{143}{756}P_1Q_1Q_3\bar{P}_2\bar{Q}_1\bar{Q}_3^8-\frac{11}{14}P_1Q_1^2Q_2Q_3\bar{P}_3\bar{Q}_3^7+\frac{1}{2}P_3Q_1^2Q_2\bar{P}_3\bar{Q}_1\bar{Q}_3^7+P_1P_3Q_1^2Q_2\bar{P}_3^2\bar{Q}_3^6\bigg)$$
$$+\frac{A_2}{X^{11}}\bigg(\frac{31}{54}Q_1\bar{Q}_1^2\bar{Q}_2\bar{Q}_3^9-\frac{11}{18}P_1\bar{P}_1\bar{Q}_1\bar{Q}_2\bar{Q}_3^9+\frac{11}{16}P_1^2Q_3\bar{P}_1\bar{P}_2\bar{Q}_3^8-\frac{10}{27}Q_1^2Q_2Q_3\bar{Q}_1\bar{Q}_3^8-\frac{143}{216}P_1Q_1Q_3\bar{P}_2\bar{Q}_1\bar{Q}_3^8$$
$$-\frac{1}{4}P_2Q_1^2\bar{P}_3\bar{Q}_1\bar{Q}_3^8-\frac{1}{4}P_1Q_1^2Q_2Q_3\bar{P}_3\bar{Q}_3^7+P_1P_3Q_1Q_2\bar{P}_1\bar{P}_3\bar{Q}_3^7-\frac{3}{4}P_3Q_1^2Q_2\bar{P}_3\bar{Q}_1\bar{Q}_3^7\bigg)$$
$$+\frac{A_3}{X^{11}}\bigg(\frac{2}{3}Q_1\bar{Q}_1^2\bar{Q}_2\bar{Q}_3^9-P_1\bar{P}_1\bar{Q}_1\bar{Q}_2\bar{Q}_3^9+P_1Q_1Q_2Q_3\bar{P}_1\bar{Q}_3^8+P_1^2Q_3\bar{P}_1\bar{P}_2\bar{Q}_3^8-\frac{2}{3}Q_1^2Q_2Q_3\bar{Q}_1\bar{Q}_3^8-\frac{2}{3}P_1Q_1Q_3\bar{P}_2\bar{Q}_1\bar{Q}_3^8\bigg)$$
$$+\frac{A_4}{X^{11}}\bigg(\frac{43}{27}Q_1\bar{Q}_1^2\bar{Q}_2\bar{Q}_3^9-\frac{8}{9}P_1\bar{P}_1\bar{Q}_1\bar{Q}_2\bar{Q}_3^9+P_1P_3Q_2\bar{P}_1^2\bar{Q}_3^8+P_1^2Q_3\bar{P}_1\bar{P}_2\bar{Q}_3^8-\frac{97}{54}Q_1^2Q_2Q_3\bar{Q}_1\bar{Q}_3^8-2P_3Q_1Q_2\bar{P}_1\bar{Q}_1\bar{Q}_3^8$$
$$-\frac{44}{27}P_1Q_1Q_3\bar{P}_2\bar{Q}_1\bar{Q}_3^8-\frac{1}{2}P_2Q_1^2\bar{P}_3\bar{Q}_1\bar{Q}_3^8\bigg), \tag{B3}$$

where A_i are complex coefficients. Hence we see that this three-point function is fixed up to four independent conserved structures. Recall that for $q=0$, the three-point function $\langle J_{(s_1,q)}\bar{J}'_{(s_2,q)}J''_{(s_3,0)}\rangle$ reduces to a three-point function of vectorlike currents. Hence, we should expect $2\,\min(s_1,s_2,s_3)+1=5$ independent structures. Similar results can be obtained for other values of q and s_3, which are contained in Table I.

Next, let us consider the three-point function $\langle J_{(s_1,q)}\bar{J}'_{(s_2,q)}J''_{(s_3,0)}\rangle$ with $s_1=9/2$, $s_2=11/2$, $q=7$ and $s_3=2$. All information about this correlation function is encoded in the following polynomial:

$$\mathcal{H}(X;U,V,W)=\mathcal{H}_{\alpha(1)\dot{\alpha}(8)\beta(9)\dot{\beta}(2)\gamma(2)\dot{\gamma}(2)}(X)\boldsymbol{U}^{\alpha(1)\dot{\alpha}(8)}\boldsymbol{V}^{\beta(9)\dot{\beta}(2)}\boldsymbol{W}^{\gamma(2)\dot{\gamma}(2)}. \tag{B4}$$

In this case there are also 13 possible linearly independent structures:

$$\{P_1P_3Q_1^2Q_2\bar{P}_3^2\bar{Q}_3^5, P_1Q_1^2Q_2Q_3\bar{P}_3\bar{Q}_3^6, P_1P_3Q_2\bar{P}_1^2\bar{Q}_3^7, P_3Q_1^2Q_2\bar{P}_3\bar{Q}_1\bar{Q}_3^6, P_1Q_1Q_2Q_3\bar{P}_1\bar{Q}_3^7, P_1Q_1Q_3\bar{P}_2\bar{Q}_1\bar{Q}_3^7,$$
$$Q_1^2Q_2Q_3\bar{Q}_1\bar{Q}_3^7, P_3Q_1Q_2\bar{P}_1\bar{Q}_1\bar{Q}_3^7, P_1P_3Q_1Q_2\bar{P}_1\bar{P}_3\bar{Q}_3^6, P_2Q_1^2\bar{P}_3\bar{Q}_1\bar{Q}_3^7, P_1\bar{P}_1\bar{Q}_1\bar{Q}_2\bar{Q}_3^8, P_1^2Q_3\bar{P}_1\bar{P}_2\bar{Q}_3^7, Q_1\bar{Q}_1^2\bar{Q}_2\bar{Q}_3^8\}. \tag{B5}$$

We now impose conservation on all three points, and the following solution is obtained:

$$\begin{aligned}
&\frac{A_1}{X^{10}}\bigg(\frac{5}{36}Q_1\bar{Q}_1^2\bar{Q}_2\bar{Q}_3^8 - \frac{5}{24}P_1\bar{P}_1\bar{Q}_1\bar{Q}_2\bar{Q}_3^8 + \frac{5}{21}P_1^2Q_3\bar{P}_1\bar{P}_2\bar{Q}_3^7 - \frac{25}{72}Q_1^2Q_2Q_3\bar{Q}_1\bar{Q}_3^7\\
&-\frac{115}{504}P_1Q_1Q_3\bar{P}_2\bar{Q}_1\bar{Q}_3^7 - \frac{5}{6}P_1Q_1^2Q_2Q_3\bar{P}_3\bar{Q}_3^6 + \frac{1}{2}P_3Q_1^2Q_2\bar{P}_3\bar{Q}_1\bar{Q}_3^6 + P_1P_3Q_1^2Q_2\bar{P}_3^2\bar{Q}_3^5\bigg)\\
&+\frac{A_2}{X^{10}}\bigg(\frac{7}{12}Q_1\bar{Q}_1^2\bar{Q}_2\bar{Q}_3^8 - \frac{5}{8}P_1\bar{P}_1\bar{Q}_1\bar{Q}_2\bar{Q}_3^8 + \frac{5}{7}P_1^2Q_3\bar{P}_1\bar{P}_2\bar{Q}_3^7 - \frac{3}{8}Q_1^2Q_2Q_3\bar{Q}_1\bar{Q}_3^7 - \frac{115}{168}P_1Q_1Q_3\bar{P}_2\bar{Q}_1\bar{Q}_3^7\\
&-\frac{1}{4}P_2Q_1^2\bar{P}_3\bar{Q}_1\bar{Q}_3^7 - \frac{1}{4}P_1Q_1^2Q_2Q_3\bar{P}_3\bar{Q}_3^6 + P_1P_3Q_1Q_2\bar{P}_1\bar{P}_3\bar{Q}_3^6 - \frac{3}{4}P_3Q_1^2Q_2\bar{P}_3\bar{Q}_1\bar{Q}_3^6\bigg)\\
&+\frac{A_3}{X^{10}}\bigg(\frac{2}{3}Q_1\bar{Q}_1^2\bar{Q}_2\bar{Q}_3^8 - P_1\bar{P}_1\bar{Q}_1\bar{Q}_2\bar{Q}_3^8 + P_1Q_1Q_2Q_3\bar{P}_1\bar{Q}_3^7 + P_1^2Q_3\bar{P}_1\bar{P}_2\bar{Q}_3^7 - \frac{2}{3}Q_1^2Q_2Q_3\bar{Q}_1\bar{Q}_3^7 - \frac{2}{3}P_1Q_1Q_3\bar{P}_2\bar{Q}_1\bar{Q}_3^7\bigg)\\
&+\frac{A_4}{X^{10}}\bigg(\frac{19}{12}Q_1\bar{Q}_1^2\bar{Q}_2\bar{Q}_3^8 - \frac{7}{8}P_1\bar{P}_1\bar{Q}_1\bar{Q}_2\bar{Q}_3^8 + P_1P_3Q_2\bar{P}_1^2\bar{Q}_3^7 + P_1^2Q_3\bar{P}_1\bar{P}_2\bar{Q}_3^7 - \frac{43}{24}Q_1^2Q_2Q_3\bar{Q}_1\bar{Q}_3^7 - 2P_3Q_1Q_2\bar{P}_1\bar{Q}_1\bar{Q}_3^7\\
&-\frac{13}{8}P_1Q_1Q_3\bar{P}_2\bar{Q}_1\bar{Q}_3^7 - \frac{1}{2}P_2Q_1^2\bar{P}_3\bar{Q}_1\bar{Q}_3^7\bigg),
\end{aligned} \tag{B6}$$

where A_i are complex coefficients. Hence we see that this three-point function is fixed up to four independent conserved structures. Recall that for the $q = 1$ case we expect $2\,\min(s_1, s_2, s_3) + 1 = 5$ independent structures. Similar results are obtained for other values of q and s_3, and with further testing we obtain Table II.

APPENDIX C: THREE-POINT FUNCTIONS INVOLVING SCALARS AND SPINORS

In this appendix we provide some examples of three-point functions involving scalars, spinors and a conserved tensor operator. The results here serve as a consistency check against those presented in [1,16].

1. Correlation function $\langle OO'J_{(s,0)}\rangle$

Let O, O' be scalar operators of dimension Δ_1 and Δ_2 respectively. We consider the three-point function $\langle OO'J_{(s,0)}\rangle$. According to the formula (2.61), a three-point function can be constructed only if J is in the (s, s) representation. Using the general formula, the ansatz for this three-point function is:

$$\langle O(x_1)O'(x_2)J_{\gamma(s)\dot{\gamma}(s)}(x_3)\rangle = \frac{1}{(x_{13}^2)^{\Delta_1}(x_{23}^2)^{\Delta_2}}\mathcal{H}_{\gamma(s)\dot{\gamma}(s)}(X_{12}). \tag{C1}$$

All information about this correlation function is encoded in the following polynomial:

$$\mathcal{H}(X;W) = \mathcal{H}_{\gamma(s)\dot{\gamma}(s)}(X)\boldsymbol{W}^{\gamma(s)\dot{\gamma}(s)}. \tag{C2}$$

We recall that $\mathcal{H}$ satisfies the homogeneity property $\mathcal{H}(X) = X^{s+2-\Delta_1-\Delta_2}\hat{\mathcal{H}}(X)$, where $\hat{\mathcal{H}}(X)$ is homogeneous degree 0. The only possible structure for $\hat{\mathcal{H}}(X)$ is

$$\hat{\mathcal{H}}(X;W) = AZ_3^s, \tag{C3}$$

where A is a complex coefficient. After imposing conservation on x_3 using the methods outlined in Sec. II B 2, we find

$$D_3\tilde{\mathcal{H}}(X;W) = A(\Delta_1-\Delta_2)(-1)^{s+1}s(s+1)Z_3^{s-1}X^{\Delta_1-\Delta_2-s-3}. \tag{C4}$$

Hence, we find that this three-point function is compatible with conservation on x_3 only for $\Delta_1 = \Delta_2$. When the scalars O, O' coincide, then the solution satisfies the point-switch symmetry associated with exchanging x_1 and x_2 only for even s. This result is in agreement with [16].

2. Correlation function $\langle\psi\bar{\psi}'J_{(s,q)}\rangle$

Let ψ, $\bar{\psi}'$ be spinor operators of dimension Δ_1 and Δ_2 respectively. We now consider the three-point function $\langle\psi\bar{\psi}'J_{(s,q)}\rangle$. According to the formula (2.61), a three-point function can be constructed only if J belongs to the representations (s, s), $(s-1, s+1)$ or $(s+1, s-1)$

(the latter two corresponding to $q = 2$). First consider the (s, s) representation. Using the general formula, the ansatz for this three-point function is

$$\langle \psi_\alpha(x_1)\bar{\psi}'_{\dot\beta}(x_2)J_{\gamma(s)\dot\gamma(s)}(x_3)\rangle = \frac{\mathcal{I}_\alpha{}^{\dot\alpha'}(x_{13})\bar{\mathcal{I}}_{\dot\beta}{}^{\beta'}(x_{23})}{(x_{13}^2)^{\Delta_1}(x_{23}^2)^{\Delta_2}}\mathcal{H}_{\dot\alpha'\beta'\gamma(s)\dot\gamma(s)}(X_{12}). \quad \text{(C5)}$$

All information about this correlation function is encoded in the following polynomial:

$$\mathcal{H}(X;U,V,W) = \mathcal{H}_{\dot\alpha\beta\gamma(s)\dot\gamma(s)}(X)\boldsymbol{U}^{\dot\alpha}\boldsymbol{V}^{\beta}\boldsymbol{W}^{\gamma(s)\dot\gamma(s)}. \quad \text{(C6)}$$

We recall that $\mathcal{H}$ satisfies the homogeneity property $\mathcal{H}(X) = X^{s+2-\Delta_1-\Delta_2}\hat{\mathcal{H}}(X)$, where $\hat{\mathcal{H}}(X)$ is homogeneous degree 0. In this case there are two possible linearly independent structures for $\hat{\mathcal{H}}(X)$:

$$\hat{\mathcal{H}}(X;U,V,W) = A_1 P_2\bar{P}_1 Z_3^{s-1} + A_2 Q_1\bar{Q}_2 Z_3^{s-1}, \quad \text{(C7)}$$

where A_1 and A_2 are complex coefficients. After imposing conservation on x_3 using the methods outlined in Sec. II B 2, we find

$$\begin{aligned} &D_3\tilde{\mathcal{H}}(X;U,V,W) \\ &= (\Delta_1-\Delta_2)(-1)^{s+1}\{(A_1+(s^2+s-1)A_2)Q_2\bar{P}_1 \\ &\quad + ((s^2+s-1)A_1+A_2)\bar{P}_2\bar{Q}_1\}Z_3^{s-2}X^{\Delta_1-\Delta_2-s-3}. \quad \text{(C8)}\end{aligned}$$

Hence, we find that this three-point function is automatically compatible with conservation on x_3 for $\Delta_1 = \Delta_2$. For $\Delta_1 \neq \Delta_2$ it is simple to see that conservation is satisfied only for $s = 1$, which results in $A_1 = -A_2$ and, hence, the solution is unique. However, for $s > 1$ there is no solution in general. In the case where $\psi = \psi'$, we also have to impose the combined point-switch/reality condition, which results in the coefficients A_i being purely real/imaginary for s even/odd. This result is consistent with [16].

Now let us consider the $(s+1, s-1)$ representation, with $s > 1$. Note that the analysis for $(s-1, s+1)$ is essentially identical and will be omitted. Using the general formula, the ansatz for this three-point function is

$$\langle \psi_\alpha(x_1)\bar{\psi}'_{\dot\beta}(x_2)J_{\gamma(s+1)\dot\gamma(s-1)}(x_3)\rangle = \frac{\mathcal{I}_\alpha{}^{\dot\alpha'}(x_{13})\bar{\mathcal{I}}_{\dot\beta}{}^{\beta'}(x_{23})}{(x_{13}^2)^{\Delta_1}(x_{23}^2)^{\Delta_2}}\mathcal{H}_{\dot\alpha'\beta'\gamma(s+1)\dot\gamma(s-1)}(X_{12}). \quad \text{(C9)}$$

All information about this correlation function is encoded in the following polynomial:

$$\mathcal{H}(X;U,V,W) = \mathcal{H}_{\dot\alpha\beta\gamma(s+1)\dot\gamma(s-1)}(X)\boldsymbol{U}^{\dot\alpha}\boldsymbol{V}^{\beta}\boldsymbol{W}^{\gamma(s+1)\dot\gamma(s-1)}. \quad \text{(C10)}$$

We recall that $\mathcal{H}$ satisfies the homogeneity property $\mathcal{H}(X) = X^{s+2-\Delta_1-\Delta_2}\hat{\mathcal{H}}(X)$, where $\hat{\mathcal{H}}(X)$ is homogeneous degree 0. In this case there is only one possible structure for $\hat{\mathcal{H}}(X)$:

$$\hat{\mathcal{H}}(X;U,V,W) = AP_2\bar{Q}_1 Z_3^{s-1}, \quad \text{(C11)}$$

where A is a complex coefficient. After imposing conservation on x_3 using the methods outlined in Sec. II B 2, we find

$$\begin{aligned} D_3\tilde{\mathcal{H}}(X;U,V,W) &= A(\Delta_1-\Delta_2-1)(-1)^{s+1}(s^2+s-2) \\ &\quad \times \bar{P}_1\bar{P}_2 Z_3^{s-2}X^{\Delta_1-\Delta_2-s-3}. \quad \text{(C12)}\end{aligned}$$

Hence, we find that this three-point function is automatically compatible with conservation on x_3 for $\Delta_2 = \Delta_1 - 1$. For $\Delta_2 \neq \Delta_1 - 1$ there is no solution in general (recall that $s > 1$). This result is also consistent with [16].

[1] H. Osborn and A. C. Petkou, Implications of conformal invariance in field theories for general dimensions, Ann. Phys. (N.Y.) **231**, 311 (1994).
[2] J. Erdmenger and H. Osborn, Conserved currents and the energy momentum tensor in conformally invariant theories for general dimensions, Nucl. Phys. **B483**, 431 (1997).
[3] A. M. Polyakov, Conformal symmetry of critical fluctuations, JETP Lett. **12**, 381 (1970).
[4] E. J. Schreier, Conformal symmetry and three-point functions, Phys. Rev. D **3**, 980 (1971).
[5] A. A. Migdal, On hadronic interactions at small distances, Phys. Lett. **37B**, 98 (1971).
[6] A. A. Migdal, Conformal invariance and bootstrap, Phys. Lett. **37B**, 386 (1971).
[7] S. Ferrara, A. F. Grillo, and R. Gatto, Manifestly conformal-covariant expansion on the light cone, Phys. Rev. D **5**, 3102 (1972).
[8] S. Ferrara, A. F. Grillo, and R. Gatto, Tensor representations of conformal algebra and conformally covariant operator product expansion, Ann. Phys. (N.Y.) **76**, 161 (1973).

[9] K. Koller, The significance of conformal inversion in quantum field theory, Commun. Math. Phys. **40**, 15 (1974).
[10] G. Mack, Convergence of operator product expansions on the vacuum in conformal invariant quantum field theory, Commun. Math. Phys. **53**, 155 (1977).
[11] E. S. Fradkin and M. Y. Palchik, Recent developments in conformal invariant quantum field theory, Phys. Rep. **44**, 249 (1978).
[12] Y. S. Stanev, Stress—energy tensor and U(1) current operator product expansions in conformal QFT, Bulg. J. Phys. **15**, 93 (1988).
[13] Y. S. Stanev, Correlation functions of conserved currents in four dimensional conformal field theory, Nucl. Phys. **B865**, 200 (2012).
[14] Y. S. Stanev, Correlation functions of conserved currents in four dimensional conformal field theory with higher spin symmetry, Bulg. J. Phys. **40**, 147 (2013).
[15] Y. S. Stanev, Constraining conformal field theory with higher spin symmetry in four dimensions, Nucl. Phys. **B876**, 651 (2013).
[16] E. Elkhidir, D. Karateev, and M. Serone, General three-point functions in 4D CFT, J. High Energy Phys. 01 (2015) 133.
[17] E. I. Buchbinder and B. J. Stone, Three-point functions of a fermionic higher-spin current in 4D conformal field theory, Phys. Rev. D **105**, 125004 (2022).
[18] A. Zhiboedov, A note on three-point functions of conserved currents, arXiv:1206.6370.
[19] J. H. Park, $N = 1$ superconformal symmetry in four-dimensions, Int. J. Mod. Phys. A **13**, 1743 (1998).
[20] H. Osborn, $N = 1$ superconformal symmetry in four-dimensional quantum field theory, Ann. Phys. (N.Y.) **272**, 243 (1999).
[21] J.-H. Park, Superconformal symmetry in six-dimensions and its reduction to four-dimensions, Nucl. Phys. **B539**, 599 (1999).
[22] J. H. Park, Superconformal symmetry and correlation functions, Nucl. Phys. **B559**, 455 (1999).
[23] J.-H. Park, Superconformal symmetry in three-dimensions, J. Math. Phys. (N.Y.) **41**, 7129 (2000).
[24] S. M. Kuzenko and S. Theisen, Correlation functions of conserved currents in $N = 2$ superconformal theory, Classical Quantum Gravity **17**, 665 (2000).
[25] A. A. Nizami, T. Sharma, and V. Umesh, Superspace formulation and correlation functions of 3d superconformal field theories, J. High Energy Phys. 07 (2014) 022.
[26] E. I. Buchbinder, S. M. Kuzenko, and I. B. Samsonov, Superconformal field theory in three dimensions: Correlation functions of conserved currents, J. High Energy Phys. 06 (2015) 138.
[27] E. I. Buchbinder, S. M. Kuzenko, and I. B. Samsonov, Implications of $\mathcal{N} = 4$ superconformal symmetry in three spacetime dimensions, J. High Energy Phys. 08 (2015) 125.
[28] S. M. Kuzenko and I. B. Samsonov, Implications of $\mathcal{N} = 5$, 6 superconformal symmetry in three spacetime dimensions, J. High Energy Phys. 08 (2016) 084.
[29] E. I. Buchbinder and B. J. Stone, Mixed three-point functions of conserved currents in three-dimensional superconformal field theory, Phys. Rev. D **103**, 086023 (2021).
[30] E. I. Buchbinder, J. Hutomo, and S. M. Kuzenko, Correlation functions of spinor current multiplets in $\mathcal{N} = 1$ superconformal theory, J. High Energy Phys. 07 (2021) 165.
[31] E. I. Buchbinder, J. Hutomo, and S. M. Kuzenko, Three-point functions of higher-spin spinor current multiplets in $\mathcal{N} = 1$ superconformal theory, J. High Energy Phys. 10 (2021) 058.
[32] E. I. Buchbinder and B. J. Stone, Three-point functions of a superspin-2 current multiplet in 3D, $N = 1$ superconformal theory, Phys. Rev. D **104**, 106004 (2021).
[33] A. Jain and A. A. Nizami, Superconformal invariants and spinning correlators in 3d $\mathcal{N} = 2$ SCFTs, Eur. Phys. J. C **82**, 1065 (2022).
[34] E. I. Buchbinder, J. Hutomo, and G. Tartaglino-Mazzucchelli, Three-point functions of higher-spin supercurrents in 4D $N = 1$ superconformal field theory, Fortschr. Phys. **70**, 2200133 (2022).
[35] E. I. Buchbinder and B. J. Stone, Three-point functions of conserved supercurrents in 3D $N = 1$ SCFT: General formalism for arbitrary superspins, Phys. Rev. D **107**, 106001 (2023).
[36] E. I. Buchbinder and B. J. Stone, Grassmann-odd three-point functions of conserved supercurrents in 3D $\mathcal{N} = 1$ SCFT, Phys. Rev. D **108**, 046001 (2023).
[37] S. Giombi, S. Prakash, and X. Yin, A note on CFT correlators in three dimensions, J. High Energy Phys. 07 (2013) 105.
[38] J. Maldacena and A. Zhiboedov, Constraining conformal field theories with a higher spin symmetry, J. Phys. A **46**, 214011 (2013).
[39] S. Giombi, V. Gurucharan, V. Kirilin, S. Prakash, and E. Skvortsov, On the higher-spin spectrum in large N Chern-Simons vector models, J. High Energy Phys. 01 (2017) 058.
[40] S. Jain, R. R. John, A. Mehta, and D. K. S, Constraining momentum space CFT correlators with consistent position space OPE limit and the collider bound, J. High Energy Phys. 02 (2022) 084.
[41] S. Jain and R. R. John, Relation between parity-even and parity-odd CFT correlation functions in three dimensions, J. High Energy Phys. 12 (2021) 067.
[42] S. Jain, R. R. John, A. Mehta, A. A. Nizami, and A. Suresh, Higher spin 3-point functions in 3d CFT using spinor-helicity variables, J. High Energy Phys. 09 (2021) 041.
[43] E. I. Buchbinder and B. J. Stone, Three-point functions of conserved currents in 3D CFT: General formalism for arbitrary spins, Phys. Rev. D **107**, 046007 (2023).
[44] N. S. Craigie, V. K. Dobrev, and I. T. Todorov, Conformally covariant composite operators in quantum chromodynamics, Ann. Phys. (N.Y.) **159**, 411 (1985).
[45] P. S. Howe, K. S. Stelle, and P. K. Townsend, Supercurrents, Nucl. Phys. **B192**, 332 (1981).

[46] S. M. Kuzenko and E. S. N. Raptakis, Extended superconformal higher-spin gauge theories in four dimensions, J. High Energy Phys. 12 (2021) 210.

[47] S. M. Kuzenko and E. S. N. Raptakis, On higher-spin $\mathcal{N} = 2$ supercurrent multiplets, J. High Energy Phys. 05 (2023) 056.

[48] S. M. Kuzenko and E. S. N. Raptakis, Symmetries of supergravity backgrounds and supersymmetric field theory, J. High Energy Phys. 04 (2020) 133.

[49] S. M. Kuzenko, R. Manvelyan, and S. Theisen, Off-shell superconformal higher spin multiplets in four dimensions, J. High Energy Phys. 07 (2017) 034.

[50] I. Buchbinder and S. Kuzenko, *Ideas and Methods of Supersymmetry and Supergravity: Or a Walk Through Superspace* (Institute of Physics Publishing, Bristol and Philadelphia, 1998).