# Approximate symmetries, pseudo-Goldstones, and the second law of thermodynamics

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We propose a general hydrodynamic framework for systems with spontaneously broken approximate symmetries. The second law of thermodynamics naturally results in relaxation in the hydrodynamic equations and enables us to derive a universal relation between damping and diffusion of pseudo-Goldstones. We discover entirely new physical effects sensitive to explicitly broken symmetries. We focus on systems with approximate U(1) and translation symmetries, with direct applications to pinned superfluids and charge density waves. We also comment on the implications for chiral perturbation theory.

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Symmetry has proved to be a powerful organizational tool in physics for characterizing and classifying phases of matter. Knowledge about the symmetries of a physical system, and whether these are spontaneously broken by the low-energy ground state, is often sufficient to develop an effective theory describing its long-distance late-time behavior. Symmetries are useful even when they are only approximate. The canonical example of this is the extremely successful effective theory for pions as Goldstones of spontaneously broken SU(2) chiral symmetry. In this context, due to nonzero quark masses, the symmetry is only approximate and the effective theory can be systematically corrected to account for the pion mass.

In this paper, we draw general lessons about effective theories featuring this *pseudospontaneous* pattern of symmetry breaking. We are interested in physical systems where an approximate global symmetry is spontaneously broken by the low-energy ground state, which can be modeled by a slightly massive pseudo-Goldstone field $\phi(x)$. The explicitly broken symmetry also means that the associated Noether charge conservation is weakly violated, giving rise to physical effects such as relaxation, damping, and pinning. Pseudospontaneous symmetry breaking is common across the phase space of matter due to inherent defects, inhomogeneities, and impurities in materials. Examples include pinned crystals [1,2], charge density waves [3–6], pinned superfluids [7,8], electrons in graphene [9], pinned nematics [10,11], and pions in chiral perturbation theory [12–16], among many others.

In recent years, there have been several efforts toward developing hydrodynamic techniques for systems with spontaneously broken approximate symmetries, aimed at explaining experimental and holographic results; see, e.g., [5,6,17–27]. The goal of this paper is to formulate a complete hydrodynamic theory for thermal systems exhibiting pseudospontaneous symmetry breaking based on the second law of thermodynamics. A similar entropic construction appeared in [14,15] for chiral perturbation theory, however, the authors only focused on the pion mass and did not consider more general dissipative effects induced by explicitly broken SU(2) chiral symmetry.

The key accomplishment of our construction is to show that damping of pseudo-Goldstones and charge (or momentum) relaxation follow from the second law of thermodynamics. In particular, we derive the relation $\Omega = D_\phi k_0^2$ among the pseudo-Goldstone damping rate $\Omega$, attenuation $D_\phi$, and correlation length $1/k_0$, first noted in holographic models [18,20,22]. We emphasize that our derivation only relies on the second law; see [11] for a derivation using the Schwinger-Keldysh effective field theory or locality of the equations of motion [28].

Surprisingly, we find that dissipative effects also lead to certain new transport coefficients in the hydrodynamic theory that have not been identified in past literature. Most significantly, the well-known Josephson relation for a U(1) superfluid $\partial_t\phi = c_\phi\mu + \cdots$ [31–34], where $\mu$ is the chemical potential and $c_\phi$ is the charge of the condensate,

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modifies to $\partial_t\phi = \lambda c_\phi \mu + \cdots$ when the U(1) symmetry is explicitly broken. The charge renormalization factor $\lambda$ is a dissipative transport coefficient that is an artifact of explicit symmetry breaking. In particular, $\lambda$ is a physical parameter and can be measured by its own Kubo formula. Similar physics also arise in the context of pinned crystals, where the Josephson equation for the displacement field $\delta\phi^i$ takes the form $\partial_t\delta\phi^i = \lambda c_\phi u^i + \cdots$, where $u^i$ is the fluid velocity and $c_\phi$ signifies the inverse lattice spacing scale. Another such coefficient $\lambda_T$ enters the entropy/heat flux of pinned crystals at leading order as $s^i = (s + \lambda_T)u^i + \cdots$. Physically, the coefficients of this type result in a modification of the speed of sound dependent on the strength of explicit symmetry breaking.

## I. PINNED SIMPLE DIFFUSION

To highlight the striking features of our construction, we start with a simple toy model with a conserved density $n$ associated with a global U(1) symmetry. In a phase where the symmetry is spontaneously broken, the low-energy equilibrium configurations of the system can be described by the free energy $F = \int \mathrm{d}^d x(\mathcal{F}(\phi) - K_{\mathrm{ext}}\phi)$, written as a functional of the Goldstone field $\phi(x)$. Here $K_{\mathrm{ext}}$ is an external source coupled to $\phi$. The free energy obeys a shift symmetry $\phi(x) \to \phi(x) - c_\phi\Lambda$, where $c_\phi$ denotes the charge of the condensate. This forbids a mass term like $\mathcal{F} \sim \frac{1}{2}m^2\phi^2$ in the free energy, rendering the Goldstone massless.

The situation is qualitatively different when the U(1) symmetry is only approximate, because the shift symmetry need not be respected. In practice, we find it convenient to artificially manifest the symmetry by introducing a background field $\Phi(x)$ that transforms as $\Phi(x) \to \Phi(x) - \Lambda$. We can say that the background explicitly breaks the U(1) symmetry by picking out a preferred phase $\Phi$. We can now write a mass term in $\mathcal{F}$, i.e.,

$$\mathcal{F} = -p + \frac{1}{2}f_s\partial_i\phi\partial^i\phi + \frac{1}{2}\ell^2 m^2(\phi - c_\phi\Phi)^2, \qquad (1)$$

where $\ell$ is a bookkeeping parameter that controls the strength of explicit symmetry breaking. The thermodynamic pressure $p$ and superfluid density $f_s$ can generically depend on the thermodynamic parameters such as temperature and chemical potential. The free energy $F$ with (1) can be understood as a generalized Ginzburg-Landau model that accounts for explicit symmetry breaking with an arbitrary source $\Phi$; see, e.g., [35]. The mass term can be thought of as an "elastic potential" that tends to align the phase $\phi$ with the background phase $\Phi$. Varying (1) results in a configuration equation for $\phi$, i.e.,

$$f_s(\partial_i\partial^i\phi - k_0^2\phi) + c_\phi m^2\ell^2\Phi + K_{\mathrm{ext}} = 0, \qquad (2a)$$

where $k_0 = \ell m/\sqrt{f_s}$ is the finite inverse correlation length for $\phi$, demoting it to a massive pseudo-Goldstone. Typically, this massive field can be integrated out from the long-wavelength effective theory. However, if the symmetry is still approximately preserved, i.e., $\ell$ is sufficiently small, the pseudo-Goldstone can still affect the long-wavelength spectrum. The static Ward identity associated with the restored U(1) symmetry in (1) reads

$$\partial_i j^i = c_\phi K_{\mathrm{ext}} - c_\phi m^2\ell^2(\phi - c_\phi\Phi), \qquad (2b)$$

where $j^i = -c_\phi f_s\partial^i\phi$. This can be derived using the usual Noether procedure or coupling the system to an external U(1) gauge field; see the Appendix for further details. As expected, the mass term results in a violation of charge conservation even in the absence of external sources.

When we leave thermal equilibrium, we can no longer start with a free energy and must rely on the framework of hydrodynamics to proceed. First, we have a Josephson equation giving dynamics to $\phi$ which, generalizing (2a), we take to have the schematic form

$$K + K_{\mathrm{ext}} = 0, \qquad (3a)$$

for some yet-unknown operator $K$. We also have a U(1) conservation equation, generalizing (2b), describing the dynamics of charge density $n$, i.e.,

$$\partial_t n + \partial_i j^i = -c_\phi K - \ell L, \qquad (3b)$$

where $L$ is some operator causing explicit symmetry breaking. We will also need a new energy conservation equation implementing the first law of thermodynamics,

$$\partial_t\epsilon + \partial_i\epsilon^i = -K\partial_t\phi - \ell L\partial_t\Phi, \qquad (3c)$$

where $\epsilon, \epsilon^i$ are the energy density and flux, respectively. A derivation of these conservation laws can be found in the Appendix. To complete these equations, we must give a set of constitutive relations for $j^i, \epsilon^i, K, L$ in terms of $n, \epsilon, \phi, \Phi$, arranged order by order in gradients. We implement the gradient counting scheme where $\phi \sim \mathcal{O}(\partial^{-1})$, making its gradients $\mathcal{O}(\partial^0)$; see [36,37]. We ascribe the scaling $\mathcal{O}(\partial)$ to the symmetry breaking parameter $\ell$ and require that all dependence on the background phase $\Phi$ must appear with a factor of $\ell$, so that setting $\ell = 0$ restores the symmetry. It is also convenient to define the phase misalignment $\psi = \ell(\phi - c_\phi\Phi) \sim \mathcal{O}(\partial^0)$.

An important ingredient in hydrodynamics is the local second law of thermodynamics. It necessitates the existence of an entropy density $s^t$ and flux $s^i$ such that

$$\partial_t s^t + \partial_i s^i \geq 0, \qquad (4)$$

is satisfied for *every* solution of the conservation equations; see [38]. Despite being an inequality, this requirement is extremely powerful and is known to give strong constraints on the constitutive relations [39]. At leading order in gradients, we simply have $s^t = s(\epsilon, n, \partial_i\phi\partial^i\phi, \psi)$. We can define the temperature $T$, chemical potential $\mu$, "superfluid density" $f_s$, pseudo-Goldstone mass parameter $m$, and the grand-canonical free-energy density $\mathcal{F}$ via

$$T\mathrm{d}s = \mathrm{d}\epsilon - \mu\mathrm{d}n - \frac{1}{2}f_s\mathrm{d}(\partial_i\phi\partial^i\phi) - m^2\psi\mathrm{d}\psi,$$
$$\mathcal{F} = \epsilon - Ts^t - \mu n. \tag{5}$$

The entropy density will, in general, admit gradient corrections. We consider these in the Appendix.

For clarity, let us assume the dynamics to be isothermal, i.e., $T = T_0$, so that energy conservation decouples from the charge conservation and Josephson equations. In this case, the second law constraints result in

$$j^i = -c_\phi f_s\partial^i\phi - \sigma_n\partial^i\mu,$$
$$K = \partial_i(f_s\partial^i\phi) - \ell m^2\psi - \sigma_\phi(\partial_t\phi - c_\phi\mu) - \ell\sigma_\times(\partial_t\Phi - \mu),$$
$$L = c_\phi m^2\psi - \ell\sigma_\Phi(\partial_t\Phi - \mu) - \sigma_\times(\partial_t\phi - c_\phi\mu). \tag{6}$$

Setting $\mu = \mu_0$ and $\partial_t\phi/c_\phi = \partial_t\Phi = \mu_0$, at leading order in gradients, we recover the equilibrium version of these equations derived from (1). We also find four dissipative coefficients $\sigma_n, \sigma_\phi, \sigma_\Phi, \sigma_\times$ that satisfy the inequality relations $\sigma_n, \sigma_\phi \geq 0$ and $\sigma_\Phi \geq \sigma_\times^2/\sigma_\phi$. We provide a detailed derivation in the Appendix.

## II. LINEARIZED FLUCTUATIONS

To highlight the physical implication of this model, we set $\Phi = \mu_0 t$, $K_{\text{ext}} = 0$, and linearly expand the equations around the solution $\mu = \mu_0$, $\phi = c_\phi\mu_0 t$. We find

$$j^i = -c_\phi f_s\partial^i\delta\phi - D_n\partial^i\delta n,$$
$$\partial_t\delta\phi = \frac{\lambda c_\phi}{\chi}\delta n - \Omega\delta\phi + D_\phi\partial_i\partial^i\delta\phi,$$
$$\ell L = \frac{\chi}{\lambda c_\phi}\omega_0^2\delta\phi + \Gamma\delta n + c_\phi(1-\lambda)f_s\partial_i\partial^i\delta\phi. \tag{7a}$$

We have defined the susceptibility $\chi$, sound speed $v_s$, pinning frequency $\omega_0$, charge attenuation $D_n$, charge relaxation $\Gamma$, pseudo-Goldstone attenuation $D_\phi$, damping $\Omega$, and a new coefficient $\lambda$ as

$$\chi = \frac{\partial n}{\partial\mu}, \qquad v_s^2 = \lambda^2 c_\phi^2\frac{f_s}{\chi}, \qquad \omega_0^2 = \lambda^2 c_\phi^2\frac{\ell^2 m^2}{\chi},$$
$$D_n = \frac{\sigma_n}{\chi}, \qquad \Gamma = \frac{\ell^2}{\chi}\left(\sigma_\Phi - \frac{\sigma_\times^2}{\sigma_\phi}\right),$$
$$D_\phi = \frac{f_s}{\sigma_\phi}, \qquad \Omega = \frac{\ell^2 m^2}{\sigma_\phi}, \qquad \lambda = 1 + \frac{\ell\sigma_\times}{c_\phi\sigma_\phi}. \tag{7b}$$

Solving the equations and assuming $\ell \sim \mathcal{O}(k)$, we find a damped sound mode with dispersion relations

$$\omega = \pm\sqrt{\omega_0^2 + v_s^2k^2} - \frac{i}{2}(k^2(D_n + D_\phi) + \Gamma + \Omega). \tag{8}$$

The second law constraints imply that $D_n, D_\phi, \Omega, \Gamma \geq 0$, ensuring that the sound mode remains stable.

From (7), it is possible to make a few interesting observations. For instance, we have proved the damping-attenuation relation

$$\Omega = D_\phi k_0^2, \tag{9}$$

where $k_0 = \omega_0/v_s$ [40]. We also see that our model naturally gives rise to charge relaxation $\Gamma$, without needing to introduce it by hand. Interestingly, we also find a new coefficient $\lambda \neq 1$ appearing in front of the $\delta n$ term in the Josephson equation. It renormalizes the Goldstone charge $c_\phi$ and modifies the speed of sound in the presence of small explicit symmetry breaking.

Since $\lambda$ renormalizes the charge $c_\phi$, it might be tempting to brush it off as unphysical by rescaling $c_\phi \to c_\phi/\lambda$. However, the flux $j^i$ in (7) depends on the bare charge $c_\phi$ directly and hence retains information about $\lambda$ following the rescaling. This can be culminated into an independent Kubo formula for $\lambda$, i.e.,

$$\lambda^2 = -v_s^2\frac{G^R_{nn}(\omega = 0, k = 0)}{G^R_{j^xj^x}(\omega = 0, k = 0)}, \tag{10}$$

where $v_s$ is read off using the singularity structure of the dispersion relations (8). Therefore, $\lambda$ is a physical observable for a U(1) superfluid with explicit symmetry breaking. More details regarding the correlation functions can be found in the Appendix.

This discussion can be extended to account for temperature and momentum fluctuations, leading to a theory of explicitly broken superfluids. This theory was recently considered in the holographic context in [8]. It will be interesting to revisit their results in the view of our new $\lambda$ coefficient, along with other similar coefficients that can appear in the energy flux and stress tensor. We will discuss this in more detail in another publication.

## III. PINNED VISCOELASTIC CRYSTALS

The hydrodynamic theory for pinned crystals can be constructed similar to the U(1) case. In $d$ spatial dimensions, a static configuration of a crystal can be described by the spatial distribution of its lattice sites $\phi^{I=1,\ldots,d}(x)$, called the "crystal fields." We can define the strain tensor as $u_{IJ} = \frac{1}{2}(h_{IJ} - \delta_{IJ}/c_\phi^2)$, where $h_{IJ}$ is the inverse of $h^{IJ} = \partial^i\phi^I\partial_i\phi^J$ and $c_\phi$ is a constant parametrizing the "inverse lattice spacing."

When the crystal is homogeneous, the theory obeys a global spatial shift symmetry $\phi^I(x) \to \phi^I(x) + c_\phi a^I$, and $\phi^I$ can be understood as Goldstones of spontaneously broken translations. However, when the crystal has slight inhomogeneities, possibly due to defects or impurities, this shift symmetry can be violated. Analogous to the U(1) case, we artificially manifest the symmetry by introducing a set of background fields $\Phi^I(x)$, shifting as $\Phi^I(x) \to \Phi^I(x) + a^I$. In the present case, $\Phi^I(x)$ can be interpreted as describing the spatial configuration of a fixed background lattice coupled to our physical crystal of interest. This allows us to introduce a mass term in the free-energy density

$$\mathcal{F} = -p + \frac{1}{2}\left(B - \frac{2}{d}G\right)(u^I{}_I)^2 + G u^{IJ} u_{IJ} + \frac{m^2}{2}\psi_I\psi^I, \tag{11}$$

where $\psi^I = \ell(\phi^I - c_\phi \Phi^I)$ is the "misalignment tensor." Here $p$ is the thermodynamic pressure, while $B$ and $G$ are bulk and shear moduli, respectively; all these coefficients can arbitrarily depend on the thermodynamic parameters such as temperature and chemical potential. $I, J, \ldots$ indices are raised/lowered using $h^{IJ}$, $h_{IJ}$.

To describe the dynamical evolution of this system, we need to formulate the theory of pinned viscoelastic hydrodynamics following the construction of [36,37]. First, analogous to (3a), we have a set of Josephson equations for the crystal fields

$$K_I + K_I^{\text{ext}} = 0, \tag{12a}$$

where $K_I$ is an unknown operator and $K_I^{\text{ext}}$ are sources coupled to $\phi^I$. Assuming the crystal to exhibit Galilean symmetry, we also have momentum conservation and continuity equations

$$\begin{aligned} &\partial_t \pi^i + \partial_j \tau^{ij} = K_I \partial^i \phi^I + \ell L_I \partial^i \Phi^I, \\ &\partial_t \rho + \partial_i \pi^i = 0, \end{aligned} \tag{12b}$$

where $\pi^i$ is the momentum density, $\tau^{ij}$ is the stress tensor, $\rho$ is the mass density, and $L_I$ is an operator causing explicitly broken translations. These have to be supplemented with the energy conservation equation arising from the first law of thermodynamics,

$$\partial_t \epsilon + \partial_i \epsilon^i = -K_I \partial_t \phi^I - \ell L_I \partial_t \Phi^I. \tag{12c}$$

We can now proceed and derive a set of constitutive relations for $\tau^{ij}, \epsilon^i, K_I, L_I$ in terms of $\pi^i, \epsilon, \rho, \phi^I, \Phi^I$, arranged in a gradient expansion, and obtain constraints due to the second law of thermodynamics.

At leading order in gradients, the entropy density is given by $s^t = s(\varepsilon, \rho, h^{IJ}, \psi^I)$, where $\varepsilon = \epsilon - \frac{1}{2}\rho \vec{u}^2$ is the "internal energy density" and $u^i = \pi^i/\rho$ is the fluid velocity. We can define the temperature $T$, chemical potential $\mu$, elastic stress tensor $r_{IJ}$, pseudo-Goldstone mass $m$, and free-energy $\mathcal{F}$ via

$$\begin{aligned} T\mathrm{d}s &= \mathrm{d}\varepsilon - \mu \mathrm{d}\rho + \frac{1}{2} r_{IJ} \mathrm{d}h^{IJ} - m^2 \psi_I \mathrm{d}\psi^I, \\ \mathcal{F} &= \varepsilon - T s^t - \mu\rho. \end{aligned} \tag{13}$$

The entropy density can also admit first order gradient corrections, which we consider in detail in the Appendix.

Similar to the U(1) case, restricting to an isothermal regime, i.e., $T = T_0$, energy conservation decouples and we obtain the allowed set of constitutive relations,

$$\begin{aligned} \tau^{ij} &= \rho u^i u^j - \mathcal{F}\delta^{ij} - r_{IJ} e^{Ii} e^{Ij} - 2\eta \partial^{\langle i} u^{j\rangle} - \zeta \partial_k u^k \delta^{ij}, \\ K_I &= -\partial_i (r_{IJ} e^{Ji}) - \ell m^2 \psi_I - \sigma_\phi h_{IJ} \frac{\mathrm{d}\phi^J}{\mathrm{d}t} - \ell \sigma_\times h_{IJ} \frac{\mathrm{d}\Phi^J}{\mathrm{d}t}, \\ L_I &= c_\phi m^2 \psi_I - \ell \sigma_\Phi h_{IJ} \frac{\mathrm{d}\Phi^J}{\mathrm{d}t} - \sigma_\times h_{IJ} \frac{\mathrm{d}\phi^J}{\mathrm{d}t}, \end{aligned} \tag{14}$$

where $e_i^I = \partial_i \phi^I$, $\mathrm{d}/\mathrm{d}t = \partial_t + u^i \partial_i$, and angular brackets denote a symmetric-traceless combination of indices. The five dissipative coefficients $\eta, \zeta, \sigma_\phi, \sigma_\Phi, \sigma_\times$ follow the inequalities $\eta, \zeta, \sigma_\phi \geq 0$ and $\sigma_\Phi \geq \sigma_\times^2/\sigma_\phi$. A detailed derivation relaxing the isothermal assumption appears in the Appendix.

## IV. LINEAR PINNED CRYSTALS

In the small strain regime, the equation of state of the crystal can be written as (11), except that $c_\phi$ in the definition of $u_{IJ}$ should be replaced by a thermodynamic coefficient $\alpha(T,\mu)$, such that $\alpha(T_0,\mu_0) = c_\phi$. The thermodynamic derivatives of $\alpha$ play an important role as expansion coefficients [37]. Setting $\Phi^I = x^I$ and $K_I^{\text{ext}} = 0$, and expanding around $\phi^I = c_\phi x^I$, $\mu = \mu_0$, $u^i = 0$, we can obtain

$$\begin{aligned} \tau^{ij} &= (p + B\alpha_m \delta\mu)\delta^{ij} - 2\eta \partial^{\langle i} u^{j\rangle} - \zeta \partial_k u^k \delta^{ij} \\ &\quad - \frac{B}{c_\phi} \partial_k \delta\phi^k \delta^{ij} - \frac{2G}{c_\phi} \partial^{\langle i} \delta\phi^{j\rangle}, \\ \partial_t \delta\phi^i &= \lambda c_\phi u^i - \Omega \delta\phi^i + \gamma_m \partial^i \mu \\ &\quad + 2D_\phi^\perp \partial_k \partial^{[k} \delta\phi^{i]} + D_\phi^\parallel \partial^i \partial_k \delta\phi^k, \\ \ell L^i &= -\frac{\rho}{\lambda c_\phi} \omega_0^2 \delta\phi^i - \Gamma \pi^i - (\lambda - 1) B\alpha_m \partial^i \mu \\ &\quad + \frac{\lambda - 1}{c_\phi} (B \partial^i \partial_k \delta\phi^k + 2G \partial_k \partial^{\langle j} \delta\phi^{i\rangle}), \end{aligned} \tag{15a}$$

where $\delta\phi^i = -\delta_I^i \delta\phi^I$. We have defined pinning frequency $\omega_0$, mass expansion coefficient $\alpha_m$, pseudo-Goldstone attenuation $D_\phi^{\perp,\parallel}$, damping $\Omega$, momentum relaxation $\Gamma$, and coefficients $\lambda, \gamma_m$ as

$$\omega_0^2 = \lambda^2 \frac{\ell^2 m^2}{\rho}, \qquad \alpha_m = -d \frac{\partial \ln \alpha}{\partial \mu}, \qquad \gamma_m = -c_\phi \frac{B\alpha_m}{\sigma_\phi},$$
$$D_\phi^\perp = \frac{G}{\sigma_\phi}, \qquad D_\phi^\parallel = \frac{B + 2\frac{d-1}{d} G}{\sigma_\phi}, \qquad \Omega = \frac{\ell^2 m^2}{\sigma_\phi},$$
$$\Gamma = \frac{\ell^2}{\rho c_\phi^2}\left(\sigma_\Phi - \frac{\sigma_\times^2}{\sigma_\phi}\right), \qquad \lambda = 1 + \frac{\ell \sigma_\times}{c_\phi \sigma_\phi}. \tag{15b}$$

Looking at the mode spectrum, we obtain a damped sound mode in the longitudinal and transverse sectors similar to (8). We also find a crystal diffusion mode in the longitudinal sector. These are given in the Appendix. Lifting the isothermal assumption leads to an energy diffusion mode coupled with crystal diffusion; see, e.g., [37].

Analogous to the U(1) case, using (15) we recover the damping-attenuation relation from [18,20,22]

$$\Omega = D_\phi^\perp k_0^2, \tag{16}$$

where $k_0 = \omega_0/v_\perp$ with $v_\perp^2 = \lambda^2 G/\rho$. The momentum relaxation $\Gamma$ also arises naturally in our model, along with the coefficient $\lambda$ affecting the Josephson equation. Upon including thermal fluctuations, we find another damping-attenuation relation similar to (16) in the energy flux. We also find a new pinning-sensitive coefficient $\lambda_T$ in the energy flux; see the Appendix for more details.

## V. DISCUSSION

In this paper we introduced a general hydrodynamic framework for dissipative systems with spontaneously broken approximate symmetries. Our construction builds upon the technology of forced fluid dynamics from [41,42], by systematically coupling the hydrodynamic equations to pseudo-Goldstone fields $\phi(x)$ and fixed background phase fields $\Phi(x)$, responsible for spontaneously and explicitly breaking the symmetries, respectively [43]. We illustrated how the interplay between the two field ingredients gives rise to physical effects such as damping, pinning, and relaxation. In particular, we showed that the elusive relation between the damping $\Omega$ and attenuation $D_\phi$ of pseudo-Goldstones follows simply by imposing the second law of thermodynamics in the presence of background fields $\Phi(x)$. The second law also requires the relaxation coefficient $\Gamma$ to be non-negative.

In addition to providing a rigorous mathematical language for systems with pseudospontaneously broken symmetries, we also found entirely new physical effects that have not been discussed in previous literature. Namely, we discovered new transport coefficients sensitive to the explicit nature of symmetry breaking that modify the hydrodynamic and Josephson equations at the thermodynamic level. These coefficients result in a modification of the speed of the damped sound mode and affect the hydrodynamic correlators in a nontrivial way.

We primarily focused on systems with approximate U(1) or approximate spatial translation symmetry. However, the framework developed here is equally relevant for other physical situations exhibiting a pseudospontaneous pattern of symmetry breaking. For instance, a hydrodynamic theory for pions recently appeared in [14], featuring a pseudospontaneously broken SU(2) chiral symmetry [12,13,49]. In particular, [14] noted that the damping-attenuation relation (9) for pions follows from the second law of thermodynamics. However, their analysis does not include additional pinning-sensitive coefficients such as $\lambda$. It is straightforward to generalize the U(1) case analyzed here to an SU(2) pion field $\phi^a$ coupled to a fixed background SU(2) phase $\Phi^a$, where $a, b, \ldots$ denote SU(2) Lie algebra indices. The linearized Josephson equation will take the schematic form

$$\partial_t \delta\phi^a = \lambda^a{}_b \delta\mu^b - \Omega^a{}_b \delta\phi^b + D_\phi{}^a{}_b \partial_i \partial^i \delta\phi^b, \tag{17}$$

with the damping-attenuation relation $\Omega^a{}_b = D_\phi{}^a{}_b k_0^2$. The coefficient $\lambda^a{}_b$ is equal to $\delta^a_b$ in the absence of explicit symmetry breaking, but can acquire corrections when the symmetry is weakly broken, modifying the mode spectrum of chiral perturbation theory.

Holographic models with pseudospontaneous pattern of symmetry breaking have been discussed in multiple works; see, e.g., [18–27]. It would be interesting to develop the relativistic versions of the hydrodynamic theories formulated in this paper and revisit their holographic applications in light of the new transport coefficients that we have identified. We leave this direction for future work.

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## APPENDIX

### 1. Details of pinned simple diffusion

In this appendix we provide details of the second law analysis in the pinned simple diffusion model, including the coupling of conserved currents to external sources. We do not assume the system to be isothermal as in the bulk of the paper.

*Static Ward identities.*—We start with a quick derivation of the static U(1) Ward identity (2b). We can gauge the restored U(1) symmetry in the free energy (1) by introducing a spatial gauge field $A_i$ transforming as usual, i.e. $A_i \to A_i + \partial_i \Lambda$. The gauged free energy is given by

$$\mathcal{F} = -p + \frac{1}{2} f_s \xi_i \xi^i + \frac{1}{2} m^2 \psi^2, \tag{A1}$$

where $\xi_i = \partial_i \phi + c_\phi A_i$. The U(1) flux $j^i$ can be read off as the response of the free energy $F = \int \mathrm{d}^d x (\mathcal{F} - K_{\text{ext}} \phi)$ to fluctuations in the background gauge field, $j^i = -\delta F / \delta A_i$, which reduces to $-c_\phi f_s \partial^i \phi$ used in (2b) when the gauge field is switched off.

More generally, we can parametrise the infinitesimal variation of the free energy density $\mathcal{F}$ as

$$\delta \int \mathrm{d}^d x \mathcal{F} = -\int \mathrm{d}^d x (j^i \delta A_i + K \delta \phi + \ell L \delta \Phi), \tag{A2}$$

which defines the operators $K$ and $L$. Accounting for the source term $K_{\text{ext}} \phi$, it immediately follows that the configuration equation for $\phi$ is simply

$$K + K_{\text{ext}} = 0. \tag{A3}$$

The Ward identity (2b) follows from requiring the variation (A2) to vanish under a symmetry transformation $\delta A_i = \partial_i \Lambda$, $\delta \phi = -c_\phi \Lambda$, and $\delta \Phi = -\Lambda$. This results in

$$\partial_i j^i = -c_\phi K - \ell L, \tag{A4}$$

which leads to (2b) after using the configuration equations and substituting the explicit form of $L$.

*Conservation laws.*—Next, let us derive the conservation laws (3) in the presence of explicit symmetry breaking, in particular the energy conservation equation that is absent in equilibrium. Let us consider that our system of interest is described by an effective action $S[\phi, \Phi]$. We couple the action to the external gauge field $A_t$, $A_i$ to gauge the U(1) symmetry. Using this, we can define the gauge covariant derivatives of $\phi$ and $\Phi$ as

$$\begin{aligned} \xi_t &= \partial_t \phi + c_\phi A_t, \qquad & \xi_i &= \partial_i \phi + c_\phi A_i, \\ \Xi_t &= \partial_t \Phi + A_t, \qquad & \Xi_i &= \partial_i \Phi + A_i. \end{aligned} \tag{A5}$$

We also introduce the "clock form" $n_t$, $n_i$ to gauge the time-translation symmetry. The clock form is the non-relativistic analogue of the time-component of the relativistic metric field $g_{tt}$, $g_{ti}$; see e.g. [50,51] for a detailed discussion. In flat space, the clock form takes the values $n_t = 1$, $n_i = 0$.

These external sources couple to the charge, energy densities and the respective fluxes via the action variation

$$\begin{aligned} \delta S = \int \mathrm{d}t \mathrm{d}^d x (& n \delta A_t + j^i \delta A_i - \epsilon \delta n_t - \epsilon^i \delta n_i \\ & + K \delta \phi + \ell L \delta \Phi). \end{aligned} \tag{A6}$$

We have included the variation with respect to $\phi$ and $\Phi$ for completeness, which defines the non-equilibrium versions of the operators $K$ and $L$. The action is required to be invariant under infinitesimal gauge transformations

$$\begin{aligned} \delta A_t &= \partial_t \Lambda, \qquad & \delta A_i &= \partial_i \Lambda, \\ \delta \phi &= -c_\phi \Lambda, \qquad & \delta \Phi &= -\Lambda. \end{aligned} \tag{A7a}$$

It is also required to be invariant under infinitesimal time-translations parametrised by some parameter $\chi^t$, i.e.

$$\begin{aligned} \delta A_t &= \chi^t \partial_t A_t + A_t \partial_t \chi^t, \\ \delta A_i &= \chi^t \partial_t A_i + A_t \partial_i \chi^t, \\ \delta n_t &= \chi^t \partial_t n_t + n_t \partial_t \chi^t, \\ \delta n_i &= \chi^t \partial_t n_i + n_t \partial_i \chi^t, \\ \delta \phi &= \chi^t \partial_t \phi, \\ \delta \Phi &= \chi^t \partial_t \Phi. \end{aligned} \tag{A7b}$$

These are nothing but Lie derivatives of the respective fields along the diffeomorphism vector $\chi^t \partial_t$. The invariance of $S$ under the gauge transformations (A7) result in the U(1) conservation equation

$$\partial_t n + \partial_i j^i = -c_\phi K - \ell L. \tag{A8}$$

On the other hand, the invariance under (A7b) results in the energy conservation equation

$$\partial_t \epsilon + \partial_i \epsilon^i = E_i j^i - K \xi_t - \ell L \Xi_t, \tag{A9}$$

where we have identified the background electric field $E_i = \partial_i A_t - \partial_t A_i$, and tuned the background to flat spacetime by setting $n_t = 1$, $n_i = 0$.

Finally, adding the source term $K_{\text{ext}} \phi$ to the action, the classical equation of motion for $\phi$ reads

$$K + K_{\text{ext}} = 0. \tag{A10}$$

*Second law constraints.*—Let us parametrise the entropy density as

$$s^t = s + \mathcal{S}, \tag{A11}$$

where $\mathcal{S}$ represent the possible gradient corrections. Using the thermodynamic relations (5) and the conservation equations, we find

$$\partial_t s^t + \partial_i s^i = -\frac{1}{T^2}\mathcal{E}^i\partial_i T - \mathcal{J}^i\left(\partial_i\frac{\mu}{T} - \frac{E_i}{T}\right) - \frac{1}{T}\mathcal{K}(\xi_t - c_\phi\mu) - \frac{\ell}{T}\mathcal{L}(\Xi_t - \mu) + \partial_t\mathcal{S} + \partial_i\mathcal{S}^i, \tag{A12}$$

where we have identified the constitutive relations

$$\begin{aligned}
\epsilon^i &= -f_s\xi^i\xi_t + \mathcal{E}^i,\\
j^i &= -c_\phi f_s\xi^i + \mathcal{J}^i,\\
K &= \partial_i(f_s\xi^i) - \ell m^2\psi + \mathcal{K},\\
L &= c_\phi m^2\psi + \mathcal{L},\\
s^i &= \frac{1}{T}\mathcal{E}^i - \frac{\mu}{T}\mathcal{J}^i + \mathcal{S}^i.
\end{aligned} \tag{A13}$$

The right-hand side of (A12) is required to be a positive semi-definite quadratic form. Truncating at first order in gradients, there are two kinds of solutions to (A12). First, we have the "non-hydrostatic sector", where $\mathcal{S}$, $\mathcal{S}^i$ are identically zero and we simply have

$$\begin{pmatrix} \frac{1}{T}\mathcal{E}^i_{\rm nhs}\\ \mathcal{J}^i_{\rm nhs}\\ \mathcal{K}_{\rm nhs}\\ \mathcal{L}_{\rm nhs}\end{pmatrix} = -\begin{pmatrix} \frac{1}{T}\kappa^{ij} & \gamma^{ij} & \gamma^i_{\epsilon\phi} & \gamma^i_{\epsilon\Phi}\\ \gamma'^{ij} & \sigma^{ij}_n & \gamma^i_{n\phi} & \gamma^i_{n\Phi}\\ \gamma'^i_{\epsilon\phi} & \gamma'^i_{n\phi} & \sigma_\phi & \sigma_\times\\ \gamma'^i_{\epsilon\Phi} & \gamma'^i_{n\Phi} & \sigma'_\times & \sigma_\Phi\end{pmatrix}\begin{pmatrix}\partial_j T\\ T\partial_j\frac{\mu}{T} - E_i\\ \xi_t - c_\phi\mu\\ \ell(\Xi_t - \mu)\end{pmatrix}. \tag{A14}$$

The objects appearing in the matrix here have to be constructed out of the zero-gradient structures $\delta^{ij}$, $\epsilon^{ij\cdots}$, and $\xi^i$, supplemented with coefficients that are arbitrary functions of $T$, $\mu$, $\partial^i\phi\partial_i\phi$, $\psi$. If we were only interested in the terms that contribute linearly to the constitutive relations, we can ignore any dependence on $\xi^i$, $\partial^i\phi\partial_i\phi$, and $\psi$. Further imposing parity-symmetry, we have the allowed coefficients

$$\begin{aligned}
&\kappa^{ij} = \kappa\delta^{ij}, \quad \sigma^{ij}_n = \sigma_n\delta^{ij}, \quad \gamma^{ij} = \gamma\delta^{ij}, \quad \gamma'^{ij} = \gamma'\delta^{ij},\\
&\sigma_\phi, \quad \sigma_\Phi, \quad \sigma_\times, \quad \sigma'_\times,
\end{aligned} \tag{A15}$$

while all vector coefficients vanish. All coefficients are functions of $T$ and $\mu$. Onsager's reciprocity relations [52,53] further impose the off-diagonal coefficients to be the same, $\gamma' = \gamma$ and $\sigma'_\times = \sigma_\times$. The entropy production rate (or the dissipative function) can be obtained by substituting (A14) into (A12); we find

$$\begin{aligned}
\partial_t s^t + \partial_i s^i = {}& \frac{\kappa}{T^2}\partial_i T\partial^i T + 2\gamma\partial_i T\left(\partial_i\frac{\mu}{T} - \frac{E_i}{T}\right)\\
&+ T\sigma_n\left(\partial_i\frac{\mu}{T} - \frac{E_i}{T}\right)\left(\partial^i\frac{\mu}{T} - \frac{E^i}{T}\right)\\
&+ \frac{\sigma_\phi}{T}(\xi_t - c_\phi\mu)^2 + \frac{2\ell\sigma_\times}{T}(\xi_t - c_\phi\mu)(\Xi_t - \mu)\\
&+ \frac{\ell^2\sigma_\Phi}{T}(\Xi_t - \mu)^2 \geq 0.
\end{aligned} \tag{A16}$$

The second law results in the inequality relations

$$\kappa, \sigma_\phi \geq 0, \qquad \sigma_n \geq \gamma^2/\kappa, \qquad \sigma_\Phi \geq \sigma_\times^2/\sigma_\phi. \tag{A17}$$

In addition, we have the "hydrostatic sector" that does not contribute to entropy production. It is characterised by corrections to the entropy density

$$\mathcal{S} = f_1\partial_i\xi^i - \frac{\ell\bar{f}_s}{T}\xi^i\Xi_i. \tag{A18}$$

In equilibrium, these terms show up as corrections to the free energy density $\mathcal{F}$ given in (1). The factor of $\ell$ in front of $\bar{f}_s$ is necessary because all dependence on $\Phi$ must be expressible as a combination involving $\psi$. Indeed $c_\phi\ell\Xi_i = \ell\xi_i - \partial_i\psi$. The coefficient $\bar{f}_s$ characterises the response of the system due to a background superfluid velocity due to the presence of $\Phi$. Note that we could include another similar term in the entropy/free energy density that goes as

$$\mathcal{S} \sim -\frac{\ell^2\bar{f}'_s}{T}\Xi^i\Xi_i. \tag{A19}$$

However this term comes with two powers of $\ell$, one for each occurrence of $\Phi$, and hence is counted at second derivative order in our counting scheme. Hence, we will drop it in our following discussion.

If we only focus on linear corrections to the constitutive relations, (A12) means that we only need to consider the entropy density up to quadratic order in fields. Therefore, without loss of generality, we can take the coefficient $\bar{f}_s$ to be constant, while $f_1$ can be taken to be a linear function of $\epsilon$ and $n$. Plugging (A18) into (A12), we derive the hydrostatic constitutive relations

$$\begin{aligned}
\mathcal{E}^i_{\rm hs} &= -c_\phi T\mu\partial_i f_1 - \ell\mu\bar{f}_s(\xi^i + c_\phi\Xi^i)\\
\mathcal{J}^i_{\rm hs} &= -c_\phi T\partial_i f_1 - \ell\bar{f}_s(\xi^i + c_\phi\Xi^i)\\
\mathcal{K}_{\rm hs} &= T\partial_i\partial^i f_1 + c_\phi T\sigma_\phi\left(\mu\frac{\partial f_1}{\partial\epsilon} + \frac{\partial f_1}{\partial n}\right)\partial_i\xi^i + \ell\bar{f}_s\partial_i\Xi^i,\\
\mathcal{L}_{\rm hs} &= \bar{f}_s\partial_i\xi^i,
\end{aligned} \tag{A20}$$

along with

$$\mathcal{S}^i = -f_1\partial_t\xi^i + \left(\partial_i f_1 + \frac{\ell\bar{f}_s}{T}\Xi_i\right)(\xi_t - c_\phi\mu) + \frac{\ell\bar{f}_s}{T}\xi_i(\Xi_t - \mu), \tag{A21}$$

where we have used the first-order equations of motion

$$\partial_t\epsilon = c_\phi\mu\sigma_\phi(\xi_t - c_\phi\mu), \qquad \partial_t n = c_\phi\sigma_\phi(\xi_t - c_\phi\mu). \tag{A22}$$

Demanding $\mathcal{S}$ to be invariant under time-reversal symmetry, the coefficient $f_1$ is not allowed. Additionally, it can be checked that the coefficient $\bar{f}_s$ does not contribute to the linearised equations of motion, when coupled to a homogeneous background $\Phi = \mu_0 t$. Note also that, linearly, these coefficients can be removed by a redefinition of the pseudo-Goldstone field $\phi \to \phi + (Tf_1 + \bar{f}_s\psi/c_\phi)/f_s$ and are only physical if one has an unambiguous macroscopic notion of the pseudo-Goldstone field. For these reasons, we have not considered these coefficients in the remainder of our discussion.

*Modes.*—Focusing on isothermal fluctuations and employing the definitions in (7), we can obtain the damped sound modes

$$\omega = \pm\sqrt{\omega_0^2 + v_s^2k^2 - \frac{1}{4}\left(\Gamma - \Omega + (D_n - D_\phi)k^2\right)^2} - \frac{i}{2}\left(k^2(D_n + D_\phi) + \Gamma + \Omega\right). \tag{A23}$$

Expanding this expression for $k^2 \ll 1$, we find

$$\omega = -\frac{i}{2}(\Gamma+\Omega) \pm \sqrt{\omega_0^2 - \frac{1}{4}(\Gamma-\Omega)^2} - \frac{i}{2}k^2\left(D_n + D_\phi \pm i\frac{v_s^2 - \frac{1}{2}(\Gamma-\Omega)(D_n - D_\phi)}{\sqrt{\omega_0^2 - \frac{1}{4}(\Gamma-\Omega)^2}}\right). \tag{A24}$$

The sound modes (8) in the main text can be obtained from here by ignoring the $\mathcal{O}(\ell^4, \ell^2k^2)$ corrections, i.e. assuming $\omega_0 \gg \Gamma, \Omega$.

*Correlation functions.*—We now consider the hydrodynamic predictions for the retarded correlation functions of various observables. To this end, we consider the equations of motion following from (7), but turning on the gauge field $A_t$, $A_i$. Using (3b), we can then obtain the equations

$$\begin{aligned}
\partial_t\delta n &= -\Gamma\delta n_A + D_n(\partial_i\partial^i\delta n_A + \chi\partial_t\partial_i A^i) + \frac{v_s^2\chi}{\lambda c_\phi}(\partial_i\partial^i\delta\phi + c_\phi\partial_i A^i) - \frac{\chi}{\lambda c_\phi}\omega_0^2\delta\phi, \\
\partial_t\delta\phi &= \frac{\lambda c_\phi}{\chi}\delta n_A - \Omega\delta\phi + D_\phi(\partial_i\partial^i\delta\phi + c_\phi\partial_i A^i),
\end{aligned} \tag{A25}$$

where $\delta n_A = \chi(\delta\mu - A_t)$. Note that in (A25), the Goldstone charge $c_\phi$ only enters via its renormalised combination $\lambda c_\phi$, except for the terms coupling to the background gauge field $A_i$. This means that in the absence of sources, one could have simply considered $\lambda c_\phi$ as the "new" charge of the Goldstone, and concluded that the renormalisation factor $\lambda$ is not independently physical. However, we can probe the original bare charge $c_\phi$ of the Goldstone by coupling the system to background sources, in which case the renormalisation factor $\lambda$ does acquire a physical meaning of its own.

To be more concrete, let us compute the retarded correlation functions of various observables by varying with respect to the respective background sources. As a function of frequency $\omega$ and zero wavevector $k$, we find

$$\begin{aligned}
G^R_{nn}(\omega) &= \chi\left(-1 + \frac{\omega(\omega + i\Omega)}{(\omega + i\Gamma)(\omega + i\Omega) - \omega_0^2}\right), \\
G^R_{\phi\phi}(\omega) &= \frac{\lambda^2c_\phi^2}{\chi\omega_0^2}\left(-1 + \frac{\omega(\omega + i\Gamma)}{(\omega + i\Gamma)(\omega + i\Omega) - \omega_0^2}\right), \\
G^R_{n\phi}(\omega) &= \frac{-i\omega\lambda c_\phi}{(\omega + i\Gamma)(\omega + i\Omega) - \omega_0^2}, \\
G^R_{j^ij^j}(\omega) &= c_\phi^2 f_s\delta^{ij} - i\sigma_n\omega\delta^{ij},
\end{aligned} \tag{A26}$$

while all other correlators vanish. Since the flux correlator is obtained by performing variations with respect to the background gauge field $A_i$, we see that it is sensitive to bare charge $c_\phi$, while the other correlators are only sensitive to the renormalised charge $\lambda c_\phi$. This allows us to isolate the Kubo formula for $\lambda$ in (10).

### 2. Second law constraints in pinned viscoelastic hydrodynamics

We now give details about the second law analysis for pinned viscoelastic hydrodynamics. We introduce a gauge field $A_t$, $A_i$, which can be used to compute the correlations of $n$, $\pi_i$ respectively. For technical simplicity, we will omit introducing sources for $\tau^{ij}$, $\epsilon^t$, $\epsilon^i$, which would require using Newton-Cartan geometry; see e.g. [54]. The energy and momentum conservation equations take the form

$$\begin{aligned}
\partial_t\epsilon + \partial_i\epsilon^i &= E_ij^i - K_I\partial_t\phi^I - \ell L_I\partial_t\Phi^I, \\
\partial_t\pi^i + \partial_j\tau^{ij} &= E^i\rho + F^{ij}j_j + K_I\partial^i\phi^I + \ell L_I\partial^i\Phi^I, \\
\partial_t n + \partial_i\pi^i &= 0,
\end{aligned} \tag{A27}$$

while the Josephson and continuity equations remains the same as (12a)–(12b). We parametrise the entropy density to be

$$s^t = s + \mathcal{S}, \tag{A28}$$

where the thermodynamic entropy density $s$ is defined in (13) and $\mathcal{S}$ denotes gradient corrections. Using the thermodynamic relations in (13), we can obtain

$$\partial_t s^t + \partial_i s^i = -\frac{1}{T^2}\mathcal{E}^i\partial_i T - \frac{1}{T}\mathcal{T}^{ij}\partial_i u_j - \frac{1}{T}\mathcal{K}_I\frac{\mathrm{d}\phi^I}{\mathrm{d}t} - \frac{\ell}{T}\mathcal{L}_I\frac{\mathrm{d}\Phi^I}{\mathrm{d}t} + \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{S} + \partial_i\mathcal{S}^i, \quad \text{(A29)}$$

where we have identified the constitutive relations

$$\begin{aligned}
\epsilon^i &= (\epsilon - \mathcal{F})u^i + r_{IJ}e^{Ii}e^J_t + \mathcal{T}^{ij}u_j + \mathcal{E}^i,\\
\tau^{ij} &= \rho u^i u^j - \mathcal{F}\delta^{ij} - r_{IJ}e^{Ii}e^{Jj} + \mathcal{T}^{ij},\\
K_I &= -\partial_i(r_{IJ}e^{Ji}) - \ell m^2\psi_I + \mathcal{K}_I,\\
L_I &= c_\phi m^2\psi_I + \mathcal{L}_I,\\
s^i &= s^t u^i + \frac{1}{T}\mathcal{E}^i + \mathcal{S}^i. \qquad \text{(A30)}
\end{aligned}$$

Here $e^I_i = \partial_i\phi^I$ and $e^I_t = \partial_t\phi^I$.

Similarly to the U(1) case, the right-hand side of (A29) is required to be a positive semi-definite quadratic form. This results in the "non-hydrostatic" constitutive relations

$$\begin{pmatrix} \frac{1}{T}\mathcal{E}^i_{\text{nhs}} \\ \mathcal{T}^{ij}_{\text{nhs}} \\ \mathcal{K}^{\text{nhs}}_I \\ \mathcal{L}^{\text{nhs}}_I \end{pmatrix} = -\begin{pmatrix} \sigma^{ij}_\epsilon & \chi^{ikl} & -\gamma_{\phi}{}^i_J & -\gamma_{\Phi}{}^i_J \\ \chi^{ijk} & \eta^{ijkl} & \chi_{\phi}{}^{ij}_I & \chi_{\Phi}{}^{ij}_I \\ \gamma_{\phi}{}^k_I & \chi_{\phi}{}^{kl}_I & \sigma^\phi_{IJ} & \sigma^\times_{IJ} \\ \gamma_{\Phi}{}^k_I & \chi_{\Phi}{}^{kl}_I & \sigma^\times_{IJ} & \sigma^\Phi_{IJ} \end{pmatrix} \begin{pmatrix} \partial_k T \\ \partial_{(k}u_{l)} \\ \frac{\mathrm{d}}{\mathrm{d}t}\phi^J \\ \ell\frac{\mathrm{d}}{\mathrm{d}t}\Phi^J \end{pmatrix}, \quad \text{(A31)}$$

where all the objects in the coefficient matrix have to be made out of $\delta^{ij}$, $\epsilon^{ij\cdots}$, $\partial_i\Phi^I$, supplemented with arbitrary transport coefficients that are functions of $T$, $\mu$, $h^{IJ}$, $\psi^I$. We have already imposed the Onsager's reciprocity relations in the matrix above. Assuming the crystal to be isotropic and parity-preserving, and focusing only on the terms that contribute to the linearised constitutive relations, we find

$$\begin{aligned}
&\sigma^{ij}_\epsilon = \sigma_\epsilon\delta^{ij}, \qquad \eta^{ijkl} = \eta(\delta^{ik}\delta^{jl} - \delta^{il}\delta^{jk}) + \left(\zeta - \frac{2}{d}\eta\right)\delta^{ij}\delta^{kl},\\
&\sigma^\phi_{IJ} = \sigma_\phi h_{IJ}, \qquad \sigma^\Phi_{IJ} = \sigma_\Phi h_{IJ},\\
&\gamma_{\phi}{}^i_J = \gamma_\phi e^i_J, \qquad \gamma_{\Phi}{}^i_J = \gamma_\Phi e^i_J, \qquad \sigma^\times_{IJ} = \sigma_\times h_{IJ}, \qquad \text{(A32)}
\end{aligned}$$

and all others zero. All coefficients are functions of $T$ and $\mu$. The second law results in a set of inequality constraints on these coefficients

$$\eta, \zeta, \sigma_\epsilon, \sigma_\phi \geq 0, \qquad \sigma_\Phi \geq \sigma_\times^2/\sigma_\phi. \quad \text{(A33)}$$

For the "hydrostatic sector", we need to consider the most general first order gradient corrections in $\mathcal{S}$. It was already found in [36,37] that, assuming the crystal to be isotropic, there are no allowed terms in the absence of the background field $\Phi^I$. However, in the presence of $\Phi^I$ we can include the term

$$\mathcal{S} = \frac{1}{T}\bar{f}_{IJ}\gamma^{IJ}, \quad \text{(A34)}$$

where

$$\gamma^{IJ} = \frac{1}{2}(-2\partial_k\phi^{(I}\partial^k\psi^{J)} + \ell h^{IJ} - \ell c_\phi^2\delta^{IJ}), \quad \text{(A35)}$$

is defined so that, linearly, we have $\gamma^{IJ} \approx \ell c_\phi^2\partial^{(I}\delta\Phi^{J)}$. We can further require that $\bar{f}_{IJ}$ is at least linear in fluctuations, because the constant contribution can be removed using a total derivative term $\partial_i\delta\Phi^i$. We hence have

$$\bar{f}_{IJ} = \bar{\alpha}h_{IJ} + \bar{C}_{IJKL}u^{KL}. \quad \text{(A36)}$$

where $\bar{\alpha}(T_0, \mu_0) = 0$. This coefficient can be understood as the response of the crystal to a background strain due to the presence of the background lattice. Ignoring non-linear terms in the constitutive relations, we have

$$\begin{aligned}
\mathcal{E}^i_{\text{hs}} &= 0,\\
\mathcal{T}^{ij}_{\text{hs}} &= -(\ell\bar{f}_{IJ} - \gamma^{KL}\bar{C}_{KLIJ})\partial^i\phi^I\partial^j\phi^J - \left((Ts + \mu\rho)\frac{\partial\bar{\alpha}}{\partial\epsilon} + \rho\frac{\partial\bar{\alpha}}{\partial\rho}\right)h_{IJ}\gamma^{IJ}\delta^{ij},\\
\mathcal{K}^{\text{hs}}_I &= \partial_i(\bar{C}_{KLIJ}\gamma^{KL}\partial^i\phi^J),\\
\mathcal{L}^{\text{hs}}_I &= -c_\phi\partial_i(\bar{f}_{IJ}\partial^i\phi^J), \qquad \text{(A37)}
\end{aligned}$$

along with

$$\mathcal{S}^i = \frac{1}{T}\bar{f}_{IJ}\left(\partial^i\psi^J\frac{\mathrm{d}\phi^I}{\mathrm{d}t} - c_\phi\ell\partial^i\phi^J\frac{\mathrm{d}\Phi^I}{\mathrm{d}t}\right) - \frac{1}{T}\bar{C}_{KLIJ}\gamma^{KL}\partial^i\phi^J\frac{\mathrm{d}}{\mathrm{d}t}\phi^I. \quad \text{(A38)}$$

We can remove one component of $\bar{C}_{IJKL}$ using the redefinition of pseudo-Goldstone fields $\phi^I \to \phi^I + a\psi^I$.

### 3. Linear pinned viscoelastic crystals

Linearising the equations on a homogeneous background $\Phi^I = x^I$, we can obtain the Josephson equation

$$\partial_t\delta\phi^i = \lambda c_\phi u^i - \Omega\delta\phi^i + \gamma_m\partial^i\mu + \gamma_T\partial^i T + 2D^\perp_\phi\partial_j\partial^{[j}\delta\phi^{i]} + D^\parallel_\phi\partial^i\partial_k\delta\phi^k, \quad \text{(A39)}$$

where

$$\gamma_m = -\frac{c_\phi}{\sigma_\phi} B\alpha_m, \qquad \gamma_T = \frac{c_\phi}{\sigma_\phi}(\gamma_\phi - B\alpha_T),$$
$$\alpha_m = -d\frac{\partial \ln\alpha}{\partial\mu}, \qquad \alpha_T = -d\frac{\partial\ln\alpha}{\partial T}. \tag{A40}$$

Here $\alpha_m$ is the mass expansion coefficient, while $\alpha_T$ is the thermal expansion coefficient. For the conservation equations, we find

$$\begin{aligned}\epsilon^i &= (\epsilon + p + T\lambda_T)u^i + T\Omega_s\delta\phi^i - \kappa_m\partial^i\mu - \kappa\partial^i T \\ &\quad - \frac{T\gamma_\phi}{c_\phi}(D^{\parallel}_\phi\partial^i\partial_k\delta\phi^k + 2D^{\perp}_\phi\partial_j\partial^{[j}\delta\phi^{i]}),\\ \tau^{ij} &= p_m\delta^{ij} - \frac{2\lambda G}{c_\phi}\partial^{\langle i}\delta\phi^{j\rangle} - \frac{\lambda B}{c_\phi}\delta^{ij}\partial_k\delta\phi^k \\ &\quad - 2\eta\partial^{\langle i}u^{j\rangle} - \zeta\partial_k u^k\delta^{ij} - \ell\mathcal{X}^{ij},\\ \ell L^i &= -\frac{\rho}{\lambda c_\phi}\omega_0^2\delta\phi^i - \Gamma\pi^i - \lambda_T\partial^i T - \ell\partial_j\mathcal{X}^{ij},\end{aligned} \tag{A41}$$

where we have further defined the mechanical pressure $p_m$, mass conductivity $\kappa_m$, thermal conductivity $\kappa$, heat damping coefficient $\Omega_s$, and a new coefficient $\lambda_T$ as

$$\begin{aligned}p_m &= p - \lambda B d\delta\ln\alpha,\\ \kappa_m &= -\frac{T\gamma_\phi}{\sigma_\phi}B\alpha_m, \qquad \kappa = T\sigma_\epsilon + \frac{T\gamma_\phi}{\sigma_\phi}(\gamma_\phi - B\alpha_T),\\ \Omega_s &= \frac{\gamma_\phi\ell^2 m^2}{c_\phi\sigma_\phi}, \qquad \lambda_T = \frac{\ell}{c_\phi}\left(\gamma_\Phi - \frac{\sigma_\times}{\sigma_\phi}\gamma_\phi\right),\end{aligned} \tag{A42}$$

as well as

$$\begin{aligned}c_\phi\mathcal{X}^{ij} &= 2\left(\bar G - \frac{\sigma_\times}{c_\phi\sigma_\phi}G\right)\partial^{\langle i}\delta\phi^{j\rangle}\\ &\quad + \delta_{ij}\left(\left(\bar B - \frac{\sigma_\times}{c_\phi\sigma_\phi}B\right)(\partial_k\delta\phi^k + d\delta\alpha) + c_\phi\delta\bar\alpha\right).\end{aligned} \tag{A43}$$

Note that $\mathcal{X}^{ij}$ identically drops out from the equations of motion, and is the only contribution that contains $\bar f_{IJ}$. However, $\mathcal{X}^{ij}$ still non-trivially affects the stress tensor and respective correlation functions. For the record, let us also note the heat/entropy flux

$$\begin{aligned}s^i &= (s + \lambda_T)u^i + \Omega_s\delta\phi^i - \frac{\kappa_m}{T}\partial^i\mu - \frac{\kappa}{T}\partial^i T\\ &\quad - \frac{\gamma_\phi}{c_\phi}(D^{\parallel}_\phi\partial^i\partial_k\delta\phi^k + 2D^{\perp}_\phi\partial_j\partial^{[j}\delta\phi^{i]}).\end{aligned} \tag{A44}$$

From here, we derive another damping-attenuation relation in energy/entropy/heat flux

$$\Omega_s = \frac{\gamma_\phi}{c_\phi}D^{\perp}_\phi k_0^2, \tag{A45}$$

where $k_0^2 = \ell^2 m^2/G$. This relation recently appeared in [11], where the authors derived it using the locality of hydrodynamic constitutive relations. We also find a new coefficient $\lambda_T$ that modifies the energy and entropy flux at thermodynamic level, and contributes to sourcing momenta.

### 4. Mode spectrum of pinned viscoelastic crystals

We can use the linearised equations of motion to derive the mode spectrum of pinned viscoelastic hydrodynamics. In the transverse sector, we find a phonon sound mode with the dispersion relation similar to the U(1) case, namely

$$\omega = \pm\sqrt{\omega_0^2 + v_\perp^2 k^2} - \frac{i}{2}(k^2(D^{\perp}_\pi + D^{\perp}_\phi) + \Gamma + \Omega), \tag{A46}$$

where

$$v_\perp^2 = \frac{\lambda^2 G}{\rho}, \qquad D^{\perp}_\pi = \frac{\eta}{\rho}. \tag{A47}$$

The longitudinal sector is considerably more involved. Focusing on isothermal configurations, we find a damped sound mode and a crystal diffusion mode

$$\begin{aligned}\omega &= \pm\sqrt{\omega_0^2 + v_\parallel^2 k^2}\\ &\quad - \frac{i}{2}\left(D^{\parallel}_s k^2 - \frac{\rho_m^2}{v_\parallel^2\chi^2}\frac{\Omega k^2}{\omega_0^2 + v_\parallel^2 k^2} + \Gamma + \Omega\right),\\ \omega &= -\frac{ik^2\rho/\chi}{\omega_0^2 + v_\parallel^2 k^2}(D^{\parallel}_\phi k^2 + \Omega),\end{aligned} \tag{A48}$$

where we have defined

$$\begin{aligned}v_\parallel^2 &= \frac{\rho_m^2/\chi + \lambda^2(B + 2\frac{d-1}{d}G)}{\rho},\\ D^{\parallel}_s &= \frac{\rho(v_\parallel^2 - \rho_m/\chi)^2}{\sigma_\phi v_\parallel^2} + \frac{\zeta + 2\frac{d-1}{d}\eta}{\rho},\end{aligned} \tag{A49}$$

along with mechanical mass density $\rho_m = \rho + \lambda B\alpha_m$ and susceptibility $\chi = \partial\rho/\partial\mu$. We have taken $\ell \sim \mathcal{O}(\partial)$ in the expressions above. We again note that $\lambda$ non-trivially affects the various speeds of mode propagation. While solving the linearised equations, it is useful to note that in the isothermal limit, $\chi\delta\mu = \delta\rho - B\alpha_m/c_\phi\partial_k\delta\phi^k$.

Turning back on the background fields, we can compute the following correlation functions at zero momentum

$$G^R_{\pi^i\pi^j}(\omega,k=0)=\rho\delta^{ij}\left(-1+\frac{\omega(\omega+i\Omega)}{(\omega+i\Gamma)(\omega+i\Omega)-\omega_0^2}\right),$$
$$G^R_{\phi^i\phi^j}(\omega,k=0)=\frac{\delta^{ij}\lambda^2c_\phi^2}{\rho\omega_0^2}\left(-1+\frac{\omega(\omega+i\Gamma)}{(\omega+i\Gamma)(\omega+i\Omega)-\omega_0^2}\right),$$
$$G^R_{\pi^i\phi^j}(\omega,k=0)=\frac{i\omega\lambda c_\phi\delta^{ij}}{(\omega+i\Gamma)(\omega+i\Omega)-\omega_0^2}. \tag{A50}$$

Computing $\tau^{ij}$ correlators is beyond the scope of this work and would require coupling the system to curved space. Note that the coefficients (A36) do not affect these three correlators above.

### 5. Comparison with previous works

In the previous version of our paper, we pointed out certain discrepancies with the work of [11]. These have now been resolved by the authors of [11] in an updated version of their paper; we present a detailed comparison below.

Let us start with the U(1) model. Our constitutive relations in (7) trivially reduce to those in [11] upon setting the transport coefficients $\sigma_\times=\bar{f}_s=0$ (resulting in $\lambda=1$) and matching the conventions $\phi\to-\phi$, $A_t\to-A_t$, $A_i\to-A_i$ and $c_\phi=1$. Our results also match with [8] in this limit upon matching the conventions $\phi\to-\phi$. In an updated version of their paper, the authors of [11] verified that their formalism does allow for nonzero $\sigma_\times$ and $\bar{f}_s$ in the presence of background gauge fields. We find the new mapping between various coefficients

$$\hat{\chi}_{nn}=\chi,\qquad \hat{c}_s^2=\frac{f_s+2\ell\bar{f}_s}{\chi},\qquad \hat{\omega}_0^2=\frac{\ell^2m^2}{\chi},$$
$$\hat{D}_n=\frac{\sigma}{\chi},\qquad \hat{D}_\phi=\frac{(1-\ell\sigma_\times/\sigma_\phi)(f_s+2\ell\bar{f}_s)}{\sigma_\phi},$$
$$\hat{\Gamma}=\frac{\ell^2}{\chi}\left(\sigma_\Phi-\frac{\sigma_\times^2}{\sigma_\phi}\right),$$
$$\hat{\kappa}=\frac{-\ell\bar{f}_s}{f_s+2\ell\bar{f}_s},\qquad \hat{\sigma}=\frac{\ell\sigma_\times}{\sigma_\phi}=\lambda-1. \tag{A51}$$

For clarity, we have denoted all the coefficients in [11] with a hat and kept $c_\phi=1$.

To compare our results with [11] in the pinned crystal case, we need to perform the following transformations to the constitutive relations

$$\tau^{ij}\to\tau^{ij}+\ell\mathcal{X}^{ij}+2\lambda G(\partial^j\delta\phi^i-\delta^{ij}\partial_k\delta\phi^k),$$
$$L^i\to L^i+\partial_j\mathcal{X}^{ij}, \tag{A52}$$

which leave the equations of motion invariant at the linearised level. $\mathcal{X}^{ij}$ was defined in (A43) and we have set $c_\phi=1$. Note, however, that the transformed quantities should not be used to reliably predict the hydrodynamic correlation functions involving stress. Focusing on $d=2$ spatial dimensions, this results in

$$s^i=(s+\lambda_T)u^i+\Omega_s\delta\phi^i-\frac{\kappa_m}{T}\partial^i\mu-\frac{\kappa}{T}\partial^iT$$
$$-\gamma_\phi D_\phi^{\parallel}\partial^i\partial_k\delta\phi^k-2\gamma_\phi D_\phi^{\perp}\partial_j\partial^{[j}\delta\phi^{i]},$$
$$\tau^{ij}=p_m\delta^{ij}-2\lambda G\partial^{[i}\delta\phi^{j]}-\lambda(B+G)\delta^{ij}\partial_k\delta\phi^k$$
$$-2\eta\partial^{\langle i}u^{j\rangle}-\zeta\partial_ku^k\delta^{ij},$$
$$\partial_t\delta\phi^i=\lambda u^i-\Omega\delta\phi^i+\gamma_m\partial^i\mu+\gamma_T\partial^iT$$
$$+2D_\phi^{\perp}\partial_j\partial^{[j}\delta\phi^{i]}+D_\phi^{\parallel}\partial^i\partial_k\delta\phi^k. \tag{A53}$$

These should be compared to [11] in the Galilean setting, i.e. upon setting $j^i=\pi^i$. Note that the displacement field $u^i$ of [11] is identified with our $\delta\phi^i$, their fluid velocity $v^i$ is our $u^i$, their heat current $j_Q^i$ is our $Ts^i$. Firstly, the Galilean constraint implies for their transport coefficients

$$\hat{n}=\hat{\chi}_{\pi\pi},\qquad \hat{\gamma}_{3c}=-\hat{\chi}_{n\lambda_\parallel}\hat{\xi}(B+G),$$
$$\hat{\Omega}_n=\hat{\sigma}_0=\hat{\alpha}_0=\hat{\bar{\alpha}}_0=\hat{\gamma}_{1l}=0. \tag{A54}$$

We have again used a hat for the coefficients in [11] to avoid confusion. The mapping between the remaining coefficients follows as

$$\hat{p}=p_m,\qquad \hat{\chi}_{\pi\pi}=\rho,$$
$$\hat{\chi}_{n\lambda_\parallel}=\frac{B\alpha_\mu}{B+G},\qquad \hat{\chi}_{s\lambda_\parallel}=\frac{B\alpha_T}{B+G},$$
$$\hat{\bar{\kappa}}_0=\kappa+\frac{T\gamma_\phi B\alpha_T}{\sigma_\phi},\qquad \hat{\gamma}_{2l}=\frac{\gamma_\phi}{\sigma_\phi},$$
$$\hat{\xi}=\frac{1}{\sigma_\phi},\qquad \hat{\gamma}_{3h}=\gamma_T. \tag{A55}$$

With these identifications, we find that the results of [11] exactly match our (A53), modulo the new coefficients $\lambda$ and $\lambda_T$ due to explicit symmetry breaking.

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