Freeze-free cosmological evolution with a nonmonotonic internal clock

Luis Martinez,[*] Martin Bojowald,[†] and Garrett Wendel[‡]
Institute for Gravitation and the Cosmos, The Pennsylvania State University, University Park, Pennsylvania 16802, USA

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Given the lack of an absolute time parameter in general relativistic systems, quantum cosmology often describes the expansion of the universe in terms of relational changes between 2 degrees of freedom, such as matter and geometry. However, if clock degrees of freedom (self-)interact nontrivially, they in general have turning points where their momenta vanish. At and beyond a turning point, the evolution of other degrees of freedom is no longer described directly by changes of the clock parameter because it stops and then turns back, while time is moving forward. Previous attempts to describe quantum evolution relative to a clock with turning points have failed and led to frozen evolution in which degrees of freedom remain constant while the clock parameter, interpreted directly as a substitute for monotonic time, is being pushed beyond its turning point. Here, a new method previously used in oscillator systems is applied to a tractable cosmological model, given by an isotropic universe with spatial curvature and scalar matter. The recollapsing scale factor presents an example of a clock with a single turning point. The method succeeds in defining unitary and freeze-free evolution by unwinding the turning point of the clock, introducing an effective monotonic time parameter that is related to but not identical with the nonmonotonic clock degree of freedom. Characteristic new quantum features are found around the turning point, based on analytical and numerical calculations.

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I. INTRODUCTION

General relativistic systems such as cosmological models are time reparametrization invariant and therefore lack a physically defined energy scale. Their canonical description requires a generalization from Hamilton's equations to constrained evolution, in which both time reparametrizations and evolution are generated by a single object, the Hamiltonian constraint. A common method to describe (physically observable) evolution within this setting and to distinguish it from mere reparametrizations of a time coordinate, going back to [1], consists in deriving changes of the dynamical variables with respect to changes of a distinguished one among them, identified as an internal time parameter or a clock variable. Similar questions have been analyzed recently in the context of quantum reference frames [2–7].

Classically, the transition from the usual time coordinate t to an internal clock ϕ locally consists in a simple substitution of the local inverse of a solution $\phi(t)$ for t. However, the local nature of this procedure implies obstacles at the quantum level. In practice, the procedure has therefore required the choice of rather special matter systems as candidates for internal clocks, such as a free, massless scalar field [8], dust [9,10], or, more generally, systems with purely kinetic energy and no potential or (self-)interactions. These choices classically imply monotonic solutions for the relevant fields as functions of coordinate time, which can be inverted globally. They also have conserved momenta, which can then serve as simple Hamiltonians that globally generate classical or quantum evolution with respect to the canonically conjugate variable. While such models therefore have consistent quantizations, their highly restrictive and nonfundamental nature means that the physical viability of their implications should be tested by eliminating the strong underlying assumptions. An analysis of models with nonmonotonic matter solutions or nonglobal internal clocks is therefore required, but previous attempts in quantum cosmology were unable to extend evolution across a turning point of a nonmonotonic clock [11,12]: evolution froze because all degrees of freedom remained constant beyond the turning point of the clock.

Based on our earlier work on oscillating clocks in quantum mechanics [13–15], we here perform a new analysis of a cosmological model using a nonglobal clock variable. The variable we choose as a clock, given by the scale factor in a closed isotropic model, has a single turning point and is not oscillating. Our previous methods are seen to apply nonetheless and show that the resulting dynamics

[*]lxm471@psu.edu
[†]bojowald@psu.edu
[‡]gmw5164@psu.edu

of a scalar field is consistent. Unlike in previous attempts, the use of an internal clock with turning points does not lead to freezing cosmological evolution in our treatment. This outcome relies on the construction of an effective monotonic time parameter from a nonmonotonic fundamental clock. After an introduction of the classical model in Sec. II, we present a detailed construction of global evolution in Sec. III. Semiclassical evolution far from the turning point, before as well as after, is described well by what is expected classically if one uses the same clock for local evolution. There are new quantum effects around the turning point which we will derive analytically and confirm by numerical solutions. Some of these results will be used in a comparison with Dirac observables in Sec. IV.

II. CLASSICAL MODEL

We begin with the Friedmann equation

$$\left(\frac{\dot{a}}{a}\right)^2+\frac{k}{a^2}=\frac{8\pi G}{3}\rho \tag{1}$$

for an isotropic model with positive spatial curvature, the dot indicating a derivative by proper time. The constant k is positive and equals $k=1$ if the full volume of a spatial 3-sphere is evolved, but it will be convenient to keep it as a variable constant in our quantum model. In particular, k may be smaller than one if only a subset of the 3-sphere is considered. While the choice of this value makes no difference classically (as long as it remains positive), quantum effects usually depend on the size of the region in which they are computed, as known from fluctuation energies or Casimir forces. Similarly, quantum cosmology is sensitive to a change of k by subdividing the spatial volume as a consequence of infrared renormalization [16].

In preparation for our quantization, we first introduce the canonical momentum

$$p_a=-\frac{3}{4\pi G}a\dot{a} \tag{2}$$

and specialize the energy density ρ to the kinetic term $\rho=\frac{1}{2}p_\phi^2/a^6$ of a free, massless scalar ϕ with momentum p_ϕ:

$$\frac{4\pi G}{3}\left(\frac{p_a}{a^2}\right)^2+\frac{3k}{4\pi G a^2}=\frac{p_\phi^2}{a^6}. \tag{3}$$

A first canonical transformation from (a,p_a) to

$$\tilde{a}=\sqrt{\frac{3}{4\pi G}}a \quad \text{and} \quad \tilde{p}_a=\sqrt{\frac{4\pi G}{3}}p_a \tag{4}$$

and from (ϕ,p_ϕ) to

$$\tilde{\phi}=\sqrt{\frac{4\pi G k}{3}}\phi \quad \text{and} \quad \tilde{p}_\phi=\sqrt{\frac{3}{4\pi G k}}p_\phi \tag{5}$$

eliminates several constant factors, such that

$$\left(\frac{\tilde{p}_a}{\sqrt{k}\tilde{a}^2}\right)^2+\frac{1}{\tilde{a}^2}=\frac{\tilde{p}_\phi^2}{\tilde{a}^6}. \tag{6}$$

A second canonical transformation from $(\tilde{a},\tilde{p}_a)$ to

$$\alpha=\sqrt{k}\ln\tilde{a} \quad \text{and} \quad p_\alpha=\frac{\tilde{a}\tilde{p}_a}{\sqrt{k}} \tag{7}$$

then implies the constraint

$$\mathcal{C}=\tilde{p}_\phi^2-p_\alpha^2-e^{\gamma\alpha}=0 \tag{8}$$

with $\gamma=4/\sqrt{k}\geq 4$. From now on, we will drop the tilde on ϕ and p_ϕ for convenience.

Classically, in terms of some gauge variable ϵ, Hamilton's equations generated by $\mathcal{C}$ are

$$\frac{\mathrm{d}\alpha}{\mathrm{d}\epsilon}=\{\alpha,\mathcal{C}\}=-2p_\alpha\approx-2\mathrm{sgn}(p_\alpha)\sqrt{p_\phi^2-e^{\gamma\alpha}} \tag{9}$$

and

$$\frac{\mathrm{d}p_\alpha}{\mathrm{d}\epsilon}=\{p_\alpha,\mathcal{C}\}=\gamma e^{\gamma\alpha}\approx\gamma(p_\phi^2-p_\alpha^2). \tag{10}$$

As indicated by the weak equalities, these equations decouple on the constraint surface $\mathcal{C}=0$, also using the fact that p_ϕ is constant thanks to the ϕ-independence of $\mathcal{C}$. Therefore,

$$\alpha(\epsilon)=-\frac{2}{\gamma}\ln\frac{\cosh(\gamma|p_\phi|\epsilon)}{|p_\phi|}, \qquad p_\alpha(\epsilon)=p_\phi\tanh(\gamma p_\phi\epsilon) \tag{11}$$

if we eliminate the sole integration constant that remains after imposing the constraint by choosing ϵ such that $p_\alpha(0)=0$.

For the scalar field we obtain the simple monotonic solution $\phi(\epsilon)=2p_\phi\epsilon+c$ with a constant c, as well as constant p_ϕ. The model therefore has a global internal time ϕ, in which case evolution is generated by the strictly positive Hamiltonian $-p_\phi=\sqrt{p_\alpha^2+e^{\gamma\alpha}}$. The equations of motion for ϕ-evolution generated by $-p_\phi$ can be solved as well, and they agree with a simple substitution of $p_\phi\epsilon=\frac{1}{2}(\phi-c)$ in (11). Inverting the resulting $\alpha(\phi)$, we obtain the double-valued function

$$\phi(\alpha)=c+\frac{2}{\gamma}\cosh^{-1}(|p_\phi|e^{-\gamma\alpha/2}). \tag{12}$$

The branch of the inverse cosh should be switched when α starts decreasing (if it has been increasing initially) at its turning point

$$\alpha_{\rm t}(p_\phi) = \frac{2\ln|p_\phi|}{\gamma}. \tag{13}$$

An application of α as an internal clock instead of ϕ is therefore challenging, in particular when the system is to be quantized. We will present the necessary steps for such a construction in the next section, but first note that classical evolution with respect to α is locally generated by the Hamiltonian

$$-p_\alpha = \pm\sqrt{p_\phi^2 - e^{\gamma\alpha}}. \tag{14}$$

Unlike the ϕ-Hamiltonian, this expression is explicitly clock (α) dependent, and it is not positive definite. It equals zero at the turning point (13) where α has to start decreasing (again, if it has been increasing initially) for the square root to remain real. Correspondingly, the sign chosen in (14) has to be flipped for α to turn around. The formalism of [13–15] allows us to implement this turning point of α even within unitary quantum evolution determined by an effective time parameter related to (but not identical with) α as an internal clock.

III. GLOBAL QUANTUM EVOLUTION

Equation (14) can formally be quantized to

$$i\hbar\frac{\partial\psi(p_\phi,\alpha)}{\partial\alpha} = \pm\sqrt{p_\phi^2 - e^{\gamma\alpha}}\psi(p_\phi,\alpha). \tag{15}$$

However, the Hamiltonian on the right-hand side is not self-adjoint on the kinematical Hilbert space $L^2(\mathbb{R}^2, {\rm d}p_\phi {\rm d}\alpha)$ and therefore does not define global evolution with respect to α. To ensure that α does not surpass the turning-point value $\alpha_{\rm t}(p_\phi)$ given in (13), we replace α in evolution equations with an effective time parameter τ that locally follows the changes of α but implements the condition that α start decreasing (with respect to this new τ) right after it reaches the turning point. These conditions, together with continuity of $\alpha(\tau)$, determine this relationship as

$$\alpha(\tau) = \begin{cases} +\tau & \text{if } \tau < \alpha_t(p_\phi) \\ -\tau + 2\alpha_t(p_\phi) & \text{if } \tau > \alpha_t(p_\phi) \end{cases} \tag{16}$$

up to a constant shift. (If a different clock rate is preferred, τ may be transformed nonlinearly in our final solutions; see Sec. III C.) While the clock variable α is allowed to be nonmonotonic around the turning point, time τ continues to increase. This crucial step, clearly distinguishing between clock and time, is sufficient for a consistent definition of global evolution [13]. In our treatment, time is an effective parameter constructed from a fundamental clock according to the prescription (16) that keeps track of turning points, just as time labels used in daily life are constructed from clock readings as well as turning points. Since the time parameter is not fundamental, we make no attempt to complete (16) to a canonical transformation, which due to multivaluedness would require an extension of the phase space.

An important feature of quantum evolution is that a generic state, as a superposition of $\hat{p}_\phi$-eigenstates, has contributions that go through the turning point at different values of the clock, thanks to the p_ϕ-dependence of $\alpha_{\rm t}(p_\phi)$. We will nevertheless continue to speak of a single turning point because it is unique for a given p_ϕ, unlike the examples of oscillating clocks studied in [14,15].

A. Evolution through a turning point

Using τ as time, we have to rewrite (15) in terms of a time derivative by τ, applying the chain rule and the parametrization (16):

$$\begin{aligned} i\hbar\frac{\partial\psi(p_\phi,\tau)}{\partial\tau} &= i\hbar\frac{{\rm d}\alpha}{{\rm d}\tau}\frac{\partial\psi(p_\phi,\tau)}{\partial\alpha} \\ &= \pm\frac{{\rm d}\alpha}{{\rm d}\tau}\sqrt{p_\phi^2 - e^{\gamma\alpha(\tau)}}\psi(p_\phi,\tau). \end{aligned} \tag{17}$$

The sign choice has to be made such that $\pm{\rm d}\alpha/{\rm d}\tau = 1$, ensuring stability by a positive τ-Hamiltonian. With this condition, evolution with respect to τ is unitary and stable (positive Hamiltonian) on the Hilbert space $L^2(\mathbb{R}, {\rm d}p_\phi)$.

The variable α is quantized to an operator only before the constraint or the evolution equation is imposed (on the kinematical Hilbert space). In (17), it has been replaced completely by an effective time parameter without quantum properties. Since (16) is not part of a canonical transformation, Eq. (17) is not directly constructed from a quantized momentum of τ, but rather by applying the change rule to the original Eq. (15), which was based on a quantized momentum of α on the kinematical Hilbert space. This treatment is consistent with our definition of τ as an effective time parameter, distinct from the values of a fundamental clock. Looking back, Eq. (15) presents the first step of our construction in which α is introduced as the clock variable. If there were now turning points, α would be an internal time and (15) would present our evolution equation for states in the physical Hilbert space $L^2(\mathbb{R}, {\rm d}p_\phi)$. Since there are turning points in our model, we need to perform the second step of our construction, introducing time by (16) and formulating evolution by (17). Solutions to this equation correspond to physical states in systems without turning points.

The sign choice implies that we should solve the evolution equation separately for the two cases in (16), depending on whether τ is before or after the turning point

$\alpha_t(p_\phi)$ for a given p_ϕ. A state written in the p_ϕ-representation evolves as

$$\psi_{p_\phi}(\tau) = f(p_\phi)\exp\left(-i\Theta(\alpha(\tau), p_\phi)\right) \tag{18}$$

with some function $f(p_\phi)$ for $\tau < \alpha_t(p_\phi)$, where $d\alpha/d\tau = 1$ for the given p_ϕ, with the phase

$$\Theta(\alpha, p_\phi) = \frac{2}{\hbar\gamma}\left(\sqrt{p_\phi^2 - e^{\gamma\alpha}} - |p_\phi|\tanh^{-1}\sqrt{1 - e^{\gamma\alpha}/p_\phi^2}\right) \le 0. \tag{19}$$

This function vanishes at the turning point $\alpha_t(p_\phi)$. Evolving onwards in the same state with respect to τ after this value is reached, α then decreases such that $d\alpha/d\tau = -1$ in (17), and we now choose the minus sign of $\pm$. The corresponding solution for the wave function is obtained from (18) by a simple sign change:

$$\psi_{p_\phi}(\tau) = f(p_\phi)\exp\left(i\Theta(\alpha(\tau), p_\phi)\right). \tag{20}$$

The phase, now given by $-\Theta(\alpha(\tau), p_\phi)$, therefore becomes positive and continues to increase after it goes through zero at the turning point. For $\alpha \to -\infty$ and using $\tanh^{-1}(1-x) \sim -\frac{1}{2}\ln(x/2)$ for $0 < x = \frac{1}{2}e^{\gamma\alpha}/p_\phi^2 \ll 1$, we have $\Theta(\alpha, p_\phi) \sim \alpha|p_\phi|/\hbar$ or $\mathrm{sgn}(d\alpha/d\tau)\Theta(\alpha(\tau), p_\phi) \sim \tau|p_\phi|/\hbar$. The combination of (18) and (20) together with (16) therefore implies that we approximate the standard linear phase in stationary states $\psi_{p_\phi}(\tau) \propto \exp(-i\tau|p_\phi|/\hbar)$ far from the turning point. There are, however, effects of a nonlinear phase around the turning point implied by the time-dependent Hamiltonian as well as superposition effects in states that are not eigenstates of $\hat{p}_\phi$.

B. Evolving expectation value

We are interested in analyzing the influence of turning points on the evolving expectation value of $\hat{\phi}$ in order to show that evolution does not freeze at a turning point and respects (at least semiclassically) the monotonic behavior (12). To this end, we should use a suitable normalizable superposition of $\hat{p}_\phi$-eigenstates. As usual, such a superposition is determined by a corresponding function for the coefficients $f(p_\phi)$ of these eigenstates. If this function has support on all p_ϕ, such as a Gaussian state, at any finite time τ where we may impose an initial state, there will be some $\hat{p}_\phi$-eigenstates that have already crossed their turning points, for which (20) should be used, and some which are still approaching their turning points, for which (18) should be used. If the initial time is τ_0 with state $\psi(p_\phi, \tau_0) = f(p_\phi)$, the evolved state for any $\tau \ge \tau_0$ is given by

$$\psi(p_\phi, \tau) = \begin{cases} f(p_\phi)e^{-i\Theta_0(p_\phi)}\exp\left(+i\Theta(\alpha(\tau), p_\phi)\right) & \text{if } \alpha_t(p_\phi) \le \tau_0 \le \tau \\ f(p_\phi)e^{+i\Theta_0(p_\phi)}\exp\left(+i\Theta(\alpha(\tau), p_\phi)\right) & \text{if } \tau_0 \le \alpha_t(p_\phi) \le \tau \\ f(p_\phi)e^{+i\Theta_0(p_\phi)}\exp\left(-i\Theta(\alpha(\tau), p_\phi)\right) & \text{if } \tau_0 \le \tau \le \alpha_t(p_\phi) \end{cases}, \tag{21}$$

where $\Theta_0(p_\phi) = \Theta(\alpha(\tau_0), p_\phi)$. The initial state determines this constant phase contribution, and sign choices are uniquely fixed by the relationship between τ and $\alpha_t(p_\phi)$ as well as continuity in τ. In particular, when τ crosses $\alpha_t(p_\phi)$ for a given p_ϕ, we move from the second case in (21) to the third case. Since $\Theta(\alpha_t(p_\phi), p_\phi) = 0$, continuity requires that the constant phase is the same in these two cases, without a sign change. The first case in (21) requires the opposite sign in the constant phase in order to be consistent with the initial state $\psi(p_\phi, \tau_0) = f(p_\phi)$ when $\tau_0 \ne \alpha_t(p_\phi)$. By changing τ at fixed τ_0 and p_ϕ, the first case can never be turned into the second or third one; therefore, there it is not subject to a condition on the phase from continuity in τ.

Similarly, for $\tau \le \tau_0$ we have

$$\psi(p_\phi, \tau) = \begin{cases} f(p_\phi)e^{-i\Theta_0(p_\phi)}\exp\left(+i\Theta(\alpha(\tau), p_\phi)\right) & \text{if } \alpha_t(p_\phi) \le \tau \le \tau_0 \\ f(p_\phi)e^{-i\Theta_0(p_\phi)}\exp\left(-i\Theta(\alpha(\tau), p_\phi)\right) & \text{if } \tau \le \alpha_t(p_\phi) \le \tau_0 \\ f(p_\phi)e^{+i\Theta_0(p_\phi)}\exp\left(-i\Theta(\alpha(\tau), p_\phi)\right) & \text{if } \tau \le \tau_0 \le \alpha_t(p_\phi) \end{cases} \tag{22}$$

with all sign choices determined by the same conditions as before. In our analytical examples, we will restrict ourselves to the case of $\tau > \tau_0$ with wave packets that are supported mainly (but not completely) on p_ϕ such that $\tau_0 < \alpha_{\rm t}(p_\phi)$. The second and third cases in (21) will then be sufficient in an approximate analysis.

The expectation value of ϕ in such an evolving state is given by

$$\langle\hat{\phi}\rangle(\tau) = i\hbar\langle\psi, \mathrm{d}\psi/\mathrm{d}p_\phi\rangle(\tau) = i\hbar\left\langle\frac{f'(\hat{p}_\phi)}{f(\hat{p}_\phi)}\right\rangle + \hbar\left\langle \mathrm{sgn}(\mathrm{d}\hat{\alpha}/\mathrm{d}\tau)\frac{\mathrm{d}\Theta(\hat{\alpha}(\tau),\hat{p}_\phi)}{\mathrm{d}p_\phi} - \frac{\mathrm{d}\Theta_0(p_\phi)}{\mathrm{d}p_\phi}\right\rangle$$
$$= i\hbar\left\langle\frac{f'(\hat{p}_\phi)}{f(\hat{p}_\phi)}\right\rangle + \hbar\langle \mathrm{sgn}(\mathrm{d}\hat{\alpha}/\mathrm{d}\tau)\left(\frac{\partial\Theta(\hat{\alpha}(\tau),\hat{p}_\phi)}{\partial p_\phi} + \frac{\partial\hat{\alpha}}{\partial p_\phi}\frac{\partial\Theta(\hat{\alpha}(\tau),\hat{p}_\phi)}{\partial\alpha}\right) - \frac{\partial\Theta_0(p_\phi))}{\partial p_\phi}\right\rangle. \quad (23)$$

[According to our assumption that $\tau_0 < \alpha_{\rm t}(p_\phi)$ for most p_ϕ, the constant phase $\Theta_0(p_\phi)$ does not depend on $\alpha_{\rm t}$ within the approximation used.] The first term is τ-independent and equals the initial expectation value ϕ_0 of ϕ at $\tau = \tau_0$, where $\psi(p_\phi, \tau_0) = f(p_\phi)$. The second term is evaluated in the same state, but has an explicitly τ-dependent operator, which we are applying in the p_ϕ-representation. It is important to note that $\hat{\alpha}(\tau)$ is now an operator because its classical expression depends on p_ϕ through $\alpha_{\rm t}(p_\phi)$. The sign change in the phase therefore depends on the p_ϕ-eigenstate in a superposition given by a general state, and the α-dependence of the phase contributes to the $\hat{\phi}$-expectation value along with the p_ϕ-dependence.

Using our solution for the phase $\Theta(\alpha(\tau), p_\phi)$, we obtain

$$\langle\hat{\phi}\rangle(\tau) = \phi_0 + \frac{2}{\gamma}\left\langle \tanh^{-1}\sqrt{1 - e^{\gamma\hat{\alpha}(\tau_0)}/\hat{p}_\phi^2}\right\rangle - \frac{2}{\gamma}\left\langle \mathrm{sgn}(\mathrm{d}\hat{\alpha}/\mathrm{d}\tau)\tanh^{-1}\sqrt{1 - e^{\gamma\hat{\alpha}(\tau)}/\hat{p}_\phi^2}\right\rangle - \frac{4}{\gamma}\left\langle\theta(-\mathrm{d}\hat{\alpha}/\mathrm{d}\tau)\sqrt{1 - e^{\gamma\hat{\alpha}(\tau)}/\hat{p}_\phi^2}\right\rangle \quad (24)$$

with the Heaviside step function $\theta(x)$ from

$$\frac{\partial\alpha(\tau)}{\partial p_\phi} = 2\theta(-\mathrm{d}\alpha/\mathrm{d}\tau)\frac{\mathrm{d}\alpha_{\rm t}}{\mathrm{d}p_\phi} = \frac{4}{\gamma|p_\phi|}\theta(-\mathrm{d}\alpha/\mathrm{d}\tau), \quad (25)$$

combining (16) and (13). The τ-dependent parts are continuous in τ in spite of the sign and the Heaviside function because the latter are multiplied by functions that vanish at the step where the sign of $\mathrm{d}\alpha/\mathrm{d}\tau$ changes.

The functional dependence of the third term in (24) can be seen to equal (12), using $\tanh(x) = \sqrt{\cosh^2(x) - 1}/\cosh(x)$, but now the branch of $\cosh^{-1}$ is explicitly determined by $\mathrm{sgn}(\mathrm{d}\alpha/\mathrm{d}\tau)$ for each p_ϕ-eigenstate in a superposition. The fourth contribution to the expectation value is entirely determined by the turning points and has no classical analog. The inverse tanh in the third term implies that the fourth term is relevant only during a time interval when the majority of the $\hat{p}_\phi$-eigenstates are crossing the turning point.

C. General features of the method

Before we continue with a detailed analysis of evolution in our specific model, we briefly discuss several aspects related to the applicability of our general method and its time parametrization.

The specific example considered here describes a simple cosmological model in which most equations can be solved analytically. A large class of generalizations is available within the same method, but there are also models that require further developments that are still in progress. Looking at the constraint, it may be generalized in two ways, by changing the clock Hamiltonian or the Hamiltonian of the system interpreted as evolving with respect to the clock.

If we first fix the clock Hamiltonian, it is easy to see that our methods can be applied to any system Hamiltonian $H(\phi, p_\phi)$ such that the initial evolution equation (15) is replaced by

$$i\hbar\frac{\partial|\psi\rangle(\alpha)}{\partial\alpha} = \pm\sqrt{\hat{H} - e^{\gamma\alpha}}|\psi\rangle(\alpha). \quad (26)$$

In this general case, it is not convenient to work in the p_ϕ-representation, but our equations can easily be adjusted if we work in the representation spanned by eigenstates of $\hat{H}$. Given the positive clock Hamiltonian and the constraint, we can restrict the spectrum of $\hat{H}$ to its positive part. Assuming the corresponding eigenstates are labeled by some number k (and additional labels in case of degeneracies in the positive part of the spectrum), which may be discrete or continuous, and the $\hat{H}$-eigenvalues are $E_k \geq 0$, the evolution equation

$$i\hbar\frac{\partial\psi_k(\alpha)}{\partial\alpha} = \pm\sqrt{E_k - e^{\gamma\alpha}}\psi_k(\alpha) \quad (27)$$

can be solved simply by replacing $|p_\phi|$ with $\sqrt{E_k}$ in our previous phase.

The same substitution can be used in the globally evolved wave function, given for $\tau \geq \tau_0$ by

$$\psi_k(\tau) = \begin{cases} f(k)e^{-i\Theta_0(k)} \exp\left(+i\Theta(\alpha(\tau),k)\right) & \text{if } \alpha_t(k) \le \tau_0 \le \tau \\ f(k)e^{+i\Theta_0(k)} \exp\left(+i\Theta(\alpha(\tau),k)\right) & \text{if } \tau_0 \le \alpha_t(k) \le \tau \\ f(k)e^{+i\Theta_0(k)} \exp\left(-i\Theta(\alpha(\tau),k)\right) & \text{if } \tau_0 \le \tau \le \alpha_t(k) \end{cases}, \tag{28}$$

where $\alpha_t(k) = \gamma^{-1}\ln(E_k)$ and a corresponding version for $\tau < \tau_0$. Since $|\psi_k(\tau)|^2 = |f(k)|^2$ for all τ, normalization of this state in the $\hat{H}$-representation is preserved by evolution. The same property is then true in any other representation, where the evolving state is

$$|\psi\rangle(\tau) = \int_k \psi_k(\tau)|k\rangle \mathrm{d}k \tag{29}$$

if $|k\rangle$ are the $\hat{H}$-eigenstates in the desired representation. In this expression, integration over k (or summation in the discrete case) at fixed τ takes into account the nontrivial k-dependence of the phase in (28) and may therefore be more complicated than the corresponding transformation in a model with an absolute time. Nevertheless, τ-independence of normalization follows from general properties, in particular the fact that the $\hat{H}$-eigenbasis is orthonormal for self-adjoint $\hat{H}$ together with preserved normalization in the $\hat{H}$-representation based on (28).

It is also possible to apply our methods to a different clock Hamiltonian. We will not go into details here because doing so would require a new parametrization of $\alpha(\tau)$, possibly with multiple turning points. The methods used here can easily be adjusted to other clock potentials if there is still only one turning-point value, α_t. The case of two turning-point values of the form $\pm\alpha_t$ has been discussed in detail in the clock model of [14,15].

Further generalizations that include direct interaction terms between clock and system, such as a constraint $C = p_\alpha^2 + I(\alpha,\phi) - H(\phi,p_\phi)$ with an interaction term $I(\alpha,\phi)$ that depends on both α and ϕ, remain challenging. In this case, since $\{I(\alpha,\phi), H(\phi,p_\phi)\} \neq 0$, we are not able to diagonalize the operators $\hat{I}$ and $\hat{H}$ simultaneously. The $\hat{H}$-representation then does not sufficiently simplify the dynamics, and without analytical solutions, it is more difficult to implement suitable phase changes at turning points. Preliminary investigations suggest that at least a numerical treatment is possible in principle, but slowed down by the requirement to transform back and forth between the $\hat{H}$ and the $\hat{I}$-representations. A generalization of our methods to systems with direct system-clock interactions would certainly be important for cosmological models.

Within a specific model such as the one used here, one may be interested in considering different time parametrizations in which the rate of change of τ does not agree with the rate of change of the clock degree of freedom, α. For instance, in our cosmological model, α is the logarithmic scale factor, but one may want to write evolution with respect to proper time, given by a different function of the scale factor, depending on the model. Our method allows for such reparametrizations. Instead of (16), we may use

$$\alpha(\tau) = \begin{cases} +R(\tau) & \text{if } R(\tau) < \alpha_t(p_\phi) \\ -R(\tau) + 2\alpha_t(p_\phi) & \text{if } R(\tau) > \alpha_t(p_\phi) \end{cases} \tag{30}$$

with a monotonic reparametrization function $R(\tau)$. The reparametrization function is not applied to $\alpha_t(p_\phi)$ in the branch conditions and in the second line of (30) because these terms are determined by α or $R(\tau)$ (but not τ) reaching the turning point. Therefore, derivative operators acting on the p_ϕ-dependence of the wave function through α_t, such as (24), and in particular the coefficient in (25), are unchanged by the reparametrization. The only effect of the reparametrization is to shift the location of turning points in terms of τ [through $\mathrm{sgn}(\mathrm{d}\hat{\alpha}/\mathrm{d}\tau$ and $\theta(-\mathrm{d}\hat{\alpha}/\mathrm{d}\tau)$ in (24)], and to function as a standard reparametrization of time away from turning points.

D. Shifts around the turning point

The functional form of (24) suggests that there may be significant quantum effects around the turning point, but at much earlier or later times we have nearly classical behavior provided the initial state is sufficiently semiclassical. A useful way to express quantum effects is by computing additional shifts they imply in the asymptotic behavior of $\phi(\tau)$ for $\tau \to \pm\infty$ in addition to the classical shift. For $\alpha \to -\infty$, and therefore for both $\tau \to -\infty$ and $\tau \to \infty$, we have

$$\frac{\mathrm{d}\phi}{\mathrm{d}\tau} = -\mathrm{sgn}(\mathrm{d}\alpha/\mathrm{d}\tau)\frac{p_\phi}{p_\alpha} = \frac{|p_\phi|}{\sqrt{p_\phi^2 - e^{\gamma\alpha}}} \to 1 \tag{31}$$

such that $\phi(\tau)$ asymptotically approaches a straight line at an angle of 45°. However, the behavior around the turning point is not linear, which implies a constant shift in the ϕ-direction between the asymptotic past and the

asymptotic future. This shift is sensitive to quantum effects around the turning point and can therefore be used to quantify them.

We first compute the classical shift as a reference basis. We can introduce in the classical solution $\phi(\alpha)$, given in Eq. (12), the same parametrization $\alpha(\tau)$ given in (16) as used in the unwinding of α as a quantum clock. Classically, the resulting $\phi(\tau)$ can be viewed as a reparametrization of the gauge orbit $\phi(\epsilon)$ with a nonlinear transformation $\tau(\epsilon)$ such that $\mathrm{d}\alpha/\mathrm{d}\epsilon = \pm\mathrm{d}\tau/\mathrm{d}\epsilon$. If the initial value $\phi = \phi_0$ is chosen at some $\tau_0 < \alpha_{\mathrm{t}}(p_\phi)$ for a given p_ϕ, continuity of $\phi(\tau)$ requires that

$$\phi(\tau) = \begin{cases} \phi_0 + 2\gamma^{-1}\tanh^{-1}\sqrt{1-e^{\gamma\tau_0}/p_\phi^2} - 2\gamma^{-1}\tanh^{-1}\sqrt{1-e^{\gamma\alpha(\tau)}/p_\phi^2} & \text{if } \tau < \alpha_{\mathrm{t}}(p_\phi) \\ \phi_0 + 2\gamma^{-1}\tanh^{-1}\sqrt{1-e^{\gamma\tau_0}/p_\phi^2} + 2\gamma^{-1}\tanh^{-1}\sqrt{1-e^{\gamma\alpha(\tau)}/p_\phi^2} & \text{if } \tau > \alpha_{\mathrm{t}}(p_\phi) \end{cases}. \tag{32}$$

This function is consistent with (11) for all τ if we use $\phi(\tau(\epsilon)) = 2p_\phi\epsilon + \phi_0 + 2\gamma^{-1}\tanh^{-1}\sqrt{1-e^{\gamma\tau_0}/p_\phi^2}$ to reparametrize from τ to ϵ. Using again $\tanh^{-1}(1-x) \sim -\frac{1}{2}\ln(x/2)$ as already applied in our analysis of the phase, we have

$$-\frac{2}{\gamma}\mathrm{sgn}(\mathrm{d}\alpha/\mathrm{d}\tau)\tanh^{-1}\sqrt{1-e^{\gamma\alpha}/p_\phi^2} \sim \mathrm{sgn}(\mathrm{d}\alpha/\mathrm{d}\tau)\alpha - \frac{2}{\gamma}\mathrm{sgn}(\mathrm{d}\alpha/\mathrm{d}\tau)\ln(2|p_\phi|). \tag{33}$$

If we simply combine the two α-branches, their asymptotic linear curves are separated by a ϕ-shift of $4\gamma^{-1}\ln(2|p_\phi|)$. However, the parametrization in terms of τ changes this result because of an additional shift of τ by $2\alpha_{\mathrm{t}}(p_\phi) = 4\gamma^{-1}\ln(|p_\phi|)$ according to (16):

$$\begin{aligned} \mathrm{sgn}(\mathrm{d}\alpha/\mathrm{d}\tau)\alpha(\tau) - \frac{1}{\gamma}\mathrm{sgn}(\mathrm{d}\alpha/\mathrm{d}\tau)\ln(2p_\phi^2) &= \begin{cases} \tau - 2\gamma^{-1}\ln(2|p_\phi|) & \text{if } \tau < \alpha_{\mathrm{t}} \\ \tau - 2\alpha_{\mathrm{t}}(p_\phi) + 2\gamma^{-1}\ln(2|p_\phi|) & \text{if } \tau > \alpha_{\mathrm{t}} \end{cases} \\ &= \begin{cases} \tau - 2\gamma^{-1}\ln(2|p_\phi|) & \text{if } \tau < \alpha_{\mathrm{t}} \\ \tau - 2\gamma^{-1}\ln(2|p_\phi|) + 4\gamma^{-1}\ln 2 & \text{if } \tau > \alpha_{\mathrm{t}} \end{cases}. \end{aligned} \tag{34}$$

The classical shift is therefore independent of p_ϕ and is given by

$$\Delta\phi_{\mathrm{classical}} = \frac{4}{\gamma}\ln 2. \tag{35}$$

The same shift appears in the quantum case, but there are additional contributions as well. For $\tau \to -\infty$, the fourth term in (24) can be ignored because the majority of $\hat{p}_\phi$-eigenstates still has to cross their turning points. The third term then guarantees nearly classical behavior. For $\tau \to \infty$ when $\alpha(\tau) \to -\infty$ and most eigenstates have gone through their turning points, the fourth term only implies a constant negative shift of $\langle\hat{\phi}\rangle(\tau)$ by

$$\Delta\phi_1 = -\frac{4}{\gamma}, \tag{36}$$

while the inverse tanh in the second term continues to increase with linear asymptotic τ-dependence as $\sqrt{1-e^{\gamma\alpha(\tau)}/p_\phi^2}$ approaches one. The full quantum shift equals

$$\Delta\phi = \Delta\phi_{\mathrm{classical}} + \Delta\phi_1 = \Delta\phi_{\mathrm{classical}} - \frac{4}{\gamma} \tag{37}$$

provided the initial state is posed sufficiently far ahead of the turning point. For $\gamma = 4$, the shifts simplify to $\Delta\phi_{\mathrm{classical}} = \ln(2)$ and $\Delta\phi = \Delta\phi_{\mathrm{classical}} - 1$.

As a function of τ, the classical-type contribution to (24), depending on $\tanh^{-1}$, provides a monotonic asymptotic contribution to $\langle\hat{\phi}\rangle(\tau)$: While $\sqrt{1-e^{\gamma\alpha(\tau)}/p_\phi^2}$ has a local minimum of zero at $\tau = \alpha_{\mathrm{t}}(p_\phi)$, and so does $\tanh^{-1}\sqrt{1-e^{\gamma\alpha(\tau)}/p_\phi^2}$, multiplying this expression with $-\mathrm{sgn}(\mathrm{d}\alpha/\mathrm{d}\tau)$ turns it into a monotonically increasing function of τ. The τ-dependent expectation value of monotonic functions of τ is also monotonic. Since the $\tanh^{-1}$ is dominant well before and after the turning point, the classical monotonic behavior of $\phi(\tau)$ is maintained in this regime.

The last contribution to (24) does not respect this behavior because it subtracts an increasing function after the turning point, thanks to the factor of $\theta(-\mathrm{d}\alpha/\mathrm{d}\tau)$. Since this term approaches a constant at late times, it can change the monotonic behavior of $\phi(\tau)$ only around the turning point, the more so far larger p_ϕ because the slope of the subtracted square root is then larger close to the turning

point. The quantum shift $\Delta\phi_1$ is implied by the same term, but it is independent of p_ϕ because it refers to the asymptotic value.

While there may therefore be noticeable quantum effects around the turning point of α, these observations demonstrate that our quantum evolution does not freeze there and has the correct asymptotic behavior, before and after the turning point. The eigenstate dependence of the turning points makes these features more involved than in standard quantum mechanics. We now turn to numerics in order to show specific examples of evolving states.

E. Numerical treatment

For a numerical analysis of equation (24) we have to choose an initial state, which we take to be a Gaussian

$$\psi_0(\phi) = (2\pi d^2)^{-1/4} \exp\left(\frac{i\phi p_\phi}{\hbar} - \frac{(\phi-\phi_0)^2}{4d^2}\right) \tag{38}$$

centered at (ϕ_0, p_ϕ). Numerically, we restrict the ϕ-range to be a finite interval between $-L$ and L for some L sufficiently large to contain a given stretch of evolving $\phi(\tau)$. In addition, we express the evolved state as a truncated sum of $\hat{p}_\phi$-eigenstates,

$$\psi(\tau,\phi) = \sum_{n=-N}^{+N} c_n \exp\left(-i\mathrm{sgn}(\mathrm{d}\alpha(\tau, p_\phi^{(n)})/\mathrm{d}\tau)\Theta(\alpha(\tau, p_\phi^{(n)}), p_\phi^{(n)})\right) \exp\left(i\phi p_\phi^{(n)}/\hbar\right), \tag{39}$$

where $p_\phi^{(n)} = n\pi/L$. In our examples, the normalization of $\psi(\tau,\phi)$ is approximately conserved within a relative accuracy of 10^{-4} for $N/L > 10$.

Since the turning point (13) depends on p_ϕ, the truncated superposition (39) experiences a turning point at $2N+1$ different values of τ. A nontruncated state supported on the infinite set of momenta p_ϕ will never completely cross all the turning points since at any finite τ there will be terms in the superposition of $\hat{p}_\phi$-eigenstates that have yet to encounter their time of the turning point. However, these terms contribute less and less as the wave function approaches zero for large $|p_\phi|$. The finite truncation is therefore expected to be reliable.

Examples of the expectation value $\langle\hat{\phi}\rangle(\tau)$ for different values of the initial time τ_0 where the state is Gaussian are shown in Fig. 1. The initial time does not matter much, even if it is set close to the main turning point. The only changes in the resulting curves are implied by a slightly different initial $\phi_0(\tau_0)$, which was chosen to follow the classical behavior. Confirming our analytical results based on Eq. (24), the field expectation value retains its monotonic behavior in asymptotic regimes and, for the specific values of γ and p_ϕ chosen in this figure, even while our local clock α goes through its turning point. The slope of $\langle\hat{\phi}\rangle(\tau)$ asymptotically approaches the value one if the state remains semiclassical because for $\alpha \to -\infty$,

$$\frac{\mathrm{d}\phi}{\mathrm{d}\tau} = \mathrm{sgn}(\mathrm{d}\alpha/\mathrm{d}\tau)\frac{\mathrm{d}\phi/\mathrm{d}\epsilon}{\mathrm{d}\alpha/\mathrm{d}\epsilon} = -\mathrm{sgn}(\mathrm{d}\alpha/\mathrm{d}\tau)\frac{p_\phi}{p_\alpha} = \frac{1}{\sqrt{1-e^{\gamma\alpha}/p_\phi^2}} \to 1 \tag{40}$$

is implied by the classical constraint.

As further support of freeze-free evolution, we show the probability density of $\hat{\phi}$ in the evolved state in Fig. 2. Our initial state is Gaussian, but it does not strictly keep this form because the α-Hamiltonian is not harmonic. Most of the turning points happen in the region close to $\tau = 0$ in this case where the magnitude $|\psi(\phi,\tau)|^2$ is largest. The wave function continues to evolve as expected while going through the turning point of α for several p_ϕ-eigenstates.

The dependence on γ is illustrated in Fig. 3. As expected from our analytical expression for the quantum shift, the future asymptotic behavior of the quantum curve is closer

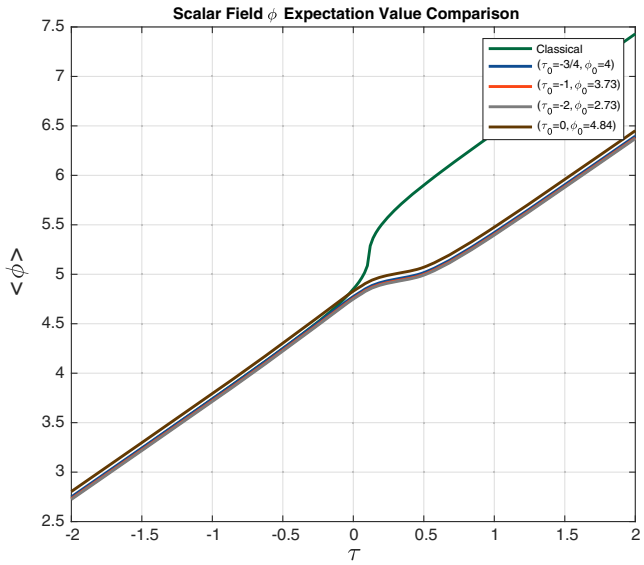

FIG. 1. Expectation value of ϕ as a function of τ for different initial times τ_0 at which the state is assumed to be Gaussian, centered at $\phi_0(\tau_0)$ following the classical curve. The differences between the future and past asymptotic behaviors in the classical and quantum cases, respectively, are consistent with the values $\Delta\phi_{\mathrm{classical}} \approx 0.7$, $\Delta\phi = \Delta\phi_{\mathrm{classical}} - 1 \approx -0.3$ implied by our formulas with the value $\gamma = 4$. The momentum in this case equals $p_\phi = 1.25$.

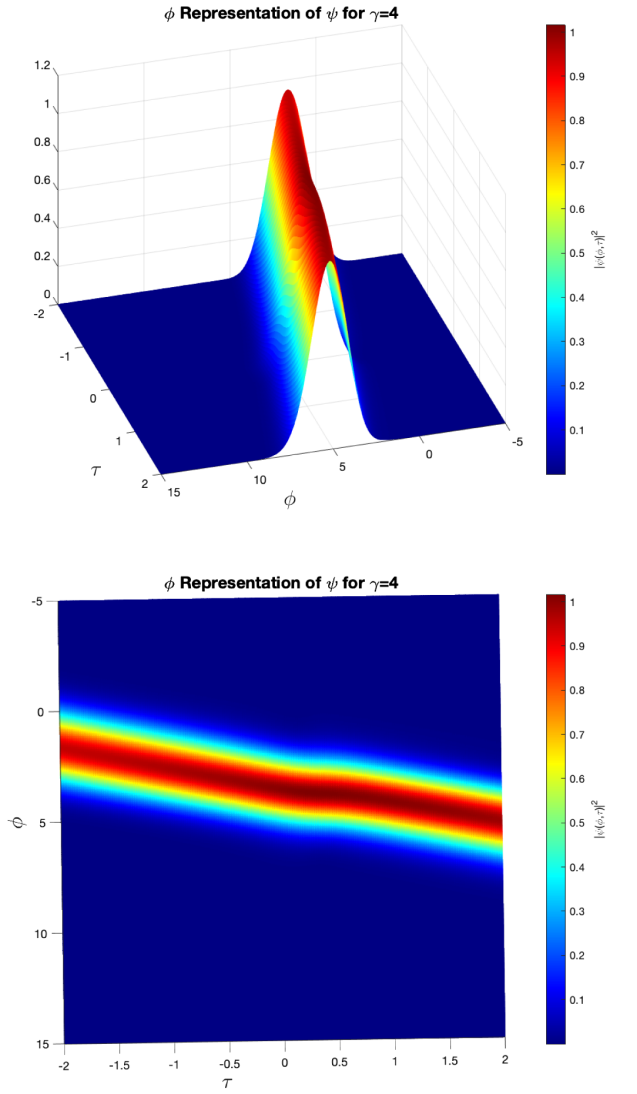

FIG. 2. Evolution of the probability density for the Gaussian wave packet under the constraint Hamiltonian for $\gamma = 4$ and $p_\phi = 1.25$.

to the classical curve for larger γ, but, in contrast to the classical solution, it is always below an extension from the asymptotic past to the future as a straight line.

A new feature is shown in the dependence on p_ϕ in Fig. 4. The asymptotic parts are not affected by changing p_ϕ, but the behavior around the turning point does change noticeably when p_ϕ is increased from the value $p_\phi = 1.25$ used in the previous plots. In particular, the expectation value $\langle\hat{\phi}\rangle(\tau)$ is no longer monotonic in a small range around the turning point. A closer look at the curves reveals that this feature is a consequence of a property seen in the analytical expression (24): Starting in the asymptotic past, the expectation value follows the classical curve longer for larger p_ϕ, but then approaches the shifted future behavior more quickly. This behavior is implied by the appearance of p_ϕ in $\sqrt{1 - e^{\gamma\alpha}/p_\phi^2}$, which leads to a more abrupt transition

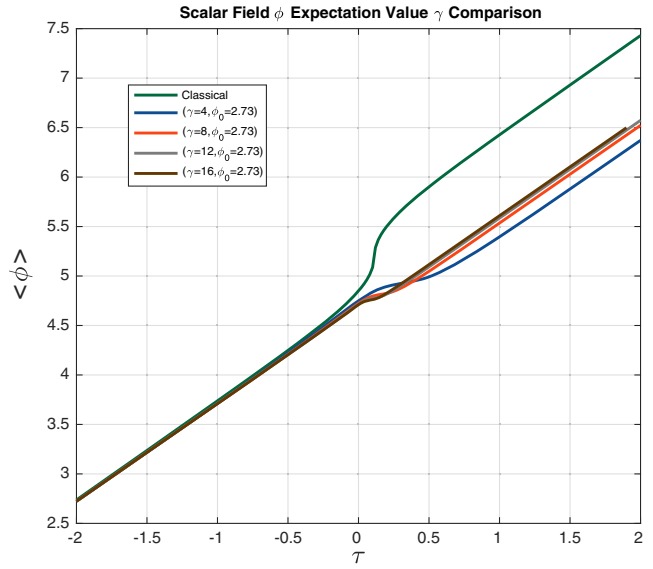

FIG. 3. Expectation value of ϕ as a function of τ for three values of $\gamma \geq 4$. Each curve corresponds to the same Gaussian initial state ($d = 1$) at the main turning point $\tau = 2\gamma^{-1} \ln|p_\phi|$, where $p_\phi = 1.2$ has been chosen for the numerics.

for larger p_ϕ and makes quantum effects of turning points more noticeable. The numerical solutions show that a sufficiently rapid transition can lead to a nonmonotonic $\langle\hat{\phi}\rangle(\tau)$ in a small interval around the turning point.

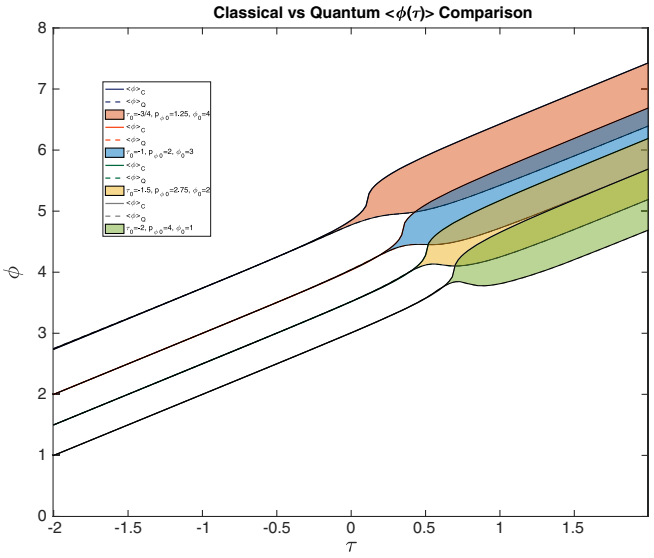

FIG. 4. Expectation value of ϕ as a function of τ for four values of p_ϕ. For the sake of clarity, the curves have been plotted with different initial $\phi_0(\tau_0)$ in order to shift them apart. While the asymptotic behavior does not depend on p_ϕ, the range around the turning point where the transition between the asymptotic past and future happens shows a new feature for larger p_ϕ: The expectation value $\langle\hat{\phi}\rangle(\tau)$ is no longer monotonic for sufficiently large values.

IV. COMPARISON WITH DIRAC QUANTIZATION

Our classical constraint $C = p_\phi^2 - p_\alpha^2 - e^{\gamma\alpha}$ can easily be turned into an operator on the kinematical Hilbert space $L^2(\mathbb{R}^2, \mathrm{d}\alpha\mathrm{d}\phi)$. According to standard Dirac quantization, the physical Hilbert space on which the constraint is solved would then be a suitable Hilbert-space completion of (generalized) states ψ satisfying

$$\hat{C}\psi = (\hat{p}_\phi^2 - \hat{p}_\alpha^2 - e^{\gamma\hat{\alpha}})\psi = 0. \tag{41}$$

A. Observables and double-valuedness

A common way [8,17] to introduce a physical inner product interprets (41) in the (α, ϕ)-representation $\psi(\alpha, \phi)$ as a Klein-Gordon equation and uses the bilinear form

$$(\psi_1, \psi_2) = i \int \left(\psi_1^* \frac{\partial\psi_2}{\partial\phi} - \frac{\partial\psi_1^*}{\partial\phi}\psi_2\right)\mathrm{d}\alpha \tag{42}$$

conserved in evolution with respect to ϕ. To obtain a positive definite inner product, one restricts the solution space to positive-frequency solutions with positive eigenvalues of $\hat{p}_\phi$, or combines positive-frequency solutions with the inner product (ψ_1, ψ_2) and negative-frequency solutions with the inner product $-(\psi_1, \psi_2)$.

Irrespective of the specific choice, the procedure makes use of the factorization

$$\hat{C} = \hat{C}_-\hat{C}_+ = \left(\hat{p}_\phi - \sqrt{\hat{p}_\alpha^2 + e^{\gamma\hat{\alpha}}}\right)\left(\hat{p}_\phi + \sqrt{\hat{p}_\alpha^2 + e^{\gamma\hat{\alpha}}}\right) \tag{43}$$

into two commuting factors. Both positive-frequency solutions, solving $\hat{C}_-\psi = 0$, and negative-frequency solutions, solving $\hat{C}_+\psi = 0$, are therefore part of the solution space of the constraint. The two types of states can then be interpreted as evolving with respect to ϕ, such that (42) together with the correct sign choice provides an inner product given by integration over α at some fixed ϕ. The value chosen for ϕ does not matter because the inner product is conserved in ϕ.

Dirac observables provide physically motivated operators acting on the physical Hilbert space. They are defined as expressions that (classically) have vanishing Poisson brackets with the constraint or (in a quantization) commute with the constraint operator. The classical version of such an expression is conserved by the gauge flow generated by the constraint on phase space, and the quantum version maps the solution space of the constraint to itself. A simple example in the present case is the conserved momentum p_ϕ, which locally has a canonically conjugate Dirac observable

$$\Phi = \phi - \frac{2}{\gamma}\cosh^{-1}(|p_\phi|e^{-\gamma\alpha/2}). \tag{44}$$

The conserved nature of Φ can be interpreted as describing evolution of α relative to ϕ, such that Φ remains constant. The double-valuedness of the inverse cosh implies that Φ is only locally defined on phase space. Once α reaches a turning point $\alpha_t(p_\phi)$, the other branch of the inverse cosh should be followed in order to have agreement between this relational evolution $\alpha(\phi)$ and the gauge orbits given by $\alpha(\epsilon)$ and $\phi(\epsilon)$ discussed in Sec. II. [A global Dirac observable other than p_ϕ is given by

$$D = e^{-\gamma\alpha/2}(p_\phi \cosh(\gamma\phi/2) - p_\alpha \sinh(\gamma\phi/2)). \tag{45}$$

It cannot be solved globally for $\phi(\alpha)$.]

B. Disambiguations

The double-valuedness of Φ means that this expression does not have a straightforward quantization. An additional choice is required that specifies how the two values are assigned to different states. For instance, the classical expression $\pm\sqrt{|\alpha|}$, as a simpler version of $\cosh^{-1}(|p_\phi|e^{-\gamma\alpha/2})$, may be disambiguated by assigning the plus choice to positive values of α and the minus choice to negative values of α, such that $\pm\sqrt{|\alpha|} = \mathrm{sgn}(\alpha)\sqrt{|\alpha|}$. In a quantization, this choice can be implemented by using the same disambiguation on the spectral decomposition of $\hat{\alpha}$ when defining $\pm\sqrt{|\hat{\alpha}|}$. The specific disambiguation depends on the physical meaning and use of the resulting operator.

For instance, $\pm\sqrt{|\alpha|}$ could appear in a Dirac observable $\phi \pm \sqrt{|\alpha|}$ of the constraint $C = p_\phi^2 - 4|\alpha|p_\alpha^2$, defining relational evolution $\phi(\alpha)$ by setting the observable to a constant value. (The constraint surface has two components. For a given sign choice, in $\phi \pm \sqrt{|\alpha|}$, a Dirac observable on one of the components is obtained.) Not only quantization but also well-defined classical evolution then requires a disambiguation. As in our main example, a suitable choice can be derived by comparing relational evolution with the behavior on gauge orbits, which is easier in this model because here α does not have a turning point.

The gauge equations can easily be solved for $\phi(\epsilon) = \phi_0 + 2p_\phi\epsilon$, $|\alpha(\epsilon)| = 4p_\phi\epsilon^2$, and $p_\alpha(\epsilon) = -\mathrm{sgn}(\alpha)/(4\epsilon)$ while p_ϕ is constant. Instead of a turning point, $\alpha(\epsilon)$ has a stationary point at $\alpha = 0$ where $\mathrm{d}\alpha/\mathrm{d}\epsilon = 0$. Positive and negative α therefore need not be connected to a single gauge orbit. However, $\phi(\epsilon)$ clearly moves through all positive and negative values. The relational description based on $\phi \pm \sqrt{|\alpha|} = \mathrm{const}$ can correctly describe this behavior only if the disambiguation $\pm\sqrt{|\alpha|} = \mathrm{sgn}(\alpha)\sqrt{|\alpha|}$ is used (up to an overall sign choice).

The suitability of this disambiguation can also be seen from the fact that the model system can be obtained by a

canonical transformation from the pair (x, p_x) with constraint $p_\phi^2 - p_x^2$ and Dirac observable $\phi - x$ (on one component of the constraint surface). A simple interpretation of $\pm\sqrt{|\alpha|} = \sqrt{|\alpha|}$, ignoring the sign choice, would follow from a local canonical transformation $x = \sqrt{|\alpha|}$, $p_x = 2\sqrt{|\alpha|}p_\alpha$ that is not defined on negative x. The global transformation $x = \mathrm{sgn}\alpha\sqrt{|\alpha|}$, $p_x = 2\sqrt{|\alpha|}p_\alpha$ (up to an overall sign choice in x) corresponds to our disambiguation.

Back to our main example, the unwinding of the local clock α to a global time parameter τ is an example of a disambiguation, determined as in the simple model by the condition that there should be states in which ϕ has the expected asymptotic semiclassical behavior according to which it goes to negative infinity at early times and positive infinity at late times. This variable classically has a strictly monotonic gauge flow for nonzero p_ϕ but inherits double-valuedness when it is constructed from the Dirac observable Φ. The double-valuedness is resolved by the disambiguation used throughout the paper.

In some cases, double-valuedness can be resolved in quantum mechanics if one requires that states are always superpositions of states suitably supported on both values. For instance, working only with even wave functions $\psi(\alpha)$ treats positive and negative α on the same footing. For operators to respect this symmetry, a specific sign choice must be made, such as using the same expression $\sqrt{|\alpha|}$ for positive and negative α. In our case and in other models of relational evolution, however, this procedure is not suitable for various reasons: (i) The symmetry of states that helps to disambiguate possible operators is imposed by hand and not derived from the dynamics. (ii) The condition is not guaranteed to be consistent with the required semiclassical behavior in asymptotic regimes. (iii) The model would be strongly restricted because any choice of initial states would have to follow the imposed symmetry condition. For instance, it would be impossible to set up a semiclassical state that compares quantum evolution with a single classical trajectory going through a given pair (α, ϕ). (iv) As we have seen in our main discussion, the symmetry condition that connects different α before and after a turning point is usually not universal but, just as α_t, depends on other phase-space degrees of freedom such as p_ϕ. (v) The strict symmetry condition ignores the possibility that quantum physics on phase spaces with nontrivial topology, as implied here by branch cuts on the solution space of the constraint, usually makes use of the universal covering space on which discrete classical symmetries are not necessarily respected. Accordingly, our solutions are not reflection symmetric around the turning point, which may not be obvious from the plot but can easily be seen from the fact that the Gaussian form of our initial state, posed on one side of the turning point, is, generically, not recovered on the other side by the nonharmonic evolution of our model.

C. Dirac observables in evolution with respect to a local clock

The standard procedure of relational evolution breaks down if one is interested in developing a picture of quantum evolution in terms of α rather than ϕ. Since $\hat{p}_\phi^2 - e^{\gamma\hat{\alpha}}$ is not positive definite, the definition of its square root is not clear. Even if this problem can be solved, an exact factorization such as (43) is no longer available because $\hat{p}_\alpha$ does not commute with $\hat{p}_\phi^2 - e^{\gamma\hat{\alpha}}$. It is possible to define the constraint operator as

$$\begin{aligned}\hat{\mathcal{C}} &= \left(\hat{p}_\alpha - \sqrt{\hat{p}_\phi^2 - e^{\gamma\hat{\alpha}}}\right)\left(\hat{p}_\alpha + \sqrt{\hat{p}_\phi^2 - e^{\gamma\hat{\alpha}}}\right)\\ &= \hat{p}_\alpha^2 - \hat{p}_\phi^2 + e^{\gamma\hat{\alpha}} + \left[\hat{p}_\alpha, \sqrt{\hat{p}_\phi^2 - e^{\gamma\hat{\alpha}}}\right] \qquad (46)\end{aligned}$$

in this specific ordering, with a commutator term that indicates quantum corrections to the classical expression. However, since the two factors do not commute, acting on a state by $\hat{\mathcal{C}}\psi$, as in standard Dirac quantization, can give only solutions with the negative sign of p_α.

Our procedure makes use of the same factorization, but since we explicitly include phases of time reversal when the clock α runs in the opposite direction of time τ, we are acting not only to the right [with a positive α-Hamiltonian $H = -p_\alpha$ from the second factor of (46)] but also on the left when $\mathrm{d}\alpha/\mathrm{d}\tau < 0$ where the α-Hamiltonian $H = -\mathrm{sgn}(\mathrm{d}\alpha/\mathrm{d}\tau)p_\alpha > 0$ is still positive if we use the first factor of (46). The p_ϕ-dependence of time reversals implied by the p_ϕ-dependence of α_t means that the standard definition of a physical Hilbert space does not apply. Nevertheless, we were able to define unitary α-evolution on a physical Hilbert space (defined as a Hilbert space on which gauge degrees of freedom are not represented) with a conserved inner product.

Moreover, our expectation value (24) faithfully models the Dirac observable Φ far from the turning points. The $\tanh^{-1}$ part of (24), evaluated in a semiclassical state, is equivalent to constant Φ with a specific branch choice of the inverse cosh determined by $\mathrm{sgn}(\mathrm{d}\alpha/\mathrm{d}\tau)$. The last term in (24) shows an additional quantum correction starting at the turning point, thanks to the theta function. This correction leads to stronger deviations between classical and quantum behavior, such as the nonmonotonic $\langle\hat{\phi}\rangle(\tau)$ seen in Fig. 4 for larger values of p_ϕ.

Asymptotically, the term implies the quantum contribution $\Delta\phi_1$ to the shift of ϕ derived in (36). Quantum evolution through the turning point therefore connects two semiclassical evolutions with respect to two different Dirac observables, Φ_1 and $\Phi_2 = \Phi_1 + \Delta\phi_1$. The classical Dirac observable Φ is not strictly conserved, but since it is not globally defined it does not have a direct quantum analog anyway. Our construction implies a successful implementation of Dirac observables that are conserved

at least in semiclassical regimes far from turning points. As our analytical expressions show, the overall shift $\Delta\phi = 4\gamma^{-1}(\ln 2 - 1)$ between the asymptotic past and the asymptotic future of a quantum trajectory decreases for larger γ, in which case the exponential potential is steeper and most p_ϕ-contributions to a state reach their turning point at almost the same time. Additional quantum corrections implied by the p_ϕ-dependence of the turning point are then suppressed.

V. CONCLUSION

We applied the methods of [13–15] to a simple cosmological model that, after rescalings and simplifications, has a Hamiltonian constraint with a standard kinetic energy for two variables, and an exponential potential for one of them. One of the degrees of freedom therefore can be used as a global internal time, while the other one encounters one turning point in its classical evolution and constitutes a local clock variable. Traditional methods of deparametrization are able to implement only the global time, while we obtained consistent quantum evolution also with the local clock.

Unlike in previous proposals our quantum evolution does not freeze when the classical turning point is reached, as we demonstrated both analytically by Eq. (24) and numerically in Figs. 1 and 2. Our construction also gives a successful description of the fact that a quantum superposition of different energy eigenstates encounters a number of turning points at different times, in contrast to the classical system at a fixed energy. A visible implication of this new behavior can be seen in the possibility that the variable that could be used as a global internal time may behave nonmonotonically when it evolves with respect to the local clock, as shown in Fig. 4. The same feature demonstrates that the two choices, global clock or local clock, are not equivalent upon quantization, adding another example to the list of models that have confirmed the inequivalence between different choices of internal times in quantum cosmology [18–23]. [Our results are, however, invariant with respect to nonlinear reparametrizations of the effective time parameter τ if they are performed after the local clock α has been chosen. The resolution of the turning point and the characteristic shift therefore do not depend on the parametrization (16) as long as it presents a full disambiguation of the flow of α.]

The energy dependence of classical turning points also implies a characteristic shift between semiclassical relational trajectories in the asymptotic past and future. The corresponding analytical expressions allowed us to demonstrate that the classical Dirac observables are approximately conserved asymptotically far from the turning point, but undergo specific changes during the transition through a turning point. This feature is remarkable because the corresponding Dirac observable does not have a well-defined quantization in this case, owing to multivaluedness of the classical expression on phase space. (See also [24,25] for a discussion of Dirac observables on nontrivial phase spaces.) Multivaluedness is related precisely to the behavior around turning points that implies the nonconservation of classical Dirac observables at the quantum level. Our procedure therefore preserves the expected classical features in regimes where turning points are not relevant, and at the same time provides a consistent freeze-free quantum evolution through turning points.

While the dependence of the turning point on the energy of a state is a characteristic feature, for a given energy the turning point is unique, given the monotonic nature of the potential used here. In this respect, the model is rather different from the first application of our methods in [14], where the clock variable was oscillating and encountered its turning points an infinite number of times even in a single energy eigenstate. The transition through the turning point remains hard to analyze without numerical input because it is sensitive to the various energy contributions of a wave packet that transit the turning point at different times. But our detailed investigation of a model with a single turning point has led to a clear analytical description of the relationship between states before and after the turning point, which might also help to understand long-term quantum evolution in the presence of local oscillating clocks.

A common feature can be seen in the observation that quantum evolution with local clocks is closer to what is expected from global deparametrization when the potential is very steep in the range of variables where turning points occur. For an oscillating clock ϕ, this happened for large clock frequencies in a potential $\lambda^2\phi^2$ with large λ. In the present case, the condition is large γ, such that the potential $e^{\gamma\alpha}$ for the local clock α with a single turning point gets very steep.

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