D-term uplifts in nonsupersymmetric heterotic string models

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Recently, we proposed that the one-loop tadpole diagram in perturbative nonsupersymmetric heterotic string vacua that contain an anomalous $U(1)$ symmetry, leads to an analog of the Fayet-Iliopoulos D-term in $\mathcal{N}=1$ supersymmetric models, and may uplift the vacuum energy from negative to positive value. In this paper, we extend this analysis to new types of vacua, including those with stringy Scherk-Schwarz (SSS) spontaneous supersymmetry breaking versus those with explicit breaking. We develop a criteria that facilitates the extraction of vacua with Scherk-Schwarz breaking. We develop systematic tools to analyze the T-duality property of some of the vacua and demonstrate them in several examples. The extraction of the anomalous $U(1)$ D-terms is obtained in two ways. The first utilizes the calculation of the $U(1)$-charges from the partition function, whereas the second utilizes the free fermionic classification methodology to classify large spaces of vacua and analyze the properties of the massless spectrum. The systematic classification method also ensures that the models are free from physical tachyons. We provide a systematic tool to relate the free fermionic basis vectors and one-loop generalized GSO phases that define the string models, to the one-loop partition function in the orbifold representation. We argue that a D-term uplift, while rare, is possible for both the SSS class of models, as well as in those with explicit breaking. We discuss the steps needed to further develop the arguments presented here.

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I. INTRODUCTION

The Standard Model (SM) of particle physics provides an effective parametrization for all observational subatomic data to date. It is even possible that it remains viable up to the grand unified theory (GUT) scale, or Planck scale, where gravitational effects become prominent. In this case, gaining insight into the fundamental origin of the SM parameters can only be gleaned by fusing it with gravity. String theory provides the most advanced contemporary framework to pursue the synthesis of gravity with the subatomic gauge interactions. For that purpose, Standard-like Models were constructed in the free fermionic formulation of the heterotic string and provide a laboratory to study how the Standard Model parameters may arise in a theory of quantum gravity [1–7]. The free fermionic heterotic string models are $\mathbb{Z}_2\times\mathbb{Z}_2$ orbifolds of six dimensional tori at enhanced symmetry points in the moduli space [8–12].

While the majority of phenomenological string models constructed to date possess $\mathcal{N}=1$ spacetime supersymmetry (SUSY), absence of SUSY at observable energy scales mandates that it has to be broken. Spacetime SUSY in string models can be broken by nonperturbative effects in the effective field theory limit of the string vacua, or it may be broken directly at the string scale. In the string constructions, we may distinguish between explicit and spontaneous breaking, where, in the former, the remaining gravitino is projected from the spectrum, whereas spontaneous breaking can arise through the Scherk-Schwarz mechanism [13–17], in which case the gravitino mass is proportional to the inverse of an internal radius of the six dimensional compactified torus.

It is clear that addressing many of the questions in string phenomenology mandates the breaking of SUSY. In particular when it comes to the cosmological evolution and string dynamics near the Planck scale. The non-SUSY string vacua typically contain physical tachyons in their spectra that indicate that they are unstable. However, also those configurations that are free of physical tachyons, in general have nonvanishing tadpoles and vacuum energy that in general lead to instability.

One of the recurring features in supersymmetric string derived models is the existence of an anomalous $U(1)$

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($U(1)_A$) symmetry. The $U(1)_A$ is canceled by an analogue of the Green-Schwarz mechanism, which generates a Fayet-Iliopoulos (FI) D-term that breaks SUSY near the Planck scale [18,19]. The nonvanishing D-term gives rise to a nonvanishing vacuum energy at two-loop [20]. SUSY can be restored by assigning nonvanishing Vacuum Expectation Values (VEV) to some Standard Model singlet fields along F- and D-flat directions. The anomalous $U(1)$ symmetry in string construction plays a pivotal role in many of the phenomenological studies of string compactifications [21–24].

An anomalous $U(1)$ symmetry is a recurring feature also in non-SUSY string vacua. The same diagram in string perturbation theory that generates the FI term in the SUSY configurations is also present in the non-SUSY configurations, i.e., both in those with explicit SUSY breaking, as well as those in which it is broken by the SSS mechanism. Similarly, the two-loop diagram contributing to the vacuum energy is also present in the case of non-SUSY vacua, either with explicit or SSS SUSY breaking. Thus, it imperative to take into account this contribution to the vacuum energy also in these cases.

This contribution to the vacuum energy is particularly pertinent to the question of the existence of a de-Sitter vacuum in string theory. Astrophysical and cosmological data indicate that the universe is accelerating. The existence of a positive vacuum energy is one of the possible explanations. However, the existence of string vacua with positive vacuum energy and stable moduli is currently under intense scrutiny and doubt. For instance, for the non-SUSY heterotic constructions examined through effective field theory methods in Ref. [25] only AdS vacua are found to be possible. However, through exact world sheet evaluation of the one-loop potentials of non-SUSY heterotic string orbifolds, Florakis and Rizos demonstrated the existence of string vacua with positive vacuum energy [26]. However, many open issues remain in the study of string vacua without SUSY and with respect to (related) issues around moduli stabilization. Additionally, it remains important for this analysis to be extended to models in which more phenomenological criteria are satisfied.

One direction in which progress can be made to the evaluation of vacuum energy for non-SUSY heterotic string vacua is through exploring the contribution of the would-be FI D-term. Since this contribution is positive definite it may uplift an *a priori* negative vacuum energy to a positive one, an idea discussed in [27]. Recently, we demonstrated this possibility in a particular string vacuum [28]. The analysis utilizes the free fermionic classification methodology to extract tachyon free non-SUSY string vacua. It then calculates the traces of the $U(1)$ gauge symmetries and extracts the tachyon free vacua with a $U(1)_A$. The one-loop vacuum amplitude was analyzed in comparison to the $U(1)_A$ would-be D-term contribution. Following Refs. [26,29,30], a numerical analysis of the potential and its dependence on the moduli in specific string models was performed in the neighborhood of the a local minimum. It was then found that a D-term uplift to a positive value may indeed be possible in a model with explicit supersymmetry breaking.

In this paper we extend the analysis of [28]. We develop a criteria to distinguish the models with SSS supersymmetry breaking that allow for the vacuum energy to be exponentially suppressed provided that the number of massless bosons and fermions is equal. We then proceed to analyze the vacuum energy and potential of a range of models with both explicit and SSS SUSY breaking. We demonstrate that a D-term uplift may indeed be possible in models with SSS breaking as well as in models with explicit breaking. We further provide some statistical measure for the frequency of tachyon free models with SSS breaking, and provide further examples of cases where a minimum of potential is not obtained for finite value of the moduli.

Our paper is organized as follows: in Sec. II we review some general aspects of the free fermion construction that are particularly relevant for the analysis in this paper, and refer to the literature for more details. In Sec. III we review the calculation of the Fayet-Iliopoulos term in $\mathcal{N} = 1$ supersymmetric string vacua and discuss its adaptation to the $\mathcal{N} = 0$ case. In Sec. IV we discuss the analysis of the one-loop partition function and potential, as well as the derivation of the anomalous $U(1)$ from the partition function that serves as a countercheck on its derivation from the massless spectrum. In Sec. V we elaborate on explicit SUSY breaking versus spontaneous SUSY breaking by the SSS-mechanism. We derive conditions that facilitate the extraction of the string vacua that utilize the SSS-mechanism and provide examples that demonstrate their utilization in Appendix B. Similarly, in Sec. V we identify the conditions on the world sheet phases that exhibit the T-duality property of the string vacua and supplement these with examples in Appendix B. In Sec. VI we discuss the conditions for the extraction of tachyon free configurations in the space of vacua, and Sec. VII elaborates on the analysis of the chiral sectors and extraction of the anomalous $U(1)_A$ symmetry from the massless spectrum. Section VIII presents our results that include examples of uplift models with SSS supersymmetry breaking as well as explicit breaking. Section IX contains our conclusions and discussion on further steps that can be taken to improve the rigour of the analysis presented in this paper as well as its predictability. In Appendix A we discuss in detail how to relate a free fermion model which is specified in term of the set of boundary condition basis vectors and one-loop GGSO phases, to the one-loop partition function in a bosonic representation. The art in this regard is in the translation of the GGSO projection coefficients to the modular invariant phase that appears in the one-loop partition function. This tool therefore facilitates the writing of the partition function, which provides access to the entire string spectrum, for any string model, which is specified in terms of the boundary condition basis vectors and one-loop phases.

II. MODEL BUILDING IN THE FREE FERMIONIC FORMULATION

In the free fermionic formulation of [31–33], all degrees of freedom are realized as free fermions propagating on the string world sheet. For the heterotic string in four dimensions, we consider holomorphic fields that realize a supersymmetric $c = 10$ conformal algebra and antiholomorphic fields that realize a nonsupersymmetric conformal algebra. Along with the spacetime bosons $X^\mu(z, \bar{z})$ we denote the fermionic fields as

$$\begin{aligned} \text{Holomorphic:}\ & \psi^{\mu=1,2}, \quad \chi^{1,\ldots,6}, \quad y^{1,\ldots,6}, \quad w^{1,\ldots,6} \quad (z) \\ \text{Antiholomorphic:}\ & \bar{y}^{1,\ldots,6}, \quad \bar{w}^{1,\ldots,6}, \quad \bar{\psi}^{1,\ldots,5}, \quad \bar{\eta}^{1,2,3}, \quad \bar{\phi}^{1,\ldots,8} \quad (\bar{z}), \end{aligned} \tag{2.1}$$

where:

(i) ψ^μ is the superpartner of the bosonic spacetime field $X^\mu(z)$ in the lightcone gauge.

(ii) $\chi^{1,\ldots,6}$ are the superpartners of the six compact directions of the bosonic coordinate fields.

(iii) $\{y^i, w^i|\bar{y}^i, \bar{w}^i\}$ realize the conformal field theory associated to the six dimensional compact geometry.

(iv) Currents associated to $\{\bar{\psi}^{1,\ldots,5}, \bar{\eta}^{1,2,3}\}$ can realize a gauge $c = 8$ conformal algebra containing an $SO(10)$ GUT from the $\bar{\psi}^{1,\ldots,5}$ that may contain Standard Model like gauge fields.

(v) Currents associated to $\{\bar{\phi}^{1,\ldots,8}\}$ can be associated to a gauge $c = 8$ conformal algebra relating to the hidden sector of the theory.

Consistent model building requires that a $N = 1$ superconformal algebra on the string world sheet is realized among the holomorphic degrees of freedom. This can be achieved via the world sheet supercurrent

$$T_F(z) = i\psi^\mu \partial X^\mu(z) + i\sum_{I=1}^{6} \chi^I y^I w^I, \tag{2.2}$$

which has conformal weight $(3/2, 0)$. This results in a local enhanced symmetry group $SU(2)^6$, the adjoint representation of which is given by the six $SU(2)$-triplets from $\{\chi^I, y^I, w^I\}$.

Models in the free fermionic formalism are then defined by considering a one-loop torus and defining a set of N boundary condition basis vectors, $\boldsymbol{v_i} \in \mathcal{B}$, specifying how each free fermion, f, propagates around the two noncontractible loops of the torus. An element $\boldsymbol{\beta}$ of the space $\Xi = \text{span}\{\mathcal{B}\}$ can then be written as

$$\boldsymbol{\beta} = \{\beta(\psi^\mu), \beta(\chi^{12}), \beta(\chi^{34}), \beta(\chi^{56}), \beta(y^1), \ldots \beta(w^6)|\beta(\bar{y}^1), \ldots, \beta(\bar{w}^6); \beta(\bar{\psi}^1), \ldots, \beta(\bar{\psi}^5), \beta(\bar{\eta}^1), \beta(\bar{\eta}^2), \beta(\bar{\eta}^3), \beta(\bar{\phi}^1), \ldots, \beta(\bar{\phi}^8)\}, \tag{2.3}$$

such that $\beta(f) \in (-1, 1]$ and Ramond (R) boundary conditions correspond to $\beta(f) = 1$, while Neveu-Schwarz (NS) is given by $\beta(f) = 0$.

The partition function in the fermionic formulation can be written as

$$Z = Z_B \sum_{\boldsymbol{\alpha},\boldsymbol{\beta} \in \Xi} C\begin{bmatrix} \boldsymbol{\alpha} \\ \boldsymbol{\beta} \end{bmatrix} Z\begin{bmatrix} \boldsymbol{\alpha} \\ \boldsymbol{\beta} \end{bmatrix}, \tag{2.4}$$

where $Z_B = 1/\eta^2\bar{\eta}^2$ is the bosonic partition function and $C[{}^{\boldsymbol{\alpha}}_{\boldsymbol{\beta}}]$ are generalized GSO (GGSO) phases which respect modular invariance. The $Z[{}^{\boldsymbol{\alpha}}_{\boldsymbol{\beta}}]$ represent the world sheet fermions and are thus products of Jacobi theta functions. The partition function for the models we explore in this work is discussed in Sec. IV.

Aside from the partition function we can also view the spectrum through the modular invariant Hilbert space, $\mathcal{H}$, of states, $|S_{\boldsymbol{\beta}}\rangle$. This is constructed through implementing the one-loop GGSO projections on each sector according to:

$$\mathcal{H} = \bigoplus_{\boldsymbol{\beta} \in \Xi} \prod_{i=1}^{N} \left\{ e^{i\pi \boldsymbol{v_i} \cdot F_{\boldsymbol{\beta}}} |S_{\boldsymbol{\beta}}\rangle = \delta_{\boldsymbol{\beta}} C\begin{bmatrix} \boldsymbol{\beta} \\ \boldsymbol{v_i} \end{bmatrix}^* |S_{\boldsymbol{\beta}}\rangle \right\} \mathcal{H}_{\boldsymbol{\beta}}, \tag{2.5}$$

where $F_{\boldsymbol{\beta}}$ is the fermion number operator and $\delta_{\boldsymbol{\beta}}$ is the spin-statistics index.

The sectors, $\boldsymbol{\beta}$, in the model can be characterized according to their holomorphic (H) and antiholomorphic (A) moving vacuum separately

$$\begin{aligned} M_H^2 &= -\frac{1}{2} + \frac{\boldsymbol{\beta}_H \cdot \boldsymbol{\beta}_H}{8} + N_H \\ M_A^2 &= -1 + \frac{\boldsymbol{\beta}_A \cdot \boldsymbol{\beta}_A}{8} + N_A, \end{aligned} \tag{2.6}$$

where N_H and N_A are sums over left and right moving oscillator frequencies, respectively

$$N_H = \sum_\lambda \nu_\lambda + \sum_{\lambda^*} \nu_{\lambda^*} \tag{2.7}$$

$$N_A = \sum_{\bar\lambda} \nu_{\bar\lambda} + \sum_{\bar\lambda^*} \nu_{\bar\lambda^*}, \tag{2.8}$$

where λ is a holomophic oscillator and $\bar\lambda$ is an antiholomorphic oscillator and the frequency is defined through the boundary condition in the sector $\boldsymbol{\beta}$

$$\nu_\lambda = \frac{1+\beta(\lambda)}{2}, \qquad \nu_{\lambda^*} = \frac{1-\beta(\lambda)}{2}. \tag{2.9}$$

Physical states must satisfy the Virasoro matching condition, $M_H^2 = M_A^2$, such that massless states are those with $M_H^2 = M_A^2 = 0$ and on-shell tachyons arise for sectors with $M_H^2 = M_A^2 < 0$.

A. Symmetric $\mathbb{Z}_2 \times \mathbb{Z}_2$ $SO(10)$ models

For this work we explore the one-loop cosmological constant and $U(1)_A$ tadpole calculations for models defined through the basis set

$$\begin{aligned}
\mathbb{1} &= \{\psi^\mu, \chi^{1,\ldots,6}, y^{1,\ldots,6}, \omega^{1,\ldots,6} \mid \bar y^{1,\ldots,6}, \bar\omega^{1,\ldots,6}, \bar\eta^{1,2,3}, \bar\psi^{1,\ldots,5}, \bar\phi^{1,\ldots,8}\},\\
\boldsymbol{S} &= \{\psi^\mu, \chi^{1,\ldots,6}\},\\
\boldsymbol{e_i} &= \{y^i, w^i \mid \bar y^i, \bar w^i\}, \quad i = 1, \ldots, 6,\\
\boldsymbol{b_1} &= \{\chi^{34}, \chi^{56}, y^{34}, y^{56} \mid \bar y^{34}, \bar y^{56}, \bar\psi^{1,\ldots,5}, \bar\eta^1\},\\
\boldsymbol{b_2} &= \{\chi^{12}, \chi^{56}, y^{12}, y^{56} \mid \bar y^{12}, \bar y^{56}, \bar\psi^{1,\ldots,5}, \bar\eta^2\},\\
\boldsymbol{z_1} &= \{\bar\phi^{1,\ldots,4}\},\\
\boldsymbol{z_2} &= \{\bar\phi^{5,\ldots,8}\}.
\end{aligned} \tag{2.10}$$

Such a basis can be associated with symmetric $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifolds [8–12] extensively classified in previous works [7,34–41]. The NS sector of the models associated to this basis produce spacetime vector bosons generating the gauge group

$$SO(10) \times U(1)_1 \times U(1)_2 \times U(1)_3 \times SO(8)^2, \tag{2.11}$$

where we note that $U(1)_{1,2,3}$ are generated by the antiholomorphic currents $\bar\eta^k \bar\eta^{k*}$. With respect to the basis (2.10) it is useful to identify the important linear combination

$$\boldsymbol{x} = \mathbb{1} + \boldsymbol{S} + \sum_{i=1}^{6} \boldsymbol{e_i} + \boldsymbol{z_1} + \boldsymbol{z_2} = \{\bar\eta^{1,2,3}, \bar\psi^{1,\ldots,5}\} \tag{2.12}$$

and $\boldsymbol{b_3} = \boldsymbol{b_1} + \boldsymbol{b_2} + \boldsymbol{x} = \{\chi^{12}, \chi^{34}, y^{12}, y^{34} | \bar y^{12}, \bar y^{34}, \bar\psi^{1,\ldots,5}, \bar\eta^3\}$, which spans the third twisted plane of the $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold and facilitates the analysis of the observable spinorial and vectorial representations as first developed in [34,35].

Models may then be defined through the choice of GGSO phases $C[^{v_i}_{v_j}]$. There are 66 free phases for this basis, with all others specified by modular invariance. The full space of models is thus of size $2^{66} \sim 10^{19.9}$. The $\mathcal{N} = 1$ supersymmetric subset of which is defined by those satisfying

$$C\begin{bmatrix}\boldsymbol{S}\\ \boldsymbol{e_i}\end{bmatrix} = C\begin{bmatrix}\boldsymbol{S}\\ \boldsymbol{z_1}\end{bmatrix} = C\begin{bmatrix}\boldsymbol{S}\\ \boldsymbol{z_2}\end{bmatrix} = -1, \tag{2.13}$$

in order to preserve one gravitino. Furthermore, we note that the phases $C[^{\mathbb{1}}_{\boldsymbol{S}}]$ and $C[^{\boldsymbol{S}}_{\boldsymbol{b_k}}]$, $k = 1, 2, 3$, determine the chirality of the degenerate Ramond vacuum $|\boldsymbol{S}\rangle$ and the gravitino is retained so long as

$$C\begin{bmatrix}\mathbb{1}\\ \boldsymbol{S}\end{bmatrix} = C\begin{bmatrix}\boldsymbol{S}\\ \boldsymbol{b_1}\end{bmatrix} C\begin{bmatrix}\boldsymbol{S}\\ \boldsymbol{b_2}\end{bmatrix} C\begin{bmatrix}\boldsymbol{S}\\ \boldsymbol{b_3}\end{bmatrix}, \tag{2.14}$$

which can, without loss of generality, be fixed to

$$C\begin{bmatrix}\mathbb{1}\\ \boldsymbol{S}\end{bmatrix} = C\begin{bmatrix}\boldsymbol{S}\\ \boldsymbol{b_1}\end{bmatrix} = C\begin{bmatrix}\boldsymbol{S}\\ \boldsymbol{b_2}\end{bmatrix} = -1, \tag{2.15}$$

for a scan of $\mathcal{N} = 1$ vacua.

Since we are interested in non-SUSY vacua in this work we will be considering the complement to this space of $\mathcal{N} = 1$ vacua. In previous work on non-SUSY heterotic string vacua from $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifolds [42–44] tachyon free configurations satisfying various phenomenological requirements and their one-loop cosmological constants are explored. In Sec. VI we will detail how we ensure that only those models free from physical tachyons are explored.

In the next section we show how the $U(1)_A$ gauge transformation manifests in a 4 dimensional theory and how the Green-Schwarz mechanism deals with it. We then compute the FI term through a string theory computation at 1-loop and we conjecture how this term can lift the vacua from an anti–de Sitter to a de Sitter.

III. FAYET-ILIOPOULOS D-TERM CALCULATION

Anomalies arise whenever one, or some, of the classical symmetries are broken by quantum effects. Some global symmetries need to be broken by anomalies in order to reproduce observable phenomenology, however a breakdown of a local symmetry indicates a symmetry current is no longer conserved and longitudinal, nonphysical modes of the gauge fields no longer may decouple from the S-matrix. This can result in the loss of unitarity and appearance of unphysical divergences.

In a four dimensional heterotic string theory the anomalies come from the one-loop triangle diagram, where the external lines can be gauge fields, gravitons or a mixture of each. In the following, for simplicity, we will consider purely $U(1)$ gauge anomalies. We start from the four dimensional effective action in the Einstein frame

$$S = \frac{1}{2\kappa^2}\int d^4x\sqrt{-G}\left[-\frac{\kappa^2 e^{-2\Phi}}{2g^2}F_{\mu\nu}F^{\mu\nu} - \frac{e^{-4\Phi}}{12}H_{\mu\nu\rho}H^{\mu\nu\rho}\right] \tag{3.1}$$

with H the field strength of the B field such that

$$H = dB - \frac{\kappa^2}{g^2}\Omega_3^{\rm YM} = dB - \frac{\kappa^2}{g^2}{\rm Tr}\left[A\wedge dA - \frac{2i}{3}A\wedge A\wedge A\right]. \tag{3.2}$$

Under a Gauge transformation, the gauge fields transform as $A_\mu \to A_\mu + \partial_\mu\Lambda$ and the effective action varies as follows

$$\delta S_{\text{one-loop}} = \frac{1}{4}\frac{1}{96\pi^2}\int d^4x{\rm Tr}[Q^3]\epsilon^{\mu\nu\rho\sigma}\Lambda F_{\mu\nu}F_{\rho\sigma}. \tag{3.3}$$

When the sum of these $U(1)$ charges is not zero this anomalous triangle diagram contribution will be present, with massless particles circulating in the loop. The Green-Schwarz mechanism [45] provides a way to cancel these one loop anomalies through the introduction of an antisymmetric 2-form coupled at one loop to the $U(1)$ 2-form field strength in the effective Lagrangian

$$-\frac{\zeta}{2}\epsilon^{\mu\nu\rho\sigma}B_{\mu\nu}F_{\rho\sigma}, \tag{3.4}$$

imposing the B field to vary under the $U(1)$ Gauge transformation as $\delta B = \frac{\kappa^2}{g^2}\Lambda F$, with the condition $4{\rm Tr}[Q^3] = {\rm Tr}[Q]$, such that it compensates the anomalous triangle diagram. The action can also be written in terms of the axion field, a, dual to the antisymmetric B field. The axion field is introduced as a Lagrange multiplier term into the Lagrangian

$$\frac{1}{\sqrt{-G}}\mathcal{L} = -\frac{e^{-2\Phi}}{4g^2}F_{\mu\nu}F^{\mu\nu} - \frac{e^{-4\Phi}}{24\kappa^2}H_{\mu\nu\rho}H^{\mu\nu\rho} - \frac{\zeta}{2}\epsilon^{\mu\nu\rho\sigma}B_{\mu\nu}F_{\rho\sigma} + \frac{a}{6}\epsilon^{\mu\nu\rho\sigma}\partial_\mu H_{\nu\rho\sigma} + \frac{a}{4}\frac{\kappa^2}{g^2}\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}, \tag{3.5}$$

such that its equations of motion give the definition of the H tensor

$$\frac{1}{6}\epsilon^{\mu\nu\rho\sigma}\partial_\mu H_{\nu\rho\sigma} + \frac{\kappa^2}{4g^2}\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma} = 0$$
$$\Rightarrow H = dB - \frac{\kappa^2}{g^2}A\wedge F. \tag{3.6}$$

Using this result and integrating out the H tensor in the Lagrangian (3.5) gives

$$\frac{1}{\sqrt{-G}}\mathcal{L} = -\frac{e^{-4\Phi}}{4g^2}F_{\mu\nu}F^{\mu\nu} - \kappa^2 e^{4\Phi}(\partial_\mu a + 2\zeta A_\mu)^2 + \frac{a}{4}\frac{\kappa^2}{g^2}\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}, \tag{3.7}$$

while the gauge transformation of the axion field can be found through the gauge invariance of H to give

$$\delta a = -2\zeta\Lambda, \tag{3.8}$$

such that when $\zeta \neq 0$ the anomalous gauge field acquires a mass and the gauge symmetry is spontaneously broken.

In order to compute the FI coefficient ζ we have to evaluate the 2-point function of the antisymmetric B field and the $U(1)_A$ gauge boson at one loop, where only chiral fermions with odd spin structure give a nonvanishing contribution.

For the ghosts b, c their zero modes are saturated by inserting $\langle b\bar{b}c\bar{c}\rangle$ into the path integral. For the ghost superpartners β, γ in order to project out of integration their zero modes one of the vertex operators has to be put in the 0 picture, the other in the -1 and an insertion of the picture changing operator $e^\phi T_F$ is needed, where the scalar ϕ arises by the bosonization of the $\beta-\gamma$ fields

$$\beta = e^{-\phi}\partial\xi, \qquad \gamma = e^{\phi}\eta. \tag{3.9}$$

The amplitude then reads

$$\int\frac{d\tau_1 d\tau_2}{2\tau_2}\int d^2z d^2w\Big\langle b(0)\bar{b}(0)c(0)\bar{c}(0)e^\phi T_F(0) \times V^\mu_{A,0}(z)V^{\nu\rho}_{B,-1}(w)\Big\rangle \tag{3.10}$$

with the denominator needed to fix the translation Killing symmetries of the torus and the residual discrete symmetry $z\to -z$. The vertex operators for the gauge boson and the antisymmetric field in the -1 and 0 pictures are the following

$$V^{\mu,i}_{A,-1} = g_c \hat{k}^{-1/2} \psi^\mu \bar{J}^i e^{ik\cdot X}$$
$$V^{\mu}_{A,0} = \sqrt{\frac{2}{\alpha'}} g_c \hat{k}^{-1/2} \left(i\partial X^\mu + \frac{\alpha'}{2} k\cdot\psi\psi^\mu \right) \bar{J} e^{ik\cdot X}$$
$$V^{\mu\nu}_{B,-1} = i\sqrt{\frac{2}{\alpha'}} g_c \bar{\partial} X^\mu e^{-\phi} \psi^\nu e^{ip\cdot X}$$
$$V^{\mu\nu}_{B,0} = \frac{2i}{\alpha'} g_c \bar{\partial} X^\mu \left(i\partial X^\nu + \frac{\alpha'}{2} p\cdot\psi\psi^\nu \right) e^{ik\cdot X}, \quad (3.11)$$

with $g_c = \kappa/2\pi$ and $\hat{k} = 1/2$. These normalizations are chosen such that the string amplitudes match with the field theory calculation and with the vertex operators in the two pictures related via $V_{q+1}(z) = \lim_{w\to z} e^\phi T_F(w) V_q(z)$, where the matter supercurrent takes the form

$$T_F = i\sqrt{\frac{2}{\alpha'}} \psi^\mu \partial X_\mu. \quad (3.12)$$

We note that we included the α' dependence and that the supercurrent is composed of only the uncompactified fields since vertex operators do not involve internal lattice excitations. When $\mathcal{N} \geq 2$ the amplitude vanishes due to the fermion zero modes. However, for $\mathcal{N} = 1$ they can be soaked up by the fermion correlator. To see this, we can take the $\mathcal{O}(p)$ linear approximation of the amplitude

$$-\frac{k_\alpha}{\alpha'^{1/2}} g_c^2 \int \frac{d\tau_1 d\tau_2}{\tau_2} \int d^2z d^2w \langle b\bar{b}c\bar{c}\rangle \langle \psi^\sigma\psi^\alpha\psi^\mu\psi^\rho\rangle \times \langle \bar{J}^i\rangle \langle \partial X^\gamma \bar{\partial} X^\nu\rangle \langle e^\phi e^{-\phi}\rangle \eta_{\sigma\gamma}, \quad (3.13)$$

where the fermion correlator can be written in terms of the fermion current correlator

$$\langle \psi^i\psi^j\rangle = \langle J^{ij}\rangle = \frac{\epsilon^{ij}}{2\pi i} \partial_\nu \mathrm{Tr}[(-1)^F e^{2\pi i\nu J^{ij}}]_{\nu=0}, \quad (3.14)$$

such that the $\langle \psi^\sigma\psi^\alpha\psi^\mu\psi^\rho\rangle$ term acts as follows on the four noncompact fermions

$$\langle \psi^\sigma\psi^\alpha\psi^\mu\psi^\rho\rangle = \epsilon^{\sigma\alpha\mu\rho} \frac{\partial_\nu}{2\pi i}\frac{\partial_\omega}{2\pi i} \left(-\frac{1}{2}\frac{\vartheta_1(\nu)}{\eta^2}\frac{\vartheta_1(\omega)}{\eta} \right) = \frac{1}{2}\epsilon^{\sigma\alpha\mu\rho}\eta^4, \quad (3.15)$$

making use of $\partial_\nu \vartheta_1(\nu) = 2\pi\eta^3$. The other one-loop correlators are the following

$$\langle b\bar{b}c\bar{c}\rangle = \eta^2\bar{\eta}^2$$
$$\langle X^\gamma(w,\bar{w}) X^\nu(0)\rangle = \left(-\frac{\alpha'}{2} \ln \vartheta_1\left(\frac{w}{2\pi}\right) \bar{\vartheta}_1\left(\frac{\bar{w}}{2\pi}\right) + \frac{\alpha'}{4\pi\tau_2} (\mathrm{Im} z)^2 \right) \eta^{\gamma\nu}$$
$$\langle e^\phi e^{-\phi}\rangle = \frac{1}{\eta^2}, \quad (3.16)$$

and for the current

$$\langle \bar{J}^i\rangle = \frac{\partial_\nu}{2\pi i} \mathrm{Tr}[(-1)^F q^H \bar{q}^{\bar{H}} e^{2\pi i\nu \bar{J}^i}]|_{\nu=0} = \bar{\eta}^2 \times \mathrm{Tr}[(-1)^F q^H \bar{q}^{\bar{H}-\frac{1}{12}} q_i], \quad (3.17)$$

where the derivative acts on the states charged under $U(1)_i$, $i = 1, 2, 3$. In the massless limit only chiral fermions contribute with $(H, \bar{H}) = (0, \frac{1}{12})$ so we get

$$\langle \bar{J}^i\rangle = \bar{\eta}^2 \mathrm{Tr}[(-1)^F q_i] = \bar{\eta}^2 \mathrm{Tr} Q_i, \quad (3.18)$$

with $\mathrm{Tr}Q_i = \sum n q_i h$ summed over all states in the spectrum such that n, q_i and h are the number of massless fermions, their $U(1)_i$ charges and their chirality, respectively. Note that the anomalous $U(1)_A$ charge is given as combination of the three $U(1)_{1,2,3}$ produced by the worldsheet currents $:\bar{\eta}^{i*}\bar{\eta}^i:$, $i = 1, 2, 3$, according to

$$U(1)_A = \sum_{i=1}^{3} \frac{aU_1 + bU_2 + cU_3}{\sqrt{a^2+b^2+c^2}} \quad (3.19)$$

where $(a,b,c) = \frac{1}{k}(\mathrm{Tr}U_1, \mathrm{Tr}U_2, \mathrm{Tr}U_3)$ and $k = \gcd(\mathrm{Tr}U_1, \mathrm{Tr}U_2, \mathrm{Tr}U_3)$. Adding the contribution of the four noncompact bosons from the partition function and their zero modes allows us to write the amplitude as

$$\frac{ig_c^2}{256\pi^5\alpha'^{3/2}} k_\alpha \epsilon^{\nu\alpha\mu\rho} \mathrm{Tr}Q_A \int \frac{d\tau_1 d\tau_2}{\tau_2^4} \int d^2z d^2w = \frac{ig_c^2}{12\alpha'^{3/2}} k_\alpha \epsilon^{\nu\alpha\mu\rho} \mathrm{Tr}Q_A, \quad (3.20)$$

using that $\int d^2z = 2(2\pi)^2\tau_2$ and $\pi/3$ is the volume of the fundamental domain. Reducing the constants and doing further contractions of this $\mathcal{N} = 1$ amplitude allows us to finally write the FI term as

$$\zeta = M_s^2 \frac{\mathrm{Tr} Q_A}{192\pi^2}, \tag{3.21}$$

such that, when $\mathrm{Tr}Q_A \neq 0$, there is an additional positive contribution in the effective D-term potential

$$V_D = \frac{1}{2} g_s^2 \zeta^2 \tag{3.22}$$

that corresponds to a two-loop dilaton tadpole. This additional term was originally computed both from the low energy effective action in Ref. [18], as well as through explicit two-loop string calculations [20].

As stated above, only for $\mathcal{N}=1$ unbroken supersymmetry is the FI term nonvanishing. In the free fermionic models, the breaking $\mathcal{N}=4 \to \mathcal{N}=1$ is achieved by the introduction of the $\boldsymbol{b_1}$ and $\boldsymbol{b_2}$ vectors (2.10) associated to $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold twists. The breaking of the last supersymmetry $\mathcal{N}=1 \to \mathcal{N}=0$ is achieved by setting properly the GGSO phases as delineated in Sec. II. We will discuss the contribution of the FI D-term associated with $\mathrm{Tr}U(1)_A$ in these $\mathcal{N}=0$ models in the following sections.

Our statement is that once the last supersymmetry is broken, either spontaneously or explicitly, the D-term contribution (3.22) will still be present. The same tadpole diagram leading to the FI term in the $\mathcal{N}=1$ supersymmetric case is also present in the $\mathcal{N}=0$ case. The analysis outlined above follows through irrespective of whether the model has $\mathcal{N}=1$ or $\mathcal{N}=0$ supersymmetry. The supersymmetric case does guarantee a measure of stability whereas the nonsupersymmetric case is fraught with further uncertainties. For example, nonsupersymmetric string vacua contain additional tadpole diagrams for the dilaton, which indicate that the string equations of motion are not satisfied in Minkowski four dimensional spacetime with constant dilaton, and the computational stability from higher loops is not preserved. Furthermore, in the following, analysis of the potential is performed with respect to a single internal moduli and all other internal moduli are set at the free fermionic point but are not fixed. One direction of improvement on the analysis that we present here is to use the Kiritsis-Kounnas modular invariant regularization scheme [46] that can regulate the infrared divergences. We note that these caveats are relevant in general in nonsupersymmetric string vacua that have been of some old and recent interest in the literature [47–55]. For our purposes here we note that the contribution of Eq. (3.22), when nonzero, which has the same mass dimension, M_s^4, as the cosmological constant, will destabilize the vacua adding a positive contribution and lift the minima of the one-loop potential.

For the purpose of getting numerical results we fix the string coupling $\mathcal{O}(g_s) \sim 1$, which corresponds to $\alpha' = g_s/4\pi \sim 0.1$. This order of magnitude can be justified by reference to work on gauge coupling unification from string model building, for example in Ref. [56]. We observe that the D-term contribution goes with g_s^2 so smaller values will quickly make an uplift less likely. In Sec. VIII, we will see that we obtain an uplift from AdS to dS only very rarely in our setup and so choosing this order of magnitude for the string coupling, rather than a smaller one, helps provide a proof of concept.

In the next sections we study some heterotic string models through the analysis of the partition function and its potential behavior and we explicitly show how the FI term is used in order to uplift the minima.

IV. PARTITION FUNCTION AND ONE-LOOP POTENTIAL

The generic form of the partition function is given in Eq. (2.4) applied to the models defined through the basis (2.10). Using the techniques developed in Appendix A, the partition function written in the free fermionic construction can then be written in the following form

$$\begin{aligned} Z = {}& \frac{1}{\eta^{10}\bar{\eta}^{22}} \frac{1}{2^2} \sum_{\substack{a,k\\b,l}} \frac{1}{2^6} \sum_{\substack{H_i\\G_i}} \frac{1}{2^4} \sum_{\substack{h_1,h_2,P_i\\g_1,g_2,Q_i}} (-1)^{a+b+P_1Q_1+P_2Q_2+\Phi\begin{bmatrix} a & k & H_i & h_1 & h_2 & P_i \\ b & l & G_i & g_1 & g_2 & Q_i \end{bmatrix}} \\ & \times \vartheta\begin{bmatrix} a \\ b \end{bmatrix}_{\psi^\mu} \vartheta\begin{bmatrix} a+h_1 \\ b+g_1 \end{bmatrix}_{\chi^{12}} \vartheta\begin{bmatrix} a+h_2 \\ b+g_2 \end{bmatrix}_{\chi^{34}} \vartheta\begin{bmatrix} a-h_1-h_2 \\ b-g_1-g_2 \end{bmatrix}_{\chi^{56}} \Gamma_{2,2}^{(1)}\left[\begin{matrix} H_1 & H_2 \\ G_1 & G_2 \end{matrix}\middle|\begin{matrix} h_1 \\ g_1 \end{matrix}\right]\left(T^{(1)},U^{(1)}\right) \\ & \times \Gamma_{2,2}^{(2)}\left[\begin{matrix} H_3 & H_4 \\ G_3 & G_4 \end{matrix}\middle|\begin{matrix} h_2 \\ g_2 \end{matrix}\right]\left(T^{(2)},U^{(2)}\right) \Gamma_{2,2}^{(3)}\left[\begin{matrix} H_5 & H_6 \\ G_5 & G_6 \end{matrix}\middle|\begin{matrix} h_1+h_2 \\ g_1+g_2 \end{matrix}\right]\left(T^{(3)},U^{(3)}\right) \\ & \times \bar{\vartheta}\begin{bmatrix} k \\ l \end{bmatrix}^5_{\bar{\psi}^{1-5}} \bar{\vartheta}\begin{bmatrix} k+h_1 \\ l+g_1 \end{bmatrix}_{\bar{\eta}^1} \bar{\vartheta}\begin{bmatrix} k+h_2 \\ l+g_2 \end{bmatrix}_{\bar{\eta}^2} \bar{\vartheta}\begin{bmatrix} k-h_1-h_2 \\ l-g_1-g_2 \end{bmatrix}_{\bar{\eta}^3} \bar{\vartheta}\begin{bmatrix} k+P_1 \\ l+Q_1 \end{bmatrix}^4_{\bar{\phi}^{1-4}} \bar{\vartheta}\begin{bmatrix} k+P_2 \\ l+Q_2 \end{bmatrix}^4_{\bar{\phi}^{5-8}}. \end{aligned} \tag{4.1}$$

For symmetric $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifolds the moduli space is generally parametrized by three complex structure and three Kähler moduli, one for each torus associated to the $\Gamma_{2,2}$ lattices. The moduli space is then $SO(2,2)/SO(2)\times SO(2)$. At the maximally symmetric (free fermionic) point (T_*, U_*), at which bosonic degrees of freedom can be fermionized, the lattices admit a factorized form which can be written entirely in terms of theta functions

$$\begin{aligned}
\Gamma_{2,2}^{(1)}\begin{bmatrix} H_1 & H_2 & h_1 \\ G_1 & G_2 & g_1 \end{bmatrix}\left(T_*^{(1)}, U_*^{(1)}\right) &= \left|\vartheta\begin{bmatrix} H_1 \\ G_1 \end{bmatrix}\vartheta\begin{bmatrix} H_1+h_1 \\ G_1+g_1 \end{bmatrix}\vartheta\begin{bmatrix} H_2 \\ G_2 \end{bmatrix}\vartheta\begin{bmatrix} H_2+h_1 \\ G_2+g_1 \end{bmatrix}\right| \\
\Gamma_{2,2}^{(2)}\begin{bmatrix} H_3 & H_4 & h_2 \\ G_3 & G_4 & g_2 \end{bmatrix}\left(T_*^{(2)}, U_*^{(2)}\right) &= \left|\vartheta\begin{bmatrix} H_3 \\ G_3 \end{bmatrix}\vartheta\begin{bmatrix} H_3+h_2 \\ G_3+g_2 \end{bmatrix}\vartheta\begin{bmatrix} H_4 \\ G_4 \end{bmatrix}\vartheta\begin{bmatrix} H_4+h_2 \\ G_4+g_2 \end{bmatrix}\right| \\
\Gamma_{2,2}^{(3)}\begin{bmatrix} H_5 & H_6 & h_1+h_2 \\ G_5 & G_6 & g_1+g_2 \end{bmatrix}\left(T_*^{(3)}, U_*^{(3)}\right) &= \left|\vartheta\begin{bmatrix} H_5 \\ G_5 \end{bmatrix}\vartheta\begin{bmatrix} H_5-h_1-h_2 \\ G_5-h_1-h_2 \end{bmatrix}\vartheta\begin{bmatrix} H_6 \\ G_6 \end{bmatrix}\vartheta\begin{bmatrix} H_6-h_1-h_2 \\ G_6-h_1-h_2 \end{bmatrix}\right|.
\end{aligned} \tag{4.2}$$

We furthermore note that the modular invariant phase $\Phi\begin{bmatrix} a & k & H_i & h_1 & h_2 & P_i \\ b & l & G_i & g_1 & g_2 & Q_i \end{bmatrix}$ in (4.1) implements the various GGSO projections. A choice of phase is equivalent to a choice of GGSO matrix and hence there is a unique one-to-one map between them. The factor of $a+b$ ensures correct spin statistics, while the explicit inclusion of the extra phase $P_1Q_1 + P_2Q_2$ means that $\Phi = 0$ is a valid modular invariant choice.

The summation indices used to write the fermionic partition function (4.1) correspond to various features of the model. The indices a, b correspond to the spin structures of the spacetime fermions ψ^μ, while k, l are associated to the 16 right-moving complex fermions giving the gauge degrees of freedom of the heterotic string. The nonfreely acting $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold twists are associated to the parameters h_1, g_1 and h_2, g_2. One of the key features of models defined by the basis (2.10) is the inclusion of the basis vectors $\boldsymbol{e_i}$ which generate freely acting orbifold shifts in the internal dimensions of the compact torus. In the partition function, these are realized by the indices H_i, G_i parametrizing each of the six independent shifts. The additional twists P_i, Q_i correspond to the basis vectors $\boldsymbol{z_1}$ and $\boldsymbol{z_2}$ acting on the hidden sector of our model.

The moduli-dependent form of the twisted/shifted lattice requires closer attention. We know that all dependence on the geometric moduli is contained in the untwisted sector of the model and hence

$$\begin{aligned}
&\Gamma_{2,2}\begin{bmatrix} H_1 & H_2 & h \\ G_1 & G_2 & g \end{bmatrix}(T,U)\Big|_{h,g\neq 0} \\
&\quad = \Gamma_{2,2}\begin{bmatrix} H_1 & H_2 & h \\ G_1 & G_2 & g \end{bmatrix}(T_*,U_*).
\end{aligned} \tag{4.3}$$

This means that for nonzero twists the lattice is precisely given by its factorized form in (4.2). In the case of the untwisted sector, the shifted lattice can be written in a Poisson resummed Hamiltonian form as

$$\begin{aligned}
&\Gamma_{2,2}\begin{bmatrix} H_1 & H_2 & 0 \\ G_1 & G_2 & 0 \end{bmatrix}(T,U) \\
&\quad = \sum_{m_i,n_i\in\mathbb{Z}} q^{\frac{1}{2}|\mathcal{P}_L(T,U)|^2}\bar{q}^{\frac{1}{2}|\mathcal{P}_R(T,U)|^2} e^{i\pi\sum_i((m_i+n_i+H_i)G_i},
\end{aligned} \tag{4.4}$$

where the left and right-moving momenta are

$$\begin{aligned}
\mathcal{P}_L &= \frac{1}{\sqrt{2T_2U_2}}\left[\frac{U}{2}(m_1+n_1) - \frac{1}{2}(m_2+n_2)\right. \\
&\qquad \left. + T(m_1-n_1+H_1) + TU(m_2-n_2+H_2)\right] \\
\mathcal{P}_R &= \frac{1}{\sqrt{2T_2U_2}}\left[\frac{U}{2}(m_1+n_1) - \frac{1}{2}(m_2+n_2)\right. \\
&\qquad \left. + \bar{T}(m_1-n_1+H_1) - TU(m_2-n_2+H_2)\right].
\end{aligned} \tag{4.5}$$

Written in this form, it is easy to extract the q-expansion of the partition function at any given point in the moduli space which is crucial for calculating the one-loop potential. It can be shown that the twisted/shifted lattice sums (4.3) and (4.4) evaluated at the special point $(T_*, U_*) = (i/2, i)$ indeed reproduce the free fermionic form of the partition function (4.2).

Given the fermionic partition function (4.1), the one-loop potential is evaluated by summing over all inequivalent world sheet tori via the modular invariant integral

$$V_{\text{one-loop}}(T^{(i)}, U^{(i)}) = -\frac{1}{2}\frac{M_s^4}{(2\pi)^4}\int_{\mathcal{F}}\frac{d^2\tau}{\tau_2^2} Z\left(\tau,\bar{\tau},T^{(i)},U^{(i)}\right), \tag{4.6}$$

where in $Z(\tau,\bar{\tau},T^{(i)},U^{(i)})$ we include the spacetime bosonic degrees of freedom arising from the world sheet. In models with a $U(1)_A$, an additional contribution to the potential V_D is generated as discussed in Sec. III. Since this term is independent of the geometric moduli it provides a constant shift of the potential throughout moduli space. Hence we use this to write the vacuum energy as

$$V_{\text{total}} = V_{\text{one-loop}}(T^{(i)}, U^{(i)}) + V_D, \tag{4.7}$$

where V_D is given in term of the trace of the anomalous $U(1)_A$ via (3.22).

In order to calculate the one-loop potential of our models we must be able to move away from this special point in the moduli space. The details of the translation of a free fermionic model into a $\mathbb{Z}_2^N$ orbifold was developed in [57] and used in [26,29] to calculate one-loop potentials. Some details of this translation are given in Appendix A.

The motivation for this procedure is that it enables us to move away from the free fermionic point in the moduli space. Although being fixed at this point allows for a generic analysis of many important features of a string model, for issues such as SUSY breaking and one-loop stability, analysis across the moduli space is required.

In general, we can perturb away from the free fermionic point using marginal operators given by the Thirring interactions [58] which take the form $J^i(z)\bar{J}^j(\bar{z}) =: y^i w^i :\, :\bar{y}^j\bar{y}^j:$. Writing these currents in bosonized form we identify the geometric moduli $J^i(z)\bar{J}^j(\bar{z}) = \partial X^i \overline{\partial X}^j$. For symmetric $\mathbb{Z}_2\times\mathbb{Z}_2$ orbifolds we parametrize the moduli space by a complex structure and Kähler modulus for each torus associated to the $\Gamma_{2,2}$ lattices: $(T^{(1)}, U^{(1)})$, $(T^{(2)}, U^{(2)})$ and $(T^{(3)}, U^{(3)})$. These moduli span the familiar $SO(2,2)/SO(2)\times SO(2)$ moduli space of $\mathbb{Z}_2\times\mathbb{Z}_2$ symmetric orbifolds.

Once we have installed this moduli dependence and followed the translation procedure we can calculate the one-loop potential numerically at specific moduli values using

$$V_{\text{one-loop}}(T^{(i)}, U^{(i)}) = -\frac{1}{2}\frac{M_s^4}{(2\pi)^4}\int_{\mathcal{F}}\frac{d^2\tau}{\tau_2^2} Z\big(\tau,\bar{\tau},T^{(i)},U^{(i)}\big) \tag{4.8}$$

where we integrate over the fundamental domain

$$\mathcal{F} = \{\tau\in\mathbb{C}\,|\,|\tau|>1, |\tau_1|<1/2\}. \tag{4.9}$$

Calculating the one-loop potential is then an exercise in solving modular integrals.

One important observation is that we have 6 complex moduli inside this integral rendering an analysis of the potential in all directions impractical. A logical approach used in [26,29], is to take the volume of the first torus associated to $\mathrm{Im}(T^{(1)}) = T_2$ and analyze the potential solely in this direction, with the other moduli all fixed at their values at the free fermionic point. This choice is somewhat arbitrary except in the case of an SSS breaking (discussed further in the Sec. V) where an internal shift in the first torus means that T_2 parametrizes the SUSY-breaking and SUSY will be restored in the large volume limit $T_2\to\infty$.

A. Calculating Tr $U(1)_A$ from the partition function

In Sec. III we discussed how the FI *D*-term contributed at 2-loop to the potential of our model. Its magnitude depended on the trace of the fields charged under the $U(1)_A$ in our model, that propagate in the anomalous triangle diagram in four dimensions. There are two equivalent ways to calculate this trace. One way is to extract those states of the massless spectrum charged under $U(1)_A$ and add up their charges. This approach utilizes free fermionic classification tools that are easily computerized. The details of this approach are given in Sec. VII. The second way to calculate Tr$U(1)_A$ is directly from the partition function, which will be explained in this subsection. In order to perform this calculation it helps to rewrite the partition function (4.1) as follows

$$\begin{aligned}\tilde{Z} = {}& \frac{1}{\eta^{11}\bar{\eta}^{22}}\frac{1}{2^2}\sum_{\substack{a,k\\ b,l}}\frac{1}{2^6}\sum_{\substack{H_i\\ G_i}}\frac{1}{2^4}\sum_{\substack{h_1,h_2,P_i\\ g_1,g_2,Q_i}}(-1)^{a+b+P_1Q_1+P_2Q_2+\Phi\begin{bmatrix} a & k & H_i & h_1 & h_2 & P_i\\ b & l & G_i & g_1 & g_2 & Q_i\end{bmatrix}}\\
&\times\vartheta\begin{bmatrix}a\\ b\end{bmatrix}(\nu)_{\psi^\mu}\vartheta\begin{bmatrix}a\\ b\end{bmatrix}(\omega)_{\psi^\mu}\left(\frac{1}{\eta^2}\right)\vartheta\begin{bmatrix}a+h_1\\ b+g_1\end{bmatrix}_{\chi^{12}}\vartheta\begin{bmatrix}a+h_2\\ b+g_2\end{bmatrix}_{\chi^{34}}\vartheta\begin{bmatrix}a-h_1-h_2\\ b-g_1-g_2\end{bmatrix}_{\chi^{56}}\\
&\times\Gamma_{2,2}^{(1)}\left[\begin{matrix}H_1 & H_2\\ G_1 & G_2\end{matrix}\middle|\begin{matrix}h_1\\ g_1\end{matrix}\right](T^{(1)},U^{(1)})\Gamma_{2,2}^{(2)}\left[\begin{matrix}H_3H_4\\ G_3G_4\end{matrix}\middle|\begin{matrix}h_2\\ g_2\end{matrix}\right](T^{(2)},U^{(2)})\Gamma_{2,2}^{(3)}\left[\begin{matrix}H_5H_6\\ G_5G_6\end{matrix}\middle|\begin{matrix}h_1+h_2\\ g_1+g_2\end{matrix}\right](T^{(3)},U^{(3)})\\
&\times\bar{\vartheta}\begin{bmatrix}k\\ l\end{bmatrix}^5_{\bar{\psi}^{1-5}}\bar{\vartheta}\begin{bmatrix}k+h_1\\ l+g_1\end{bmatrix}(u_1)_{\bar{\eta}^1}\bar{\vartheta}\begin{bmatrix}k+h_2\\ l+g_2\end{bmatrix}(u_2)_{\bar{\eta}^2}\bar{\vartheta}\begin{bmatrix}k-h_1-h_2\\ l-g_1-g_2\end{bmatrix}(u_3)_{\bar{\eta}^3}\bar{\vartheta}\begin{bmatrix}k+P_1\\ l+Q_1\end{bmatrix}^4_{\bar{\phi}^{1-4}}\bar{\vartheta}\begin{bmatrix}k+P_2\\ l+Q_2\end{bmatrix}^4_{\bar{\phi}^{5-8}}.\end{aligned} \tag{4.10}$$

where we have reinserted the orthogonal component of ψ^μ and its ghost contribution and the theta functions corresponding to the $\bar{\eta}^i$ fields acquire a nonzero argument u_i. The total $U(1)_{i=1,2,3}$ traces will then be given as

$$\mathrm{Tr}U(1)_i = \frac{\partial_\nu}{2\pi}\frac{\partial_\omega}{2\pi}\frac{\partial_{u_i}}{2\pi}\tilde{Z}\Big|_{\nu,\omega,u_i=0} \tag{4.11}$$

where the derivatives $\partial_\nu\partial_\omega$ correspond to the correlator $\langle\psi\psi\psi\psi\rangle$, while ∂_{u_i} corresponds to the $U(1)_i$ contribution of $\langle\bar{J}^i\rangle$ in (3.13) acting on the partition function. The total anomalous $U(1)_A$ will be the combination of the three $U(1)_i$ according to (3.19).

V. EXPLICIT VS SPONTANEOUS SUSY BREAKING

The SUSY breaking $\mathcal{N}=1\to 0$ can happen in two ways: explicit or spontaneous (SSS) breaking. In the latter case, the GGSO phases can be set such that the gravitino acquires a mass and supersymmetry is broken spontaneously. The requirement is that the partition function, which will be nonzero in any point in the T_2 moduli space, vanishes when $T_2\to\infty$ such that the potential in that limit vanishes and supersymmetry is restored. Spontaneous breaking certainly has attractive features compared with the explicit case. As discussed in various works [13–15,54,59–61], when accompanied by massless Bose-Fermi degeneracy at some point in the moduli space, $N_b^0 = N_f^0$, we have the so-called "super no-scale" models in which the cosmological constant is exponentially suppressed according to [62]

$$\Lambda \propto \left(N_b^0 - N_f^0\right)\frac{1}{T_2^2} + \mathcal{O}\left(e^{-c\sqrt{T_2}}\right), \tag{5.1}$$

We note that without the $N_b^0 = N_f^0$ condition, SSS models have polynomial, rather than exponential, suppression of their one-loop cosmological constant. However, as noted in [26,29,30] such super no-scale conditions are merely necessary, not sufficient, conditions on the global structure of the effective potential, which will be crucially dependent on the full mass tower of states, including the non level-matched ones around special self-dual points in moduli space.

Despite the attractive features of such super no-scale models, we note that the cosmological constant problem remains an issue, also in these models. Even in the case of spontaneous SUSY-breaking, there are sectors in the additive group Ξ generated by the basis vectors that produce equal numbers of bosons and fermions [53]. The states from these sectors do not reside in supermultiplets as supersymmetry is broken. As usual the respective bosonic and fermionic states arise from the generic sectors, e.g., $\alpha\in\Xi$ and $S+\alpha\in\Xi$, and the bosonic and fermionic states from these sectors differ in some of their $U(1)$ charges, reflecting the fact that supersymmetry is broken. However, the phenomenological requirement still demands that the bosonic states from these sectors, that may, for example, correspond to the would be superpartners of the chiral generations, receive mass of the order of 1 TeV. Generating this mass splitting between will produce a cosmological constant. Similarly, the other mass scales in the Standard Model, e.g., the QCD scale, will contribute to the cosmological constant. The cosmological constant problem is therefore much more profound, indicating a fundamental dichotomy between quantum field theories expectations and gravitational observations, and it is naive to expect that the suppression observed in Eq. (5.1) can address the problem. We would ideally also consider higher loop contributions, e.g., the two-loop cosmological constant that should ideally be incorporated into this analysis. The fact is that the cosmological constant problem remains regardless of the SUSY-breaking mechanism. For this reason, we also explore some models with explicitly broken supersymmetry in our analysis. However, we also detail in the following how to identify the SSS models, which have their distinct phenomenological characteristics.

In the SSS SUSY-breaking the gravitino acquires a mass proportional to $\frac{1}{T_2}$, such that SUSY is restored at the border of the moduli space when $T_2\to\infty$ and the partition function vanishes. For this to happen, we require that in that limit the modular block in the partition function relating to the $\{\psi^\mu,\chi^{12},\chi^{34},\chi^{56}\}$ fermions gives rise to the Jacobi identity. To explore when this can happen, we set (h_i,g_i) indices to zero since for all cases when SUSY is restored from $\mathcal{N}=0$ to $\mathcal{N}\to 1, 2, 4$, the partition function Z will vanish.

The momenta in (4.5), through a redefinition of the summation variables $n_i\to n_i+m_i+H_i$ and fixing T_1, U_1 and U_2 at the free fermionic point, can be written as follows

$$\begin{aligned}\mathcal{P}_L &= \frac{1}{\sqrt{2T_2}}\left[\left(m_1+\frac{n_1}{2}+\frac{H_1}{2}-n_1T_2\right)\right.\\ &\qquad\left.+\,i\left(m_2+\frac{n_2}{2}+\frac{H_2}{2}-n_2T_2\right)\right]\\ \mathcal{P}_R &= \frac{1}{\sqrt{2T_2}}\left[\left(m_1+\frac{n_1}{2}+\frac{H_1}{2}+n_1T_2\right)\right.\\ &\qquad\left.+\,i\left(m_2+\frac{n_2}{2}+\frac{H_2}{2}+n_2T_2\right)\right].\end{aligned} \tag{5.2}$$

We can now observe that when $T_2\to\infty$ the only nonzero term in the moduli-dependent lattice sum (4.4) is for $n_i=0$ and in this limit it contributes according to

$$\Gamma_{2,2}^{(1)}\left[\begin{matrix}H_1 & H_2\\ G_1 & G_2\end{matrix}\middle|\begin{matrix}h_1\\ g_1\end{matrix}\right]\left(T_2^{(1)}\to\infty, T_{1*}^{(1)}, U_*^{(1)}\right)\to\sum_{m_i\in\mathbb{Z}}1, \tag{5.3}$$

where the sum, up to a normalization that will not affect our discussion, can be set to 1. Then the partition function (4.1) will factorize as

$$Z\big(T_2^{(1)}\to\infty,T_{1*}^{(1)},U_*^{(1)}\big)=\frac{1}{\eta^{10}\bar\eta^{22}}\frac{1}{2^{12}}\sum_{\substack{k,H_i,P_i\\ l,G_i,Q_i}}(-1)^{P_1Q_1+P_2Q_2}\bar\vartheta\begin{bmatrix}k\\ l\end{bmatrix}^8\bar\vartheta\begin{bmatrix}k+P_1\\ l+Q_1\end{bmatrix}^4\bar\vartheta\begin{bmatrix}k+P_2\\ l+Q_2\end{bmatrix}^4\Gamma_{2,2}^{(2)}\left[\begin{matrix}H_3 & H_4\\ G_3 & G_4\end{matrix}\middle|\begin{matrix}0\\ 0\end{matrix}\right]\big(T_*^{(2)},U_*^{(2)}\big)$$
$$\times\Gamma_{2,2}^{(3)}\left[\begin{matrix}H_5 & H_6\\ G_5 & G_6\end{matrix}\middle|\begin{matrix}0\\ 0\end{matrix}\right]\big(T_*^{(3)},U_*^{(3)}\big)\left((-1)^{\Phi\begin{bmatrix}0 & k & H_i & 0 & 0 & P_i\\ 0 & l & G_i & 0 & 0 & Q_i\end{bmatrix}}\vartheta_3^4\right.$$
$$\left.+(-1)^{1+\Phi\begin{bmatrix}1 & k & H_i & 0 & 0 & P_i\\ 0 & l & G_i & 0 & 0 & Q_i\end{bmatrix}}\vartheta_2^4+(-1)^{1+\Phi\begin{bmatrix}0 & k & H_i & 0 & 0 & P_i\\ 1 & l & G_i & 0 & 0 & Q_i\end{bmatrix}}\vartheta_4^4\right). \tag{5.4}$$

The SSS condition requires

$$Z(T_2^{(1)}\to\infty,T_{1*}^{(1)},U_*^{(1)})=0 \tag{5.5}$$

In order to vanish the phase has to satisfy

$$\sum_{\substack{H_1,H_2\\ G_1,G_2}}(-1)^{\Phi\begin{bmatrix}0 & k & H_i & 0 & 0 & P_i\\ 0 & l & G_i & 0 & 0 & Q_i\end{bmatrix}}=\sum_{\substack{H_1,H_2\\ G_1,G_2}}(-1)^{\Phi\begin{bmatrix}1 & k & H_i & 0 & 0 & P_i\\ 0 & l & G_i & 0 & 0 & Q_i\end{bmatrix}}=\sum_{\substack{H_1,H_2\\ G_1,G_2}}(-1)^{\Phi\begin{bmatrix}0 & k & H_i & 0 & 0 & P_i\\ 1 & l & G_i & 0 & 0 & Q_i\end{bmatrix}} \tag{5.6}$$

which translates into a set of intricate conditions and relations between the GGSO phases. In Appendix B we will show explicitly two examples, one satisfying the SSS conditions and the other with explicit SUSY breaking.

A. T-duality

For our choice of models specified by the basis set (2.10) T-duality is not always preserved. In particular, the symmetric shifts represented by the $\boldsymbol{e_i}$ basis vectors may spoil the original $SL(2;\mathbb{Z})_T$ symmetry associated to the moduli T. We will now show how, for an SSS model, T-duality can be broken. As already specified in Sec. IV, we will vary T_2 only associated to the first 2-torus.

The left and right momenta of the shifted lattice in (4.5) are left invariant under the following transformation

$$T\to-\frac{1}{4T}\leftrightarrow T_2\to\frac{1}{4T_2}. \tag{5.7}$$

However, the phase in the lattice (4.4) gets an additional term according to

$$e^{i\pi\sum_i((m_i+n_i+H_i)G_i}\to e^{i\pi\sum_i\left((m_i+n_i+H_i)G_i\right.}\times e^{i\pi(H_1G_1+H_2G_2\big)}, \tag{5.8}$$

which will generally break T-duality. The same result can also be obtained following the discussion of Sec. V. In the $T_2\to0$ limit, with the other moduli fixed, the only nonzero contributions from the lattice (4.4) are 1, setting $m_i=n_i=H_i=0$ in the lattice sum, $\sum_{m_i}1$, setting $n_i=-2m_i$, $H_i=0$, and $e^{i\pi(H_1G_1+H_2G_2)}$ with $m_i=0,n_i=-H_i$. The first two terms correspond, up to a normalization constant, to (5.3) which for an SSS model give a vanishing contribution. Meanwhile, the third contribution generates an additional phase

$$\Gamma_{2,2}^{(1)}\left[\begin{matrix}H_1 & H_2\\ G_1 & G_2\end{matrix}\middle|\begin{matrix}h_1\\ g_1\end{matrix}\right]\big(T_2^{(1)}\to0,T_{1*}^{(1)},U_*^{(1)}\big)\to e^{i\pi(H_1G_1+H_2G_2)}, \tag{5.9}$$

In order to impose T-duality (5.7), in addition to (5.5) for the SSS condition, we must then require

$$Z\big(T_2^{(1)}\to\infty,T_{1*}^{(1)},U_*^{(1)}\big)=Z\big(T_2^{(1)}\to0,T_{1*}^{(1)},U_*^{(1)}\big)=0, \tag{5.10}$$

with

$$Z(T_2^{(1)}\to 0, T_{1*}^{(1)}, U_*^{(1)}) = \frac{1}{\eta^{10}\bar\eta^{22}}\frac{1}{2^{12}}\sum_{\substack{k,H_i,P_i\\ l,G_i,Q_i}}(-1)^{P_1Q_1+P_2Q_2+H_1G_1+H_2G_2}\bar\vartheta\begin{bmatrix}k\\ l\end{bmatrix}^8\bar\vartheta\begin{bmatrix}k+P_1\\ l+Q_1\end{bmatrix}^4\bar\vartheta\begin{bmatrix}k+P_2\\ l+Q_2\end{bmatrix}^4$$
$$\times\Gamma_{2,2}^{(2)}\begin{bmatrix}H_3 & H_4 & 0\\ G_3 & G_4 & 0\end{bmatrix}(T_*^{(2)},U_*^{(2)})\Gamma_{2,2}^{(3)}\begin{bmatrix}H_5 & H_6 & 0\\ G_5 & G_6 & 0\end{bmatrix}(T_*^{(3)},U_*^{(3)})\left((-1)^{\Phi\begin{bmatrix}0 & k & H_i & 0 & 0 & P_i\\ 0 & l & G_i & 0 & 0 & Q_i\end{bmatrix}}\vartheta_3^4\right.$$
$$\left.+(-1)^{1+\Phi\begin{bmatrix}1 & k & H_i & 0 & 0 & P_i\\ 0 & l & G_i & 0 & 0 & Q_i\end{bmatrix}}\vartheta_2^4+(-1)^{1+\Phi\begin{bmatrix}0 & k & H_i & 0 & 0 & P_i\\ 1 & l & G_i & 0 & 0 & Q_i\end{bmatrix}}\vartheta_4^4\right). \tag{5.11}$$

As in (5.6), T-duality now requires

$$\sum_{\substack{H_1,H_2\\ G_1,G_2}}(-1)^{\Phi\begin{bmatrix}0 & k & H_i & 0 & 0 & P_i\\ 0 & l & G_i & 0 & 0 & Q_i\end{bmatrix}+H_1G_1+H_2G_2} = \sum_{\substack{H_1,H_2\\ G_1,G_2}}(-1)^{\Phi\begin{bmatrix}1 & k & H_i & 0 & 0 & P_i\\ 0 & l & G_i & 0 & 0 & Q_i\end{bmatrix}+H_1G_1+H_2G_2}$$
$$= \sum_{\substack{H_1,H_2\\ G_1,G_2}}(-1)^{\Phi\begin{bmatrix}0 & k & H_i & 0 & 0 & P_i\\ 1 & l & G_i & 0 & 0 & Q_i\end{bmatrix}+H_1G_1+H_2G_2}, \tag{5.12}$$

which again will correspond to specific constraints on the GGSO phases.

Models which satisfy (5.10) will then exhibit a SSS SUSY breaking with unbroken T-duality and the one-loop potential will then have the following behavior

$$V_{\text{one-loop}}(T_2) = V_{\text{one-loop}}\left(\frac{1}{4T_2}\right) \tag{5.13}$$

In particular the extrema of the potential, either a maximum or a minimum, will lie at the self-dual point $T_2 = \frac{1}{2}$.

We note that if instead of having $\boldsymbol{e_i}$ in our basis, we had $\boldsymbol{T_j} = \boldsymbol{e_{2j-1}} + \boldsymbol{e_{2j}}$, $j = 1, 2, 3$, as used in [26,29], the additional phase in (5.8) is modified according to $e^{i\pi(H_1G_1+H_2G_2)} \to e^{i\pi(H_1G_1+H_1G_1)} = 1$. Thus for these models, with the indices $H_i, G_i \in \mathbb{Z}$, T-duality (5.7) will always be satisfied. In Appendix B we will show how for an SSS model the T-duality conditions can be implemented.

VI. TACHYON PROJECTION

The tachyonic sectors in the models defined by the basis (2.10) and their projection conditions are much the same as detailed in Refs. [42,43]. The sectors and their mass levels are summarized in Table I.

In order to determine whether a sector survives the GGSO projections and remains in the spectrum we can construct a projector. For example, taking a sector with no oscillators, $|\boldsymbol{\beta}\rangle$, the survival/projection condition is encapsulated in the generalized projector

$$\mathbb{P}_{\boldsymbol{\beta}} = \prod_{\boldsymbol{\xi}\in\Upsilon(\boldsymbol{\beta})}\frac{1}{2}\left(1+\delta_{\boldsymbol{\beta}}C\begin{bmatrix}\boldsymbol{\beta}\\ \boldsymbol{\xi}\end{bmatrix}\right), \tag{6.1}$$

where

$$\delta_{\boldsymbol{\beta}} = \begin{cases} +1 & \text{if } \beta(\psi^\mu) = 0 \Leftrightarrow \text{sector is bosonic} \\ -1 & \text{if } \beta(\psi^\mu) = 1 \Leftrightarrow \text{sector is fermionic.} \end{cases} \tag{6.2}$$

TABLE I. Level-matched tachyonic sectors and their mass level, where $i \neq j \neq k = 1, \ldots, 6$ and $\bar\lambda^m$ is any right-moving complex fermion with NS boundary condition for the relevant tachyonic sector.

Mass level	Vectorials	Spinorials
$(-1/2, -1/2)$	$\{\bar\lambda^m\}\lvert 0\rangle$	$\boldsymbol{z_1}, \boldsymbol{z_2}$
$(-3/8, -3/8)$	$\{\bar\lambda^m\}\boldsymbol{e_i}$	$\boldsymbol{e_i}+\boldsymbol{z_1}, \boldsymbol{e_i}+\boldsymbol{z_2}$
$(-1/4, -1/4)$	$\{\bar\lambda^m\}\boldsymbol{e_i}+\boldsymbol{e_j}$	$\boldsymbol{e_i}+\boldsymbol{e_j}+\boldsymbol{z_1}, \boldsymbol{e_i}+\boldsymbol{e_j}+\boldsymbol{z_2}$
$(-1/8, -1/8)$	$\{\bar\lambda^m\}\boldsymbol{e_i}+\boldsymbol{e_j}+\boldsymbol{e_k}$	$\boldsymbol{e_i}+\boldsymbol{e_j}+\boldsymbol{e_k}+\boldsymbol{z_1}, \boldsymbol{e_i}+\boldsymbol{e_j}+\boldsymbol{e_k}+\boldsymbol{z_2}$

The $\Upsilon(\boldsymbol{\beta})$ is defined as a minimal linearly independent set of vectors $\boldsymbol{\xi}$ such that $\boldsymbol{\xi}\cap\boldsymbol{\beta}=\emptyset$. To check whether the sector $\boldsymbol{\beta}$ is projected simply amounts to checking $\mathbb{P}_{\boldsymbol{\beta}}=0$.

In the presence of a single right-moving oscillator $\bar{\lambda}$ with $\nu_f=\frac{1}{2}$, the generalized projector is modified to

$$\mathbb{P}_{\boldsymbol{\beta}}=\prod_{\boldsymbol{\xi}\in\Upsilon(\boldsymbol{\beta})}\frac{1}{2}\left(1+\delta_{\boldsymbol{\beta}}\delta_{\boldsymbol{\xi}}^{\bar{\lambda}}C\begin{bmatrix}\boldsymbol{\beta}\\ \boldsymbol{\xi}\end{bmatrix}\right), \tag{6.3}$$

such that

$$\delta_{\boldsymbol{\xi}}^{\bar{\lambda}}=\begin{cases}+1 & \text{if } \bar{\lambda}\in\boldsymbol{\xi}\\ -1 & \text{if } \bar{\lambda}\notin\boldsymbol{\xi}\end{cases}. \tag{6.4}$$

In order to build tachyon-free models we simply require that $\mathbb{P}_{\boldsymbol{t}}=0$ for all tachyonic sectors, $\boldsymbol{t}$. This requires defining the projection sets $\Upsilon(\boldsymbol{t})$ for each tachyonic sector of Table I. For example, the tachyonic states from $\boldsymbol{z_1}$ have the projection set

$$\Upsilon(\boldsymbol{z_1})=\{\boldsymbol{S},\boldsymbol{e_1},\boldsymbol{e_2},\boldsymbol{e_3},\boldsymbol{e_4},\boldsymbol{e_5},\boldsymbol{e_6},\boldsymbol{x},\boldsymbol{z_2}\}. \tag{6.5}$$

Checking for the absence of tachyonic sectors then amounts to checking the GGSO phases associated to such projection sets for all tachyonic sectors.

VII. CHIRAL SECTOR ANALYSIS

Having explained how to get $\mathrm{Tr}U(1)_A$ directly from the partition function in subsection (IV A) we now explain how this can be calculated more efficiently by analysis of the sectors that produce massless states that can be charged under $U(1)_A$. This is equivalent to inspecting chiral sectors giving rise to states that are charged under the complex $\bar{\eta}^{1,2,3}$ world sheet fermion fields, where we recall that the charge of a free fermion is given by

$$Q(f)=\frac{1}{2}\alpha(f)+F(f), \tag{7.1}$$

and the action of the fermion number operator is

$$F:\begin{cases}f|0\rangle_{NS}=+1\\ f^*|0\rangle_{NS}=-1\\ |+\rangle=0\\ |-\rangle=-1\end{cases}, \tag{7.2}$$

where we write the two helicities of the degenerate Ramond vacuum as $|\pm\rangle$.

The first sectors we can identify with nontrivial chirality come from

$$\begin{aligned}\boldsymbol{F}^1_{pqrs}&=\boldsymbol{S}+\boldsymbol{b_1}+p\boldsymbol{e_3}+q\boldsymbol{e_4}+r\boldsymbol{e_5}+s\boldsymbol{e_6}\\ &=\{\psi^\mu,\chi^{1,2},(1-p)y^3\bar{y}^3,pw^3\bar{w}^3,(1-q)y^4\bar{y}^4,qw^4\bar{w}^4\\ &\quad(1-r)y^5\bar{y}^5,rw^5\bar{w}^5,(1-s)y^6\bar{y}^6,sw^6\bar{w}^6,\bar{\eta}^1,\bar{\psi}^{1,\dots,5}\}\\ \boldsymbol{F}^2_{pqrs}&=\boldsymbol{S}+\boldsymbol{b_2}+p\boldsymbol{e_1}+q\boldsymbol{e_2}+r\boldsymbol{e_5}+s\boldsymbol{e_6}\\ \boldsymbol{F}^3_{pqrs}&=\boldsymbol{S}+\boldsymbol{b_3}+p\boldsymbol{e_1}+q\boldsymbol{e_2}+r\boldsymbol{e_3}+s\boldsymbol{e_4},\end{aligned} \tag{7.3}$$

which we note are the sectors that generate the spinorial $\mathbf{16}/\overline{\mathbf{16}}$'s of our $SO(10)$ GUT, although we will not be interested in this aspect of our models in this work.

Associated to these sectors are the projection sets

$$\begin{aligned}\Upsilon(\boldsymbol{F}^1_{pqrs})&=\{\boldsymbol{z_1},\boldsymbol{z_2},\boldsymbol{e_1},\boldsymbol{e_2}\}\\ \Upsilon(\boldsymbol{F}^2_{pqrs})&=\{\boldsymbol{z_1},\boldsymbol{z_2},\boldsymbol{e_3},\boldsymbol{e_4}\}\\ \Upsilon(\boldsymbol{F}^3_{pqrs})&=\{\boldsymbol{z_1},\boldsymbol{z_2},\boldsymbol{e_5},\boldsymbol{e_6}\},\end{aligned} \tag{7.4}$$

which are used to determine whether a sector remains in the Hilbert space of the model, just as explained for the tachyonic sectors in the previous section.

Once we have checked the survival of a particular sector, we can then determine the chirality of the $\bar{\eta}^{1,2,3}$ for the resultant states through the chirality projections defined for the three orbifold planes as follows

$$\begin{aligned}\chi(\boldsymbol{F}^1_{pqrs})&=\mathrm{ch}(\bar{\eta}^1)=-\mathrm{ch}(\psi^\mu)C\begin{bmatrix}\boldsymbol{F}^1_{pqrs}\\ \boldsymbol{S}+\boldsymbol{b_2}+\boldsymbol{x}+(1-r)\boldsymbol{e_5}+(1-s)\boldsymbol{e_6}\end{bmatrix}^*\\ \chi(\boldsymbol{F}^2_{pqrs})&=\mathrm{ch}(\bar{\eta}^2)=-\mathrm{ch}(\psi^\mu)C\begin{bmatrix}\boldsymbol{F}^2_{pqrs}\\ \boldsymbol{S}+\boldsymbol{b_1}+\boldsymbol{x}+(1-r)\boldsymbol{e_5}+(1-s)\boldsymbol{e_6}\end{bmatrix}^*\\ \chi(\boldsymbol{F}^3_{pqrs})&=\mathrm{ch}(\bar{\eta}^3)=-\mathrm{ch}(\psi^\mu)C\begin{bmatrix}\boldsymbol{F}^3_{pqrs}\\ \boldsymbol{S}+\boldsymbol{b_1}+\boldsymbol{x}+(1-r)\boldsymbol{e_3}+(1-s)\boldsymbol{e_4}\end{bmatrix}^*.\end{aligned} \tag{7.5}$$

Without loss of generality we can choose $\mathrm{ch}(\psi^\mu)=|+\rangle$ since the CPT-conjugates are necessarily present with the opposite chirality choice. This then fully determines the charges under the $U(1)_{1,2,3}$ for these sectors.

Along with these 3 groups of 16 sectors we have a further 12 such groups $\boldsymbol{F}^{4,5,6}_{pqrs},\boldsymbol{F}^{7,8,9}_{pqrs}$ and $\boldsymbol{V}^{1,2,3}_{pqrs}$ but we relegate the details of how to extract their charge contributions to Appendix C. As with the analysis of the tachyonic sectors, the projection conditions that determine the overall $\mathrm{Tr}U(1)_A$ can then be computerized to allow for efficient scans of large spaces of different GGSO phase configurations. The results from such a scan is presented in the next section along with the results of our analysis of the D-term uplifted potentials.

As mentioned above, in this work we will not consider extra phenomenological issues in our models such as the number of spinorial 16's or vectorial 10's classified in previous works for these symmetric $\mathbb{Z}_2 \times \mathbb{Z}_2$ models [7,34–41]. One may wonder about the relationship between such characteristics and the value of $\mathrm{Tr}U(1)_A$. Since viable $SO(10)$ phenomenology requires a condition such as $N_{\mathbf{16}} - N_{\overline{\mathbf{16}}} \geq 6$ we could expect some relationship between the trace values and the presence of an appropriate number of these representations. However, since $\boldsymbol{F}^{4,5,6,7,8,9}_{pqrs}$ are hidden sectors and the vectorials $\boldsymbol{V}^{1,2,3}_{pqrs}$ would only be constrained by the presence of at least vectorial 10 we do not expect there to be any significant change to whether we can find D-term uplifted models once we incorporate such phenomenological considerations into the analysis.

VIII. RESULTS

The methodology we use for the extraction of D-term uplifted models defined through the basis (2.10) follows the 5 step procedure:

(1) Extract $\mathcal{N} = 0$ models, by checking whether Eq. (2.13) and/or Eq. (2.14) are violated. Subsequently, we check that the models are free from physical tachyons by checking the projection conditions outlined in Sec. VI.
(2) Compute the values of $\mathrm{Tr}U(1)_A$ using the efficient analysis of chiral sectors explained in Sec. VII and Appendix C for these $\mathcal{N} = 0$ tachyon-free models.
(3) Extract out those models with larger $\mathrm{Tr}U(1)_A$ values and satisfying the SSS SUSY breaking conditions discussed in Sec. V.
(4) Perform the numerical analysis of the one-loop potentials and check for an uplift from AdS to dS for these SSS models with large $\mathrm{Tr}U(1)_A$.
(5) The models with explicit SUSY breaking but large $\mathrm{Tr}U(1)_A$ can then be analyzed and checked for an uplift.

We note that the key bottleneck in this methodology is performing the numerical analysis of the one-loop potential integral(s). Depending on how many points with respect to T_2 are evaluated, the number of models we can analyze the potentials for in reasonable computing time is approximately only $\mathcal{O}(10^3)$. This helps to motivate the 5 step procedure above that seeks to maximize the probability we find an uplifted model, with or without SSS breaking.

A. Distribution of $\mathrm{Tr}U(1)_A$ for $\mathcal{N} = 0$ models

For our purpose of finding an uplifted model it was sufficient in step 1. to take a random scan of 10^9 $\mathcal{N} = 0$ GGSO configurations and checking them for the absence of physical tachyons using the conditions explained in Sec. VI. This scan resulted in $\sim 1.6 \times 10^7$ tachyon-free models. Following step 2. of the methodology, we then calculated the values of $\mathrm{Tr}U(1)_A$ for these models. The results for the distribution of these values are shown in Fig. 1.

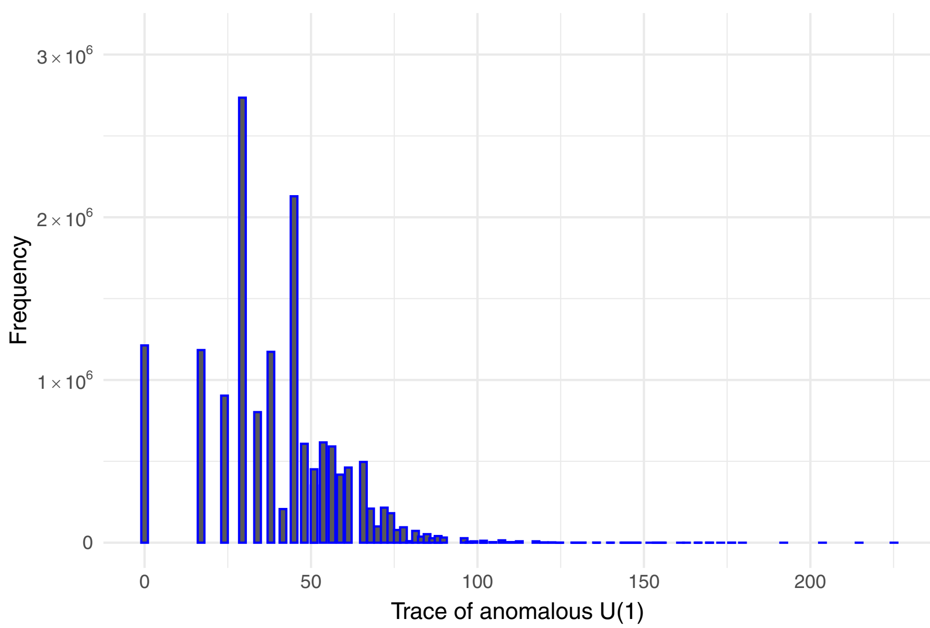

FIG. 1. The distribution of Tr $U(1)_A$.

B. Example Scherk-Schwarz models

Once the $\mathrm{Tr}U(1)_A$'s for tachyon-free models were founds, we moved to step 3. to begin the analysis of the one-loop potential starting with those models that had a larger value for $\mathrm{Tr}U(1)_A$ and also satisfied the SSS condition derived in Sec. V. Then we perform step 4. to search for the D-term uplift for these larger $\mathrm{Tr}U(1)_A$ SSS models.

In a scan of approximately 10^3 such models we did indeed find a single model that exhibited the desired uplift. This model is defined through the following set of GGSO phases

$$C\begin{bmatrix} v_i \\ v_j \end{bmatrix} = \begin{array}{c} \\ \mathbf{1} \\ \boldsymbol{S} \\ \boldsymbol{e_1} \\ \boldsymbol{e_2} \\ \boldsymbol{e_3} \\ \boldsymbol{e_4} \\ \boldsymbol{e_5} \\ \boldsymbol{e_6} \\ \boldsymbol{b_1} \\ \boldsymbol{b_2} \\ \boldsymbol{z_1} \\ \boldsymbol{z_2} \end{array} \begin{array}{c} \begin{array}{cccccccccccc} \mathbf{1} & \boldsymbol{S} & \boldsymbol{e_1} & \boldsymbol{e_2} & \boldsymbol{e_3} & \boldsymbol{e_4} & \boldsymbol{e_5} & \boldsymbol{e_6} & \boldsymbol{b_1} & \boldsymbol{b_2} & \boldsymbol{z_1} & \boldsymbol{z_2} \end{array} \\ \begin{pmatrix} -1 & -1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & -1 & -1 & 1 \\ -1 & -1 & 1 & 1 & -1 & -1 & -1 & -1 & 1 & -1 & 1 & -1 \\ -1 & 1 & 1 & -1 & -1 & -1 & 1 & 1 & 1 & -1 & -1 & -1 \\ 1 & 1 & -1 & -1 & 1 & 1 & 1 & 1 & 1 & -1 & -1 & 1 \\ -1 & -1 & -1 & 1 & 1 & 1 & 1 & -1 & 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & 1 & 1 & -1 & 1 & -1 & 1 & 1 & 1 & 1 \\ -1 & -1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 & -1 & -1 & 1 & -1 & -1 & -1 & 1 & 1 \\ -1 & -1 & 1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & 1 \\ -1 & 1 & -1 & -1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & 1 \\ -1 & 1 & -1 & -1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & 1 \\ 1 & -1 & -1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix} \end{array} \tag{8.1}$$

and has a one-loop cosmological constant value of $\Lambda = -0.000212496$ at the free fermionic point and $\mathrm{Tr}U(1)_A = 72\sqrt{2}$, with a FI contribution of 0.00144365. Performing the numerical analysis allowed us to graph the potential and demonstrate its uplift, as shown in Fig. 2.

Through an analysis of $\mathcal{O}(10^3)$ GGSO configurations, we also evaluated the one-loop potential shapes for SSS broken models. We summarize these possibilities in Figs. 3–8. We find two examples of SSS models with unbroken T-duality, see Figs. 3 and 4, and two with broken T-duality, see Figs. 5 and 6, where the difference can be clearly seen from the behavior of the potential when the modulus approaches zero. Figures 7 and 8 instead show two SSS models with broken T-duality and with no minima or maxima.

The models we are especially interested in for our purpose are the ones with a negative minima. In these cases the additional D-term contribution (3.22) typically is not sufficient to uplift the minima. It is only in very rare,

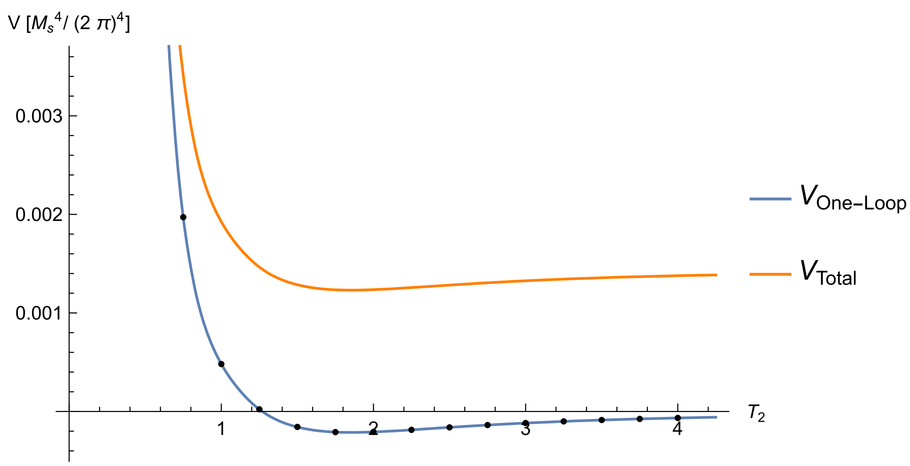

FIG. 2. One-loop potential of SSS example model with uplift.

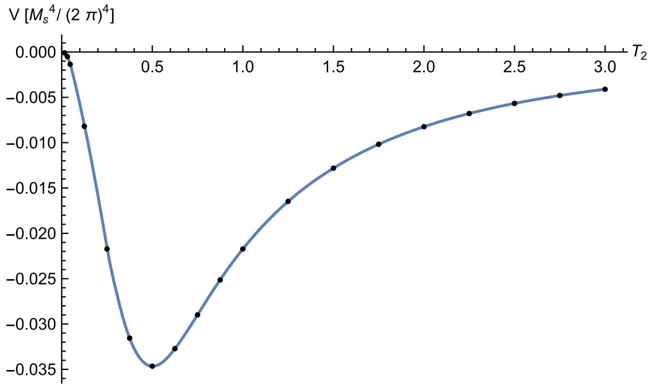

FIG. 3. One-loop Scherk-Schwarz potential with local minimum and unbroken T-duality.

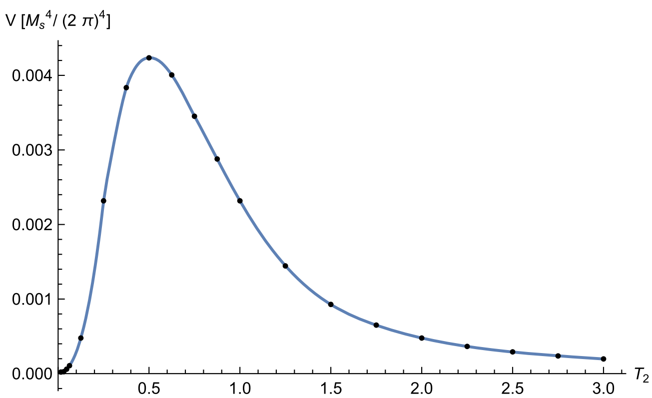

FIG. 4. One-loop Scherk-Schwarz potential with local maximum and unbroken T-duality.

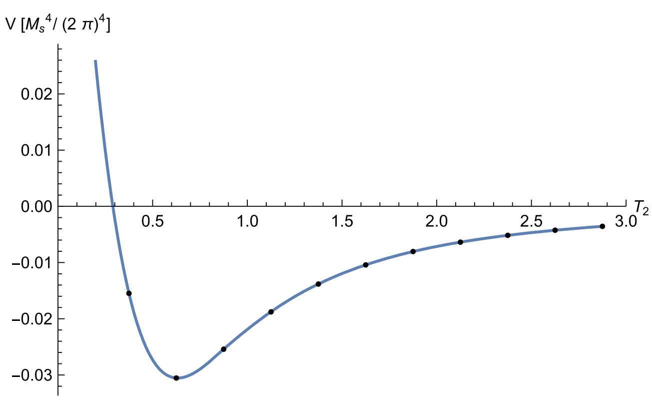

FIG. 5. One-loop Scherk-Schwarz potential with local minimum and broken T-duality.

finely tuned, examples that this uplift is observed. Finally, we note that although we projected out physical tachyons at the free fermionic point, in general there may be tachyons at other points in the moduli space. However, through analysis across the range of the modulus T_2 we find that none of our potentials suffer tachyonic instabilities.

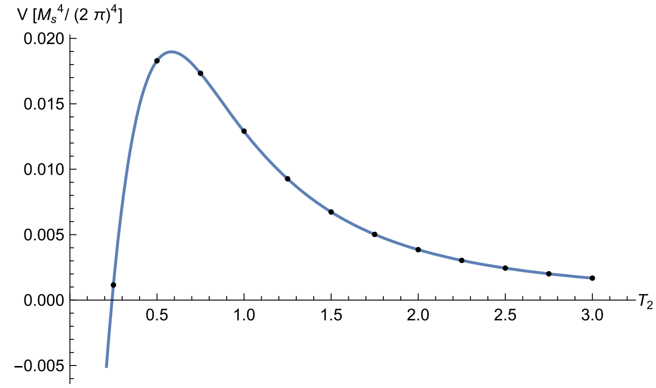

FIG. 6. One-loop Scherk-Schwarz potential with local maximum and broken T-duality.

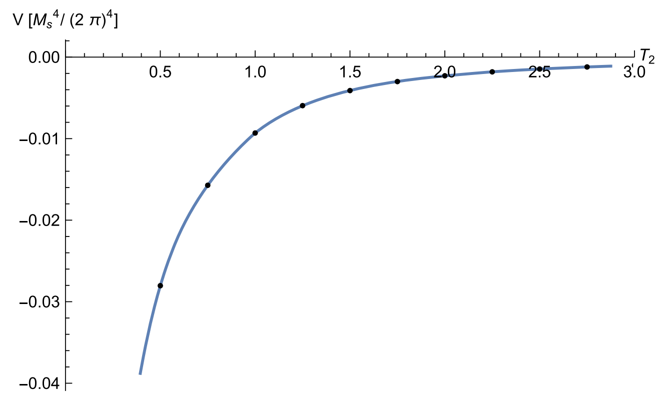

FIG. 7. One-loop Scherk-Schwarz potential without any extreme point and broken T-duality.

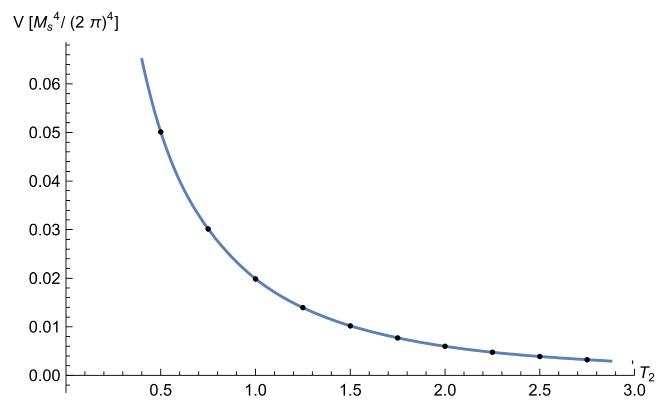

FIG. 8. One-loop Scherk-Schwarz potential without any extreme point and broken T-duality.

C. Explicitly broken model examples

Performing a similar scan of large $\mathrm{Tr}U(1)_A$ models but with explicit breaking in step 5, we also find models in which the FI D-term uplifts the one-loop potential. An example of such a model with explicit SUSY-breaking is given by the GGSO phases

$$C\begin{bmatrix} v_i \\ v_j \end{bmatrix} = \begin{array}{c} \\ \mathbf{1} \\ \boldsymbol{S} \\ \boldsymbol{e}_1 \\ \boldsymbol{e}_2 \\ \boldsymbol{e}_3 \\ \boldsymbol{e}_4 \\ \boldsymbol{e}_5 \\ \boldsymbol{e}_6 \\ \boldsymbol{b}_1 \\ \boldsymbol{b}_2 \\ \boldsymbol{z}_1 \\ \boldsymbol{z}_2 \end{array} \begin{array}{c} \begin{array}{cccccccccccc} \mathbf{1} & \boldsymbol{S} & \boldsymbol{e}_1 & \boldsymbol{e}_2 & \boldsymbol{e}_3 & \boldsymbol{e}_4 & \boldsymbol{e}_5 & \boldsymbol{e}_6 & \boldsymbol{b}_1 & \boldsymbol{b}_2 & \boldsymbol{z}_1 & \boldsymbol{z}_2 \end{array} \\ \left(\begin{array}{cccccccccccc} -1 & -1 & 1 & 1 & -1 & 1 & -1 & 1 & 1 & 1 & -1 & 1 \\ -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & 1 & -1 & -1 & 1 \\ 1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & 1 & -1 & 1 & 1 \\ -1 & -1 & -1 & -1 & 1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\ 1 & -1 & 1 & -1 & -1 & -1 & 1 & 1 & -1 & -1 & -1 & -1 \\ -1 & -1 & 1 & -1 & -1 & 1 & 1 & -1 & 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 & -1 & 1 & -1 & -1 & 1 & -1 & -1 & -1 \\ 1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & 1 & 1 & 1 & -1 \\ 1 & 1 & 1 & -1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 & -1 \\ -1 & -1 & -1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 & -1 & -1 \\ 1 & 1 & -1 & 1 & -1 & -1 & 1 & -1 & -1 & -1 & -1 & 1 \end{array}\right) \end{array} \tag{8.2}$$

corresponding to a model with one-loop cosmological constant $\Lambda = -0.000785598$ and $\mathrm{Tr}U(1)_A = 72\sqrt{2}$ that generates a FI contribution of 0.00144365 to the one-loop potential, ensuring a positive minimum as depicted in Fig. 9.

Also in this case, through analysis of $\mathcal{O}(10^3)$ models with explicitly broken SUSY, we find only certain possibilities for the shapes of the potential. We summarize these possibilities in Figs. 10–15. In all these graphs, when the moduli $T_2 \to \infty$, the potential diverges so SUSY is broken explicitly.

In particular in Figs. 10 and 12 the minima is positive while in Fig. 11 only a negative maxima is present. In Figs. 13 and 15 instead there are no extremal points at all. These are not the cases we are interested in. Only the shapes of the potential as in Fig. 13 present a negative minima required for the uplift.

IX. DISCUSSION AND CONCLUSIONS

One important, and generic, issue with non-SUSY theories is the issue of a nonvanishing dilaton tadpole within such models with a nonzero cosmological constant. This means that the string equations of motion are not satisfied for our theory on a Minkowskian 4D spacetime with a constant dilaton. If we want to find the true perturbative quantum vacuum, we would need to solve the string equations of motion to all loop levels. However, for non-SUSY strings we generically lose computational control at higher orders in the string loop expansion.

We note that in that respect the analysis performed in this paper is rather heuristic. The dilaton VEV was fixed by hand and most of the moduli are set at the free fermionic point but are not fixed, i.e., they can be varied away from the free fermionic point. Similarly, the two-loop

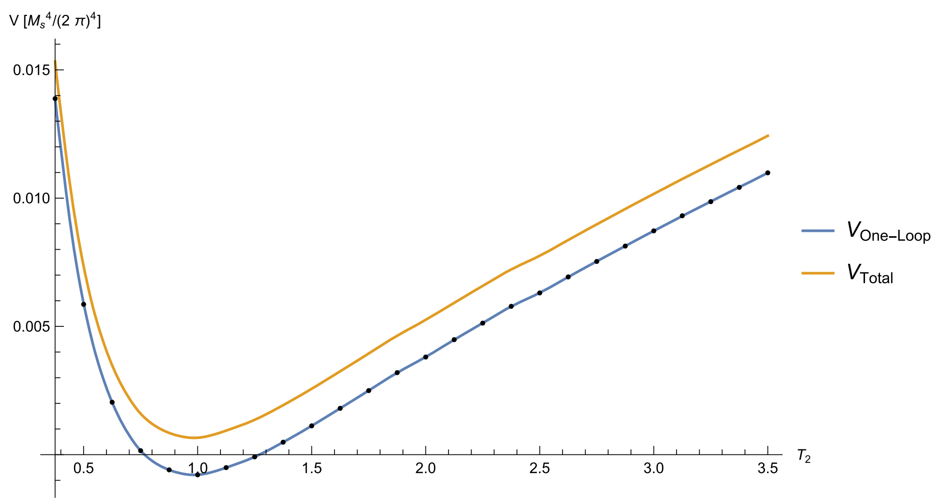

FIG. 9. One-loop potential of example model with explicitly broken SUSY with uplift.

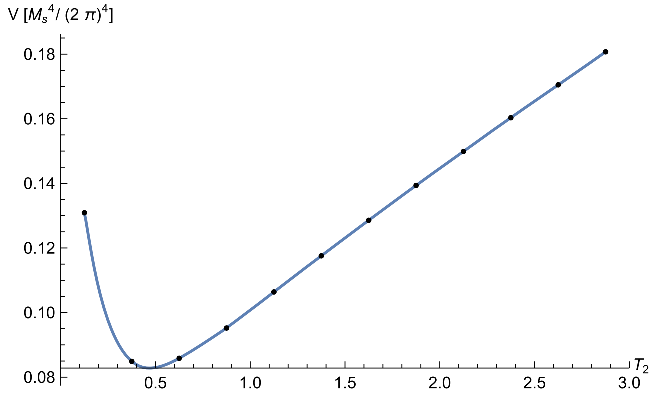

FIG. 10. One-loop potential with explicitly broken SUSY and local minimum.

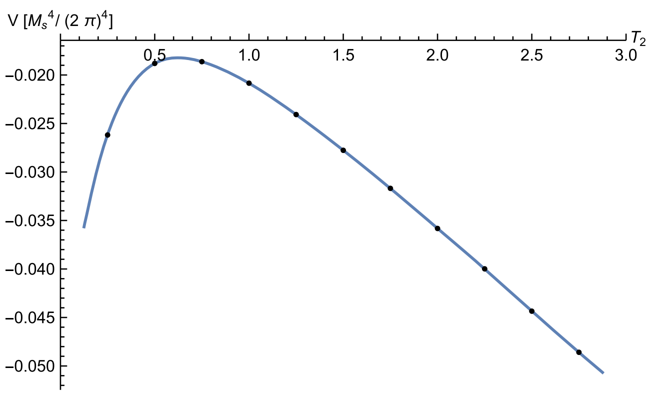

FIG. 11. One-loop potential with explicitly broken SUSY and local maximum.

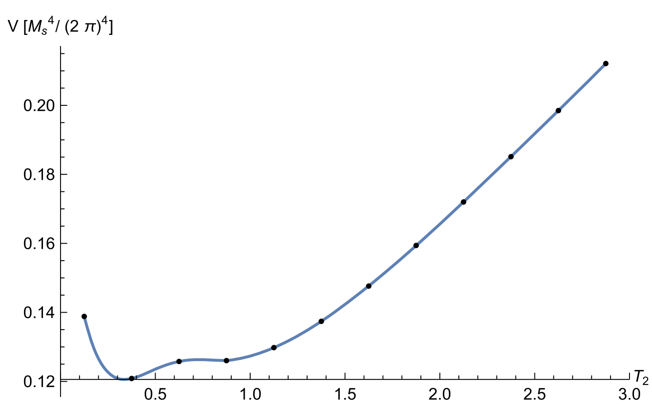

FIG. 12. One-loop potential with explicitly broken SUSY and local minima and maximum.

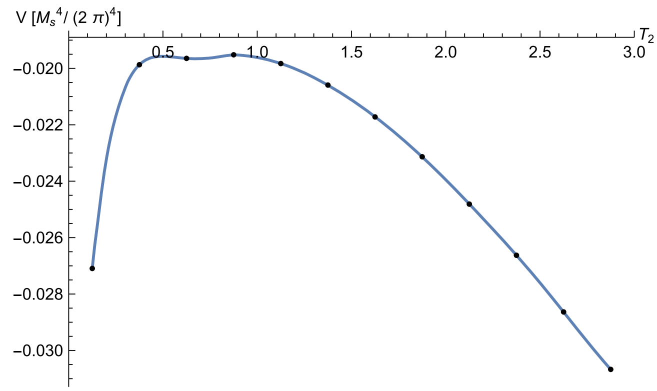

FIG. 13. One-loop potential with explicitly broken SUSY and local minimum and maxima.

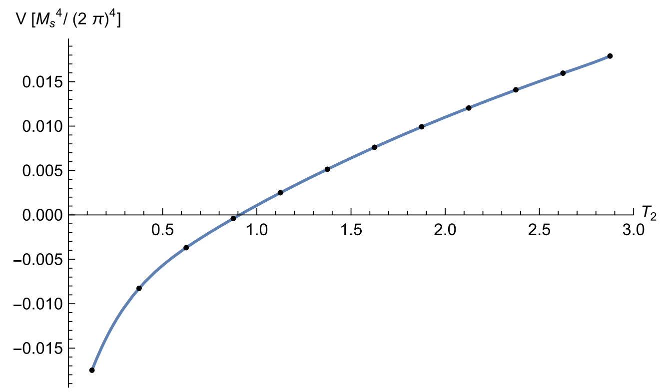

FIG. 14. One-loop potential with explicitly broken SUSY without any extreme point.

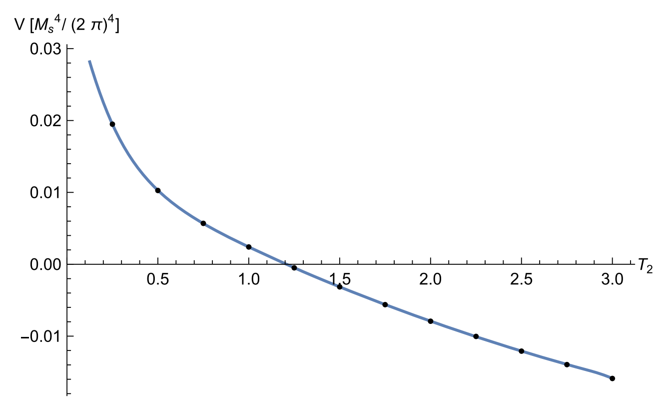

FIG. 15. One-loop potential with explicitly broken SUSY without any extreme point.

contribution to the vacuum energy was adopted from the SUSY cases and not carried our explicitly. We remark that while our arguments here are, in these respects, not conclusive, the analysis can be made more rigorous by adopting the Kiritsis-Kounnas modular invariant regularization method of one-loop string amplitudes [46]. In that scheme, the four dimensional spacetime curvature is used as a infrared regulator, and is achieved by compactifying the four dimensional spacetime on a Wess-Zumino-Witten current algebra sigma model. The simplest solution is obtained with the conformal theory $W_k^{(4)} = SU(2)_k \times U(1)$, which has the asymptotic (large k) geometry $S^3 \times R^+$. The heterotic string constructions with a mass gap for the massless spectrum is then constructed by substituting the world sheet coordinate and the spin-fields of the uncompactified Minkowski spacetime conformal and superconformal blocks of appropriate world sheet supersymmetry and central charge. It should be noted that while this regularization methodology was developed for vacua with $\mathcal{N}=1$ spacetime supersymmetry, its adaptation to non-SUSY string vacua should be possible. This scheme therefore provide the tools to construct non-SUSY string solutions with regulated infrared divergences. Similarly, while perhaps presenting a substantial technical challenge, the two-loop costribution to the vacuum energy can in principle be calculated directly in string theory [20,61]. Thus, the somewhat heuristic arguments made here can, in principle, be put on firmer grounds.

We also remark that the question of moduli stabilization can also be partially addressed directly in the heterotic string world sheet constructions. String theory allows for asymmetric boundary conditions of the left- and right-moving world sheet internal fermions, which the corresponding bosonic representation entail an asymmetric action on the internal coordinates of the compactified six dimensional space. As a consequence, some or even all of the moduli fields that parametrize the properties of the internal manifold can be projected out [63] and be frozen at a specific value, typically at the self-dual point in the moduli space. In that case, the space of unfixed moduli is substantially reduced. One can envision that stabilization of the remaining unfixed moduli and extraction of the global minimum is possible. The question whether it is possible to obtain a model with positive vacuum energy and fixed moduli with or without a D-term uplift may be further investigated.

In this paper we explored the contribution of the would be FI D-term to the vacuum energy in non-SUSY heterotic string vacua. This contribution can uplift the vacuum energy from a negative to positive value and give rise to a positive constant cosmological background. We distinguished in our analysis between string vacua with explicit SUSY breaking versus vacua in which SUSY is broken spontaneously by the Scherk-Schwarz mechanism. We found that while rare, a D-term uplift to a positive cosmological constant might indeed be possible.

ACKNOWLEDGMENTS

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APPENDIX A: TRANSLATION OF FERMIONIC PARTITION FUNCTION

The goal of this appendix is to find a one-to-one correspondence between the partition functions written in the free fermionic construction, (2.4), and in the orbifold language. An outline of this procedure can be found in [57] and is also presented in [64].

The aim of this procedure is to find an equality of the form

$$\begin{aligned} Z_F &= \frac{1}{2^N}\sum_{\boldsymbol{\alpha},\boldsymbol{\beta}} C\begin{bmatrix}\boldsymbol{\alpha}\\ \boldsymbol{\beta}\end{bmatrix} Z\begin{bmatrix}\boldsymbol{\alpha}\\ \boldsymbol{\beta}\end{bmatrix} \\ &= \sum_{\substack{a,k,\cdots\\ b,l,\cdots}} (-1)^{\Psi\begin{bmatrix}a & k & \cdots\\ b & l & \cdots\end{bmatrix}} Z\begin{bmatrix}a & k & \cdots\\ b & l & \cdots\end{bmatrix}, \end{aligned} \tag{A1}$$

where the product over the fermions is now implicit and contained within $Z[^{\boldsymbol{\alpha}}_{\boldsymbol{\beta}}]$. The right-hand side requires further comments. The term $Z\begin{bmatrix}a & k & \cdots\\ b & l & \cdots\end{bmatrix}$ represents the theta functions in terms of the summation indices $a, b, k, l, \ldots$. The phase $(-1)^{\Psi\begin{bmatrix}a & k & \cdots\\ b & l & \cdots\end{bmatrix}}$ is the analog of the GGSO phase in this new formulation.

1. The modular invariant phase

To see all possible choices of indices, which in turn fix the form of $Z\begin{bmatrix}a & k & \cdots\\ b & l & \cdots\end{bmatrix}$, we note that to represent a partition function of a model with N basis vectors requires the use of N summation indices. This can be seen by matching the number of terms on each side of (A1). Thus, the translation of the form of the partition function is uniquely determined by the choice of a change of basis matrix, S, which encodes the correspondence between the basis vectors and the summation indices. For our choice of models specified by the basis vectors (2.10) and the summation indices in the partition function as in (4.1), the S matrix is given by

$$S = \begin{pmatrix} 1&1&1&1&1&1&1&1&0&0&0&0\\ 1&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0&0&0&0&0\\ 0&0&0&1&0&0&0&0&0&0&0&0\\ 0&0&0&0&1&0&0&0&0&0&0&0\\ 0&0&0&0&0&1&0&0&0&0&0&0\\ 0&0&0&0&0&0&1&0&0&0&0&0\\ 0&0&0&0&0&0&0&1&0&0&0&0\\ 0&1&0&0&0&0&0&0&0&1&1&1\\ 0&1&0&0&0&0&0&0&1&0&1&1\\ 0&0&0&0&0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&0&0&0&0&1 \end{pmatrix} \tag{A2}$$

All invertible $N \times N$ matrices whose entries take values in $\mathbb{Z}_2$ are valid choices. However, the above choice is the one which best illuminates the geometry of the underlying compactification. Choosing $S = I_N$, the N-dimensional identity matrix, would render the translation trivial and the form of $Z\begin{bmatrix} a & k & \cdots \\ b & l & \cdots \end{bmatrix}$ and $(-1)^{\Psi\begin{bmatrix} a & k & \cdots \\ b & l & \cdots \end{bmatrix}}$ would match that of $Z[^{\alpha}_{\beta}]$ and $C(^{\alpha}_{\beta})$, respectively, modulo some subtleties we discuss in the following section.

Once S is specified, the partition function is written in its index form and we can start making the connection between the GGSO phases $C[^{\alpha}_{\beta}]$ and the modular invariant phase Ψ. We assume that Ψ can be expressed as a polynomial in the summation variables. Then, two-loop modular invariance imposed on the GGSO phases via the rule

$$C\begin{bmatrix} \boldsymbol{v_i} \\ \boldsymbol{v_j} + \boldsymbol{v_k} \end{bmatrix} = \delta_{\boldsymbol{v_i}} C\begin{bmatrix} \boldsymbol{v_i} \\ \boldsymbol{v_j} \end{bmatrix} C\begin{bmatrix} \boldsymbol{v_i} \\ \boldsymbol{v_k} \end{bmatrix}, \tag{A3}$$

implies that $\Psi\begin{bmatrix} a & k & H_i & h_i & P_i \\ b & l & G_i & g_i & Q_i \end{bmatrix}$ is at most second order in its variables. Moreover, the presence of $\delta_{\boldsymbol{v_i}}$ restricts the first-order terms. That is Ψ must include a term a and cannot contain other terms like it. More precisely, (A3) implies

$$\begin{cases} \Psi \ni a, \\ \Psi \not\ni k, h_i, H_i, P_i, \end{cases} \tag{A4}$$

where we take "$\in$" to mean a term in the sum. These conditions can be implemented in a compact form by requiring the phase to be of the form

$$\Psi\begin{bmatrix} a & k & H_i & h_i & P_i \\ b & l & G_i & g_i & Q_i \end{bmatrix} = a + \beta_i \Delta_i + \Gamma_i \Omega_{ij} \Delta_j, \tag{A5}$$

where we defined

$$\begin{aligned} \Gamma &= (a, k, H_1, H_2, H_3, H_4, H_5, H_6, h_1, h_2, P_1, P_2), \\ \Delta &= (b, l, G_1, G_2, G_3, G_4, G_5, G_6, g_1, g_2, Q_1, Q_2), \end{aligned} \tag{A6}$$

to be the vectors containing top and bottom indices respectively.

We now impose one-loop modular invariance by requiring that the partition function (A1) remains invariant under S and T-transformations, under which the theta functions transform as

$$\begin{aligned} S\colon\ \vartheta\begin{bmatrix} a \\ b \end{bmatrix} &\to e^{i\pi ab/2}\vartheta\begin{bmatrix} b \\ -a \end{bmatrix}, \\ T\colon\ \vartheta\begin{bmatrix} a \\ b \end{bmatrix} &\to e^{i\pi a(a-2)/4}\vartheta\begin{bmatrix} a \\ a+b-1 \end{bmatrix}. \end{aligned} \tag{A7}$$

By using a compact notation for the theta and eta function terms as in (A1), i.e.

$$\begin{aligned} Z_F = \frac{1}{2^2}\sum_{\substack{a,k \\ b,l}} \frac{1}{2^{10}} \sum_{\substack{H_i,h_i,P_i \\ G_i,g_i,Q_i}} (-1)^{\Psi\begin{bmatrix} a & k & H_i & h_i & P_i \\ b & l & G_i & g_i & Q_i \end{bmatrix}} \\ \times Z\begin{bmatrix} a & k & H_i & h_i & P_i \\ b & l & G_i & g_i & Q_i \end{bmatrix}, \end{aligned} \tag{A8}$$

we can express the modular transformations more readily. In particular, under modular transformations

$$\begin{aligned} Z\begin{bmatrix} a & k & H_i & h_i & P_i \\ b & l & G_i & g_i & Q_i \end{bmatrix} &\xrightarrow{S} Z\begin{bmatrix} b & l & G_i & g_i & Q_i \\ -a & -k & -H_i & -h_i & -P_i \end{bmatrix}, \\ Z\begin{bmatrix} a & k & H_i & h_i & P_i \\ b & l & G_i & g_i & Q_i \end{bmatrix} &\xrightarrow{T} (-1)^{1+a+P_1+P_2} Z\begin{bmatrix} a & k & H_i & h_i & P_i \\ a+b-1 & k+l-1 & H_i+G_i-1 & h_i+g_i & P_i+Q_i \end{bmatrix}, \end{aligned}$$

where the extra factor of -1 in the T-transformation comes from the η-functions. By noting that the phase Ψ transforms trivially, as it is just a constant factor, we can conclude that to be modular invariant the phase must satisfy

$$\begin{aligned} \Psi\begin{bmatrix} a & k & H_i & h_i & P_i \\ b & l & G_i & g_i & Q_i \end{bmatrix} &\stackrel{S}{=} \Psi\begin{bmatrix} b & l & G_i & g_i & Q_i \\ -a & -k & -H_i & -h_i & -P_i \end{bmatrix}, \\ \Psi\begin{bmatrix} a & k & H_i & h_i & P_i \\ b & l & G_i & g_i & Q_i \end{bmatrix} &\stackrel{T}{=} 1 + a + P_1 + P_2 + \Psi\begin{bmatrix} a & k & H_i & h_i & P_i \\ a+b-1 & k+l-1 & H_i+G_i-1 & h_i+g_i & P_i+Q_i \end{bmatrix}. \end{aligned} \tag{A9}$$

The first equation, i.e., S-invariance, shows that Ψ must be symmetric under the exchange of lower and upper indices, which together with (A5), implies that

$$\Psi\begin{bmatrix} a & k & H_i & h_i & P_i \\ b & l & G_i & g_i & Q_i \end{bmatrix} = a + b + \Gamma_i \Omega_{ij} \Delta_j, \qquad \text{(A10)}$$

with $\Omega_{ij} = \Omega_{ji}$. Implementing the condition for T-invariance in (A9) further restricts the form of Ω imposing the conditions on its elements

$$\sum_{\substack{j=1 \\ j\neq i}}^{8} \Omega_{ij} = 0 \quad \text{for } i = 2, \ldots, 8,$$
$$\sum_{j=1}^{8} \Omega_{ij} = \Omega_{ii} \quad \text{for } i = 9, 10,$$
$$\sum_{j=1}^{8} \Omega_{ij} = 1 + \Omega_{ii} \quad \text{for } i = 11, 12, \qquad \text{(A11)}$$

where all equalities are understood modulo 2. These fix a further 11 components of Ω_{ij}. Together with the condition from S-invariance $\Omega_{ij} = \Omega_{ji}$, we are left with $(12^2/2 + 12/2) - 11 = 67$ independent choices for the Ω_{ij}. This precisely matches the number of independent GGSO phases for a 12 basis vector model.[1]

What we achieved here is precisely the derivation of the modular invariance conditions, for the phase Ψ. These are analogous to the well-known conditions on the GGSO coefficients in the fermionic formulation. All remaining independent components of Ω can be freely chosen as $\Omega_{ij} \in \{0, 1\}$ with each choice giving a new consistent model.

2. The translation

We have found a consistent modular invariant way of representing a model in terms of a phase Ψ, what remains is to find a translation between the GGSO phases and Ψ as set out in (A1). With the above setup, this means finding a correspondence between the independent GGSO phases $C[^{\boldsymbol{\alpha}}_{\boldsymbol{\beta}}]$ and the matrix elements Ω_{ij}. We have already established that the number of these elements is in agreement on both sides and both quantities perform the same role so such a translation should be possible in principle.

To make the connection, one has to notice that the forms of the theta functions on the left and right-hand sides of (A1) do not match. In particular the expression

$$\sum_{\substack{a,k,\cdots \\ b,l,\cdots}} (-1)^{\Psi\begin{bmatrix} a & k & \cdots \\ b & l & \cdots \end{bmatrix}} Z\begin{bmatrix} a & k & \cdots \\ b & l & \cdots \end{bmatrix} \qquad \text{(A12)}$$

involves theta functions which may take arguments such as $\vartheta[^{\,1}_{-1}], \vartheta[^3_0], \ldots$ not permitted on the free fermionic side where the arguments are either 0 or 1. We can, however, use the periodicity properties of the theta functions

$$\vartheta\begin{bmatrix} a+2 \\ b \end{bmatrix} = \vartheta\begin{bmatrix} a \\ b \end{bmatrix}$$
$$\vartheta\begin{bmatrix} a \\ b+2 \end{bmatrix} = e^{i\pi a}\vartheta\begin{bmatrix} a \\ b \end{bmatrix}, \qquad \text{(A13)}$$

to rewrite (A12) in terms of the standard theta functions. This will allow for consistent term-by-term matching. By denoting the "fundamental" form of the theta functions as

$$\vartheta_f\begin{bmatrix} a \\ b \end{bmatrix} \equiv \vartheta\begin{bmatrix} a & \bmod 2 \\ b & \bmod 2 \end{bmatrix}, \qquad \text{(A14)}$$

we can find equations using (A13) that help bring all theta functions to this reduced form, e.g.,

$$\vartheta\begin{bmatrix} a+h_1 \\ b+g_1 \end{bmatrix} = (-1)^{(a+h_1)bg_1}\vartheta_f\begin{bmatrix} a+h_1 \\ b+g_1 \end{bmatrix}$$
$$\vartheta\begin{bmatrix} a-h_1-h_2 \\ b-g_1-g_2 \end{bmatrix} = (-1)^{(a-h_1-h_2)(g_1+g_2+bg_1+bg_2+g_1g_2)} \times \vartheta_f\begin{bmatrix} a-h_1-h_2 \\ b-g_1-g_2 \end{bmatrix}. \qquad \text{(A15)}$$

These relations can always be found by writing $\vartheta\begin{bmatrix} a & \cdots \\ b & \cdots \end{bmatrix} = (-1)^{F(a,b,\cdots)}\vartheta_f\begin{bmatrix} a & \cdots \\ b & \cdots \end{bmatrix}$, with $F(a,b,\cdots)$ a suitably general polynomial, and restricting the form of F by requiring (A13) to hold.

Utilizing these expressions, we can rewrite the right-hand side of (A1), fully in terms of the reduced theta functions as

$$Z_F = \sum_{\substack{a,k,\cdots \\ b,l,\cdots}} (-1)^{\chi\begin{bmatrix} a & k & \cdots \\ b & l & \cdots \end{bmatrix} + \Psi\begin{bmatrix} a & k & \cdots \\ b & l & \cdots \end{bmatrix}} \times Z_f\begin{bmatrix} a & k & \cdots \\ b & l & \cdots \end{bmatrix}, \qquad \text{(A16)}$$

where we defined the compensating phase factor χ. For our specific model it is given by

[1]We can either count 66 or 67 independent GGSO phases depending on whether we specify that the unimportant phase $C[^1_1]$, generating an overall chirality, is fixed or not.

$$\chi\begin{bmatrix} a & k & H_i & h_i & P_i \\ b & l & G_i & g_i & Q_i \end{bmatrix} = (a+k)(g_1+g_2+g_1g_2) + (b+l)(h_1g_2+h_2g_1), \quad \text{(A17)}$$

which enforces the rules (A15). Here, by Z_f we denote that all theta functions have been brought to their mod 2 form as written in (A14). This compensating phase is crucial for the matching of the partition functions.

We are now ready to make the connection between the two formalisms. To compare the two sides of (A1) we must reexpress the GGSO matrix C in the form

$$C_{ij} = (-1)^{G_{ij}}, \quad \text{(A18)}$$

this allows for a direct comparison of Ψ and G. Furthermore, it will be convenient to separate Ψ into its first and second order terms, that is we define

$$\Psi\begin{bmatrix} a & k & H_i & h_i & P_i \\ b & l & G_i & g_i & Q_i \end{bmatrix} = a+b+\Gamma_i\Omega_{ij}\Delta_j \coloneqq a+b+\Phi\begin{bmatrix} a & k & H_i & h_i & P_i \\ b & l & G_i & g_i & Q_i \end{bmatrix}. \quad \text{(A19)}$$

We can now express the factor of $a+b+\chi$ in the basis formed by the basis vectors (2.10) as a matrix P whose elements are

$$P_{ij} = \left\{ a+b+\chi\begin{bmatrix} a & k & \cdots \\ b & l & \cdots \end{bmatrix} \middle| \Gamma_k = S_{ik} \quad \text{and} \quad \Delta_k = S_{jk} \right\}. \quad \text{(A20)}$$

All that remains is to express Φ, i.e., Ω, in the same basis so we can equate the two. We can do this by noticing that

$$\left\{ \Phi\begin{bmatrix} a & k & \cdots \\ b & l & \cdots \end{bmatrix} \middle| \Gamma_k = S_{ik} \quad \text{and} \quad \Delta_k = S_{jk} \right\} = S_{ik}\Omega_{kl}S_{jl} = S\Omega S^T, \quad \text{(A21)}$$

and so $\tilde{\Omega} = S\Omega S^T$ is the phase expressed in the basis formed by the basis vectors of the free fermionic model. Since all quantities are now expressed in the same basis we can write down the equality which implements the translation, namely

$$G + P = S\Omega S^T, \quad \text{(A22)}$$

where the equality is understood modulo 2. Solving the above equation means finding values for all Ω_{ij} and so fixing Ω. Once the solution is found to the linear system, the final phase can be expressed using (A19), that is

$$\Psi\begin{bmatrix} a & k & H_i & h_i & P_i \\ b & l & G_i & g_i & Q_i \end{bmatrix} = a+b+\Gamma\Omega\Delta. \quad \text{(A23)}$$

This gives a precise one-to-one correspondence between the modular invariant phase and the GGSO matrix.

It is important to note that the above methods only cover the case for real boundary conditions, i.e., $\mathbb{Z}_2$-models such that the fermions are either R or NS. This, in turn, implies that all GGSO phases are real. It is, however, possible to generalize this construction to allow for more general choices of boundary condition vectors and GGSO matrices.

APPENDIX B: SCHERK-SCHWARZ AND T-DUALITY CONDITIONS

In this section we will explicitly show how the techniques developed in Secs. V and V A can be applied in order to check for SSS and T-duality. In particular, we will use the two models of section VIII, specified by the GGSO phase configurations (8.1) and (8.2).

The condition for SSS is that for $T_2 \to \infty$ the partition function vanishes so SUSY is restored, which can happen only if a Jacobi identity is realized. This identity in (5.4) must hold for all indices fixed, except for H_1, H_2, G_1, G_2 which do not appear as arguments in the theta functions, since the moduli-dependent lattice $\Gamma^{(1)}_{2,2}\begin{bmatrix} H_1 & H_2 \\ G_1 & G_2 \end{bmatrix}\begin{vmatrix} h_1 \\ g_1 \end{vmatrix}$ is set to 1. Given the GGSO phases, the phase Φ in the partition function (4.1) can be calculated using (A23) following the discussion of Appendix A. For the model (8.1) this phase is given by

$$\begin{aligned}\Phi\begin{bmatrix} a & k & H_i & h_1 & h_2 & P_i \\ b & l & G_i & g_1 & g_2 & Q_i \end{bmatrix} &= b(a+H_1+h_2+H_2+P_1) + l(h_2+k+P_1) + G_1(a+h_1+H_2+H_3+H_4+P_1+P_2) \\ &\quad + G_2(a+H_1+h_2+H_2+P_1) + G_3(H_1+H_6) + G_4(H_1+H_4+H_6) \\ &\quad + G_6(h_1+h_2+H_3+H_4+H_6) + g_1(H_1+H_6+P_1) + g_2(a+H_2+H_6+k+P_1) \\ &\quad + Q_1(a+h_1+H_1+h_2+H_2+k) + Q_2(H_1+P_2). \end{aligned} \quad \text{(B1)}$$

Then, as stated in Sec. V, the Jacobi identity holds only if the following condition is satisfied

$$\sum_{\substack{H_1,H_2\\G_1,G_2}}(-1)^{\Phi\begin{bmatrix}0 & k & H_i & 0 & 0 & P_i\\0 & l & G_i & 0 & 0 & Q_i\end{bmatrix}} = \sum_{\substack{H_1,H_2\\G_1,G_2}}(-1)^{\Phi\begin{bmatrix}1 & k & H_i & 0 & 0 & P_i\\0 & l & G_i & 0 & 0 & Q_i\end{bmatrix}} = \sum_{\substack{H_1,H_2\\G_1,G_2}}(-1)^{\Phi\begin{bmatrix}0 & k & H_i & 0 & 0 & P_i\\1 & l & G_i & 0 & 0 & Q_i\end{bmatrix}}. \tag{B2}$$

Actually it is sufficient to prove just the first equality, since due to modular invariance the phase $\Phi\begin{bmatrix}a & k & H_i & h_1 & h_2 & P_i\\b & l & G_i & g_1 & g_2 & Q_i\end{bmatrix}$ is the same by exchanging lower and upper indices. Then, by performing the lattice sum, it is easy to check that the equality holds.

Alternatively, it can also be check directly by inspecting the phase $\Phi\begin{bmatrix}a & k & H_1 & H_2 & H_i & 0 & 0 & P_i\\b & l & G_1 & G_2 & G_i & 0 & 0 & Q_i\end{bmatrix}$, where we have distinguished H_1, H_2 from the other H_i's to render the discussion clearer. For $\Phi\begin{bmatrix}0 & k & H_1 & H_2 & H_i & 0 & 0 & P_i\\0 & l & G_1 & G_2 & G_i & 0 & 0 & Q_i\end{bmatrix}$ set the indices H_1 and H_2 to zero. If there are some combinations of H_1^* and H_2^* for $\Phi\begin{bmatrix}1 & k & H_1^* & H_2^* & H_i & 0 & 0 & P_i\\0 & l & G_1 & G_2 & G_i & 0 & 0 & Q_i\end{bmatrix}$ such that the following equality holds

$$\Phi\begin{bmatrix}0 & k & 0 & 0 & H_i & 0 & 0 & P_i\\0 & l & G_1 & G_2 & G_i & 0 & 0 & Q_i\end{bmatrix} = \Phi\begin{bmatrix}1 & k & H_1^* & H_2^* & H_i & 0 & 0 & P_i\\0 & l & G_1 & G_2 & G_i & 0 & 0 & Q_i\end{bmatrix}, \tag{B3}$$

then the condition (B2) will be satisfied and the Jacobi identity will hold. In particular, for the phase (B1) it is easy to check that the equality is satisfied setting $H_1^* = 0$, $H_2^* = 1$. Once this holds, for all other combinations of H_1, H_2 the equality will hold according to

$$\Phi\begin{bmatrix}0 & k & 0+i & 0+j & H_i & 0 & 0 & P_i\\0 & l & G_1 & G_2 & G_i & 0 & 0 & Q_i\end{bmatrix} = \Phi\begin{bmatrix}1 & k & H_1^*+i & H_2^*+j & H_i & 0 & 0 & P_i\\0 & l & G_1 & G_2 & G_i & 0 & 0 & Q_i\end{bmatrix}. \tag{B4}$$

This second way of checking for SSS is particularly convenient in terms of time efficiency, being easy to implement in a computational program.

We now check whether T-duality is preserved for this model, i.e., if condition (5.12) holds. To do this, we check that

$$\begin{aligned}\sum_{\substack{H_1,H_2\\G_1,G_2}}(-1)^{\Phi\begin{bmatrix}0 & k & H_i & 0 & 0 & P_i\\0 & l & G_i & 0 & 0 & Q_i\end{bmatrix}+H_1G_1+H_2G_2} &= \sum_{\substack{H_1,H_2\\G_1,G_2}}(-1)^{\Phi\begin{bmatrix}1 & k & H_i & 0 & 0 & P_i\\0 & l & G_i & 0 & 0 & Q_i\end{bmatrix}+H_1G_1+H_2G_2}\\ &= \sum_{\substack{H_1,H_2\\G_1,G_2}}(-1)^{\Phi\begin{bmatrix}0 & k & H_i & 0 & 0 & P_i\\1 & l & G_i & 0 & 0 & Q_i\end{bmatrix}+H_1G_1+H_2G_2}\end{aligned} \tag{B5}$$

is satisfied. As discussed in Sec. VA, due to modular invariance, just the first equality is sufficient. Performing the sum, we can see that the T-duality condition does not hold. Also using (B3) modified with the additional T-dual contribution

$$\Phi\begin{bmatrix}0 & k & 0 & 0 & H_i & 0 & 0 & P_i\\0 & l & G_1 & G_2 & G_i & 0 & 0 & Q_i\end{bmatrix} = \Phi\begin{bmatrix}1 & k & H_1^* & H_2^* & H_i & 0 & 0 & P_i\\0 & l & G_1 & G_2 & G_i & 0 & 0 & Q_i\end{bmatrix} + H_1^*G_1 + H_2^*G_2 \tag{B6}$$

it is easy to check that there is no possible combination of H_1^*, H_2^* such that the equality is preserved.

We thus conclude that for this particular model defined by the GGSO phases (8.1) the potential will exhibit an SSS breaking with broken T-duality, such that $V(T_2 \to \infty) \to 0$ and $V(T_2 \to 0) \not\to 0$. This behavior of the potential is demonstrated in Fig. 2.

For the second model specified by the GGSO phases (8.2), the phase Φ is given by

$$\Phi\begin{bmatrix} a & k & H_i & h1 & h2 & P_i \\ b & l & G_i & g1 & g2 & Q_i \end{bmatrix} = b(a+h_2+P_2)+l(H_2+H_3+H_5+H_6+P_1)+G_1(H_1+H_2+H_3+P_1+P_2) + G_2(H_1+h_2+H_2+H_3+H_4+H_5+H_6+k)+G_3(H_1+H_2+H_4+H_5+H_6 + k+P_1+P_2)+G_4(h_1+h_2+H_2+H_3+H_4+P_1+P_2)+G_5(h_1+H_2+H_3 + H_6+k+P_1)+G_6(h_2+H_2+H_3+H_5+H_6+k+P_1+P_2)+g_1(h_2+H_4 + H_5+P_1)+g_2(a+h_1+H_2+H_4+H_6+P_1)+Q_1(h_1+H_1+h_2+H_3+H_4 + H_5+H_6+k+P_2)+Q_2(a+H_1+H_3+H_4+H_6+P_1+P_2). \quad \text{(B7)}$$

For this model it can be checked that the SSS condition (B2), or equivalently condition (B3), is not satisfied which implies that the potential diverges at infinity, $V(T_2\to\infty)\to\pm\infty$. The shape of the potential in Fig. 9 shows what we expect.

APPENDIX C: CHIRAL SECTOR ANALYSIS

In addition to the sectors $\boldsymbol{F}^{1,2,3}_{pqrs}$ discussed in Sec. VII, the following sectors also give rise to massless states transforming under spinorial representations with chirality under the $U(1)_{1,2,3}$ gauge factors

$$\begin{aligned}
\boldsymbol{F}^4_{pqrs} &= \boldsymbol{S}+\boldsymbol{b_1}+\boldsymbol{x}+\boldsymbol{z_1}+p\boldsymbol{e_3}+q\boldsymbol{e_4}+r\boldsymbol{e_5}+s\boldsymbol{e_6}\\
\boldsymbol{F}^5_{pqrs} &= \boldsymbol{S}+\boldsymbol{b_2}+\boldsymbol{x}+\boldsymbol{z_1}+p\boldsymbol{e_1}+q\boldsymbol{e_2}+r\boldsymbol{e_5}+s\boldsymbol{e_6}\\
\boldsymbol{F}^6_{pqrs} &= \boldsymbol{S}+\boldsymbol{b_3}+\boldsymbol{x}+\boldsymbol{z_1}+p\boldsymbol{e_1}+q\boldsymbol{e_2}+r\boldsymbol{e_3}+s\boldsymbol{e_4}\\
\boldsymbol{F}^7_{pqrs} &= \boldsymbol{S}+\boldsymbol{b_1}+\boldsymbol{x}+\boldsymbol{z_2}+p\boldsymbol{e_3}+q\boldsymbol{e_4}+r\boldsymbol{e_5}+s\boldsymbol{e_6}\\
\boldsymbol{F}^8_{pqrs} &= \boldsymbol{S}+\boldsymbol{b_2}+\boldsymbol{x}+\boldsymbol{z_2}+p\boldsymbol{e_1}+q\boldsymbol{e_2}+r\boldsymbol{e_5}+s\boldsymbol{e_6}\\
\boldsymbol{F}^9_{pqrs} &= \boldsymbol{S}+\boldsymbol{b_3}+\boldsymbol{x}+\boldsymbol{z_2}+p\boldsymbol{e_1}+q\boldsymbol{e_2}+r\boldsymbol{e_3}+s\boldsymbol{e_4},
\end{aligned} \quad \text{(C1)}$$

which have the following projecting sets

$$\begin{aligned}
\Upsilon(\boldsymbol{F}^4_{pqrs}) &= \{\boldsymbol{z_2},\boldsymbol{e_1},\boldsymbol{e_2}\}\\
\Upsilon(\boldsymbol{F}^5_{pqrs}) &= \{\boldsymbol{z_2},\boldsymbol{e_3},\boldsymbol{e_4}\}\\
\Upsilon(\boldsymbol{F}^6_{pqrs}) &= \{\boldsymbol{z_2},\boldsymbol{e_5},\boldsymbol{e_6}\}\\
\Upsilon(\boldsymbol{F}^7_{pqrs}) &= \{\boldsymbol{z_1},\boldsymbol{e_1},\boldsymbol{e_2}\}\\
\Upsilon(\boldsymbol{F}^8_{pqrs}) &= \{\boldsymbol{z_1},\boldsymbol{e_3},\boldsymbol{e_4}\}\\
\Upsilon(\boldsymbol{F}^9_{pqrs}) &= \{\boldsymbol{z_1},\boldsymbol{e_5},\boldsymbol{e_6}\},
\end{aligned} \quad \text{(C2)}$$

and chirality operators

$$\begin{aligned}
\chi(\boldsymbol{F}^4_{pqrs}) &= \mathrm{ch}(\bar\eta^2)+\mathrm{ch}(\bar\eta^3) = -C\begin{bmatrix}\boldsymbol{F}^4_{pqrs}\\ \boldsymbol{S}+\boldsymbol{b_2}+(1-r)\boldsymbol{e_5}+(1-s)\boldsymbol{e_6}\end{bmatrix}^* - C\begin{bmatrix}\boldsymbol{F}^4_{pqrs}\\ \boldsymbol{S}+\boldsymbol{b_3}+(1-p)\boldsymbol{e_3}+(1-q)\boldsymbol{e_4}\end{bmatrix}^*\\
\chi(\boldsymbol{F}^5_{pqrs}) &= \mathrm{ch}(\bar\eta^1)+\mathrm{ch}(\bar\eta^3) = -C\begin{bmatrix}\boldsymbol{F}^5_{pqrs}\\ \boldsymbol{S}+\boldsymbol{b_1}+(1-r)\boldsymbol{e_5}+(1-s)\boldsymbol{e_6}\end{bmatrix}^* - C\begin{bmatrix}\boldsymbol{F}^5_{pqrs}\\ \boldsymbol{S}+\boldsymbol{b_3}+(1-p)\boldsymbol{e_1}+(1-q)\boldsymbol{e_2}\end{bmatrix}^*\\
\chi(\boldsymbol{F}^6_{pqrs}) &= \mathrm{ch}(\bar\eta^1)+\mathrm{ch}(\bar\eta^2) = -C\begin{bmatrix}\boldsymbol{F}^6_{pqrs}\\ \boldsymbol{S}+\boldsymbol{b_1}+(1-r)\boldsymbol{e_3}+(1-s)\boldsymbol{e_4}\end{bmatrix}^* - C\begin{bmatrix}\boldsymbol{F}^6_{pqrs}\\ \boldsymbol{S}+\boldsymbol{b_2}+(1-p)\boldsymbol{e_1}+(1-q)\boldsymbol{e_2}\end{bmatrix}^*\\
\chi(\boldsymbol{F}^7_{pqrs}) &= \mathrm{ch}(\bar\eta^2)+\mathrm{ch}(\bar\eta^3) = -C\begin{bmatrix}\boldsymbol{F}^7_{pqrs}\\ \boldsymbol{S}+\boldsymbol{b_2}+(1-r)\boldsymbol{e_5}+(1-s)\boldsymbol{e_6}\end{bmatrix}^* - C\begin{bmatrix}\boldsymbol{F}^7_{pqrs}\\ \boldsymbol{S}+\boldsymbol{b_3}+(1-p)\boldsymbol{e_3}+(1-q)\boldsymbol{e_4}\end{bmatrix}^*\\
\chi(\boldsymbol{F}^8_{pqrs}) &= \mathrm{ch}(\bar\eta^1)+\mathrm{ch}(\bar\eta^3) = -C\begin{bmatrix}\boldsymbol{F}^8_{pqrs}\\ \boldsymbol{S}+\boldsymbol{b_1}+(1-r)\boldsymbol{e_5}+(1-s)\boldsymbol{e_6}\end{bmatrix}^* - C\begin{bmatrix}\boldsymbol{F}^8_{pqrs}\\ \boldsymbol{S}+\boldsymbol{b_3}+(1-p)\boldsymbol{e_1}+(1-q)\boldsymbol{e_2}\end{bmatrix}^*\\
\chi(\boldsymbol{F}^9_{pqrs}) &= \mathrm{ch}(\bar\eta^1)+\mathrm{ch}(\bar\eta^2) = -C\begin{bmatrix}\boldsymbol{F}^9_{pqrs}\\ \boldsymbol{S}+\boldsymbol{b_1}+(1-r)\boldsymbol{e_3}+(1-s)\boldsymbol{e_4}\end{bmatrix}^* - C\begin{bmatrix}\boldsymbol{F}^9_{pqrs}\\ \boldsymbol{S}+\boldsymbol{b_2}+(1-p)\boldsymbol{e_1}+(1-q)\boldsymbol{e_2}\end{bmatrix}^*,
\end{aligned} \quad \text{(C3)}$$

Further to these sectors, the following can give rise to massless states when accompanied by one right-moving Neveu-Schwarz oscillator given by

$$\begin{aligned} \boldsymbol{V}^1_{pqrs} &= \boldsymbol{S} + \boldsymbol{b_1} + \boldsymbol{x} + p\boldsymbol{e_3} + q\boldsymbol{e_4} + r\boldsymbol{e_5} + s\boldsymbol{e_6} \\ \boldsymbol{V}^2_{pqrs} &= \boldsymbol{S} + \boldsymbol{b_2} + \boldsymbol{x} + p\boldsymbol{e_1} + q\boldsymbol{e_2} + r\boldsymbol{e_5} + s\boldsymbol{e_6} \\ \boldsymbol{V}^3_{pqrs} &= \boldsymbol{S} + \boldsymbol{b_3} + \boldsymbol{x} + p\boldsymbol{e_1} + q\boldsymbol{e_2} + r\boldsymbol{e_3} + s\boldsymbol{e_4}, \end{aligned} \tag{C4}$$

which have the following projecting sets

$$\begin{aligned} \Upsilon(\boldsymbol{V}^1_{pqrs}) &= \{\boldsymbol{z_1}, \boldsymbol{z_2}, \boldsymbol{e_1}, \boldsymbol{e_2}\} \\ \Upsilon(\boldsymbol{V}^2_{pqrs}) &= \{\boldsymbol{z_1}, \boldsymbol{z_2}, \boldsymbol{e_3}, \boldsymbol{e_4}\} \\ \Upsilon(\boldsymbol{V}^3_{pqrs}) &= \{\boldsymbol{z_1}, \boldsymbol{z_2}, \boldsymbol{e_5}, \boldsymbol{e_6}\}, \end{aligned} \tag{C5}$$

and chirality operators

$$\begin{aligned} \chi(\boldsymbol{V}^1_{pqrs}) &= \mathrm{ch}(\bar{\eta}^2) + \mathrm{ch}(\bar{\eta}^3) = -C\left[\begin{matrix} \boldsymbol{V}^1_{pqrs} \\ \boldsymbol{S} + \boldsymbol{b_2} + (1-r)\boldsymbol{e_5} + (1-s)\boldsymbol{e_6} \end{matrix}\right]^* - C\left[\begin{matrix} \boldsymbol{V}^1_{pqrs} \\ \boldsymbol{S} + \boldsymbol{b_3} + (1-p)\boldsymbol{e_3} + (1-q)\boldsymbol{e_4} \end{matrix}\right]^* \\ \chi(\boldsymbol{V}^2_{pqrs}) &= \mathrm{ch}(\bar{\eta}^1) + \mathrm{ch}(\bar{\eta}^3) = -C\left[\begin{matrix} \boldsymbol{V}^2_{pqrs} \\ \boldsymbol{S} + \boldsymbol{b_1} + (1-r)\boldsymbol{e_5} + (1-s)\boldsymbol{e_6} \end{matrix}\right]^* - C\left[\begin{matrix} \boldsymbol{V}^2_{pqrs} \\ \boldsymbol{S} + \boldsymbol{b_3} + (1-p)\boldsymbol{e_1} + (1-q)\boldsymbol{e_2} \end{matrix}\right]^* \\ \chi(\boldsymbol{V}^3_{pqrs}) &= \mathrm{ch}(\bar{\eta}^1) + \mathrm{ch}(\bar{\eta}^2) = -C\left[\begin{matrix} \boldsymbol{V}^3_{pqrs} \\ \boldsymbol{S} + \boldsymbol{b_1} + (1-r)\boldsymbol{e_3} + (1-s)\boldsymbol{e_4} \end{matrix}\right]^* - C\left[\begin{matrix} \boldsymbol{V}^3_{pqrs} \\ \boldsymbol{S} + \boldsymbol{b_2} + (1-p)\boldsymbol{e_1} + (1-q)\boldsymbol{e_2} \end{matrix}\right]^*. \end{aligned} \tag{C6}$$

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