Renormalization group flows connecting a $4-\epsilon$ dimensional Hermitian field theory to a $\mathcal{PT}$-symmetric theory for a fermion coupled to an axion

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The renormalization group flow of a Hermitian field theory is shown to have trajectories which lead to a non-Hermitian parity-time ($\mathcal{PT}$) symmetric field theory for an axion coupled to a fermion in spacetime dimensions $D = 4 - \epsilon$, where $\epsilon > 0$. In this renormalizable field theory, the Dirac fermion field has a Yukawa coupling g to a pseudoscalar (axion) field and there is quartic pseudoscalar self-coupling u. The robustness of this finding is established by considering flows between ϵ dependent Wilson-Fisher fixed points and also by working to *three loops* in the Yukawa coupling and to *two loops* in the quartic scalar coupling. The flows in the neighborhood of the nontrivial fixed points are calculated using perturbative analysis, together with the ϵ expansion. The global flow pattern indicates flows from positive u to negative u; there are no flows between real and imaginary g. Using summation techniques we demonstrate a possible nonperturbative $\mathcal{PT}$-symmetric saddle point for $D = 3$.

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I. INTRODUCTION

Non-Hermitian $\mathcal{PT}$-symmetric field theories are effective theories, which may describe aspects of Beyond-the-Standard Model physics (BSM) [1–15]. $\mathcal{P}$ is a linear operator (such as parity) and $\mathcal{T}$ is an antilinear operator (such as time-reversal). A quantum mechanical system with unbroken $\mathcal{PT}$-symmetry [16,17] has a completely real spectrum which leads to unitary dynamics [18]. Our aim is not to pursue phenomenological aspects of BSM physics, but to investigate in depth an intriguing behavior noticed in a recent study of a field theory developed for gravitational axion phenomenology and dynamical mass generation [13,14,19,20]. We noticed a renormalization group flow [19] from Hermitian values of the coupling to those of a non-Hermitian but $\mathcal{PT}$-symmetric version of the field theory in a one-loop analysis. We examine here the robustness of these findings by working with beta functions with nonzero ϵ and by working to *three loops* in the Yukawa coupling and *two loops* in the quartic scalar coupling [21–25]. The quantum theory is performed using path integrals [26]. The issues dealing with path integrals for $\mathcal{PT}$-symmetric theories has been studied at length recently [19,27].

In spacetime dimensions D, Hermitian quantum mechanical systems are treated either in the language of path integrals [28] or of operators acting on a Hilbert space [29]. The bridge between path integrals and operator descriptions is understood for Hermitian theories [30,31]. For $\mathcal{PT}$-symmetric quantum theories in $D = 1$ the observables are self-adjoint with respect to an inner product [16,17,32] which is different from the usual Dirac inner product and is specific to the theory being considered. The path integral formulation of $\mathcal{PT}$-symmetric theories in $D = 1$ has been shown in detailed examples to give the same Green's functions [19,33,34] as the operator treatment. The general argument [33] justifying this in $D = 1$ is extended to $D > 1$ in [19]. In [19,27] it was shown that the Feynman rules which describe the weak coupling behavior of the theory around the trivial saddle point of the path integral follow just from the Lagrangian of the theory and produce the correct asymptotic series at weak coupling of the theory.

An early example providing an indication that a Hermitian field theory, when renormalized, may need a reinterpretation as a $\mathcal{PT}$-symmetric field theory [35,36] is provided by the Lee model [37]. The Lee model has been solved explicitly in $D = 1$ and $D = 4$. It has mass, wave function and coupling constant renormalization in $D = 4$. However, the model does not have crossing symmetry and the particles in the model do not obey the spin-statistics theorem [38]. An important feature of the model is that the bare coupling has a square root singularity in terms of the renormalized coupling. This nonanalyticity leads to ghost states in a conventional interpretation. In a $\mathcal{PT}$-symmetric

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interpretation the Hamiltonian is self-adjoint with respect to a different inner product [35]. A second example is the emergence of unstable but $\mathcal{PT}$-symmetric effective potential for the Higgs field in the Standard Model (discussed in a $D = 1$ approximation [19]). This effective potential arises from renormalized one-loop effects [39,40].

It is known that there is an asymptotic weak coupling perturbation theory [19,27] of a $\mathcal{PT}$-symmetric field theory in $D = 4$. The key to this is the existence of path integrals in $\mathcal{PT}$-symmetric theories, which are steepest descent paths and are associated with boundary conditions on the complex-valued paths or Lefscchetz thimbles [41,42] used in the path integral. When we come to consider $D = 4$, we have the additional issues of regularization and renormalization associated with Feynman perturbation theory around the trivial saddle point. Dimensional regularization with $D = 4 - \epsilon$, where $\epsilon > 0$ enables the study of Wilson-Fisher fixed points [43]. Flow between such fixed points remain perturbatively small because ϵ is small.

We consider a renormalizable field-theory for axion physics, which is a massive Yukawa model [13,14] and is also one of the simplest renormalizable field theories [44]. The interaction terms have a conventional form but can be tuned to have values which render the QFT no longer Hermitian, but still $\mathcal{PT}$-symmetric (as in [35]). The model provides a framework for studying the interplay of renormalization and $\mathcal{PT}$ symmetry in the presence of a fermion and a pseudoscalar near four dimensions. Unlike the Lee model [37,45] this model is a conventional crossing-symmetric field theory. Our principal aim is to understand, in a *controlled* way, the interplay of renormalization and $\mathcal{PT}$ symmetry in a relativistic four-dimensional QFT model, starting with a *Hermitian* theory. The massive Yukawa model we consider is given by the bare Lagrangian [19] in 3-space and 1-time dimensions in terms of bare parameters with subscript 0[1]

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi_0\partial^\mu\phi_0 - \frac{M_0^2}{2}\phi_0^2 + \bar{\psi}_0(i\not{\partial} - m_0)\psi_0 - ig_0\bar{\psi}_0\gamma^5\psi_0\phi_0 - \frac{u_0}{4!}\phi_0^4. \quad (1)$$

$\mathcal{L}$ is renormalized in four dimensions through mass, coupling constant and wave function renormalizations; the scalar self-interaction is obtained from continuation of δ to 2 in the manifestly $\mathcal{PT}$-symmetric deformation [16,17]

$$\frac{u_0}{4!}\phi_0^2(i\phi_0)^\delta, \quad (2)$$

for $u_0, \delta > 0$, in any spacetime dimension D. To be clear, the parameter being continued is δ and not u_0; this is essential for $\mathcal{PT}$ symmetry as will become clear when the reality of path integrals is considered. This is the simplest nontrivial renormalizable model of a Dirac fermion field ψ_0 interacting with a pseudoscalar field ϕ_0. In the Dirac representation of γ matrices the standard discrete transformations [46] on ψ_0 are

$$\mathcal{P}\psi_0(t,\vec{x})\mathcal{P}^{-1} = \gamma^0\psi_0(t,-\vec{x}),$$
$$\mathcal{T}\psi_0(t,\vec{x})\mathcal{T}^{-1} = i\gamma^1\gamma^3\psi_0(-t,\vec{x}), \quad (3)$$

$\mathcal{T}$ is an antilinear operator. Moreover, under the action of $\mathcal{P}$ and $\mathcal{T}$, the pseudoscalar field $\phi_0(t,\vec{x})$ transforms as

$$\mathcal{P}\phi_0(t,\vec{x})\mathcal{P}^{-1} = -\phi_0(t,-\vec{x}),$$
$$\mathcal{T}\phi_0(t,\vec{x})\mathcal{T}^{-1} = \phi_0(-t,\vec{x}). \quad (4)$$

These definitions go through in D dimensions with the Dirac gamma algebra given in (33). In dimensional regularization, expressions for Green functions from covariant perturbation theory, which are valid for integer D, are analytically continued in D [47]. Lorentz covariants such as $\gamma_\mu, p_\mu, g_{\mu\nu}$ are treated as formal entities [48] that obey prescribed algebraic identities. The specific values of indices are not used.[2] However the definition of γ_5 requires special consideration (see IV A).

If g_0 is real, then the Yukawa term in (1) is Hermitian and $g_0^2 > 0$. If g_0 is imaginary, then the Yukawa term is non-Hermitian but is $\mathcal{PT}$-symmetric and so $g_0^2 < 0$. u_0 is real but it can be positive (Hermitian) or negative ($\mathcal{PT}$-symmetric).

The plan of this paper is as follows:

(1) We briefly review the role of renormalizability in $\mathcal{PT}$-symmetric quantum field theory and the subtleties in defining the corresponding path integrals [17,19,27]. In particular we note:
   - (i) In the Lee model [36,37,45], a model of historical importance in the study of renormalization, the bare coupling has a nonanalytic dependence on the renormalized coupling. Moreover, the nonanalyticity is in terms of a branch cut. The Lee model is a quantum mechanical Hermitian model which allows for (an exact treatment of) renormalization starting with a Hilbert space with the conventional Dirac inner product. The well-known ghost problem [35], which develops due to renormalization, is removed by interpreting the model with a new inner product related to the C operator of $\mathcal{PT}$ symmetry [49].
   - (ii) In order to understand $\mathcal{PT}$-symmetric path integrals it is instructive to consider $D = 0$

[1]Our Minkowski-metric signature convention is $(+,-,-,-)$.

[2]These calculations differ from those required for the energy eigenvalues of a Dirac equation in general integer dimensions where the explicit representations of the gamma matrices are used.

$\mathcal{PT}$-symmetric integrals using standard complex analysis techniques. The related analysis of $D \geq 1$ can be found in [19,27]. The presence of fermions does not change this analysis qualitatively since massive fermions can be integrated out (at one loop) to give an effective potential contribution [50–52] to the scalar self-interaction, in terms of logarithmic factors.

(2) Perturbation theory using Feynman diagrams is applied to the Yukawa model. This gives an asymptotic series in the couplings that is valid near the trivial saddle point. The contributions from the nontrivial saddle points (due to bounces) are asymptotically subdominant in the weak coupling limit [53]. However, the bounce (instanton) solutions give rise to imaginary contributions to odd point Green's functions which would otherwise vanish [27,53]. Hence our approach, which ignores the subdominant contributions from nontrivial saddle points, is based on perturbation theory around the trivial saddle point, which is valid for renormalization group flows around all sufficiently weak-coupling fixed points. We also comment on the subtleties of using dimensional regularization in noninteger dimensions. Using a general purpose *Mathematica* program RGBeta [21], the perturbation theory is performed to three loops in g and two loops in u. RGBeta has the feature that it also accepts complex couplings. Beta functions of the renormalization group flow [44,54] can be calculated. We solve for the fixed points and determine their stability. Going from $\epsilon = 0$ to nonzero ϵ leads to the trivial fixed point spawning three new ϵ-dependent fixed points, whose magnitudes are directly controlled by ϵ. Furthermore, the flow in the neighborhood of the fixed points is joined together to give a more global flow picture. From this picture, we can see how the Hermitian and non-Hermitian fixed points interact with each other, i.e., how the flow is organized around these fixed points. For one *non-Hermitian* fixed point the ϵ expansion is stable, i.e., the coefficients do not increase rapidly with order, so resummation techniques using Padé approximants leads to a genuine fixed point in $D = 3$, which is not sensitive to variations in the form of Padé approximants used. This fixed point has the stability of a saddle point.

(3) We examine some aspects of applying finite loop-order perturbation theory, and compare our model to that presented in [55], where similar analysis is performed.

(4) In the conclusions we discuss and summarise our results. Furthermore, there are appendices giving some additional details on our findings; we give some checks of robustness of our main results related to the effects of finite loop order in perturbation theory.

II. THE LEE MODEL

The Lee model (LM) is a class of soluble simplified field theories [37] used to study renormalization, which can be carried out exactly. LM[3] involves fermionic particles N and V with operators ψ_N and ψ_V and a bosonic particle θ with operator a (in $D = 1$). The interactions in the model allow

$$V \to N + \theta \tag{5}$$

and also the reverse process

$$N + \theta \to V. \tag{6}$$

Because the field theory does not have crossing symmetry the process $N \to V + \bar{\theta}$ is not allowed where $\bar{\theta}$ is the antiparticle of θ. The fermions N and V do not have spin and so the spin-statistics theorem [38] is not satisfied. The interactions imply conservation rules for B and Q where

(i) $B = n_N + n_V$

(ii) $Q = n_V - n_\theta$,

and n_N, n_V, and n_θ are the number of quanta for N, V and θ respectively. This simplification facilitates the ability to solve the model [35]. In $D = 1$ the Hamiltonian $\mathcal{H}$ is $\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_1$ where

$$\mathcal{H}_0 = m_V \psi_V^\dagger \psi_V + m_N \psi_N^\dagger \psi_N + \mu a^\dagger a \tag{7}$$

and

$$\mathcal{H}_1 = \delta m_V \psi_V^\dagger \psi_V + g_0 (\psi_V^\dagger \psi_N a + a^\dagger \psi_N^\dagger \psi_V). \tag{8}$$

The sector with $B = 1$ and $Q = 0$ is spanned by the states $|1,0,0\rangle$ and $|0,1,1\rangle$. The eigenstates of $\mathcal{H}$ are denoted by $|V\rangle$ and $|N\theta\rangle$, with associated eigenvalues m_V and $E_{N\theta}$ given by

$$m_V = \frac{1}{2}\left(m_N + \mu + m_{V_0} - \sqrt{M_0^2 + 4g_0^2}\right)$$
$$E_{N\theta} = \frac{1}{2}\left(m_N + \mu + m_{V_0} + \sqrt{M_0^2 + 4g_0^2}\right)$$

where $M_0 \equiv m_N + \mu - m_{V_0}$ and $m_{V_0} \equiv m_V + \delta m_V$. The wave-function renormalization constant Z_V is determined [35] through the relation

$$\sqrt{Z_V} = \langle 0|\psi_V|V\rangle \tag{9}$$

which leads to [35]

$$Z_V = \frac{2g_0^2}{\sqrt{M_0^2 + 4g_0^2}(\sqrt{M_0^2 + 4g_0^2} - M_0)}. \tag{10}$$

[3] A $D = 1$ version of the Lee model suffices to show the essential effect of renormalization present in the $D = 4$ model [35].

The renormalized coupling constant g satisfies

$$g^2 = Z_V g_0^2. \tag{11}$$

In terms of $M \equiv m_N + \mu - m_V$, a renormalized quantity, it is straightforward to see that

$$M_0 = M - \frac{g_0^2}{M}. \tag{12}$$

From (11) and (12) we can deduce the nonperturbative result that

$$g_0^2 = \frac{g^2}{(1 - \frac{g^2}{M^2})}. \tag{13}$$

Hence g_0 is related to g by a square root singularity with a branch cut between $-M$ and M. If $g^2 > M^2$, then the bare coupling can become imaginary and the Hamiltonian is non-Hermitian, but $\mathcal{PT}$-symmetric [35]. Explicitly the transformations due to $\mathcal{P}$ are

$$\mathcal{P}V\mathcal{P} = -V \qquad \mathcal{P}N\mathcal{P} = -N \qquad \mathcal{P}a\mathcal{P} = -a$$
$$\mathcal{P}V^\dagger\mathcal{P} = -V^\dagger \qquad \mathcal{P}N^\dagger\mathcal{P} = -N^\dagger \qquad \mathcal{P}a^\dagger\mathcal{P} = -a^\dagger \tag{14}$$

and due to $\mathcal{T}$ are

$$\mathcal{T}V\mathcal{T} = V \qquad \mathcal{T}N\mathcal{T} = N \qquad \mathcal{T}a\mathcal{T} = a$$
$$\mathcal{T}V^\dagger\mathcal{T} = V^\dagger \qquad \mathcal{T}N^\dagger\mathcal{T} = N^\dagger \qquad \mathcal{T}a^\dagger\mathcal{T} = a^\dagger. \tag{15}$$

The non-Hermiticity of the Hamiltonian leads to states with energies that are not real. Because of the $\mathcal{PT}$-symmetry, a new inner product was constructed for the Hilbert space which removed ghost states from the spectrum [35].[4] The Lee model has some similarities with $\mathcal{L}$ in (1). The massive Yukawa model has the trilinear interaction between fermions and bosons as in the Lee model but it has also a quartic boson self-interaction. It has crossing symmetry and the spin-statistics connection, features which are essential for any realistic fundamental theory. $\mathcal{PT}$ symmetry in the Lee model emerges for a nonweak coupling strength. Non-Hermiticity in the massive Yukawa model occurs for small couplings and hence is amenable to a renormalization group analysis.

III. $\mathcal{PT}$-SYMMETRIC PATH INTEGRALS

In the modern study of field theory, quantum aspects can be explored through path integrals where the Hilbert space structure is not paramount [44]. In non-Hermitian (but $\mathcal{PT}$-symmetric) field theory, this advantage persists and simplifies calculations at weak coupling [34]. We concentrate on the modification in $D = 0$ of paths for the existence of path integrals in $\mathcal{PT}$-symmetric framework. The discussion of semi-classical analysis and steepest descent paths can be found in [19,27].

We shall focus on the bosonic part of the path integral for $\mathcal{L}$ [19].[5] and consider two forms of the bosonic path integral, one which preserves manifest $\mathcal{PT}$ symmetry and the other which does not

$$Z_i = \int \mathcal{D}\phi \exp\left(-S_i[\phi]\right), \quad i = 1, 2 \tag{16}$$

where $\mathcal{D}\phi$ is the path integral measure and the action is given by

$$S_i[\phi] = \int d^D x \left(\frac{1}{2}(\partial_\mu\phi)^2 + V_i(\phi)\right) \tag{17}$$

and

$$V_1(\phi) = \frac{1}{2}m^2\phi^2 + \frac{u}{4}\phi^2(i\phi)^\delta, \tag{18}$$

$$V_2(\phi) = \frac{1}{2}m^2\phi^2 + \frac{u}{4}\exp i\zeta\phi^4 \tag{19}$$

where we consider monotonic continuations in the parameters, with $\delta \to 2$ in the first case and $\zeta \to \pm\pi$ in the second case. In both cases we need the path integral to converge and the contours of paths have to be chosen appropriately. Although the limiting form of V_i in the parameter continuations are

$$V(\phi) = \frac{1}{2}m^2\phi^2 - \frac{u}{4}\phi^4 \tag{20}$$

the contours required with the different deformations are *distinct* and we will see that $Z_1 \neq Z_2$ in their imaginary parts. The first deformation is $\mathcal{PT}$-symmetric whereas the second deformation is not since under $\mathcal{P}$ and $\mathcal{T}$ we require

(i) $\mathcal{P}: \phi \to -\phi$;
(ii) $\mathcal{T}: \phi \to \phi^*, \{i \to -i\}$.

The δ deformation is central to $\mathcal{PT}$ symmetry. We shall show that the δ deformation keeps the partition function real while the coupling deformation leads to a Z with imaginary parts.

A. $D = 0$

We consider the $D = 0$ case[6] to illustrate the importance of $\mathcal{PT}$-preserving deformations. Then we have

[4] An analogue version of the Lee model in the non-Hermitian region has also been proposed [56].

[5] The fermions in the model give a logarithmic correction to the quartic self-interaction when integrated out [50] of the path integral and does not cause a significant change in the discussion.

[6] This case is an example of a trivial field theory at a single spacetime point. It is useful in understanding the nature of the deformations which are necessary to have a $\mathcal{PT}$-symmetric path integral.

$$Z_1 = \int_{\mathcal{C}} \mathrm{d}z \exp\left(-\left(\frac{1}{2}m^2 z^2 + \frac{u}{4}z^2(iz)^\delta\right)\right) \quad (21)$$

where $\mathcal{C}$ is a contour in the complex plane, which is a deformation of the real line interval $(-\infty, \infty)$ such that Z is finite and ϕ has been replaced by the variable z. The path integral has become an integral whose convergence is determined by the term proportional to u. On writing $z = r\exp(i\chi)$ we have

$$z^2(iz)^\delta = r^{2+\delta}\exp\left(i\left[2\chi + \delta\left(\chi + \frac{\pi}{2}\right)\right]\right) \quad (22)$$

and the integral for Z converges when

$$2n\pi - \frac{\pi}{2} < (\delta+2)\chi + \delta\frac{\pi}{2} < \frac{\pi}{2} + 2n\pi \quad (23)$$

where n is an integer defining Stokes wedges which defines an opening in χ

$$\chi_l < \chi < \chi_u \quad (24)$$

where $\chi_u = \frac{\frac{\pi}{2}(1-\delta)+2n\pi}{\delta+2}$ and $\chi_l = \frac{2n\pi - \frac{\pi}{2}(1+\delta)}{\delta+2}$. There are four distinct wedges labeled by $n = 0, 1, 2, 3$. The $n = 0$ and $n = 3$ form a $\mathcal{PT}$-symmetric set. By Cauchy's theorem, any contour in a wedge is equivalent to any other in its contribution to the integral. Our choice will be to take the contour through the *center* of the wedge. We shall call this particular contour $\mathcal{C}_{PT}$, see Fig. 1.

It is convenient to rescale $z \to z/\sqrt{u}$, for the case $\delta = 2$, which leads to

$$Z_1 = \int_{\mathcal{C}_{PT}} dz \exp\left(-\frac{1}{u}\left[\frac{1}{2}m^2 z^2 - \frac{z^4}{4}\right]\right). \quad (25)$$

We will now evaluate Z_1 over the $\mathcal{C}_{\mathcal{PT}}$ contour (for $\delta = 2$) to show that it is real. We find

$$Z_1 = \exp\left(-\frac{i\pi}{4}\right)\int_0^\infty dr\left[\cos\left(\frac{m^2r^2}{2u}\right) + i\sin\left(\frac{m^2r^2}{2u}\right)\right] \times \exp\left(-\frac{r^4}{4u}\right) + \text{c.c.} \quad (26)$$

$$= \frac{m\pi}{2^{\frac{3}{2}}}\exp\left(-\frac{m^4}{8u}\right)\left(I_{-\frac{1}{4}}\left(\frac{m^4}{8u}\right) + I_{\frac{1}{4}}\left(\frac{m^4}{8u}\right)\right). \quad (27)$$

where c.c. refers to complex conjugation and the $I_\nu(x)$ are the modified Bessel functions of the first kind. $Z_{\mathcal{C}_{PT}}$ is real and has a nonzero small u expansion since $I_\nu(x) \sim \frac{\exp(x)}{\sqrt{2\pi x}}[1 - \frac{4\nu^2-1}{8x}]$ as $x \to \infty$ and the exponential pieces cancel.

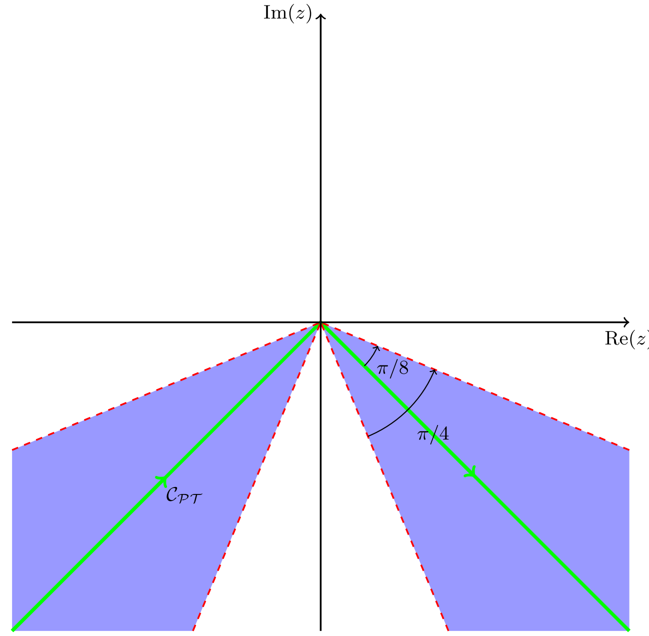


FIG. 1. Stokes wedges (shown in blue, boundaries in red) and contour $\mathcal{C}_{\mathcal{PT}}$ (shown in green) for $\delta \to 2$ in Z_1.

We will compare with the $D = 0$ version of Z_2, given by

$$Z_2 = \int_{\mathcal{C}} \exp\left(-\frac{1}{2}m^2z^2 - \frac{u}{4}e^{i\zeta}z^4\right)dz \quad (28)$$

Similarly, we let $z = re^{i\theta}$. The integral in r converges if

$$-\frac{\pi}{8}(4n+1) - \frac{\zeta}{4} < \theta < \frac{\pi}{8}(1-4n) - \frac{\zeta}{4} \quad (29)$$

The distinct Stokes wedges are for $n = 0$ and $n = 1$ when $\zeta = \pi$. This wedge pair is not $\mathcal{PT}$-symmetric. We shall call this particular contour $\mathcal{C}_{\text{rotation}}$, see Fig. 2. The Hermitian case is $\zeta = 0$ and $\mathcal{C} = (-\infty, \infty)$.

On consideration of Z_2 for the $\zeta = \pi$ wedge pair, we find that it is complex

$$Z_2 = \frac{m\pi}{2\sqrt{2}}(1-i)\exp\left(-\frac{m^4}{8u}\right)\left[I_{-1/4}\left(\frac{m^4}{8u}\right) + iI_{1/4}\left(\frac{m^4}{8u}\right)\right]. \quad (30)$$

We therefore see how the choice of contours is crucial for defining a $\mathcal{PT}$-symmetric theory and ensuring that the path integrals are real.

Furthermore, we note that Green's functions for odd functions of ϕ are purely imaginary in the $\mathcal{PT}$-deformed theory, which is characteristic of $\mathcal{PT}$ symmetry. Explicitly we have

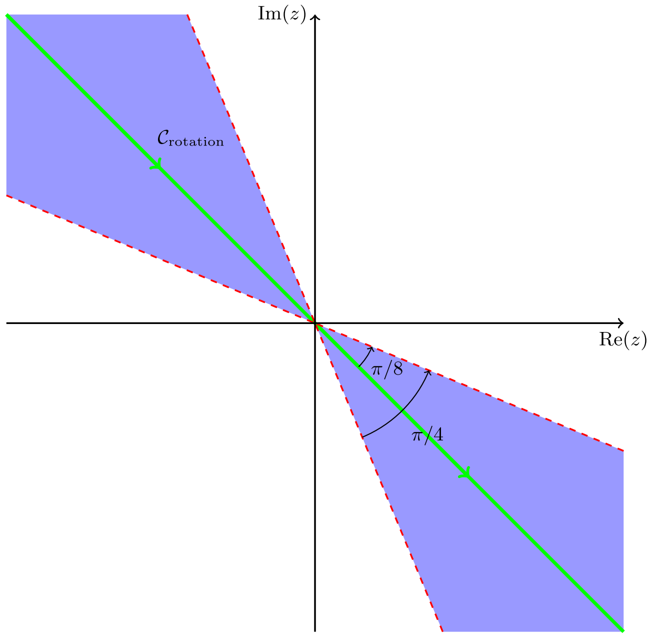


FIG. 2. Stokes wedges (shown in blue, boundaries in red) and contour $\mathcal{C}_{\text{rotation}}$ (shown in green) for $\zeta \to \pi$ in Z_2.

$$\langle z^{2n+1}\rangle = i^{n+1}\left\{\int_0^\infty drr^{2n+1}\exp\left(-\frac{r^4}{4u}\right)\times\left[(-1)^{n+1}\exp\left(i\frac{m^2r^2}{2u}\right)-\exp\left(-i\frac{m^2r^2}{2u}\right)\right]\right\} \tag{31}$$

where $n = 0, 1, 2, \ldots$. These integrals can be written in terms of modified Bessel functions.

The partition function and Green's functions cannot be calculated exactly for $D > 0$. However, we are interested in weak coupling expansions of the $\mathcal{PT}$ field theories. A way of analysing weak coupling expansions of partition functions is through a saddle point analysis of the path integral which is discussed in [19,27]. We have defined path integrals in [19,27] appropriate for $\mathcal{PT}$ symmetry in weak coupling using the method of steepest descents. The formal arguments have been illustrated in a specific case [33,34] where the Hamiltonian is

$$H = \frac{1}{2}(p^2 + x^2) + i\lambda x^3 \tag{32}$$

and Greens functions are also calculated using operator methods. The two methods agree for $D = 1$. The findings of this concrete calculation have been supported more generally by an argument for $D = 1$ [33] (based on the Schwinger construction [28] of the partition function in the operator theory). It was also stated in [33], without an explicit proof, that the arguments go through for $D > 1$. The details of the generalization for $D > 1$ are given in [19].

IV. THE YUKAWA MODEL

We have the basis for applications of path integral quantization to our $\mathcal{PT}$-symmetric model. The path integral is defined using complex deformation of paths or thimbles in complex Morse theory [41,42,57] which ensures that the integral converges. In $D = 4$ a closely related path-integral method was used to study false vacuum decay in [58,59]. The feature missing from these earlier treatments is the requirement of $\mathcal{PT}$ symmetry.

We are interested in the leading small coupling *asymptotic* expansion [60] using Feynman rules for the Yukawa model. The perturbation expansion around the trivial saddle point needs regularization and renormalization because of well-known infinities of loop Feynman diagrams [44]. The regularization is achieved by going to $D = 4 - \epsilon$ where $\epsilon > 0$, i.e., by using the method of dimensional regularization [47]. The renormalization is achieved through minimal subtraction.

A. Dimensional regularization in scalar/fermionic theories

Although dimensional regularization is a well-established technique, there are subtleties such as the consistent treatment of chiral anomalies and evanescent operators [61] in D dimensions. These, however, have been well investigated [48,62].

For our application, since we are not dealing with chiral *gauge* theories, the procedures we adopt are mathematically consistent. For Hermitian theories it is generally accepted that the continuation in dimension *preserves* unitarity and causality. Our treatment of $\mathcal{PT}$ theories involves an analytic continuation in the coupling or in a deformation parameter in the scalar self-interaction. Moreover we are following a flow from a Hermitian theory to a non-Hermitian theory and so we assume that our conclusions about flow to non-Hermitian theories is unaffected by subtle issues in dimensional regularization.

The validity of the quantum action principle [63] within the framework of dimensional regularization allows the study of symmetries of Greens functions. The consequences of symmetries such as Lorentz and gauge invariance are preserved. Nonanomalous symmetry breaking is removed by the use of evanescent operators. Explicitly for vector gauge theories, gauge invariance is preserved by dimensional regularization [64].

From the early days of dimensional regularization it was noticed that it is impossible to require the relations

$$\{\gamma_\mu, \gamma_\nu\} = 2g_{\mu\nu}, \qquad \mu = 1, \ldots, D \tag{33}$$

$$\{\gamma_5, \gamma_\nu\} = 0, \qquad \mu = 1, \ldots, D \tag{34}$$

since they imply

$$\mathrm{Tr}(\gamma_5\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma) = 0, \qquad D \neq 0, 2, 4. \tag{35}$$

This result cannot be continued to $D = 4$ where

$$Tr(\gamma_5\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma) = 4\epsilon_{\mu\nu\rho\sigma}. \tag{36}$$

We follow the resolution proposed by 't Hooft and Veltman [64] by defining

$$\gamma_5 = \frac{1}{4!}\epsilon_{\mu_1\mu_2\mu_3\mu_4}\gamma_{\mu_1}\ldots\gamma_{\mu_4} \tag{37}$$

where the indices take values in (0, 1, 2, 3). This ensures the validity of (37); however now

$$\{\gamma_5, \gamma_\mu\} = 0, \qquad \mu = 1, \ldots, 4 \tag{38}$$

$$[\gamma_5, \gamma_\mu] = 0, \qquad \mu = 5, \ldots, D \tag{39}$$

This scheme is algebraically consistent. The work in [48] has shown that Ward identities are preserved, at least when chiral gauge theories are not involved.[7] This is the relevant situation for us; for our Yukawa model different schemes of dimensional regularization have been explicitly shown to be consistent [66].

B. Renormalization of the Yukawa model

Corresponding to the bare Lagrangian of the Yukawa model, the associated renormalized Lagrangian $\mathcal{L}$ (in terms of renormalized parameters without the subscript 0 and with counterterms) is

$$\begin{aligned}\mathcal{L} = &\frac{1}{2}(1+\delta Z_\phi)\partial_\mu\phi\partial^\mu\phi - \frac{M_0^2}{2}(1+\delta Z_\phi)\phi^2 \\ &+ (1+\delta Z_\psi)\bar{\psi}(i\not{\partial} - m_0)\psi \\ &- ig_0(1+\delta Z_\psi)\sqrt{1+\delta Z_\phi}\bar{\psi}\gamma^5\psi\phi \\ &- \frac{u_0}{4!}(1+\delta Z_\phi)^2\phi^4,\end{aligned} \tag{40}$$

where we have introduced the multiplicative renormalizations Z_ϕ, Z_ψ, Z_g, Z_u, Z_m, and Z_M defined through

$$\phi_0 = \sqrt{Z_\phi}\phi \equiv \sqrt{1+\delta Z_\phi}\phi, \tag{41}$$

$$\psi_0 = \sqrt{Z_\psi}\psi \equiv \sqrt{1+\delta Z_\psi}\psi, \tag{42}$$

$$M_0^2 Z_\phi = M^2 + \delta M^2 \equiv M^2 Z_M, \tag{43}$$

$$m_0 Z_\psi = m + \delta m \equiv m Z_m, \tag{44}$$

$$g_0 Z_\psi\sqrt{Z_\phi} = g + \delta g \equiv g Z_g, \tag{45}$$

$$u_0(Z_\phi)^2 = u + \delta u \equiv u Z_u. \tag{46}$$

[7]Even for chiral gauge theories the scheme can be modified with nongauge invariant finite counterterms [65].

We use dimensional regularization to evaluate the counterterms, taking $D = 4 - \epsilon$ and μ as the renormalization scale. This leads to the perturbative renormalization group (see, for example, [44]). From the discussion in Sec. I, the perturbative renormalization group is unaffected by the nontrivial saddle points [53], which give asymptotically subdominant contributions.

The field theoretic action S generally depends on these μ dependent couplings such that

$$S[Z(\mu)^{1/2}\Phi; \mu, g_i(\mu)] = S[Z(\mu')^{1/2}\Phi; \mu', g_i(\mu')] \tag{47}$$

where $Z(\mu)$ is the wave function renormalization (generally a matrix) of the generic field Φ. As an example, for a scalar field theory, we can write

$$S[\phi;\mu,g_i] = \int d^D x\left(-\frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \sum_i \mu^{D-d_i} g_i O_i(x)\right) \tag{48}$$

where $O_i(x)$ is a local operator of mass dimension d_i and g_i is dimensionless. The μ dependence of g_i is determined through functions $\beta_i(\{g_j\})$

$$\mu\frac{dg_i(\mu)}{d\mu} = \beta_i(\{g_j\}), \tag{49}$$

which are the renormalization group equations.

C. Coupling constant analyticity

We have noted that in the Lee model, the bare coupling has a square root singularity in the renormalized coupling. The Lee model was constructed in such a way that renormalization could be performed exactly. In realistic theories, we cannot expect to obtain exact information about renormalization. We use a renormalization (or subtraction point μ) to define our theory. If we could calculate to all orders in perturbation theory then it is expected that results for physical quantities would be independent of μ. The renormalization group enforces this condition on quantities calculated to low orders in the loop expansion. In this sense, some of the important features of an exact analysis are incorporated. However, the situation is more complicated since the perturbation series are believed not to be convergent, but only asymptotic [67–69]. This led to investigations of the analyticity properties of physical quantities such as the ground state energy (related to the partition function) as a function of couplings (e.g., u) [67,70–72] using large orders in perturbation theory.

We conjecture that square root singularities of the type found in the Lee model may contribute to the emergence of $\mathcal{PT}$ theories starting with a Hermitian theory. Such a result would be extremely hard to prove. The presence of a square root singularity implies that the coupling has a different sign on either side of the cut. For the anharmonic oscillator Bender and Wu [70] found an accumulation of square root singularities in the complex coupling constant Riemann sheets for the energy levels arbitrarily close to the origin.

However, on general grounds, it may be expected that square root singularities will also be present in field theories. Higher D field theories are much more complicated than the $D = 1$ anharmonic oscillator and so square root singularities will not be expected to appear in the same way as in the single component anharmonic oscillator [71]. Eigenvalue problems are ubiquitous in field theory and it is argued persuasively[8] in [60] that square root singularities are generically the most likely singularities of eigenvalues as functions of couplings continued to the complex plane.

D. The renormalization group analysis

In terms of $t = \log \mu$ and $h = g^2$ the renormalization group beta functions for $h \geq 0$ are

$$\frac{dh}{dt} = \beta_h\ (h, u) \quad \text{and} \quad \frac{du}{dt} = \beta_u\ (h, u) \tag{50}$$

where

$$\begin{aligned}\beta_h\ (h, u) = &-\epsilon h + \frac{1}{(4\pi)^2} 10h^2 \\ &+ \frac{1}{(4\pi)^4}\left(-\frac{57}{2}h^3 - 4h^2 u + \frac{1}{6}hu^2\right) \\ &+ \frac{1}{(4\pi)^6}\left(\left[-\frac{339}{8} + 222\zeta(3)\right]h^4 + 72h^3 u \right. \\ &\left. + \frac{61}{24}h^2u^2 - \frac{1}{8}hu^3\right)\end{aligned} \tag{51}$$

and

$$\begin{aligned}\beta_u(h, u) = &-\epsilon u + \frac{1}{(4\pi)^2}(-48h^2 + 8hu + 3u^2) \\ &+ \frac{1}{(4\pi)^4}\left(384h^3 + 28h^2 u - 12hu^2 - \frac{17}{3}u^3\right).\end{aligned} \tag{52}$$

where ζ denotes the Riemann zeta function. These expressions for the beta functions have been found from a perturbative calculation to three loops for the Yukawa coupling and two loops for the quartic coupling using the *Mathematica* package RGBeta [21] and are independent of m and M.[9] When g is pure imaginary, h is negative and so h positive or negative distinguishes between Hermitian and $\mathcal{PT}$-symmetric cases, respectively. The expressions for the beta functions given here are only applicable for $h \geq 0$ (the case for which g is real). Our qualitative conclusions are unaffected by the sign of h, and the $h \geq 0$ and $h < 0$ sectors do not mix, so for brevity in the main text we restrict to $\boldsymbol{h \geq 0}$ (the Hermitian case for g). However, we give the $h < 0$ (non-Hermitian in g) results for completeness in Appendix C.

In the next subsections we shall consider:

(1) The zeros of the beta functions β_u and β_h which determine the fixed points of the renormalization group.

(2) The stability of the fixed points, which can be determined from a linearized analysis around the fixed points (except for the trivial fixed point when $\epsilon = 0$).

(3) The full nonlinear flows connecting the different fixed points. These flows are instructive, especially for the epsilon-dependent fixed points emanating from the trivial fixed point.

(4) Once we have an ϵ expansion of the fixed points it is natural to enquire about any possible resummation to determine information about fixed points and their stability at $D = 3$. We have used the method of Padé approximants and made checks on the pole structure [60] in the neighborhood of $\epsilon = 1$ to determine the trustworthiness of any $D = 3$ fixed point determined this way.

1. Fixed points for $\epsilon = 0$

It is customary to denote the fixed point of h as h^* and the fixed point of u as u^*. However, in the main text, for clarity we will use $f_{i,h}$ (the fixed point value for h) and $f_{u,h}$ (the fixed point value for u) for our numerical results for the fixed points, given to three significant figures. When $\epsilon = 0$, we have two fixed points

(1) The trivial (or Gaussian) fixed point: $f_{1,h} = 0$, and $f_{1,u} = 0$.

(2) $f_{2,h} = 0$, and $f_{2,u} \simeq 83.6$ which corresponds to a quartic coupling $\simeq 3.48$ (rescaled by $1/4!$); since the $f_{2,h}$ and $f_{2,u}$ are non-negative this is a Hermitian fixed point.

The trivial fixed point is the progenitor of the fixed points for $\epsilon \neq 0$. We perform a linearized analysis first for the fixed point f_2. A nonlinear analysis is necessary for f_1.

E. Stability analysis

A linearized analysis around fixed points h^* and u^* consists of examining the evolution of $\delta h = h - h^*$ and $\delta u = u - u^*$. A linearized stability analysis [73] is determined by

$$\frac{d}{dt}\begin{pmatrix}\delta h \\ \delta u\end{pmatrix} = \mathcal{M}(h^*, u^*)\begin{pmatrix}\delta h \\ \delta u\end{pmatrix} \tag{53}$$

[8] See Ch. 7, Sec. 7.5 of [60] for a comprehensive discussion.

[9] The flows for m and M are dependent on the flows for h and u however.

where $\mathcal{M}$ is a 2×2 matrix.[10] $\mathcal{M}$ is diagonalized to obtain eigenvalues $(\lambda_1(h^*,u^*),\lambda_2(h^*,u^*))$ and corresponding eigenvectors $(\vec{e}_1(h^*,u^*),\vec{e}_2(h^*,u^*))$.

Here, we summarize the eigenvectors and eigenvalues for $f_{2,h}$:

(i) $\lambda_1(f_{2h},f_{2u})\approx -1.59$, and $\vec{e}_1(f_{2h},f_{2u})=\binom{0}{1}$

(ii) $\lambda_2(f_{2h},f_{2u})\approx 0.0282$ and $\vec{e}_2(f_{2h},f_{2u})=\binom{1.85}{1}$

1. Nonlinear analysis around trivial fixed point

The stability of the trivial fixed point requires a nonlinear analysis, due to the vanishing of the eigenvalues of the linear stability matrix M.

For the study of renormalization group flows in the neighborhood of the trivial fixed point, $\beta_u(h,u)$ and $\beta_h(h,u)$ can be simplified to

$$\beta_u(h,u)\simeq\frac{1}{\pi^2}\left[-3h^2+\frac{1}{2}hu+\frac{3}{16}u^2\right] \tag{54}$$

and

$$\beta_h(h,u)\simeq\frac{5}{8\pi^2}h^2. \tag{55}$$

The family of flows for h, parametrized with h_0 and t_0, is given by

$$h(t)=\frac{8\pi^2h_0}{8\pi^2-5h_0(t-t_0)}. \tag{56}$$

We define $f(t)=8\pi^2-5h_0(t-t_0)$ for convenience. The accompanying flow for u is

$$u(t)=-\frac{8\pi^2h_0}{3f(t)}\left[\frac{-pf(t)^n+qc}{f(t)^n+c}\right] \tag{57}$$

where c is an integration constant, $p=1+\sqrt{145}\approx 13$, $q=-1+\sqrt{145}\approx 11$, $n=\sqrt{\frac{29}{5}}\approx 2.4$. The behavior is complicated and when h or u becomes large, which occurs due to the presence of a Landau pole, the perturbative analysis is not valid. We can write $u(t)$ in terms of $h(t)$ directly as

$$u(t)=-\frac{1}{3}h(t)\left[\frac{-ph_0^n+q\tilde{c}h(t)^n}{h_0^n+\tilde{c}h(t)^n}\right] \tag{58}$$

writing $c=(8\pi^2)^n\tilde{c}$. This allows us to relate $\tilde{c}$ to h_0 and u_0 as

$$u_0=-\frac{1}{3}h_0\left[\frac{-p+q\tilde{c}}{1+\tilde{c}}\right] \tag{59}$$

If we define $k=\frac{u_0}{h_0}$, then we find

$$\tilde{c}=\frac{p-3k}{3k+q} \tag{60}$$

This suggests that if the h_0 and u_0 are sufficiently close to the origin, then any straight line through the origin is possible.

F. Renormalization group flows

We shall examine the flow around the fixed points f_{ih} and f_{iu}, for $i=1,2$. For $\epsilon=0$ the dimensionless couplings are of $O(1)$ and are not small in any controlled fashion; hence the flows derived from perturbation theory can only be indicative of possible features of renormalization. Moreover, geometric methods are best suited to visualize the flows.[11]

In the figures, the vertical axis is the u-axis and the horizontal axis is the h-axis. The h-axis (where present) is shown in red, and any fixed points are shown in blue (color online). Some features to be noted are

(i) There are no flows from positive to negative h and vice versa.[12]

(ii) There are flows from positive u to negative u, i.e. from a Hermitian to a $\mathcal{PT}$-symmetric region.

(iii) The flows around the trivial fixed point f_1 do not show a simple source, sink or saddle-point behavior, but rather a nonlinear flow. This flow is complicated but an approximate solution is given in (58). In Figs. 3 and 4, there are approximate lines of both positive and negative slope crossing the h-axis, which are an indication of this behavior.

Given that the analysis is based on perturbation theory, flows in regions where the couplings are large compared to 1 can only be misleading. However, near the trivial fixed point, we can see evidence for flows from positive to negative u, i.e., from Hermitian to $\mathcal{PT}$-symmetric behavior. This type of behavior is discussed and investigated below in much more detail for a situation where there are four fixed points which occur at small values of u and h. In our context, this arises since there is a separate parameter which controls the size of the couplings and makes perturbation theory possible. This parameter is ϵ.

1. Fixed points for $\epsilon\neq 0$

We consider $\epsilon>0$ and examine the flows of (50). We have fixed points which we denote by F_i, $i=0,1,2,\ldots,4$. $F_0=f_1$ is the trivial fixed point. The remaining F_i are

[10] $\mathcal{M}$ will also have a dependence on ϵ in $D=4-\epsilon$.

[11] Solving individual trajectories as a function of t requires initial conditions and the description of flows requires a grid of initial conditions. A geometric method [74], whereby tangents to the flows are pieced together as streamlines, is preferable.

[12] This has been verified by performing the analysis for $h<0$, see Appendix C.

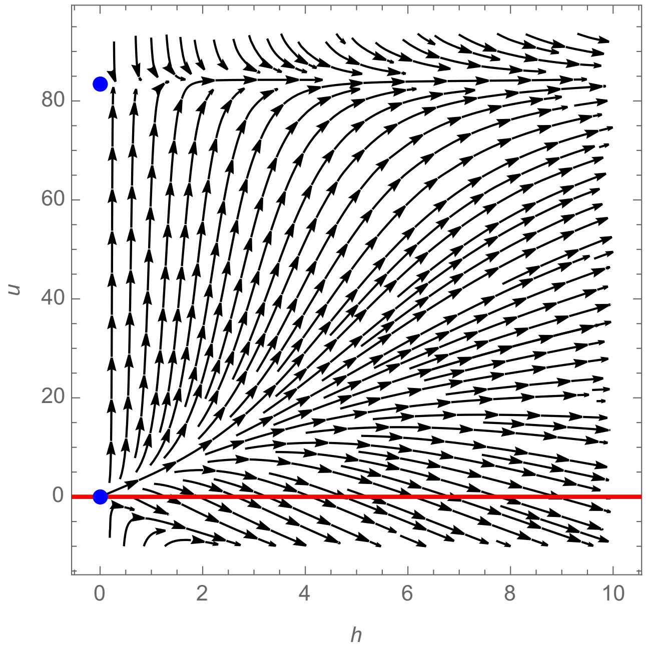


FIG. 3. Global flow for $\epsilon = 0$.

given in terms of series which are not typically convergent but asymptotic as $\epsilon \to 0$. The expressions for the fixed points are given in Appendix A. These expressions allow tracking of fixed points as a function of ϵ and also, in some circumstances, an extrapolation to $\epsilon = 1$ using the technique of Padé approximants. In the limit $\epsilon \to 0$, the fixed point $F_4 \to f_2$, and the fixed points $F_i \to f_1$ for $i = 1, 2, 3$. Hence the trivial fixed point becomes 4 fixed points for $\epsilon \neq 0$: the trivial fixed point and 3 further fixed points ($F_i, i = 1, 2, 3$) which are $O(\epsilon)$. For sufficiently small ϵ, F_2 is a non-Hermitian ($\mathcal{PT}$-symmetric) fixed point whereas F_1 and F_3 are Hermitian. The renormalization group flows in the neighborhoods of $F_i, i = 1, 2, 3$ and f_1 are described through perturbative analysis and are our main focus. Although near F_4 our analysis does indicate possible new behavior (in terms of flows between Hermitian and $\mathcal{PT}$-symmetric regions in the h coupling) these latter findings can only remain conjectural since perturbation theory is unreliable for large couplings. As such, we ignore this point in most of our analysis below. However, it is worth noting that the emergence of $\mathcal{PT}$ symmetry in the Lee model is in terms of h [35] and occurs at strong coupling.

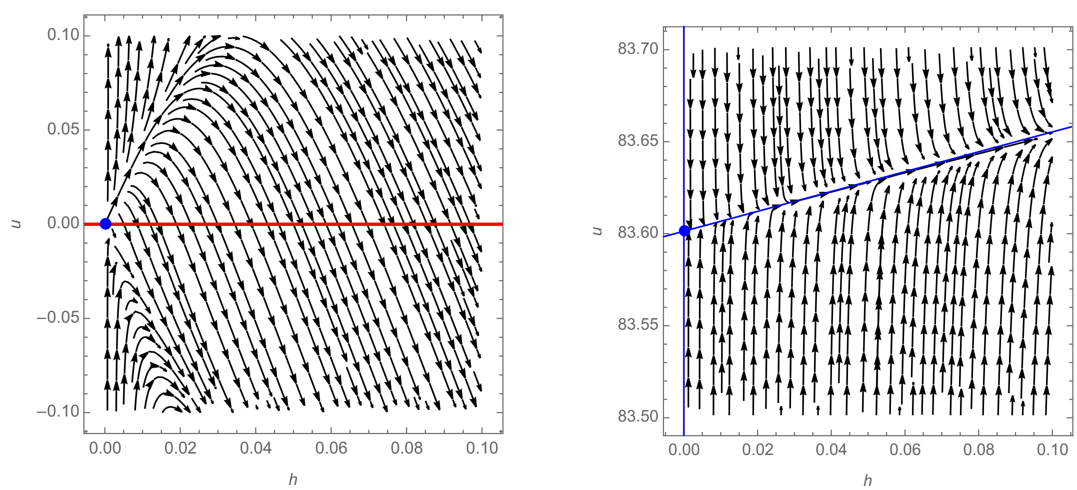
G. The stability of fixed points for $\epsilon \neq 0$

We follow the linear stability analysis of (53) for the fixed points $F_0 \equiv f_1$ and F_j, $(j = 1, 2, 3)$. $F_\alpha (\alpha = 0, 1, 2, 3)$ has two components: $F_{\alpha,u}$, the fixed point value for u and $F_{\alpha,h}$, the fixed point value for h. The eigenvalues of the stability matrix around F_α, will be denoted by $\Lambda_{\alpha,j}$, $j = 1, 2$. The corresponding 2 component eigenvectors will be denoted by $\vec{E}_{\alpha j}$, $j = 1, 2$.

1. *The renormalization group flow between fixed points for $\epsilon \neq 0$*

The renormalization group flows for $0 < \epsilon \lesssim 0.027$ are qualitatively the same and so we shall consider the case $\epsilon = 0.01$ as a representative flow. The flows are organized by the different fixed points F_α. We determine the flows numerically and nonperturbatively in ϵ.

As expected, many of the features from the $\epsilon = 0$ case persist, particularly regarding flows across the coordinate axes (see Figs. 5 and 6). However, the nonzero ϵ ensures that the behavior of the flow near the origin can now be characterized using linear stability analysis [73]; we find an ultraviolet stable stellar node there (as shown in Fig. 7a). Furthermore, three additional points emanate from the origin as ϵ has increased. If we focus on the non-Hermitian (and



(a) Non-linear saddle at trivial fixed point f_1. (b) Hermitian saddle fixed point f_2.

FIG. 4. The local flows around the fixed points for $\epsilon = 0$.

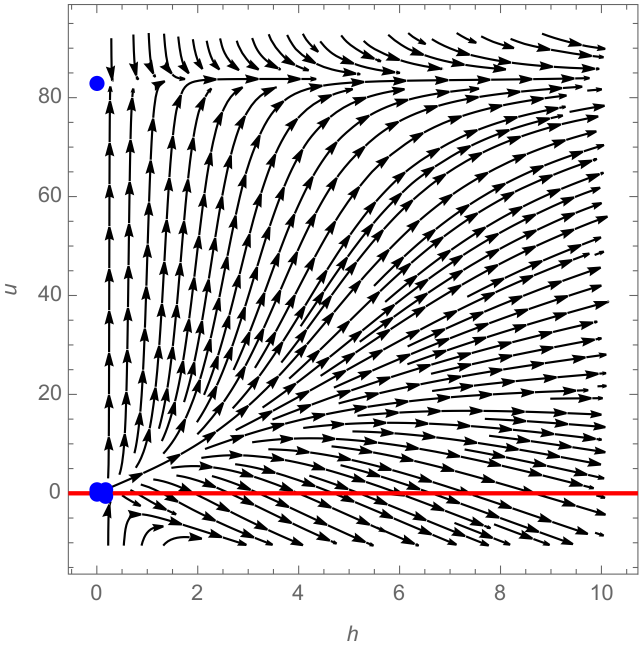


FIG. 5. Global flow for $\epsilon = 0.01$. There are a group of four fixed points that are close to the origin, and one high-u fixed point that we ignore from concerns over its validity in perturbation theory.

$\mathcal{PT}$-symmetric) saddle fixed point F_2 (Fig. 7c), we note that (by examining Fig. 6):

(i) There is a flow that originates at the Hermitian infrared fixed point F_3 (Fig. 7d) in the IR (large negative t) limit, which can flow to the non-Hermitian saddle F_2 in the UV (large positive t) limit.

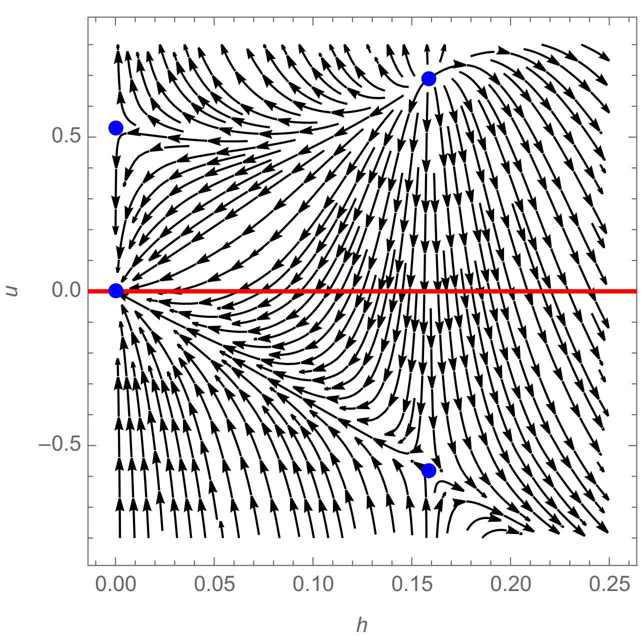


FIG. 6. Flows around the group of fixed points near the origin for $\epsilon = 0.01$.

(ii) There is a flow that originates at the stellar node at the origin F_0 (Fig. 7a) in the UV (large positive t) limit, which can flow to the non-Hermitian saddle F_2 in the IR (large negative t) limit.

Some of these features have been noted previously in the literature in the context of the Hermitian theory (for example, in [55,75]), but we are now able to interpret the flow to the non-Hermitian region for the coupling constants in the framework of $\mathcal{PT}$-symmetric theory [76]. Furthermore, we have additional control here from the use of the engineering dimension ϵ.

As ϵ continues to increase, we reach a critical value $\epsilon_c \sim 0.027$ where the behavior of the large-u fixed point changes (in terms of the eigenvalues of the linear stability analysis). However, this is not significant for our interests here, since we cannot be sure of the validity of the analysis for these fixed points in the perturbation theory of h and u. The next critical value of ϵ for which the character of a fixed point changes is $\epsilon_{c'} \sim 0.44$, but this is likely too high to trust within our perturbative expansion in ϵ. We investigate the robustness of our results in this section to changing the loop orders of the computation, as well as the effect of increasing ϵ, in Appendix B.

We note that the character of the non-Hermitian saddle fixed point F_2 seems to be preserved as we extend our analysis to $D = 3$ from above (and so $\epsilon \to 1$) with Padé approximants.

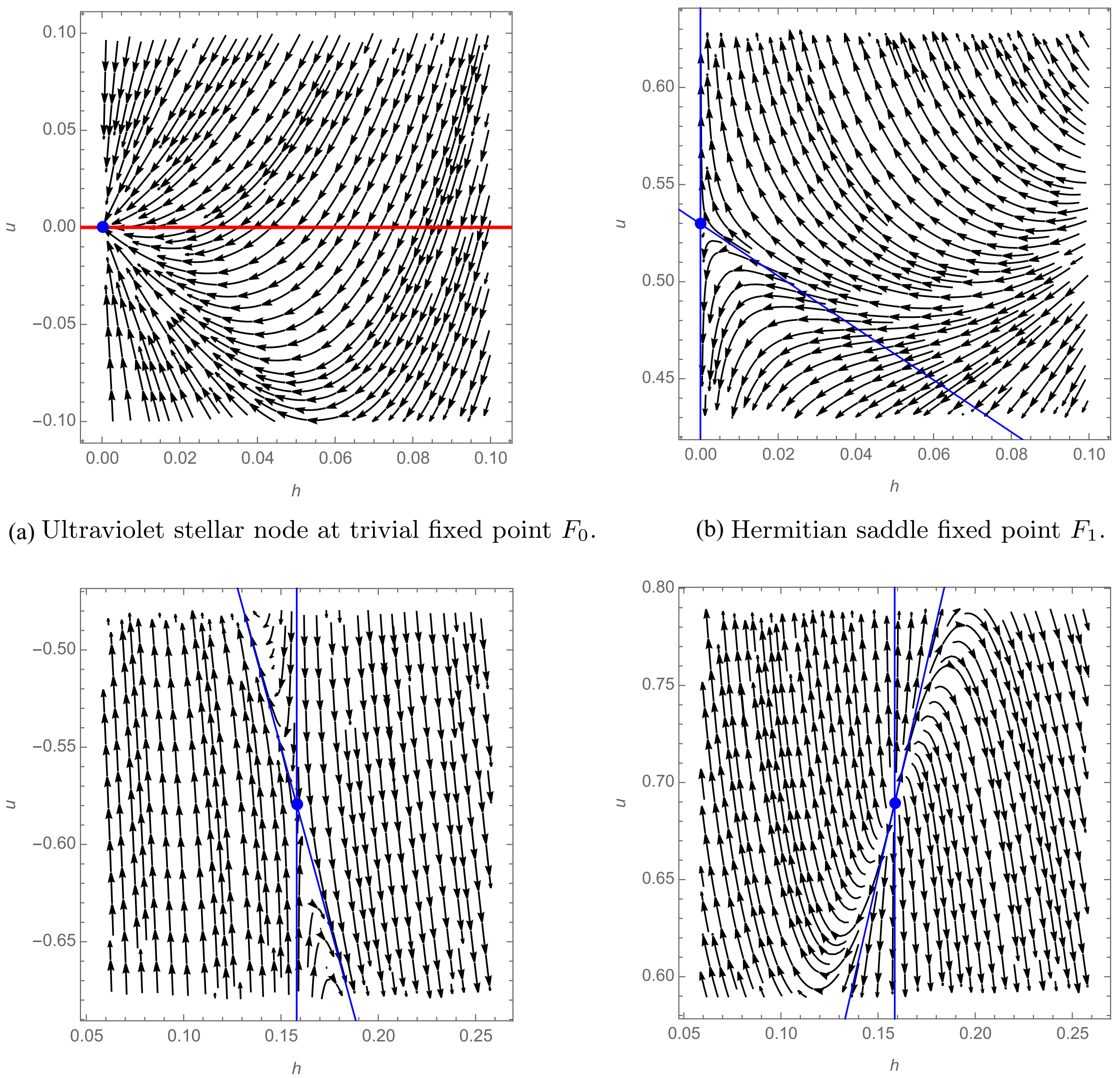
V. PADÉ APPROXIMANTS AND THE $D = 3$ FIXED POINT

The ϵ expansion is used in the study of critical phenomena [43,77], but its convergence is not understood in any systematic way. Although series using the ϵ expansion are readily generated, the series are generally divergent. Hence there is no radius of convergence ϵ_R such that the series is convergent for $|\epsilon| < \epsilon_R$. If the perturbation series is singular, it diverges for all nonzero ϵ. Padé approximants can sometimes offer a way of summing such a series. The partial sums of the ϵ series cannot be summed directly, since for fixed ϵ the sequence of partial sums diverge.

If we have a formal power series $P(\epsilon) = \sum a_n \epsilon^n$ in ϵ then the Padé approximant $P_M^N(\epsilon)$ is defined by

$$P_M^N(\epsilon) = \frac{\sum_{n=0}^{N} A_n \epsilon^n}{\sum_{n=0}^{M} B_n \epsilon^n}. \tag{61}$$

Without loss of generality we take $B_0 = 1$ and the first $M + N + 1$ coefficients of $\sum a_n \epsilon^n$ are used to determine the coefficients $A_0, A_1, \ldots, A_N, B_1, B_2, \ldots, B_M$. $P_N^N(\epsilon)$ is a diagonal Padé sequence. All Padé approximants have pole singularities from the denominator and zeros from the numerator. If there are poles in the neighborhood of $\epsilon = 1$ then an extrapolation to $\epsilon = 1$ using Padé sequences is not viable. By checking for the consistent predictions of fixed points and their stability as N and M are varied, we decide



(a) Ultraviolet stellar node at trivial fixed point F_0.

(b) Hermitian saddle fixed point F_1.

(c) Non-Hermitian saddle fixed point F_2.

(d) Hermitian infrared fixed point F_3.

FIG. 7. The four trustworthy fixed points (in perturbation theory) for $\epsilon = 0.01$.

on the validity of our extrapolation [60] to $\epsilon = 1$. This is a necessary (but not sufficient) criterion for a valid extrapolation to $D = 3$.

We consider the cases where $P(\epsilon)$ is truncated to ϵ^{2n}, for $n = 4, 5, 6, 7$; then we examine the corresponding diagonal Padé approximants $P_N^N(\epsilon)$ for $N = 4, 5, 6, 7$, as well as off-diagonal Padé sequences $P_{N-1}^{N+1}(\epsilon)$ and $P_{N+1}^{N-1}(\epsilon)$. The convergence of the various Padé approximants for the fixed points F_α is only consistent for F_2, a non-Hermitian fixed point. The resultant fixed point at $D = 3$ is

$$(h^*, u^*) = (17.6, -32.3) \tag{62}$$

whose linearized stability is characterized by eigenvalues $\Lambda_1 = -1.16$ and $\Lambda_2 = 1.08$. Hence the fixed point has saddle-like stability. The eigenvectors $\vec{E}_j$ associated with Λ_j, for $j = 1, 2$ are

$$\vec{E}_1 = (-0.0121, 1) \tag{63}$$

and

$$\vec{E}_2 = (-4.21, 1). \tag{64}$$

As ϵ has increased from small values this fixed point has retained its non-Hermitian character and its Padé approximants have been stable for diagonal and off-diagonal sequences. Hence these computations provide some confidence that this is a genuine nonperturbative fixed point for $D = 3$. The putative fixed point may be relevant to studies of UV completions of the Nambu–Jona-Lasinio and Gross-Neveu models between 2 and 4 dimensions [78] and quantum phase transitions in electronic systems [79,80], which is beyond the scope of this paper. We examine the robustness of our conclusions in this section as we change the loop orders for the computations in Appendix B.

VI. PERSPECTIVE ON THE PERTURBATIVE CALCULATIONS

The methods we apply are used in the study of critical phenomena [81,82]. It is widely recognized that they are applicable in the context of relativistic field theories in particle physics [83]. Although in this work we have focused on the emergence of a $\mathcal{PT}$-symmetric field-theory description emerging from a Hermitian theory, this Hermitian theory is a prototype theory for axion physics. The role of relativistic fermions in such models certainly distinguishes them from the scalar field theories belonging to the Ising universality class, which are influential in critical phenomena.

The presence of fermions necessitates revisiting discussions on the nature of perturbation series [67,84] and dimensional regularization [47,62]. Our calculations raise some technical issues that appear in the presence of fermions, which we will discuss below.

A. The behavior of higher orders of perturbation theory for our Yukawa model

In examining our results from IV, we ignore the high-u fixed points (for the scalar self interaction), as we expect them to be untrustworthy in perturbation theory. Here we clarify our intuition on this point.

A naive expectation of perturbation theory in a coupling u, is that for a quantity $f(u)$ (such as a beta function or partition function), there exists a sequence

$$f_N(u) = \sum_{n=0}^{N} f_n u^n \tag{65}$$

which converges to $f(u)$ as $N \to \infty$. In a field theory where the perturbation is generated by Feynman diagrams, the number of diagrams increases with n. This increases the number of terms that contribute to f_n and consequently f_n is expected to increase with higher n [67]; however in order to understand the convergence it will be insufficient to just have bounds on f_n.

Major progress on estimating f_n was made by Bender and Wu [70] for the ground state energy of the anharmonic oscillator in $D = 1$ dimensions (the ϕ^4 field theory for quantum mechanics). The wave function for the energy level with energy E satisfies the Schrödinger equation

$$-\frac{d^2}{dx^2}\psi(x)+\left(\frac{x^2}{4}+u\frac{x^4}{4}-E\right)\psi(x)=0, \quad \psi(\pm\infty)=0. \tag{66}$$

For $E = E_0$ the ground state energy has $f_n \sim -(\frac{6}{\pi^3})^{1/2}(-3)^n\Gamma(n+\frac{1}{2})$. The resulting series is divergent and is an example of an asymptotic series, where [60]

$$f(u) - f_N(u) = O(u^{N+1}) \quad \text{as } u \to 0. \tag{67}$$

If u is ϵ dependent, then ϵ is another control parameter that one can use to make u small. This gives additional confidence in the resulting fixed points.

The extension of Bender and Wu's work to higher order terms in field theory is intimately related to the contributions of instantons in false vacuum decay in a semi-classical analysis of path integrals [67,85,86]. The resulting estimates for the higher order terms are qualitatively similar to that of Bender and Wu.

This analysis has been extended to $D \geq 3$ for Yukawa field theories involving a single fermion and scalar in [82]. Qualitatively similar results were found as for the ϕ^4 theory.

Hence any finite number of higher order terms in perturbation theory would not allow us to investigate putative high-u fixed points for D near 4.

B. Comparison with a standard-model inspired Yukawa theory

There is some similarity of our work with another nongauge Yukawa model (which we denote by M2) that is obtained from a simplification of the Standard Model in the leptonic sector [55]. The fields in M2 are a left-handed fermion doublet [under $SU(2)$], a right-handed fermion $SU(2)$ singlet and a $SU(2)$ scalar doublet. There is a Yukawa coupling of the fermions and scalars consistent with the $SU(2)$ structure. The fact that there are multicomponent (flavor) fields in M2 contrasts with the single Dirac fermion and pseudoscalar field in the axion model that we consider [87–91]. For two component pseudoscalar fields, for example, it is not possible to distinguish a parity transformation from a rotation. Therefore in the presence of multicomponent fields it is not always possible to make a $\mathcal{PT}$ transformation. Our axion model is manifestly $\mathcal{PT}$-symmetric when the couplings flow away from Hermitician values.

We have two types of $\mathcal{PT}$-symmetric extensions of Hermitian theories in the axion model. One is in terms of a negative self-coupling and the other is in terms of an imaginary g (or negative h) [14]. Starting from a Hermitian value of u the renormalization group flow to negative u is possible. Such a feature was noted in the model of M2 as a possibility but issues of $\mathcal{PT}$ symmetry were not discussed there [55]. We have noted that renormalization group flows do *not* connect positive h to negative h. However, the renormalization group flows are symmetric about the axis $h = 0$ in the h–u plane. See Appendix C for more discussion.

VII. CONCLUSIONS

In terms of a simple renormalizable field theory relevant for axion physics involving a pseudoscalar field and a Dirac fermion, the role of renormalization in linking Hermitian and $\mathcal{PT}$-symmetric Hamiltonians in $D = 4 - \epsilon$ has been explored in depth. In order to carry out this investigation, it has been necessary to use path integrals, which in turn has depended on the complex deformations of path integrals within the context of steepest descent paths [27]. This deformation can be regarded as a nontrivial change in the measure employed in the definition of the path integral. It has been argued that on complexifying the bosonic path in the path integral and invoking $\mathcal{PT}$ symmetry, that it is possible to have a theory where Green's functions can be calculated in a weak coupling expansion [19]. In this limit, the path integral is defined on a steepest descent contour (or its higher dimensional generalization the Lefschetz thimble). Expansions around individual stationary points on the contour give rise to asymptotic series, of which the trivial saddle point gives the dominant contribution.

The key to our analysis is the flow pattern between ϵ-dependent fixed points which provides a degree of control over the perturbation series [43] in terms of the renormalized coupling, together with calculations of the renormalization group performed at higher loop. More recently, the possible emergence of unstable $\mathcal{PT}$-symmetric potentials in the Standard Model due to renormalization has been considered within the framework of $\mathcal{PT}$ symmetry [19,76] (but restricted to $D = 1$). This treatment can be enhanced to address the issues for $D = 4$ since we have clarified

(i) the steepest descent-like paths in the path integral, and the role of the trivial saddle points in function space within the steepest descent path, together with the sub-dominant contributions from the nontrivial fixed points.
(ii) renormalization around the trivial fixed point and introduction of Wilson-Fisher ϵ-dependent fixed points.
(iii) the significance of beta functions from Feynman perturbation theory and the renormalization group flows of couplings.
(iv) the usefulness of RGBeta, a program in the symbolic language program *Mathematica*, which can handle complex values of couplings.

Our analysis has found that Hermitian to non-Hermitian flows occur only in terms of the quartic self-couplings. These flows have been observed previously in the context of Hermitian theories, but can now be *reinterpreted in the context of $\mathcal{PT}$-symmetric theory* with full justification. We conjecture that renormalization and the emergence of $\mathcal{PT}$-symmetric theory starting with a Hermitian theory may well occur in other field theories. This conjecture is related to the possibility of square-root type singularities in the coupling appearing generically in other field theories (just as in the Lee model). The robustness of these findings in other renormalizable field theories is worthy of further study.

ACKNOWLEDGMENTS

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APPENDIX A: DATA FOR FIXED POINTS AND THEIR STABILITY FOR $\epsilon \neq 0$

In this appendix, we give the series results in ϵ for the fixed points and their linear stability eigenvalues and eigenvectors. Here, we provide these results to three decimal places (unless exact, or where this would give no significant figures).

(i) $F_{0h} = 0$, $F_{0u} = 0$. This is the trivial Hermitian fixed point. The stability matrix has degenerate eigenvalues: $\Lambda_{0,1} = \Lambda_{0,2} = -\epsilon$. For $\epsilon \neq 0$ (and sufficiently small), this is a UV-stable stellar node (so that trajectories which begin near F_0 approach F_0 on straight lines).

(ii) $F_{1h} = 0$, $F_{1u} = 52.638\epsilon + 33.142\epsilon^2 + 41.735\epsilon^3 + 65.694\epsilon^4 + 115.816\epsilon^5 + 218.763\epsilon^6 + 432.896\epsilon^7 + 885.833\epsilon^8 + 1859.156\epsilon^9 + 3979.970\epsilon^{10} + 8656.771\epsilon^{11} + 19076.958\epsilon^{12}$.

The stability matrix has eigenvalues $\Lambda_{1,1} = \epsilon - 0.630\epsilon^2 - 0.793\epsilon^3 - 1.248\epsilon^4 - 2.200\epsilon^5 - 4.156\epsilon^6 - 8.224\epsilon^7 - 16.829\epsilon^8 - 35.320\epsilon^9 - 75.610\epsilon^{10} - 164.459\epsilon^{11} - 362.419\epsilon^{12}$ and $\Lambda_{1,2} = -\epsilon + 0.019\epsilon^2 + 0.019\epsilon^3 + 0.028\epsilon^4 + 0.048\epsilon^5 + 0.090\epsilon^6 + 0.176\epsilon^7 + 0.359\epsilon^8 + 0.750\epsilon^9 + 1.601\epsilon^{10} + 3.472\epsilon^{11} + 7.636\epsilon^{12}$, with corresponding eigenvectors $\vec{E}_{1,1} = \binom{0}{1}$ and $\vec{E}_{1,2} = \binom{A_{1,2}}{1}$, with $A_{1,2} = -0.750 + 0.340\epsilon + 0.383\epsilon^2 + 0.566\epsilon^3 + 0.960\epsilon^4 + 1.765\epsilon^5 + 3.425\epsilon^6 + 6.905\epsilon^7 + 14.326\epsilon^8 + 30.386\epsilon^9 + 65.590\epsilon^{10} + 143.623\epsilon^{11} + 318.258\epsilon^{12}$.

For $\epsilon \neq 0$ (and sufficiently small), this is a Hermitian saddle fixed point.

(iii) $F_{2h} = 15.791\epsilon + 1.819\epsilon^2 + 1.646\epsilon^3 - 0.757\epsilon^4 + 0.405\epsilon^5 - 1.241\epsilon^6 + 0.643\epsilon^7 - 1.430\epsilon^8 + 1.411\epsilon^9 - 1.983\epsilon^{10} + 2.625\epsilon^{11} - 3.393\epsilon^{12}$, $F_{2u} = -58.121\epsilon + 16.812\epsilon^2 - 8.154\epsilon^3 + 16.338\epsilon^4 - 9.360\epsilon^5 + 17.343\epsilon^6 - 16.587\epsilon^7 + 23.178\epsilon^8 - 28.866\epsilon^9 + 37.721\epsilon^{10} - 50.784\epsilon^{11} + 67.832\epsilon^{12}$.

The stability matrix has eigenvalues $\Lambda_{2,1} = -2.408\epsilon - 0.601\epsilon^2 + 1.301\epsilon^3 - 0.089\epsilon^4 + 1.006\epsilon^5 - 0.593\epsilon^6 + 0.986\epsilon^7 - 1.204\epsilon^8 + 1.462\epsilon^9 - 2.076\epsilon^{10} + 2.641\epsilon^{11} - 3.691\epsilon^{12}$ and $\Lambda_{2,2} = \epsilon - 0.115\epsilon^2 - 0.159\epsilon^3 + 0.186\epsilon^4 - 0.072\epsilon^5 + 0.255\epsilon^6 - 0.173\epsilon^7 + 0.336\epsilon^8 - 0.383\epsilon^9 + 0.546\epsilon^{10} - 0.752\epsilon^{11} + 1.030\epsilon^{12}$, with corresponding eigenvectors $\vec{E}_{2,1} = \binom{A_{2,1}}{1}$ and $\vec{E}_{2,2} = \binom{A_{2,2}}{1}$, with $A_{2,1} = 0.015\epsilon - 0.013\epsilon^2 + 0.001\epsilon^3 - 0.013\epsilon^4 + 0.006\epsilon^5 - 0.015\epsilon^6 + 0.014\epsilon^7 - 0.022\epsilon^8 + 0.029\epsilon^9 - 0.039\epsilon^{10} + 0.055\epsilon^{11} - 0.077\epsilon^{12}$ and $A_{2,2} = -0.272 - 0.188\epsilon - 0.075\epsilon^2 - 0.159\epsilon^3 - 0.087\epsilon^4 - 0.178\epsilon^5 - 0.080\epsilon^6 - 0.209\epsilon^7 - 0.051\epsilon^8 - 0.262\epsilon^9 + 0.014\epsilon^{10} - 0.362\epsilon^{11} + 0.293\epsilon^{12}$.

For $\epsilon \neq 0$ (and sufficiently small), this is a non-Hermitian saddle fixed point.

(iv) $F_{3h} = 15.791\epsilon + 6.749\epsilon^2 - 3.314\epsilon^3 - 12.829\epsilon^4 - 11.559\epsilon^5 + 9.263\epsilon^6 + 37.770\epsilon^7 + 28.770\epsilon^8 - 64.624\epsilon^9 - 196.697\epsilon^{10} - 156.077\epsilon^{11} + 274.654\epsilon^{12}$, $F_{3u} = 68.648\epsilon + 29.392\epsilon^2 + 2.112\epsilon^3 - 11.144\epsilon^4 + 26.493\epsilon^5 + 143.046\epsilon^6 + 300.979\epsilon^7 + 383.667\epsilon^8 + 347.310\epsilon^9 + 566.087\epsilon^{10} + 2056.631\epsilon^{11} + 5955.454\epsilon^{12}$.

The stability matrix has eigenvalues $\Lambda_{3,1} = \epsilon - 0.427\epsilon^2 + 0.785\epsilon^3 + 1.460\epsilon^4 + 0.700\epsilon^5 - 1.668\epsilon^6 - 2.969\epsilon^7 + 2.758\epsilon^8 + 20.656\epsilon^9 + 48.759\epsilon^{10} + 86.232\epsilon^{11} + 188.086\epsilon^{12}$ and $\Lambda_{3,2} = 2.408\epsilon - 2.406\epsilon^2 - 3.775\epsilon^3 - 2.340\epsilon^4 + 1.815\epsilon^5 + 4.386\epsilon^6 - 3.621\epsilon^7 - 28.393\epsilon^8 - 59.880\epsilon^9 - 72.951\epsilon^{10} - 78.896\epsilon^{11} - 238.428\epsilon^{12}$, with corresponding eigenvectors $\vec{E}_{3,1} = \binom{A_{3,1}}{1}$ and $\vec{E}_{3,2} = \binom{A_{3,2}}{1}$, with $A_{3,1} = 0.230 - 0.000\epsilon - 0.244\epsilon^2 - 0.768\epsilon^3 - 1.951\epsilon^4 - 4.607\epsilon^5 - 10.748\epsilon^6 - 25.330\epsilon^7 - 60.213\epsilon^8 - 143.193\epsilon^9 - 339.680\epsilon^{10} - 806.636\epsilon^{11} - 1998.394\epsilon^{12}$ and $A_{3,2} = -0.018\epsilon + 0.019\epsilon^2 + 0.060\epsilon^3 + 0.141\epsilon^4 + 0.332\epsilon^5 + 0.867\epsilon^6 + 2.439\epsilon^7 + 6.966\epsilon^8 + 19.714\epsilon^9 + 55.425\epsilon^{10} + 156.092\epsilon^{11} + 441.899\epsilon^{12}$.

For $\epsilon \neq 0$ (and sufficiently small), this is a Hermitian IR-stable fixed point.

(v) $F_{4h} = 0$, $F_{4u} = 83.601 - 52.638\epsilon - 33.142\epsilon^2 - 41.735\epsilon^3 - 65.694\epsilon^4 - 115.816\epsilon^5 - 218.763\epsilon^6 - 432.896\epsilon^7 - 885.833\epsilon^8 - 1859.156\epsilon^9 - 3979.970\epsilon^{10} - 8656.771\epsilon^{11} - 19076.958\epsilon^{12}$.

The stability matrix has eigenvalues $\Lambda_{4,1} = -1.588 + 3.000\epsilon + 0.630\epsilon^2 + 0.793\epsilon^3 + 1.248\epsilon^4 + 2.200\epsilon^5 + 4.156\epsilon^6 + 8.224\epsilon^7 + 16.829\epsilon^8 + 35.320\epsilon^9 + 75.610\epsilon^{10} + 164.459\epsilon^{11} + 362.419\epsilon^{12}$ and $\Lambda_{4,2} = 0.028 - 1.024\epsilon - 0.019\epsilon^2 - 0.019\epsilon^3 - 0.028\epsilon^4 - 0.048\epsilon^5 - 0.090\epsilon^6 - 0.176\epsilon^7 - 0.359\epsilon^8 - 0.750\epsilon^9 - 1.601\epsilon^{10} - 3.472\epsilon^{11} - 7.636\epsilon^{12}$, with corresponding eigenvectors $\vec{E}_{4,1} = \binom{0}{1}$ and $\vec{E}_{4,2} = \binom{A_{4,2}}{1}$, with $A_{4,2} = 1.854 - 7.949\epsilon + 14.292\epsilon^2 - 28.867\epsilon^3 + 53.621\epsilon^4 - 107.027\epsilon^5 + 199.582\epsilon^6 - 398.417\epsilon^7 + 740.733\epsilon^8 - 1486.571\epsilon^9 + 2742.767\epsilon^{10} - 5559.741\epsilon^{11} + 10127.112\epsilon^{12}$.

For $\epsilon \neq 0$ (and sufficiently small), this is a Hermitian saddle fixed point.

APPENDIX B: ROBUSTNESS OF THE LOOP ANALYSIS

In this appendix, we examine the consistency of our results for the fixed points found in IV G, by varying the orders of loops.[13] In IV G, we gave, for example, the renormalization group flows for $\epsilon = 0.01$ as a representative flow for the case $3 + 2$ where the 3 refers to calculation of beta functions to 3 loops in the Yukawa coupling and 2 refers to 2 loops in the scalar self-coupling.

We report on the sensitivity of our results to loop order. The package RGBeta allows changes to the order of the loops. We compare the results for different loop orders: $1 + 1$, $2 + 1$, $2 + 2$ and $3 + 2$ in the Fig. 8 and focus on the fixed points that spawn from the origin in coupling constant space as ϵ is turned on.[14]

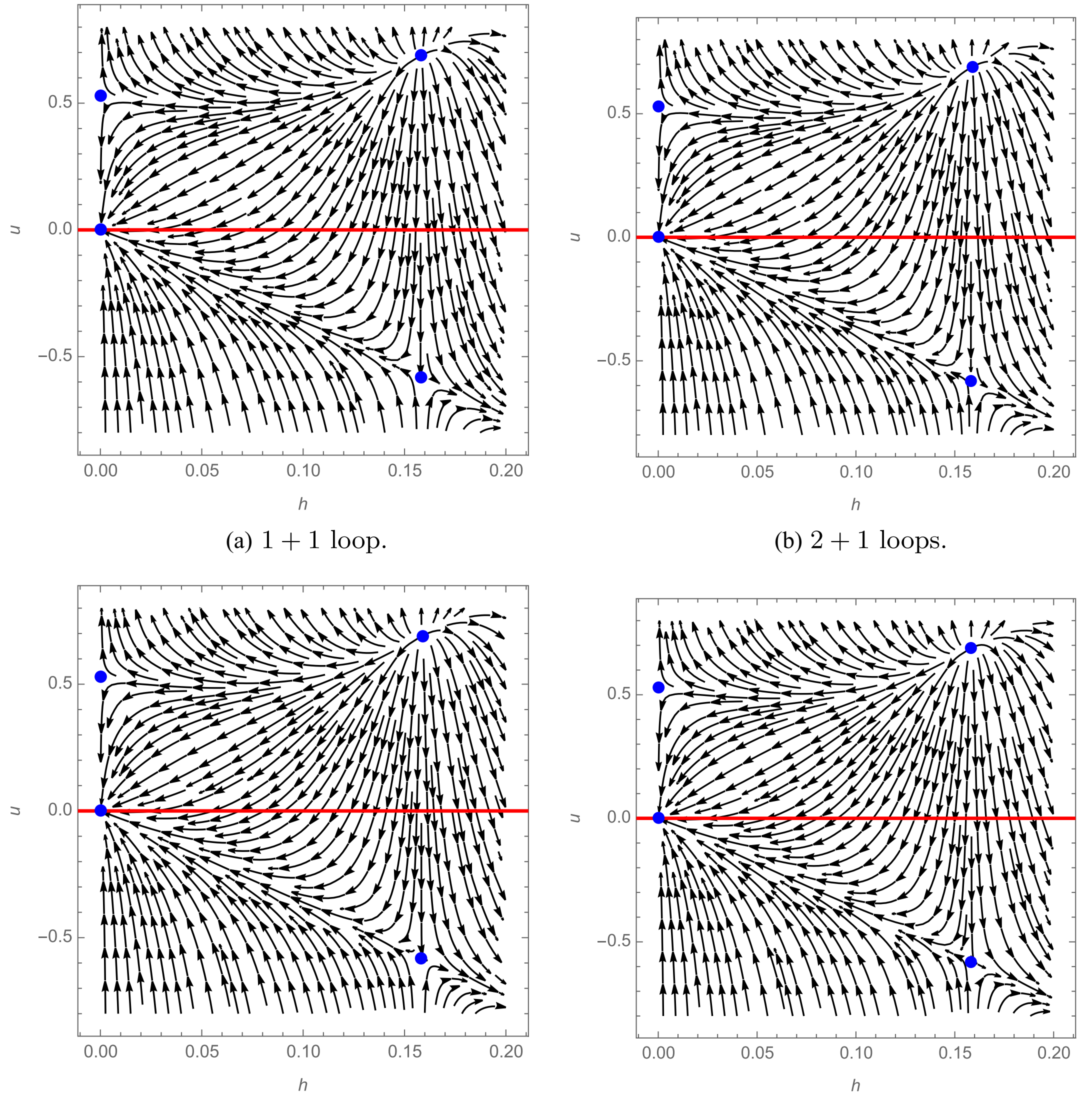
The resulting flows for $\epsilon = 0.01$ for the aforementioned loop orders are plotted in Fig. 8. Qualitatively, we observe that the flow diagrams in Fig. 8 appear very similar on changing the loop order. Quantitatively, in terms of h and u, the fixed points only vary at most with 1% relative difference, as we change the loop orders in the manner prescribed above. Since the magnitudes of the coupling constants at the fixed points are small, it is consistent that an increase of loop order only leads to small changes, i.e. the additional terms that enter into the beta functions are subdominant at this level. The changes of the fixed point couplings are more significant at the lower end of the loop orders (or equivalently the coupling constant values at the fixed points are more stable at the higher end of the loop orders).

Furthermore we can check whether this feature continues to hold as we begin to increase ϵ. To probe this, we consider $\epsilon = 0.1$ and perform the same analysis (through changing the loop orders) as above. The resulting flows are shown in Fig. 9. The flow diagrams in Fig. 9 remain similar as we change the loop order. The relative difference of the fixed point values vary at most by 5%, as we change the loop orders. This maximum relative difference is moderately strong for $\epsilon = 0.1$ compared to the corresponding result for $\epsilon = 0.01$, and so indeed the higher loop corrections to the beta function become more significant at larger ϵ (as we would expect). As before, the changes are more significant at the lower end of the loop orders.

We could continue increasing ϵ, but, as noted in Sec. IV G, there is a critical value $\epsilon_{c'} \sim 0.44$ beyond which the character of one of the ϵ-dependent fixed points change. By this point, the value of ϵ is likely too large to trust in the perturbative expansion in ϵ; and simultaneously the resulting ϵ-dependent fixed points spawning from the origin also become too large in magnitude to trust the perturbation theory.

[13]This procedure has also been advocated in [55].

[14]The other fixed points are too large for perturbation theory to be reliable.



(a) $1+1$ loop.

(b) $2+1$ loops.

(c) $2+2$ loops.

(d) $3+2$ loops.

FIG. 8. Flows (at the labeled loop orders) near the fixed points that spawn from the origin as ϵ is introduced, for $\epsilon = 0.01$.

Therefore, from these tests, we conclude that within the region of parameter space for which perturbation theory is likely to be valid, the results from Sec. IV G are robust.

We can also consider the robustness of the nonperturbative results in Sec. V. There, we find a putative nonperturbative $D = 3$ fixed point, which is non-Hermitian with saddle stability. Since we perform a Padé analysis and set $\epsilon = 1$, we can only consider the robustness of these results for different loop orders, taking $1+1$, $2+1$, $2+2$ and $3+2$ loops as above. We give the values of the coupling constants (h^*, u^*), the eigenvalues $\Lambda_{1,2}$ and the eigenvectors $E_{1,2}$, in each case as

(i) $1+1$ loops: $(h^*, u^*) = (15.8, -58.1)$, $\Lambda_1 = -2.41$, $\Lambda_2 = 1.00$, $E_1 = (0, 1)$, $E_2 = (-0.272, 1)$.

(ii) $2+1$ loops: $(h^*, u^*) = (17.9, -69.4)$, $\Lambda_1 = -3.00$, $\Lambda_2 = 0.873$, $E_1 = (0.0190, 1)$, $E_2 = (-0.250, 1)$.

(iii) $2+2$ loops: $(h^*, u^*) = (25.9, -76.7)$, $\Lambda_1 = -0.953$, $\Lambda_2 = 1.82$, $E_1 = (0.0858, 1)$, $E_2 = (-0.0826, 1)$.

(iv) $3+2$ loops: $(h^*, u^*) = (17.6, -32.3)$, $\Lambda_1 = -1.16$, $\Lambda_2 = 1.08$, $E_1 = (-0.0121, 1)$, $E_2 = (-4.21, 1)$.

The values of the coupling constants, and the eigenvalues and eigenvectors do not appear to be converging as the loop orders increase. We should bear in mind that the application to epsilon expansions of Padé approximants is long known not to be rigorous [43]. However, in the results the fixed points do remain of the same character in each case: Hermitian in h, but non-Hermitian in u. In all of the loop cases that we consider here, the relevant $D = 3$ fixed point is a non-Hermitian saddle. Hence this analysis is suggestive that there is a non-Hermitian fixed point which is of saddle type at $D = 3$. Only a nonperturbative analysis, perhaps using the functional renormalization group, can prove the existence of such a fixed point rigorously.

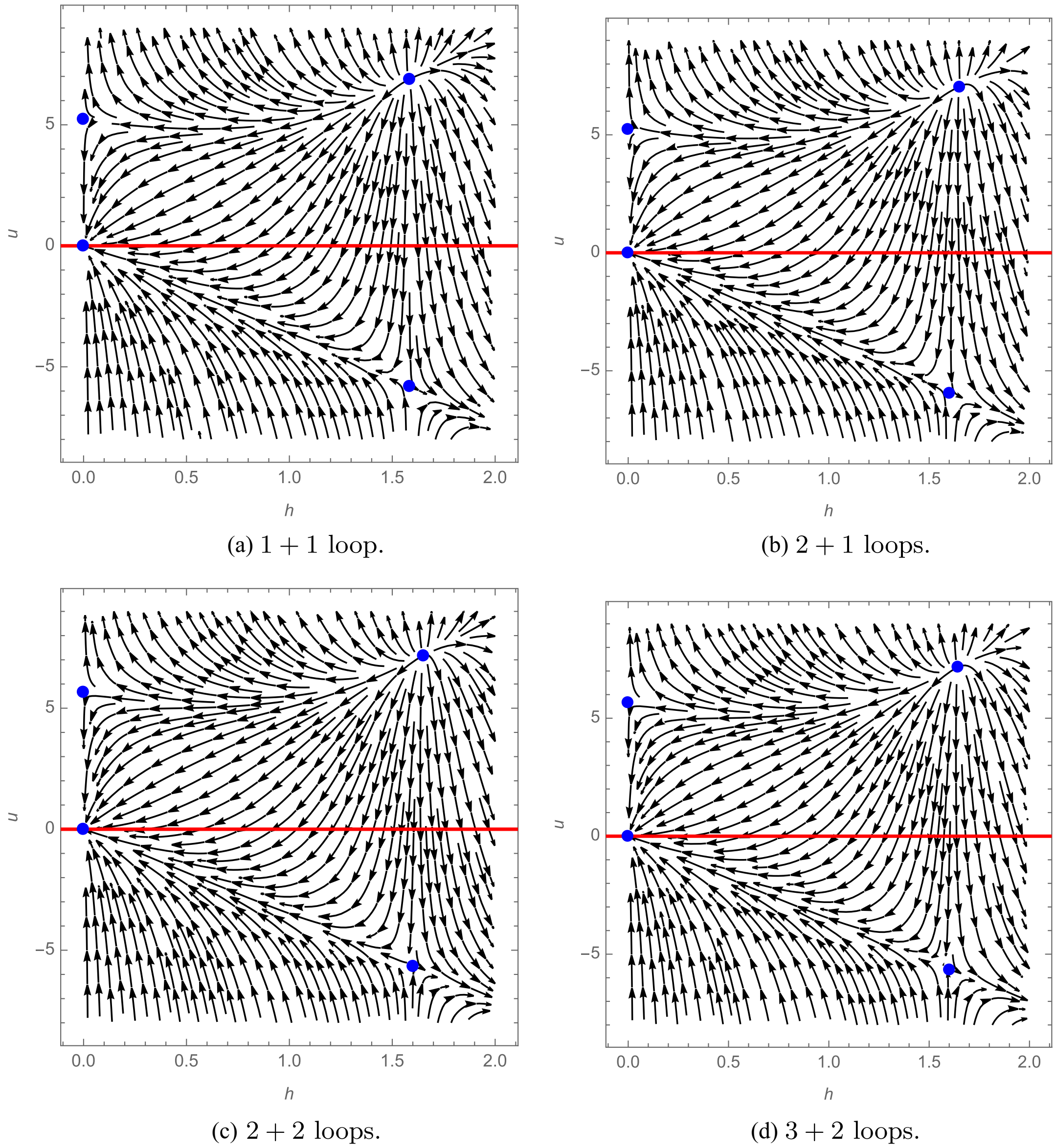


FIG. 9. Flows (at the labeled loop orders) near the fixed points that spawn from the origin as ϵ is introduced, for $\epsilon = 0.1$.

APPENDIX C: DISCUSSION OF $h < 0$

In the main text, we only consider $h \geq 0$ in our renormalization group analysis, where $h = g^2$, and g is the quartic self-coupling in the Lagrangian (1). In particular, the beta functions given in (51) and (52) are only valid for $h \geq 0$. When $h < 0$, g is imaginary, which causes alterations of the beta functions through their dependence generally on g and its complex conjugate $\bar{g}$. Instead, the beta functions in the $h < 0$ case are

$$\beta_h(h,u) = -\epsilon h - \frac{1}{(4\pi)^2} 10h^2 + \frac{1}{(4\pi)^4}\left(-\frac{57}{2}h^3 + 4h^2 u + \frac{1}{6}hu^2\right) + \frac{1}{(4\pi)^6}\left(\left[\frac{339}{8} - 222\zeta(3)\right]h^4 + 72h^3 u - \frac{61}{24}h^2u^2 - \frac{1}{8}hu^3\right) \tag{C1}$$

and

$$\beta_u(h,u) = -\epsilon u + \frac{1}{(4\pi)^2}(-48h^2 - 8hu + 3u^2) + \frac{1}{(4\pi)^4}\left(-384h^3 + 28h^2 u + 12hu^2 - \frac{17}{3}u^3\right). \tag{C2}$$

These $h < 0$ beta functions are identical to those given in (51) and (52) for $h \geq 0$, except for the relative signs between terms. However, the signs work out such that the differential equations (50) governing the renormalization group flows for $h < 0$ are the same as those for $h > 0$, but with $h \to -h$. This ultimately causes a $h \to -h$ reflection symmetry in the results.

We illustrate this in Fig. 10, showing the $h < 0$ results for

(i) The global flow for $\epsilon = 0$.
(ii) The global flow for $\epsilon = 0.01$.
(iii) The flows around the group of fixed points near the origin for $\epsilon = 0.01$.

which are the analogues of the $h \geq 0$ results presented in Figs. 3, 5, and 6, respectively. Indeed, the figures show identical results to the aforementioned $h \geq 0$ counterparts, except reflected in the vertical u-axis. Furthermore, no flows cross the vertical u-axis, so the $h \geq 0$ sector can essentially be considered independently of the $h < 0$ sector (and there is no flow from the Hermitian to non-Hermitian values of h, or vice-versa). For each fixed point with $h \geq 0$, there is an identical one with $h < 0$, with the same $|h|$, but opposite sign. The nature and stability of these fixed points are also preserved. For brevity, we therefore restrict to $h \geq 0$ in the main text.

However, some nontrivial comments should be made:

(i) In the case of $\epsilon = 0.01$, there are two fixed points with $h \neq 0$, shown in the last plot in Fig. 10. These are therefore both non-Hermitian fixed points in h. Of particular interest is the point with $h < 0$ and $u > 0$, which is non-Hermitian but also IR stable, which may be significant for dynamical mass generation [19].
(ii) The symmetry gives rise to another non-Hermitian (now both in g and u) saddle in the $D = 3$ Padé analysis with $(h^*, u^*) = (-17.6, -32.3)$.

We further note that the possibility of negative h in effective theories has been motivated previously [14] in terms of a microscopic picture. The picture is string inspired and is motivated by a mathematical ambiguity in continuing from an Euclidean to a Minkowski formulation. After compactification to four dimensions, the closed string sector of heterotic superstring theory [92,93] consists of spin 0 dilaton field Φ, spin 2 graviton field $g_{\mu\nu}$ and spin 1 antisymmetric gauge field tensor $B_{\mu\nu}$, the Kalb-Ramond field. To lowest order in the string Regge slope α', the Euclidean effective action of the closed bosonic string is

$$S_B = -\int d^4x\sqrt{-g}\left(\frac{1}{2\kappa^2}R + \frac{1}{6}\mathcal{H}_{\lambda\mu\nu}\mathcal{H}^{\lambda\mu\nu} + \ldots\right) \qquad \text{(C3)}$$

where

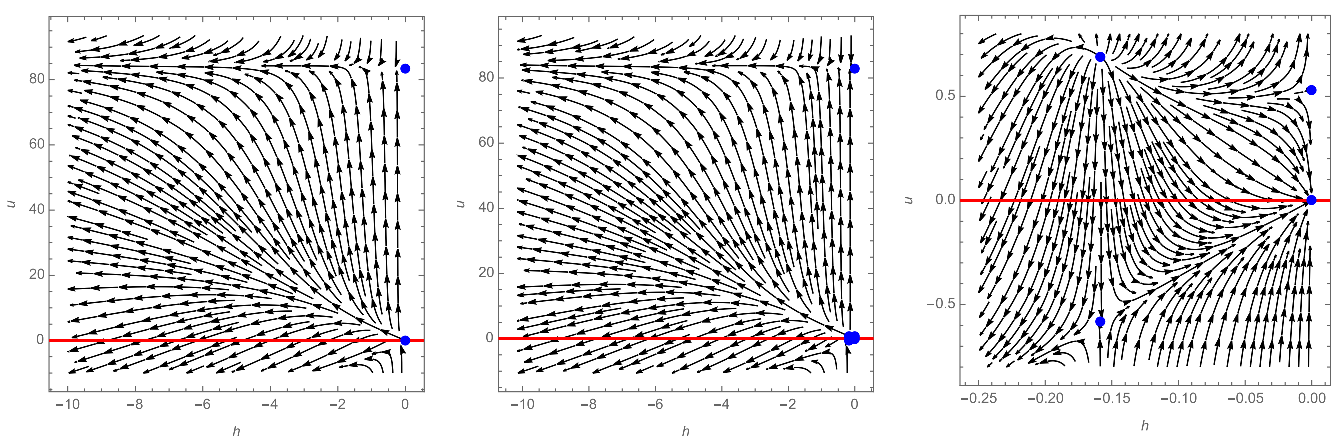
$$\mathcal{H}_{\mu\nu\rho}(x) = \partial_{[\mu}B_{\nu\rho]}, \qquad \text{(C4)}$$

R is the Ricci scalar, $\kappa = \frac{\sqrt{8\pi}}{M_P}$, M_P is the Planck mass, and g is the determinant of $g_{\mu\nu}$. To this order in the expansion in α', S_B can be interpreted as a modified gravity theory with torsion [94,95] where the usual metric based connection $\Gamma^{\rho}{}_{\mu\nu}$ is replaced by

$$\bar{\Gamma}^{\rho}{}_{\mu\nu} = \Gamma^{\rho}{}_{\mu\nu} + \frac{\kappa}{\sqrt{3}}\mathcal{H}^{\rho}{}_{\mu\nu} \neq \bar{\Gamma}^{\rho}{}_{\nu\mu}. \qquad \text{(C5)}$$

For the heterotic string the Bianchi identity is

$$\begin{aligned}\epsilon^{\mu\nu\rho\sigma}\mathcal{H}_{[\nu\rho\sigma;\mu]} &= \frac{\alpha'}{32\kappa}\sqrt{-g}(R_{\mu\nu\rho\sigma}\tilde{R}^{\mu\nu\rho\sigma} - F^a_{\mu\nu}\tilde{F}^{a\mu\nu}) \\ &\equiv \sqrt{-g}\mathcal{G}(\omega, \mathbf{A}) \qquad \text{(C6)}\end{aligned}$$



(a) The global flow for $\epsilon = 0$. (b) The global flow for $\epsilon = 0.01$. (c) The flows around the group of fixed points near the origin for $\epsilon = 0.01$.

FIG. 10. Some flow diagrams for $h < 0$.

where $\mathbf{A}^a$ is a Yang-Mills gauge field with a Latin group index a and

$$\tilde{R}_{\mu\nu\rho\sigma} = \frac{1}{2}\epsilon_{\mu\nu\lambda\pi}R^{\lambda\pi}{}_{\rho\sigma}, \qquad \tilde{F}^a{}_{\mu} = \frac{1}{2}\epsilon_{\mu\nu\lambda\pi}F^{a\lambda\pi}. \tag{C7}$$

with

$$\epsilon^{\mu\nu\rho\sigma} = \frac{\mathrm{sgn}(g)}{\sqrt{-g}}\eta^{\mu\nu\rho\sigma} \tag{C8}$$

and $\eta^{\mu\nu\rho\sigma}$ is the *flat space* Levi-Civita symbol with $\eta^{0123} = 1$. The Bianchi identity is implemented in the path integral Z_B through a delta function:

$$Z_B = \int D\mathcal{H}\exp\left(-S_B\right)\prod_x \delta(\eta^{\mu\nu\rho\sigma}\mathcal{H}_{[\nu\rho\sigma;\mu]}(x) - \mathcal{G}(\omega, \mathbf{A})). \tag{C9}$$

The axion field $b(x)$ appears as a Lagrange multiplier field implementing the delta function

$$\int Db\exp\left[-i\int d^4x\sqrt{-g(x)}\left(\frac{1}{\sqrt{3}}\partial^\mu b(x)\eta_{\mu\nu\rho\sigma}H^{\nu\rho\sigma}(x) + \frac{b}{\sqrt{3}}\mathcal{G}(\omega, \mathbf{A})\right)\right] \tag{C10}$$

On integrating over $\mathcal{H}$, Z_B becomes

$$Z_B = \int db\exp\left(-\int d^4x\sqrt{g^{(E)}}\left\{\frac{1}{2\kappa^2}R + \frac{1}{12}\eta^{(E)}_{\mu\nu\rho\lambda}\eta^{\mu\nu\rho\sigma(E)}\partial^\lambda b\partial_\sigma b + \frac{b}{\sqrt{3}}\mathcal{G}(\omega, \mathbf{A})\right\}\right). \tag{C11}$$

The Euclidean formulation is emphasized by using the superscript (E). There is an ambiguity (or ordering issue) [96] on continuing back from Euclidean to Minkowski space. In [14] it was stressed that one has two choices:

(1) Before continuing back to Minkowski space we can replace $\eta^{(E)}_{\mu\nu\rho\lambda}\eta^{\mu\nu\rho\sigma(E)}$ with $6\delta^\sigma_\lambda$.
(2) After continuing back to Minkowski space we can replace $\eta^{(E)}_{\mu\nu\rho\lambda}\eta^{\mu\nu\rho\sigma(E)}$ with $-6\delta^\sigma_\lambda (= \eta_{\mu\nu\rho\lambda}\eta^{\mu\nu\rho\sigma})$ and also redefine the phase of b by $\pi/2$ in order to get the canonical sign for the kinetic term. This leads to the redefinition $b \to ib$. A Hermitian b transforms as $\mathcal{T}: b \to -b$ [29]; hence with the field redefinition we get the transformation in (4).

On introducing fermions the above ambiguity leads to a Yukawa term

$$\mathcal{S}_{\mathrm{b-F}} = \mathrm{const}\times\int d^4x\sqrt{-g}i^\xi b(x)\nabla_\mu(\overline{\psi}\gamma^5\gamma^\mu\psi), \tag{C12}$$

with $\xi = 0$ or 1, depending on the way we analytically continue. Consequently it is not surprising that we did not find any renormalization group flow between the Hermitian and non-Hermitian sectors of the Yukawa coupling constant g.

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