Two interacting conformal Carroll particles

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In this paper we analyze two different models of two interacting conformal Carroll particles that can be obtained as the Carrollian limit of two relativistic conformal particles. The first model describes particles with zero velocity and exhibits infinite dimensional symmetries which are reminiscent of the BMS symmetries. A second model of interaction of Carrollian particles is proposed, where the particles have nonzero velocity and therefore, as a consequence of the limit $c \to 0$, are tachyons. Infinite dimensional symmetries are present also in this model.

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I. INTRODUCTION

Carroll symmetry, which was introduced by [1,2] as the limit of the Poincaré symmetry where the velocity of light is going to zero, $c \to 0$, has recently received a lot of attention, mainly in connection with its relation with the Bondi, Metzner and Sachs (BMS) group [3,4]. Some applications of Carrollian physics allow, for example, one to understand the symmetries of null hypersurfaces, such as black-hole horizons [5], boundaries of asymptotically flat spacetimes [6], and Carroll fluids [7]. For an introductory review of certain aspects of Non-Lorentzian theories, see [8].

In particular, in this paper we are interested to have a dynamical understanding of the relation between the conformal Carroll structures [6] and the symmetries of Carrollian particles.

A (massive) nonconformal Carroll particle was introduced as a Carroll limit of a relativistic massive particle [9]; it can also be obtained from a coadjoint orbit of the Carroll group [10]. The (massless) Carroll particle was also considered in [9]. A model of two nonconformal interacting particles was also considered where the individual particles move but the center of mass does not move.

The symmetry of massive and massless free Carroll particles is infinite dimensional. In the massless case it contains the finite conformal Carroll symmetry introduced in [11], see also [12]. In the case of the two nonconformal particles the symmetry algebra is finite dimensional [9].

The causal structure of the Carroll space allows spacelike intervals and therefore the existence of Carroll tachyons. In [13] a model is constructed where a tachyonic Carroll particle with zero energy moves.

In this paper we propose two models for two conformal Carroll particles as Carrollian limits of two conformal relativistic particles introduced in [14].

The first one describes particles which do not move and exhibits an infinite dimensional symmetry algebra. Any free conformal Carroll particle has vanishing energy; therefore, two free Carrollian particles have total energy equal to zero. The nice feature of this model is that the system can have a total energy different from zero and still be Carroll conformal invariant. Clearly the energy, which turns out to be constant, depends on the interaction. We also study some possible connections with the infinite dimensional conformal symmetry of [6] and the BMS symmetry.

In the second model the particles move, their individual energy is zero, and consequently they are Carrollian tachyons. Also in this case the symmetry turns out to be infinite dimensional.

II. CONFORMAL CARROLL PARTICLE

The canonical form of the action of a massless relativistic particle is given by [we assume the signature of the metric $(+,-,-,-)$]

$$S = \int d\tau \left[-p\cdot\dot{x} - \frac{e}{2}p^2\right], \tag{1}$$

where the dot denotes the differentiation with respect to an invariant parameter τ. The action is also invariant under the

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conformal group that contains dilatations and special conformal transformations (SCT). The infinitesimal transformations are given by the ordinary Poincaré transformations, plus dilatations

$$\delta x^\mu = \epsilon_D x^\mu, \qquad \delta p^\mu = -\epsilon_D p^\mu, \qquad \delta e = 2e\epsilon_D \quad (2)$$

and SCT:

$$\begin{aligned} \delta x^\mu &= b^\mu x^2 - 2(b\cdot x)x^\mu, \\ \delta p^\mu &= -2b\cdot p x^\mu + 2b\cdot x p^\mu + 2p\cdot x b^\mu, \end{aligned} \quad (3)$$

$$\delta e = -4eb\cdot x. \quad (4)$$

We next consider the Carroll limit which is given by

$$x^0 = \frac{t}{\omega}, \qquad p_0 = \omega E, \quad (5)$$

with $\omega \to \infty$. It is understood that, before taking the limit, we rescale the einbein variable like

$$e \to \frac{e}{\omega^2}. \quad (6)$$

Note that we use a dimensionless parameter ω instead of the velocity of light. Indeed, if one considers the Carrollian counterpart of the nonrelativistic limit of a string, one needs to use a dimensionless parameter [15–17]. The contractions we consider in this work correspond to the limit $c \to 0$ of a world probed by particles. There are more general possible contractions that correspond to the non-Lorentzian limits of extended objects such as strings and branes.

In the Carroll limit the action (1) becomes

$$S_C = \int d\tau \left[-E\dot{t} + \mathbf{p}\cdot\dot{\mathbf{x}} - \frac{e}{2}E^2\right]. \quad (7)$$

From the symplectic form of the canonical action we deduce the following Poisson brackets:

$$\{E,t\} = 1, \quad \{e,\pi\} = 1, \quad \{x^i, p^j\} = \delta^{ij}, \quad i = 1,2,3. \quad (8)$$

This action can also be obtained by the method of nonlinear realizations [9] and the coadjoint orbit method [10]. Since the action (1) is relativistic conformal invariant, the action (7) is also invariant under the Carroll conformal transformations. We have

$$\begin{aligned} \delta t &= \boldsymbol{\beta}\cdot\mathbf{x} + a_t, \qquad \delta x^i = \epsilon^{ijk}\theta^j x^k + a^i, \\ \delta p^i &= \epsilon^{ijk}\theta^j p^k + \beta^i E, \qquad \delta E = 0, \end{aligned} \quad (9)$$

$$\begin{aligned} &\delta t = \epsilon_D t, \qquad \delta\mathbf{x} = \epsilon_D\mathbf{x}, \qquad \delta E = -\epsilon_D E, \\ &\delta\mathbf{p} = -\epsilon_D\mathbf{p}, \qquad \delta e = 2\epsilon_D e, \end{aligned} \quad (10)$$

$$\delta t = -b^0\mathbf{x}^2 + 2\mathbf{b}\cdot\mathbf{x}t, \qquad \delta\mathbf{x} = -\mathbf{b}\mathbf{x}^2 + 2(\mathbf{b}\cdot\mathbf{x})\mathbf{x}, \quad (11)$$

$$\begin{aligned} &\delta E = -2\mathbf{b}\cdot\mathbf{x}E, \\ &\delta\mathbf{p} = -2(b^0 E - \mathbf{b}\cdot\mathbf{p})\mathbf{x} - 2\mathbf{b}\cdot\mathbf{x}\mathbf{p} + 2(Et - \mathbf{p}\cdot\mathbf{x})\mathbf{b}, \end{aligned} \quad (12)$$

$$\delta e = 4\mathbf{b}\cdot\mathbf{x}e, \quad (13)$$

where $\beta^i, a_t, a^i, \epsilon_D, b^0, b^i$ are the infinitesimal parameters associated to Carrollian boost, space-time translations, dilatation, and special Carrollian conformal transformations.

The phase space realization of the conformal Carroll generators is the following:

$$H = E, \qquad \mathbf{P} = \mathbf{p}, \qquad \mathbf{G} = E\mathbf{x} \qquad \mathbf{J} = \mathbf{x}\times\mathbf{p}, \quad (14)$$

$$D = -Et + \mathbf{p}\cdot\mathbf{x}, \quad K^0 = -E\mathbf{x}^2, \quad \mathbf{K} = 2D\mathbf{x} - \mathbf{x}^2\mathbf{p}. \quad (15)$$

The corresponding transformations are obtained as $\delta = \{G, \}$.

The conformal Carroll algebra in phase space in four dimensions is given by (see [11], and also [12])

$$\begin{aligned} &\{E,P^i\} = \{E,G^i\} = \{E,J^i\} = \{E,K^0\} = 0, \{E,D\} = -E, \{E,K^i\} = -2G^i, \\ &\{P^i,P^j\} = 0, \{P^i,G^j\} = -\delta^{ij}E, \{P^i,J^j\} = \epsilon^{ijk}P^k, \\ &\{P^i,D\} = -P^i, \{P^i,K^0\} = 2G^i, \{P^i,K^j\} = 2\epsilon^{ijk}J^k - 2D\delta^{ij}, \\ &\{G^i,G^j\} = 0, \{G^i,J^j,\} = \epsilon^{ijk}G^k, \{G^i,D\} = 0, \{G^i,K^0\} = 0, \{G^i,K^j\} = \delta^{ij}K^0, \\ &\{J^i,J^j\} = \epsilon^{ijk}J^k, \{J^i,D\} = 0, \{J^i,K^0\} = 0, \{J^i,K^j\} = \epsilon^{ijk}K^k, \\ &\{D,K^0\} = -K^0, \{D,K^i\} = -K^i, \\ &\{K^i,K^0\} = 0, \{K^i,K^j\} = 0. \end{aligned} \quad (16)$$

The conformal Carroll algebra can be obtained from the relativistic conformal algebra with generators $M^{\mu\nu}, P^\mu, K^\mu, D$ associated to Lorentz, space translations, SCT, and dilatations

$$
\begin{aligned}
\left[\tilde{M}^{\mu\nu}, \tilde{M}^{\rho\sigma}\right] &= \eta^{\nu\rho}\tilde{M}^{\mu\sigma} + \cdots, \\
\left[\tilde{P}^{\rho}, \tilde{K}^{\sigma}\right] &= 2(\eta^{\rho\sigma}\tilde{D} + \tilde{M}^{\rho\sigma}), \\
\left[\tilde{P}^{\mu}, \tilde{M}^{\rho\sigma}\right] &= \eta^{\mu[\rho}\tilde{P}^{\sigma]}, \\
\left[\tilde{K}^{\mu}, \tilde{M}^{\rho\sigma}\right] &= \eta^{\mu[\rho}\tilde{K}^{\sigma]}, \\
\left[\tilde{P}^{\rho}, \tilde{D}\right] &= -\tilde{P}^{\rho}, \\
\left[\tilde{K}^{\rho}, \tilde{D}\right] &= \tilde{K}^{\rho}, \\
\left[\tilde{P}^{\mu}, \tilde{P}^{\nu}\right] &= \left[\tilde{K}^{\mu}, \tilde{K}^{\nu}\right] = \left[\tilde{D}, \tilde{M}^{\mu\nu}\right] = 0
\end{aligned} \tag{17}
$$

by the following contractions

$$
E = \frac{1}{\omega}\tilde{P}^0, \qquad G^i = \frac{1}{\omega}\tilde{M}^{i0}, \qquad K^0 = \frac{1}{\omega}\tilde{K}^0 \tag{18}
$$

and the identification

$$
J^i = \frac{1}{2}\epsilon^{ijk}\tilde{M}^{jk}. \tag{19}
$$

A. Infinite-dimensional symmetries

Now we want to analyze which is the most general point transformation of the Carroll particle action (1) following the steps of [9].

In order to do that we want to write the Carollian Killing equations. Let us first consider the equations of motion deduced from (7),

$$
\dot{t} = -eE, \qquad \dot{\mathbf{x}} = 0, \qquad \dot{e} = \lambda(\tau), \tag{20}
$$

$$
\dot{E} = 0, \qquad \dot{\mathbf{p}} = 0, \qquad \dot{\pi} = -1/2E^2, \tag{21}
$$

where $\lambda(\tau)$ is an arbitrary function and π is the momentum associated to the einbein variable which is constrained by $\pi = 0$.

Consider the following generator of the canonical transformations,

$$
G = E\xi^0(\mathbf{x}, t) - \mathbf{p}\cdot\boldsymbol{\xi}(\mathbf{x}, t) + \gamma(\mathbf{x}, t)\pi, \tag{22}
$$

with $\xi^0(\mathbf{x}, t)$, $\xi^i(\mathbf{x}, t)$, and $\gamma(\mathbf{x}, t)$ arbitrary functions of the space-time coordinates. The transformations generated by this generator are given by

$$
\begin{aligned}
&\delta t = \{G, t\} = \xi^0(\mathbf{x}, t), \qquad \delta x^i = \{G, x^i\} = \xi^i(\mathbf{x}, t), \qquad \delta e = \{e, G\} = \gamma(\mathbf{x}, t), \\
&\delta E = \{G, E\} = \partial_t\xi^0(\mathbf{x}, t)E + \partial_t\xi^i(\mathbf{x}, t)p_i + \partial_t\gamma(\mathbf{x}, t)\pi, \\
&\delta p^i = \{G, p^i\} = \partial_i\xi^0(\mathbf{x}, t)E + \partial_i\xi^j(\mathbf{x}, t)p_j + \partial_i\gamma(\mathbf{x}, t)\pi.
\end{aligned} \tag{23}
$$

These transformations are symmetries of the free Carroll particle, provided that G is a constant of motion, i.e., $dG/d\tau = 0$.This leads to the following restriction, after use of Eqs. (20) and (21) and π that is a primary constraint:

$$
\begin{aligned}
0 &= E\left[\dot{t}\partial_t\xi^0(\mathbf{x}, t) + \dot{x}_j\partial^j\xi^0(\mathbf{x}, t)\right] \\
&\quad + p_i\left[\dot{t}\partial_t\xi^i(\mathbf{x}, t) + \dot{x}_j\partial^j\xi^i(\mathbf{x}, t)\right] + \dot{\pi}\gamma(\mathbf{x}, t) \\
&= -eE^2\partial_t\xi^0(\mathbf{x}, t) - eEp_i\partial_t\xi^i(\mathbf{x}, t) - \frac{1}{2}\gamma(\mathbf{x}, t)E^2.
\end{aligned} \tag{24}
$$

Equation (24) splits into the following two equations due to the different powers of the conjugate momenta:

$$
\partial_t\xi^0(t, \mathbf{x}) = -\frac{\gamma}{2e}, \qquad \partial_t\xi^i(t, \mathbf{x}) = 0, \tag{25}
$$

where $\gamma(\mathbf{x}, t)$ remains an arbitrary parameter. This leads to the following generator of the conformal Killing transformations [9]:

$$
G = E\xi^0(t, \mathbf{x}) + p_i\xi^i(\mathbf{x}) - 2\pi e\partial_t\xi^0(t, \mathbf{x}), \tag{26}
$$

which generates an infinite dimensional symmetry, which will be called $\mathcal{G}_1$. These transformations include Carroll conformal transformations. However they are more general. The dependence of $\xi^0(t, \mathbf{x})$ on the parameter t is arbitrary while in BMS is linear [6]. This arbitrary dependence on t is reminiscent of the Newman-Unti group, which contains BMS which is isomorphic to the infinite extension of the Carroll conformal group [6].

III. TWO CONFORMAL RELATIVISTIC PARTICLES

In this section we briefly recall the most relevant features of a model of two interacting conformal particles proposed in [14]. The general motivation to consider two relativistic conformal particles comes from the fact that, after quantization, this theory is naturally connected with nonlocal field theories appearing in the context of higher spin theories, see for example [18]. A more recent motivation comes from AdS/CFT correspondence, in particular the bilocal-holography approach of [19] and reference therein, since due to the world approach to field theory [20–22] there is a natural connection among the two particle models and bilocal scalar theories.

The canonical action is formulated, in terms of the coordinates x_1, x_2, the momenta p_1, p_2, the einbeins e_1, e_2, and the associated momenta π_1, π_2 that are primary constraints, by

$$S = \int d\tau(-p_1 \dot{x}_1 - p_2 \cdot \dot{x}_2 + \pi_1 \dot{e}_1 + \pi_2 \dot{e}_2 - H), \quad (27)$$

where the Hamiltonian H is given by

$$H = -e_1\phi_1 - e_2\phi_2 - \pi_1\mu_1 - \pi_2\mu_2, \quad (28)$$

μ_1, μ_2 are arbitrary functions of τ and the mass-shells constraints are

$$\phi_1 = \frac{1}{2}\left(p_1^2 - \frac{\alpha^2}{4}\sqrt{\frac{e_2}{e_1}}\frac{1}{r^2}\right),$$
$$\phi_2 = \frac{1}{2}\left(p_2^2 - \frac{\alpha^2}{4}\sqrt{\frac{e_1}{e_2}}\frac{1}{r^2}\right), \quad (29)$$

where $r = x_1 - x_2$ is the relative space-time coordinate. Note that when the interaction, which is regulated by the parameter α, vanishes, the model describes two massless free relativistic particles. Notice also that the model is conformal invariant with effective masses different from zero, a property that is not possible for a single conformal particle.

The equations of motion are given by

$$\dot{x}_a^\mu = \{x_a^\mu, H\} = e_a p_a^\mu, \qquad a = 1,2, \quad (30)$$

$$\dot{p}_1^\mu = \{p_1^\mu, H\} = \frac{\alpha^2}{4}\sqrt{e_1 e_2}\left\{p_1^\mu, \frac{1}{r^2}\right\}$$
$$= -\sqrt{e_1 e_2}\frac{\alpha^2}{2r^2}r^\mu = -\dot{p}_2^\mu, \quad (31)$$

$$\dot{e}_a = \{e_a, H\} = \mu_a \qquad a = 1,2, \quad (32)$$

$$\dot{\pi}_a = \{\pi_a, H\} = \phi_a, \qquad a = 1,2. \quad (33)$$

The stability of the primary constraints π_a implies the secondary constraints $\phi_a = 0$. Note the appearance of the einbeins in the constraints ϕ_a.

A. Killing equations and symmetries

Like in the free particle case we would like to find the most general point transformations of the model. We want to write the corresponding Killing equations for the two particle model. In order to do that we write the most general generator

$$G = \sum_{a=1}^{2} [\xi_{a\mu}(x_1, x_2) p_a^\mu + \gamma_a \pi_a]. \quad (34)$$

As shown in [14] the requirement $dG/d\tau = 0$ is satisfied provided the following conditions are verified:

$$\xi_a(x_1, x_2) = \xi_a(x_a), \qquad a = 1,2, \quad (35)$$

$$\frac{\partial \xi_{a\mu}}{\partial x_a^\nu} + \frac{\partial \xi_{a\nu}}{\partial x_a^\mu} - g_{\mu\nu}\tilde{\gamma}_a = 0, \qquad a = 1,2, \quad (36)$$

where

$$\tilde{\gamma}_a = \frac{\gamma_a}{e_a}, \qquad a = 1,2, \quad (37)$$

and

$$(\xi_1 - \xi_2) \cdot r = \frac{r^2}{8}\sum_{a=1}^{2} \partial_{a\nu}\xi_a^\nu. \quad (38)$$

From Eq. (36) we also obtain

$$\tilde{\gamma}_a = \frac{1}{2}\partial_{a\nu}\xi_a^\nu, \qquad a = 1,2. \quad (39)$$

We have verified that the model is invariant under the diagonal subgroup of the two conformal groups $\mathrm{SO}(4,2)_{1,2}$ acting on the two variables x_1 and x_2 respectively by checking that Eqs. (36) and (38) are satisfied for a generic infinitesimal transformation of the diagonal conformal subgroup,

$$\xi_a^\mu = \epsilon^\mu + \omega^\mu_{a\nu} x_a^\nu + \epsilon_D x_a^\mu + b^\mu x_a^2 - 2(b \cdot x_a) x_a^\mu, \qquad a = 1,2. \quad (40)$$

We can now compute the transformation properties of the variables under dilatations and SCT using

$$\delta x_a^\mu = \{G, x^\mu\} = \xi_a^\mu, \qquad \delta p_a^\mu = \{G, p_a^\mu\} = -\frac{\partial \xi_a^\nu}{\partial x_{a\mu}} p_{a\nu},$$
$$a = 1,2, \quad (41)$$

$$\delta e_a = \{e_a, G\}, \qquad a = 1,2. \quad (42)$$

Under dilatations:

$$\delta x_a^\mu = \epsilon_D x_a^\mu, \qquad \delta p_a^\mu = -\epsilon_D p_a^\mu,$$
$$\delta e_a = 2 e_a \epsilon_D, \qquad a = 1,2. \quad (43)$$

Under special conformal transformations:

$$\delta x_a^\mu = b^\mu x_a^2 - 2(b\cdot x_a)x_a^\mu,$$
$$\delta p_a^\mu = -2b\cdot p_a x_a^\mu + 2b\cdot x_a p_a^\mu + 2p_a\cdot x_a b^\mu, \qquad a=1,2, \tag{44}$$

$$\delta e_a = e_a\tilde{\gamma}_a = -e_a 4b\cdot x_a, \qquad a=1,2. \tag{45}$$

IV. TWO CONFORMAL CARROLL PARTICLES

Here we consider the Carroll limit of the conformal relativistic canonical action (27) by assuming

$$p_a{}^0 = \omega E_a, \qquad x_a^0 = \frac{1}{\omega}t_a, \qquad a=1,2. \tag{46}$$

From the mass-shell constraints given in Eq. (29) we get

$$\phi_1 = \frac{1}{2}\left(\omega^2E_1^2 - \mathbf{p}_1^2 + \frac{\alpha^2}{4}\sqrt{\frac{e_2}{e_1}}\frac{1}{\frac{1}{\omega^2}r_0^2 - \mathbf{r}^2}\right),$$
$$\phi_2 = \frac{1}{2}\left(\omega^2E_2^2 - \mathbf{p}_2^2 + \frac{\alpha^2}{4}\sqrt{\frac{e_1}{e_2}}\frac{1}{\frac{1}{\omega^2}r_0^2 - \mathbf{r}^2}\right). \tag{47}$$

By defining

$$\tilde{e}_a = e_a\omega^2, \qquad \tilde{\alpha}^2 = \frac{\alpha^2}{\omega^2}, \qquad a=1,2, \tag{48}$$

we obtain, in the limit $\omega\to\infty$,

$$e_1\phi_1 = \frac{\tilde{e}_1}{2}\left(E_1^2 - \frac{\tilde{\alpha}^2}{4}\sqrt{\frac{\tilde{e}_2}{\tilde{e}_1}}\frac{1}{\mathbf{r}^2}\right), \tag{49}$$

$$e_2\phi_2 = \frac{\tilde{e}_2}{2}\left(E_2^2 - \frac{\tilde{\alpha}^2}{4}\sqrt{\frac{\tilde{e}_1}{\tilde{e}_2}}\frac{1}{\mathbf{r}^2}\right), \tag{50}$$

so that the Carroll canonical Lagrangian is given by

$$L = -E_1\dot{t}_1 + \mathbf{p}_1\cdot\dot{\mathbf{x}}_1 - E_2\dot{t}_2 + \mathbf{p}_2\cdot\dot{\mathbf{x}}_2 + \frac{\tilde{e}_1}{2}\left(E_1^2 - \frac{\tilde{\alpha}^2}{4}\sqrt{\frac{\tilde{e}_2}{\tilde{e}_1}}\frac{1}{\mathbf{r}^2}\right) + \frac{\tilde{e}_2}{2}\left(E_2^2 - \frac{\tilde{\alpha}^2}{4}\sqrt{\frac{\tilde{e}_1}{\tilde{e}_2}}\frac{1}{\mathbf{r}^2}\right), \tag{51}$$

From now on we neglect the tilde, by renaming $\tilde{e}\to e, \tilde{\alpha}\to\alpha$.

When $\alpha=0$ the Lagrangian is invariant under two independent Carroll transformations acting on the corresponding particle coordinates. When $\alpha\neq 0$, the Lagrangian is invariant under the following diagonal Carroll transformations for each particle

$$\delta_C x_a^i = \epsilon^i - \Lambda^{ij}x_a^j, \qquad \delta_C p_a^i = -\Lambda^{ij}p_a^j + \beta^j E_a, \tag{52}$$

$$\delta_C t_a = h + \boldsymbol{\beta}\cdot\mathbf{x}_a, \qquad \delta_C E_a = 0,$$
$$\delta e_a = 0, \qquad a=1,2. \tag{53}$$

The Lagrangian equations of motion are

$$\dot{\mathbf{p}}_1 = \frac{\alpha^2}{2}\frac{\sqrt{e_1e_2}}{\mathbf{r}^4}\mathbf{r},$$
$$\dot{\mathbf{p}}_2 = -\frac{\alpha^2}{2}\frac{\sqrt{e_1e_2}}{\mathbf{r}^4}\mathbf{r},$$
$$\dot{\mathbf{x}}_a = 0, \qquad \dot{E}_a = 0, \qquad \dot{t}_a = e_aE_a,$$
$$0 = \dot{\pi}_1 = \frac{\partial L}{\partial e_1} = \frac{1}{2}\left(E_1^2 - \frac{\alpha^2}{4\mathbf{r}^2}\sqrt{\frac{e_2}{e_1}}\right),$$
$$0 = \dot{\pi}_2 = \frac{\partial L}{\partial e_2} = \frac{1}{2}\left(E_2^2 - \frac{\alpha^2}{4\mathbf{r}^2}\sqrt{\frac{e_1}{e_2}}\right). \tag{54}$$

This model describes interacting Carroll particles that have zero velocity. Since the distance among the particles is constant, it is only fixed by the initial conditions.

A. Eliminating the einbeins

The einbeins are nondynamical variables and can be eliminated through their equations of motion [23]. In fact from the equations of motion of e_1 we obtain

$$\sqrt{\frac{e_1}{e_2}} = \frac{\alpha^2}{4E_1^2\mathbf{r}^2} \tag{55}$$

and by squaring

$$e_1 = e_2\left(\frac{\alpha^2}{4E_1^2\mathbf{r}^2}\right)^2. \tag{56}$$

Then the canonical Carroll Lagrangian can be rewritten as

$$L = -E_a\dot{t}_a + \sum_{a=1,2}\mathbf{p}_a\cdot\dot{\mathbf{x}}_a + \frac{e_2}{2}\left(E_2^2 - \left(\frac{\alpha^2}{4\mathbf{r}^2}\right)^2\frac{1}{E_1^2}\right). \tag{57}$$

The equation of motion for E_2 gives

$$\dot{t}_2 = e_2E_2 \tag{58}$$

and substituting in Eq. (57) we obtain

$$L = -E_1\dot{t}_1 - \frac{1}{2}\frac{\dot{t}_2^2}{e_2} + \sum_{a=1,2}\mathbf{p}_a\cdot\dot{\mathbf{x}}_a - \frac{e_2}{2}\left(\frac{\alpha^2}{4\mathbf{r}^2}\right)^2\frac{1}{E_1^2}. \tag{59}$$

By using the equation for e_2

$$e_2 = \pm \frac{4\mathbf{r}^2}{\alpha^2}(E_1^2 t_2^2)^{1/2}, \tag{60}$$

and substituting in Eq. (59), the Lagrangian can be rewritten as

$$L = -E_1 \dot{t}_1 + \sum_{a=1,2} \mathbf{p}_a \cdot \dot{\mathbf{x}}_a \mp \frac{\alpha^2}{4\mathbf{r}^2}\left(\frac{\dot{t}_2^2}{E_1^2}\right)^{1/2}. \tag{61}$$

For the equation of motion of E_1 we get

$$E_1 = \pm \frac{\alpha}{2}(\dot{t}_2^2)^{1/4} \frac{1}{(\mathbf{r}^2 \dot{t}_1)^{1/2}}. \tag{62}$$

By substituting E_1 in Eq. (61) the final Lagrangian is (with a suitable choice of $\pm$ sign)

$$\begin{aligned} L &= \sum_{a=1,2} \mathbf{p}_a \cdot \dot{\mathbf{x}}_a - \alpha \left[\frac{\dot{t}_1^2 \dot{t}_2^2}{\mathbf{r}^4}\right]^{1/4} \\ &= \sum_{a=1,2} \mathbf{p}_a \cdot \dot{\mathbf{x}}_a - \frac{\alpha}{|\mathbf{r}|}[\dot{t}_1 \dot{t}_2]^{1/2}. \end{aligned} \tag{63}$$

We assume $\dot{t}_1 \dot{t}_2 > 0$.

Since the Lagrangian is homogeneous of degree one in the velocities, it is invariant under diffeomorphisms and gives rise to the primary constraint

$$E_1 E_2 - \frac{\alpha^2}{4\mathbf{r}^2} = 0. \tag{64}$$

In the case of two conformal free particles their individual energies are zero. Instead in this model, when the interaction is turned on ($\alpha \neq 0$), the energies are different from zero and constant. The particles are interacting, indeed; as shown in Eq. (64), the energy of each particle depends on the energy and the position of the other one. In conclusion the model is Carroll conformal even if the two particles have nonvanishing energies.

Since the energies E_1 and E_2 are constants of motion, the meaning of the Eq. (64) is that each particle stays inside its own light cone, corresponding to the lines with constant values of $\mathbf{x}_1$ and $\mathbf{x}_2$ [the velocities are zero, see Eq. (54)] and of $\mathbf{r}$. Introducing the total and the relative energies of the two particles,

$$E_T = E_1 + E_2, \qquad E_r = E_1 - E_2, \tag{65}$$

we get

$$E_T^2 = E_r^2 + \frac{\alpha^2}{\mathbf{r}^2}. \tag{66}$$

This equation shows that the minimum of the total energy is for $E_1 = E_2 = 0$ and for $\mathbf{r} \to \infty$.

The wave equation associated to Eq. (64) is

$$\left[\partial_{t_1}\partial_{t_2} + \frac{\alpha^2}{4\mathbf{r}^2}\right]\Phi(t_1, t_2, \vec{x}_1, \vec{x}_2) = 0. \tag{67}$$

B. Invariance under dilatations and special conformal transformations

The Lagrangian (51) is invariant also under the dilatations and special conformal transformations. Under dilatations we have

$$\begin{aligned} &\delta t_a = \epsilon_D t_a, \qquad \delta \mathbf{x}_a = \epsilon_D \mathbf{x}_a, \qquad \delta E_a = -\epsilon_D E_a, \\ &\delta \mathbf{p}_a = -\epsilon_D \mathbf{p}_a, \qquad \delta e_a = 2\epsilon_D e_a, \qquad a = 1, 2. \end{aligned} \tag{68}$$

Under SCT:

$$\begin{aligned} &\delta t_a = -b^0 \mathbf{x}_a^2 + 2\mathbf{b} \cdot \mathbf{x}_a t_a, \qquad \delta \mathbf{x}_a = -\mathbf{b}\mathbf{x}_a^2 + 2\mathbf{b} \cdot \mathbf{x}_a \mathbf{x}_a, \\ &a = 1, 2, \end{aligned} \tag{69}$$

$$\begin{aligned} &\delta E_a = -2\mathbf{b} \cdot \mathbf{x}_a E_a, \\ &\delta \mathbf{p}_a = -2(b^0 E_a - \mathbf{b} \cdot \mathbf{p}_a)\mathbf{x}_a - 2\mathbf{b} \cdot \mathbf{x}_a \mathbf{p}_a + 2(E_a t_a - \mathbf{p}_a \cdot \mathbf{x}_a)\mathbf{b}, \end{aligned} \tag{70}$$

$$\delta e_a = 4\mathbf{b} \cdot \mathbf{x}_a e_a, \qquad a = 1, 2. \tag{71}$$

These SCTs are obtained from the covariant ones given in Eqs. (44) and (45), considering

$$x_a^0 = \frac{1}{\omega} t_a, \qquad p_a^0 = \omega E_a, \qquad b^0 = \frac{b^0}{\omega}, \qquad a = 1, 2. \tag{72}$$

C. Infinite dimensional symmetries

Like in the previous sections, here we would like to find the most general point symmetries of the model.

Let us consider the generic Killing vector

$$\begin{aligned} G = \sum_{a=1,2} [&\xi_a^0(t_1, t_2, \mathbf{x}_1, \mathbf{x}_2) E_a - \boldsymbol{\xi}_a(t_1, t_2, \mathbf{x}_1, \mathbf{x}_2) \\ &\cdot \mathbf{p}_a + \gamma_a(t_1, t_2, \mathbf{x}_1, \mathbf{x}_2)\pi_a]. \end{aligned} \tag{73}$$

By considering the τ derivative we obtain

$$\begin{aligned} \frac{dG}{d\tau} = \sum_{a,c=1,2} \Bigg[&\frac{\partial \xi_a^0(t_1, t_2, \mathbf{x}_1, \mathbf{x}_2)}{\partial t_c} \dot{t}_c E_a - \frac{\partial \xi_a^j(t_1, t_2, \mathbf{x}_1, \mathbf{x}_2)}{\partial t_c} \dot{t}_c p_a^j \\ &- \boldsymbol{\xi}_a(t_1, t_2, \mathbf{x}_1, \mathbf{x}_2) \cdot \dot{\mathbf{p}}_a + \gamma_a(t_1, t_2, \mathbf{x}_1, \mathbf{x}_2)\dot{\pi}_a\Bigg], \end{aligned} \tag{74}$$

where we have used the equations of motion

$$\dot{E}_a = 0, \qquad \dot{\mathbf{x}}_a = 0 \tag{75}$$

and the primary constraints $\pi_a = 0$. Using Eq. (54) we get

$$\begin{aligned}\frac{dG}{d\tau} = &\sum_{a,c=1,2}\left[\frac{\partial\xi_a^0(t_1,t_2,\mathbf{x}_1,\mathbf{x}_2)}{\partial t_c}e_cE_cE_a - \frac{\partial\xi_a^j(t_1,t_2,\mathbf{x}_1,\mathbf{x}_2)}{\partial t_c}\right.\\ &\left.\times e_cE_cp_a^j + \frac{1}{2}\gamma_a(t_1,t_2,\mathbf{x}_1,\mathbf{x}_2)E_a^2\right]\\ &-\frac{\alpha^2}{2\mathbf{r}^4}\sqrt{e_1e_2}(\boldsymbol{\xi}_1-\boldsymbol{\xi}_2)\cdot\mathbf{r} - \frac{\alpha^2}{8\mathbf{r}^2}\left[\gamma_1(t_1,t_2,\mathbf{x}_1,\mathbf{x}_2)\sqrt{\frac{e_2}{e_1}}\right.\\ &\left.+\gamma_2(t_1,t_2,\mathbf{x}_1,\mathbf{x}_2)\sqrt{\frac{e_1}{e_2}}\right].\end{aligned} \tag{76}$$

Therefore the Killing condition $dG/d\tau = 0$ implies

$$\frac{\partial\xi_a^j(t_1,t_2,\mathbf{x}_1,\mathbf{x}_2)}{\partial t_c} = 0, \qquad a,c = 1,2, \tag{77}$$

or $\xi_a^j = \xi_a^j(\mathbf{x}_1,\mathbf{x}_2)$. We see that $\delta\mathbf{x}_a \equiv \boldsymbol{\xi}_a$ does not depend on the times t_1, t_2.

By defining

$$\gamma_a = e_a\tilde{\gamma}_a \tag{78}$$

from $dG/d\tau = 0$ we must have also

$$\frac{\partial\xi_a^0(t_1,t_2,\mathbf{x}_1,\mathbf{x}_2)}{\partial t_c} = -\frac{1}{2}\delta_{ac}\tilde{\gamma}_a, \qquad a,b,c = 1,2, \tag{79}$$

and

$$\frac{\alpha^2}{\mathbf{r}^2}(\boldsymbol{\xi}_1-\boldsymbol{\xi}_2)\cdot\mathbf{r} = \frac{\alpha^2}{2}\frac{\delta\mathbf{r}^2}{\mathbf{r}^2} = -\frac{\alpha^2}{4}(\tilde{\gamma}_1+\tilde{\gamma}_2), \qquad a = 1,2. \tag{80}$$

In the case $\alpha = 0$, that is when we turn off the interaction, we see that we have two sets of independent Killing equations, one set for each particle. Therefore the infinite group of symmetry transformations is $\mathcal{G}_1\times\mathcal{G}_1$. When the interaction is on and $\alpha \neq 0$, we get:

$$\frac{1}{\mathbf{r}^2}(\boldsymbol{\xi}_1-\boldsymbol{\xi}_2)\cdot\mathbf{r} = \frac{1}{2}\frac{\delta\mathbf{r}^2}{\mathbf{r}^2} = -\frac{1}{4}(\tilde{\gamma}_1+\tilde{\gamma}_2), \qquad a = 1,2. \tag{81}$$

From Eq. (79) we have

$$\frac{\partial\xi_1^0}{\partial t_2} = 0 = \frac{\partial\xi_2^0}{\partial t_1}. \tag{82}$$

Furthermore from Eq. (81) we note that since $\boldsymbol{\xi}_a$ are only functions of $\mathbf{x}_1, \mathbf{x}_2, \tilde{\gamma}_a$ depend only on $\mathbf{x}_1, \mathbf{x}_2$. In conclusion we obtain

$$\begin{aligned}&\xi_a^0 \equiv \delta t_a = -\frac{1}{2}\tilde{\gamma}_a(\mathbf{x}_1,\mathbf{x}_2)t_a + h_a(\mathbf{x}_1,\mathbf{x}_2),\\ &\delta e_a = \gamma_a = e_a\tilde{\gamma}_a(\mathbf{x}_1,\mathbf{x}_2),\end{aligned} \tag{83}$$

with the extra condition given by Eq. (81).

Summarizing the Killing generator is given by

$$\begin{aligned}G = \sum_{a=1,2}&\left[\left(-\frac{1}{2}\tilde{\gamma}_a(\mathbf{x}_1,\mathbf{x}_2)t_a + h_a(\mathbf{x}_1,\mathbf{x}_2)\right)E_a\right.\\ &\left.-\boldsymbol{\xi}_a(\mathbf{x}_1,\mathbf{x}_2)\cdot\mathbf{p}_a + e_a\tilde{\gamma}_a(\mathbf{x}_1,\mathbf{x}_2)\pi_a\right],\end{aligned} \tag{84}$$

where $h_a, \boldsymbol{\xi}_a, \tilde{\gamma}_a$ are arbitrary functions of $\mathbf{x}_1$, $\mathbf{x}_2$ that must satisfy the condition (79). Therefore the original ten arbitrary functions reduce to nine independent arbitrary functions of $\mathbf{x}_1$, $\mathbf{x}_2$.

Therefore the model of two conformal Carroll particles analyzed in this paper has an infinite dimensional group of symmetries that we will call $\mathcal{G}_2$. Notice that $\mathcal{G}_2 \subset \mathcal{G}_1\times\mathcal{G}_1$. The main difference between these two groups is the transformation laws of the times t_a that in the free case are arbitrary functions of the times and the coordinates, while in the interacting case Eq. (79) restricts the time dependence of the transformations of ξ_a^0 to be linear in t_a.

It is convenient to make a comparison with [6]. Let us generalize to two particles the expression for the generators of the conformal Carrollian algebra from [6]. We consider the diagonal subgroup of the two conformal Carroll groups.

For Carroll dilatations and SCT we have

$$\xi_a^i = (\epsilon_D + 2\mathbf{b}\cdot\mathbf{x}_a)x_a^i - b^i\mathbf{x}_a^2, \tag{85}$$

$$\xi_a^0 = (\epsilon_D + 2\mathbf{b}\cdot\mathbf{x}_a)t_a + T_a(\mathbf{x}_1,\mathbf{x}_2). \tag{86}$$

We have

$$\frac{\partial\xi_a^0}{\partial t_a} = \epsilon_D + 2\mathbf{b}\cdot\mathbf{x}_a \tag{87}$$

and

$$\frac{1}{2}\left(\frac{\partial\xi_1^0}{\partial t_1} + \frac{\partial\xi_2^0}{\partial t_2}\right) = \epsilon_D + \mathbf{b}\cdot\mathbf{x}_1 + \mathbf{b}\cdot\mathbf{x}_2, \tag{88}$$

$$\begin{aligned}(\boldsymbol{\xi}_1-\boldsymbol{\xi}_2)\cdot\mathbf{r} &= \epsilon_D\mathbf{r}^2 + 2(\mathbf{b}\cdot\mathbf{x}_1)\mathbf{x}_1\cdot\mathbf{r} - \mathbf{b}\cdot\mathbf{r}(\mathbf{x}_1)^2\\ &\quad - 2(\mathbf{b}\cdot\mathbf{x}_2)\mathbf{x}_2\cdot\mathbf{r} + \mathbf{b}\cdot\mathbf{r}(\mathbf{x}_2)^2\\ &= \epsilon_D\mathbf{r}^2 + \mathbf{r}^2(\mathbf{b}\cdot\mathbf{x}_1 + \mathbf{b}\cdot\mathbf{x}_2).\end{aligned} \tag{89}$$

In conclusion Eq. (81) is satisfied by Eqs. (85) and (86) and the so-called supertranslations $T_a(\mathbf{x}_1,\mathbf{x}_2)$ in our case are arbitrary functions of both particle positions $\mathbf{x}_1$, $\mathbf{x}_2$.

V. TWO CONFORMAL CARROLL PARTICLES: A TACHYONIC MODEL

In this section, following the approach of [13,24], we develop a model describing two interacting Carrollian tachyons. Let us consider the Lagrangian

$$\begin{aligned} L &= \sum_{a=1,2} [-E_a \dot{t}_a + \mathbf{p}\dot{\mathbf{x}}_a - e_a \phi_a - \chi_a E_a] \\ &= -E_1 \dot{t}_1 + \mathbf{p}_1 \cdot \dot{\mathbf{x}}_1 - E_2 \dot{t}_2 + \mathbf{p}_2 \cdot \dot{\mathbf{x}}_2 \\ &\quad - \frac{e_1}{2}\left(\mathbf{p}_1^2 - \frac{\alpha^2}{4}\sqrt{\frac{e_2}{e_1}}\frac{1}{\mathbf{r}^2}\right) - \frac{e_2}{2}\left(\mathbf{p}_2^2 - \frac{\alpha^2}{4}\sqrt{\frac{e_1}{e_2}}\frac{1}{\mathbf{r}^2}\right) \\ &\quad - \chi_1 E_1 - \chi_2 E_2. \end{aligned} \tag{90}$$

Notice the presence of the Lagrangian multipliers χ_1, χ_2, that implement the energies are zero; these terms are necessary in order to have Carroll invariance. An analogous situation appears for a single particle in [24] where the action is named the magnetic Carroll particle. In fact this Lagrangian is invariant under the Carroll, Eqs. (52) and (53), scale and SCT, Eqs. (68)–(71), by requiring the following transformations for the einbein χ_a:

$$\delta\chi_a = -e_a \mathbf{p}_a \cdot \boldsymbol{\beta}, \qquad a = 1,2, \qquad \text{Carroll transformations,} \tag{91}$$

$$\delta\chi_a = \epsilon_D \chi_a, \qquad a = 1,2, \qquad \text{scale transformations,} \tag{92}$$

$$\begin{aligned} \delta\chi_a &= 2\Big[e_a(b^0 \mathbf{p}_a \cdot \mathbf{x}_a - t_a \mathbf{p}_a \cdot \mathbf{b}) + \chi_a \mathbf{b} \cdot \mathbf{x}_a\Big], \\ &a = 1,2, \qquad \text{SCT transformations.} \end{aligned} \tag{93}$$

The equations of motion derived from the Lagrangian (90) by varying with respect to t_a, E_a, $\mathbf{p}_a$, $\mathbf{x}_a$, χ_a, e_a are the following:

$$\dot{E}_a = 0, \qquad \dot{t}_a = -E_a, \tag{94}$$

$$e_a \mathbf{p}_a - \dot{\mathbf{x}}_a = 0, \qquad \dot{\mathbf{p}}_1 = -\frac{\alpha^2}{2\mathbf{r}^4}\sqrt{e_1 e_2}\,\mathbf{r} = -\dot{\mathbf{p}}_2, \tag{95}$$

$$E_a = 0, \tag{96}$$

$$\phi_a = 0. \tag{97}$$

In this model the particles have nonzero velocity and therefore, as a consequence of the limit $c \to 0$, they are necessarily tachyons. Or equivalently, since $E_a = 0$, if we use the relativistic mass-shell constraint the two particle invariant squared masses are negative ($E_a^2 - \mathbf{p}_a^2 < 0$).

A. Eliminating the einbeins

Let us first study the equivalence of this model with the $c \to 0$ limit of the model for two relativistic conformal particles [14] described by the Lagrangian

$$L = -\alpha\left(\frac{\dot{x}_1^2 \dot{x}_2^2}{r^4}\right)^{1/4}. \tag{98}$$

In order to do that, we start by eliminating the nondynamical variable e_1 in Eq. (90) through its own equation of motion. We obtain

$$\sqrt{\frac{e_1}{e_2}} = \frac{\alpha^2}{4\mathbf{r}^2 \mathbf{p}_1^2} \tag{99}$$

and substituting in Eq. (90)

$$\begin{aligned} L &= -E_1 \dot{t}_1 + \mathbf{p}_1 \cdot \dot{\mathbf{x}}_1 - E_2 \dot{t}_2 + \mathbf{p}_2 \cdot \dot{\mathbf{x}}_2 \\ &\quad - \frac{e_2}{2}\left[\mathbf{p}_2^2 - \left(\frac{\alpha^2}{4\mathbf{r}^2}\right)^2 \frac{1}{\mathbf{p}_1^2}\right] - \chi_1 E_1 - \chi_2 E_2. \end{aligned} \tag{100}$$

The equations of motion of the nondynamical variables $\mathbf{p}_2$ are

$$\mathbf{p}_2 = \frac{1}{e_2}\dot{\mathbf{x}}_2 \tag{101}$$

and the Lagrangian becomes

$$\begin{aligned} L &= -E_1 \dot{t}_1 - E_2 \dot{t}_2 - \chi_1 E_1 - \chi_2 E_2 \\ &\quad + \mathbf{p}_1 \cdot \dot{\mathbf{x}}_1 + \frac{1}{2e_2}\dot{\mathbf{x}}_2^2 + \frac{e_2}{2}\left(\frac{\alpha^2}{4\mathbf{r}^2}\right)^2 \frac{1}{\mathbf{p}_1^2}. \end{aligned} \tag{102}$$

Then we eliminate the nondynamical variable e_2 through its own equation of motion (choosing the $-$ sign)

$$e_2 = -\frac{4\mathbf{r}^2}{\alpha^2}\sqrt{\dot{\mathbf{x}}_2^2 \mathbf{p}_1^2} \tag{103}$$

and substituting in L

$$L = -\sum_{a=1,2}(E_a \dot{t}_a + \chi_a E_a) + \mathbf{p}_1 \cdot \dot{\mathbf{x}}_1 - \frac{\alpha^2}{4\mathbf{r}^2}\sqrt{\frac{\dot{\mathbf{x}}_2^2}{\mathbf{p}_1^2}}. \tag{104}$$

Finally, taking into account that

$$\dot{\mathbf{x}}_1 = -\mathbf{p}_1 \frac{\alpha^2}{4\mathbf{r}^2}\sqrt{\dot{\mathbf{x}}_2^2}(\mathbf{p}_1^2)^{-3/2} \tag{105}$$

and

$$\mathbf{p}_1^2 = \frac{\alpha^2}{4\mathbf{r}^2}\sqrt{\frac{\dot{\mathbf{x}}_2^2}{\dot{\mathbf{x}}_1^2}}, \tag{106}$$

we get the following expression for the Lagrangian:

$$L = -\sum_{a=1,2}(E_a\dot{t}_a + \chi_a E_a) - \alpha\left(\frac{\dot{\mathbf{x}}_1^2\dot{\mathbf{x}}_2^2}{\mathbf{r}^4}\right)^{1/4}. \tag{107}$$

By eliminating E_a and χ_a we obtain

$$\dot{t}_a = -\chi_a, \qquad E_a = 0 \tag{108}$$

and the following expression for the Lagrangian:

$$L = -\alpha\left(\frac{\dot{\mathbf{x}}_1^2\dot{\mathbf{x}}_2^2}{\mathbf{r}^4}\right)^{1/4}, \tag{109}$$

which coincides with the $c \to 0$ limit of (98).

A primary constraint can be obtained by squaring $\mathbf{p}_a$,

$$\mathbf{p}_1^2\mathbf{p}_2^2 - \frac{\alpha^2}{16\mathbf{r}^4} = 0. \tag{110}$$

B. Killing equations and symmetries

In this section we derive the Killing vector by working directly with the Lagrangian given in Eq. (90). By taking the total derivative with respect to τ of the Killing vector

$$G = \sum_{a=1,2}(\xi_a^0 E_a - \boldsymbol{\xi}_a\cdot\mathbf{p}_a + \gamma_a\pi_a + \nu_a\pi_a^\chi), \tag{111}$$

where

$$\pi_a^\chi = \frac{\partial L}{\partial\dot{\chi}_a}, \tag{112}$$

we get

$$\begin{aligned}
\frac{dG}{d\tau} &= \sum_{a,b=1,2}\left(\frac{\partial\xi_a^0}{\partial t_b}\dot{t}_b E_a + \frac{\partial\xi_a^0}{\partial\mathbf{x}_b}\dot{\mathbf{x}}_b E_a - \frac{\partial\boldsymbol{\xi}_a}{\partial t_b}\dot{t}_b\mathbf{p}_a - \frac{\partial\xi_a^i}{\partial x_b^j}\dot{x}_b^j p_a^i\right) - \sum_{a=1,2}(\boldsymbol{\xi}_a\cdot\dot{\mathbf{p}}_a + \gamma_a\dot{\pi}_a + \nu_a\dot{\pi}_a^\chi)\\
&= \sum_{a,b=1,2}\left(-\frac{\partial\xi_a^0}{\partial t_b}\chi_b E_a + \frac{\partial\xi_a^0}{\partial\mathbf{x}_b}e_b\mathbf{p}_b E_a + \frac{\partial\boldsymbol{\xi}_a}{\partial t_b}\chi_b\mathbf{p}_a - \frac{\partial\xi_a^i}{\partial x_b^j}e_b p_b^j p_a^i\right) - \sum_{a=1,2}(\boldsymbol{\xi}_a\cdot\dot{\mathbf{p}}_a + \gamma_a\phi_a + \nu_a E_a)\\
&= \sum_{a,b=1,2}\Bigg[\left(-\frac{\partial\xi_a^0}{\partial t_b}\chi_b + \frac{\partial\xi_a^0}{\partial\mathbf{x}_b}e_b\mathbf{p}_b - \nu_a\right)E_a - \frac{\partial\boldsymbol{\xi}_a}{\partial t_b}\chi_b\mathbf{p}_a - \frac{\partial\xi_a^i}{\partial x_b^j}e_b p_b^j p_a^i - \frac{1}{2}\gamma_a\mathbf{p}_a^2\\
&\quad - \frac{\alpha^2}{2\mathbf{r}^4}\sqrt{e_1e_2}(\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2)\cdot\mathbf{r} - \frac{\alpha^2}{8r^2}\sqrt{e_1e_2}(\tilde{\gamma}_1 + \tilde{\gamma}_2)\Bigg],
\end{aligned} \tag{113}$$

where use has been made of the primary constraints $\pi_a = \pi_a^\chi = 0$, of Eqs. (94) and (95) and of

$$\dot{\pi}_a = -\phi_a, \tag{114}$$

$$\dot{\pi}_a^\chi = -E_a. \tag{115}$$

The sum over latin indices is understood.

In order to get $dG/d\tau = 0$, we must require

$$\frac{\partial\boldsymbol{\xi}_a}{\partial t_b} = 0, \tag{116}$$

so that $\boldsymbol{\xi}_a$ is a function of only $\mathbf{x}_1$, $\mathbf{x}_2$; we require also the vanishing of the coefficient of E_a,

$$\sum_b\left(-\frac{\partial\xi_a^0}{\partial t_b}\chi_b + \frac{\partial\xi_a^0}{\partial\mathbf{x}_b}e_b\mathbf{p}_b\right) - \nu_a = 0. \tag{117}$$

We also require, assuming $\boldsymbol{\xi}_a = \boldsymbol{\xi}_a(\mathbf{x}_a)$, the vanishing of the coefficient of $p_a^i p_a^j$

$$\left(\frac{\partial\xi_a^i}{\partial x_a^j} + \frac{\partial\xi_a^j}{\partial x_a^i}\right) = -\delta^{ij}\tilde{\gamma}_a(\mathbf{x}_a), \tag{118}$$

so that $dG/d\tau$ implies, when $\alpha \neq 0$,

$$-\frac{1}{4}(\tilde{\gamma}_1 + \tilde{\gamma}_2) = \frac{1}{6}\left(\frac{\partial\xi_1^i}{\partial x_1^i} + \frac{\partial\xi_2^i}{\partial x_2^i}\right) = \frac{(\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2)\cdot\mathbf{r}}{\mathbf{r}^2}. \tag{119}$$

In conclusion the general solution for the Killing vector is

$$G = \sum_{a=1,2} \left[\xi^0_a(t_1, t_2, \mathbf{x}_1, \mathbf{x}_2) E_a - \boldsymbol{\xi}_a(\mathbf{x}_a) \cdot \mathbf{p}_a + e_a \tilde{\gamma}_a \pi_a + \nu_a \pi^\chi_a\right], \tag{120}$$

where

$$\tilde{\gamma}_a = -\frac{2}{3} \frac{\partial \xi^i_a}{\partial x^i_a}, \tag{121}$$

$$\nu_a = \sum_b \left(-\frac{\partial \xi^0_a}{\partial t_b} \chi_b + \frac{\partial \xi^0_a}{\partial \mathbf{x}_b} e_b \mathbf{p}_b\right), \tag{122}$$

with ξ^0_a an arbitrary function of t_1, t_2, $\mathbf{x}_1$, $\mathbf{x}_2$, and $\boldsymbol{\xi}_a(\mathbf{x}_a)$ satisfying Eq. (119). As expected from the tachyonic Carroll particle case discussed in Sec. V, while $\xi^0_a, \boldsymbol{\xi}_a, \tilde{\gamma}_a$ are arbitrary point functions of t_a and $\boldsymbol{\xi}_a$, in general ν_a depend also on the momenta $\mathbf{p}_a$. Carroll, scale transformations, and conformal transformations satisfy Eq. (119).

We have checked that, by using the expression of ξ^0_a for Carroll, scale and SCT in Eq. (122), we recover the correct transformations for χ_a given in Eqs. (91)–(93), by computing

$$\delta\chi_a = \{\chi_a, G\} \tag{123}$$

and assuming

$$\{\chi_a, \pi^\chi_b\} = -\delta_{ab} \tag{124}$$

Note also that the dependence of ν_a on $\mathbf{x}_a$, $\mathbf{p}_a$ in principle could give additional contributions to the transformations of $\mathbf{p}_a$, $\mathbf{x}_a$, which however are vanishing because they are proportional to the primary constraints π^χ_a.

Notice also that in the case of two nonconformal particles [9] the individual particles move with energies different from zero and they have only a diagonal finite conformal symmetry.

VI. CONCLUSIONS

Carroll symmetries have recently received a lot of attention because of several applications in different domains of theoretical physics, from theory of gravity and strings to studies of fractons in condensed matter.

In this paper, after reexamining the case of a single conformal Carroll particle, we derive the conformal Carroll generators, their algebra, and the infinite extension of this algebra [9]. Then we propose two different models of Carroll interacting particles that provide dynamical realizations of the Carroll conformal algebra. The models are obtained from the relativistic model of interacting conformal particles proposed in [14] through suitable limits for $c \to 0$.

Concerning the dynamics, the first model describes particles with zero velocity but with nonvanishing energy as a consequence of the interaction. Free conformal Carroll particles have zero energy; here, when the interaction is on, the energy of one particle depends on the energy of the second one and on the particle relative distance which is constant.

The second model is a tachyonic one: the particles have nonzero velocity and therefore, due to the limit $c \to 0$, they are necessarily tachyons. Nevertheless their energy is vanishing.

Both models exhibit, after a complete analysis of the most general point symmetry transformations, infinite dimensional symmetries which include supertranslations as in the case of the BMS group [3,4], the symmetry that arises in asymptotically flat space-time at null infinity. These infinite dimensional algebras contain the Carroll conformal one [6], which is equivalent to BMS ones. The infinite symmetries emerging from these models look like accidental symmetries as the global symmetries that appear in the particle Standard Model. Therefore it is possible that the Lagrangian could contain other terms invariant under conformal Carroll but breaking the infinite dimensional symmetries.

In the future several directions could be investigated: on the one hand it would be interesting to extend the analysis of the two conformal Carroll models to include corrections to the next order in $c \to 0$ expansion, see for example [24]; on the other hand a deeper analysis of the infinite symmetries could be performed to check similarities with the BMS group. The conformal two Carroll particle model potentially could also be useful to study bilocal holography for flat space-times.

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