Landau singularities and higher-order polynomial roots

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Landau's work on the singularities of Feynman diagrams suggests that they can only be of three types: either poles, logarithmic divergences, or the roots of *quadratic* polynomials. On the other hand, many Feynman integrals exist whose singularities involve arbitrarily higher-order polynomial roots. We investigate this apparent paradox using concrete examples involving cube roots and roots of a degree-eight polynomial in four dimensions and roots of a degree-six polynomial in two dimensions and suggest that these higher-order singularities can only be approached via kinematic limits of higher codimension than one, thus evading Landau's argument.

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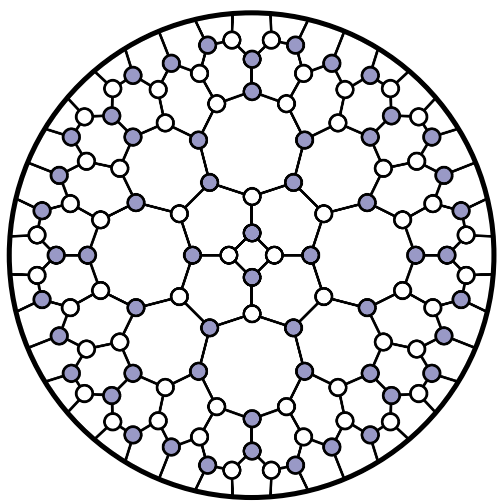
I. INTRODUCTION

This paper tries to reconcile two seemingly contradictory facts. First, there is the folk wisdom that Feynman diagrams have only three types of singularities: poles, square roots, and logarithms. This folk wisdom can be traced back to Landau, who demonstrated it in Ref. [1], though his argument included one oft-forgotten caveat: the restriction only holds for singularities approachable at kinematic configurations of *codimension one*. Second, we now have something Landau did not: an extensive body of work on the singularities of perturbative scattering amplitudes—especially in the case of planar, maximally supersymmetric ($\mathcal{N}=4$) super Yang-Mills (sYM) [2–10]—where examples are known to involve roots of arbitrarily high-order polynomials in the kinematics. Consider for example the following on-shell diagram (Fig. 1) involving 40 external particles at 37 loops:



(1)

This diagram encodes a function where all internal lines are taken to be on shell, with the particular parities of each three-point vertex indicated in the diagram. But it turns out that there are 2^{10}(!) *distinct* solutions to the cut conditions with this particular set of three-point parities—with each encoding a different, specific on-shell function involving the various roots of some 1024th-degree polynomial [9,10].

Simpler examples of higher-than-quadratic roots arise at much lower loop orders. In massless theories, nested quadratic roots make their first appearance at two loops, but the first example of an irreducible cubic root appears at three loops involving 11 massless particles. For massive Feynman integrals, integrals involving sextic roots also appear at three loops. This raises a natural question: do these examples contradict Landau's analysis? Or if not, why not?

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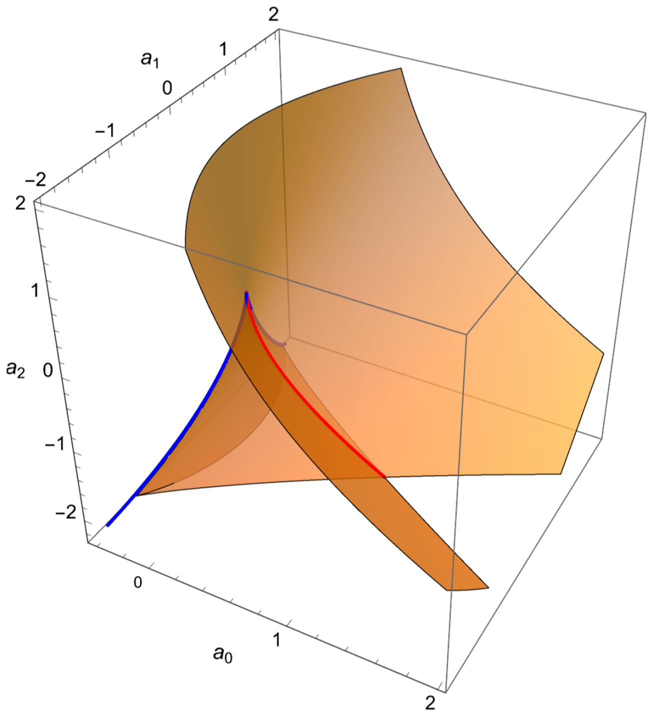


FIG. 1. The swallow-tail singularity of Eq. (12). The blue curves depict the locus where three roots coincide, while the red curve depicts the locus where two pairs of roots coincide. At the point where these curves intersect, all roots coincide.

In this work, we suggest a simple resolution to the apparent paradox: cubic- (or higher-) root singularities can and do occur in Feynman integrals but correspond to singularities of amplitudes at codimension two or higher. Specifically, this means that such roots cannot be accessed by any *codimension-one* kinematic limit. To access such higher-order singularities, one may first consider starting from a set of restricted kinematics in which some discriminant locus vanishes. This may be possible in some cases but is not in the cases we investigate here: the discriminant locus is not merely vanishing, but singular in these restricted kinematics, which makes further monodromies ill defined.

Our work is organized as follows. We begin in Sec. II with some useful background material and review; specifically, we review how one finds singularities via the Landau equations in Sec. II A, and revisit Landau's argument regarding the types of singularities that arise in Sec. II B, before finishing the section by discussing some useful facts about the roots of higher-degree polynomials. We then study the implications of these ideas for three concrete examples in considerable detail. In Sec. III we describe the first example of a cube root arising in the case of theories of massless particles in four dimensions. We show how this root is required when one tries to represent the on-shell space of the diagram in terms of a rational parametrization of the external kinematics and how any codimension-one kinematic limit will nonetheless isolate a pole, not a cubic root. We also discuss how one could set up the Landau equations in this kinematic parametrization. In Sec. IV we discuss a similar example with more generic kinematics, where the on-shell space is described by two degree-eight polynomials. In this example the Landau equations can be fully solved, and we describe a parametrization of the kinematics which rationalizes the roots present, analogously to the way one can rationalize the square root in the one-loop box. Finally, in Sec. V we describe an example involving sextic roots that arises for massive theories in two dimensions, where we can once again solve the Landau equations.

II. PRELIMINARIES

A. Where do amplitudes have singularities?

It has been known since the work of Landau (see Ref. [1] but also Ref. [11]) that the singularities of scattering amplitudes or Feynman integrals are given by the solutions to Landau equations. The scalar Feynman integral

$$I(p) = \int \frac{d^n k}{s_1(p,k)\dots s_m(p,k)} \tag{2}$$

has singularities when a subset of the denominators vanish *and* there exist a not-all-vanishing set $\alpha_1, \dots, \alpha_m$ such that

$$\sum_{e=1}^{m} \alpha_e d s_e(p,k) = 0, \tag{3}$$

where the differential is taken with respect to the integration variables k only.

The perhaps more familiar form of the Landau equations is obtained by using $s_e(p,k) = q_e^2 - m_e^2$ where q_e is a linear combination of independent external momenta $p \coloneqq \{p_a\}$ and independent loop momenta k which depends on the graph we are studying. Then Eq. (3) yields a vector equation for each independent loop momentum, which are normally called the Landau equations.

Mathematically, the condition in Eq. (3) arises as a critical value condition for a map between the on-shell space defined by the vanishing of a subset of denominators and the space of external momenta (see Refs. [12–14]). Reference [15] introduced a physical way of understanding Landau diagrams as real scattering processes of on-shell particles—where the Landau equations arise as the condition for the closure of the loops.

B. What kind of singularities can amplitudes have?

In the original paper Ref. [1], Landau studied not only the location of the singularities, but also their nature: how scattering amplitudes behave when approaching these singularities. Similar results were obtained by Leray in

Ref. [11] (see also a review of Leray's results in Ref. [12], p. 109, Chap. VI). A more detailed derivation following Landau's original idea was presented in Ref. [16]. The asymptotic expansion around these singularities has been used recently in Ref. [17] to impose constraints on the symbol of polylogarithmic integrals.

The derivation of the behavior of a Feynman integral in the neighborhood of a Landau singularity is slightly too technical for our current needs, so we will not present it in detail. Instead, we refer the reader to the result of Ref. [12] (which uses slightly different notation from ours).

Consider an integral of the form [18]

$$I(p) = \int \frac{\omega}{\prod_{e=1}^{m} s_e(p,k)^{\delta_e}}, \tag{4}$$

where $\delta_1, \ldots, \delta_m > 0$ are integers and ω is some holomorphic differential n-form on the k's independent of the external momenta p. We will find it useful to define $\delta \coloneqq \sum_{e=1}^{m} \delta_e$ [19].

The integral is regular unless the contour of integration is trapped (or pinched) by the singularity hypersurfaces $s_1(p,k) = \cdots = s_m(p,k) = 0$, so that it cannot be deformed away from them. A detailed discussion of the geometry of the contour pinching can be found in Ref. [12]. The integral is potentially singular for p such that we have a set of not-all-vanishing variables $\alpha_1, \ldots, \alpha_m$ such that there exists a function $\ell(p)$ with

$$d\ell(p) = \sum_{e=1}^{m} \alpha_e ds_e(p,k), \tag{5}$$

where now on the right-hand side the differential is taken with respect to both p and k. The left-hand side is independent of k by virtue of Eq. (3). We can then choose a constant factor so that the equation of the Landau singularities is $\ell(p) = 0$ (see Ref. [14] for more details).

Some technical assumptions are necessary here. First, we assume that the Landau locus $\ell(p) = 0$ arises from a *single* Landau diagram.

Second, we take $p \to p^*$ where p^* is smooth point of the Landau locus $[\ell(p^*) = 0]$. This is necessary to be able to define homotopy paths going around the complexified Landau locus. Roughly, one needs to be able to define a complex transversal space where the Landau locus sits at the origin. Then in this transversal space one can construct a homotopy path as a loop around the origin. However, at a singular point the tangent space is not well defined and therefore one can not define a transversal space either. The Landau loci are themselves generically singular so this restriction is necessary (we present an example at the end of Sec. II D). Third, the pinching has to happen for a unique value of internal momenta. Related to this, a certain Hessian (see Ref. [17], Appendix E, for a more detailed discussion) which appears in the denominator of the coefficient A below must not vanish.

Finally, we will approach the Landau locus only at generic points, which are not also singular points for *other* Landau singularities. Indeed, Landau loci generically intersect other Landau loci. These intersections have been the subject of study, for example in connection with Steinmann relations (see Ref. [17]).

In this notation and under the conditions described above, we have the following asymptotic behavior for $I(p)$ when $\ell(p) \to 0$:

(1) if $n+m-1$ is odd, then

$$I(p) = -\frac{N}{2}\frac{(2\pi i)^m A(p) \prod_{i=1}^{m}(-\alpha_i)^{\delta_i}}{\prod_{i=1}^{m}(\alpha_i - 1)!}\frac{\ell(p)^{\frac{n+m-1}{2}-\delta}}{\Gamma(1+\frac{n+m-1}{2}-\delta)}(1+o(\ell(p))) + \mathrm{hf}; \tag{6}$$

(2) if $n+m-1$ is even and $n+m-1 \geq 2\delta$, then

$$I(p) = N\frac{(2\pi i)^{m-1} A(p) \prod_{i=1}^{m}(-\alpha_i)^{\delta_i}}{\prod_{i=1}^{m}(\alpha_i - 1)!}\frac{\ell(p)^{\frac{n+m-1}{2}-\delta}}{(\frac{n+m-1}{2}-\delta)!}\log(\ell(p))(1+o(\ell(p))) + \mathrm{hf}; \tag{7}$$

(3) if $n+m-1$ is even and $n+m-1 < 2\delta$, then

$$I(p) = -N\frac{(2\pi i)^{m-1} A(p) \prod_{i=1}^{m}(-\alpha_i)^{\delta_i}}{\prod_{i=1}^{m}(\alpha_i - 1)!}\frac{(-\frac{n+m-1}{2}+\delta-1)!}{(-\ell(p))^{-\frac{n+m-1}{2}+\delta}}(1+o(\ell(p))) + N\log(\ell(p))\mathrm{hf} + \mathrm{hf}'; \tag{8}$$

(4) if $m = n+1$, we have the more specific result that

$$I(p) = (-1)^{n+1} N\frac{(2\pi i)^{n} A(p) \prod_{i=1}^{m}\alpha_i^{\delta_i}}{\prod_{i=1}^{m}(\alpha_i - 1)!}\frac{(\delta-n-1)!}{\ell(p)^{\delta-n}}(1+o(\ell(p))) + \mathrm{hf}. \tag{9}$$

Here, n is the number of integrations [see Eq. (2)], "hf" (and hf$'$) denote any holomorphic function and $o(\ell(p))$ is any holomorphic function which vanishes at the Landau locus. The quantity $A(p)$ is essentially the inverse of the square root of a Hessian, while N is an intersection index which is purely numerical. Explicit expressions can be given for N and $A(p)$, but we will not need their detailed form in what follows.

From this discussion we may conclude that there are three basic types of singularities that may arise in the neighborhood of a Landau locus: polar, square root, and logarithmic. The ramification type of each in the neighborhood of a Landau locus can also be determined from a homological analysis, as was done in Ref. [12], p. 96, Sec. 2.6.

C. Nonexamples

Let us list a number of cases which might look like they violate the result in the previous subsection but do not satisfy the conditions we require.

First, one might think about removing the square-root singularities by a change of coordinates, which amounts to going to a double cover. Such transformations are sometimes useful but we want to study the singularities in the original coordinates, which can be either Mandelstam invariants or momentum components in a special frame.

We do not study second-type singularities, for which the pinch happens at infinite values of loop momenta. In principle this case can also be studied after an appropriate compactification of the internal kinematic space has been made. Then one can change coordinates to parametrize the points at infinity and do the same analysis. The numerator enters in an essential way in this analysis; for pinches at finite values of momenta the numerators can only cancel singularities, not create new ones.

Another seeming counterexample is that of the massless box in six dimensions. This is a finite integral, which up to a global factor can be computed to be

$$\frac{\log^2 \frac{s}{t} + \pi^2}{s+t}. \tag{10}$$

It might look like terms of type $\log^2$ contradict the general results in Eqs. (6)–(9).

However, this example violates several of the requirements we have imposed above. First, the kinematics is such that the momentum of each external massless particle is at the threshold. Therefore, the integral has a permanent pinch. For integrals which have such permanent pinches even the *existence* of an analytic continuation in kinematic variables is not guaranteed. Another problem is that $s = 0$ or $t = 0$ correspond to multiple Landau loci. They arise in bubble Landau singularities as well as in triangle and box singularities. The bubble and triangle singularities are particularly troublesome since the pinching in internal momenta does not happen for some fixed values of the momenta, but for a one-parameter family of values. This precludes the usual construction of vanishing homology classes which arise in the application of Picard-Lefschetz theory (see Ref. [12] and also Ref. [17]).

Despite these problems, at least the prefactor of the integral (which is a second-type singularity in six dimensions) can be analyzed in dimensional regularization and computed to all orders in ϵ. It is also interesting to note that, since this second-type singularity cannot appear in the physical region, when $s = -t$ the numerator vanishes so as to cancel the pole. This uniquely fixes the π^2 term once we know there is a $\log^2 \frac{s}{t}$ term. We believe this kind of constraint should provide a handle on "beyond the symbol terms" (or "initial conditions" in a differential equations language).

In general, we expect that cases such as these should give rise to higher powers of the singularities described in the previous section (higher poles, powers of square roots, and powers of logarithms), as they correspond to factorizable singularities. We do not expect them to give rise to new types of singularities, such as higher roots.

D. Roots of polynomial equations

Consider what happens to the roots of an algebraic equation when taking its coefficients along some closed path. As the coefficients change, the various roots follow always continuous paths themselves and return to the same locations as at the beginning—but possibly up to permutation.

If we have a square-root singularity at codimension one, this means that going *twice* around the singularity must always return the roots to where they began: if they had been exchanged in going once around, then going twice would return them to their initial locations.

When the discriminant of their defining polynomial vanishes, the two roots coincide, and the singularity is therefore at codimension one.

A cubic-root singularity would correspond to a situation where *three* roots simultaneously coincide. This however can only happen at higher than codimension one for an intuitive reason: the equality of any pair of roots imposes one constraint on the coefficients of the polynomial, and collapsing the pair with the remaining one requires one further constraint. Thus, all three roots collide only at codimension two.

Consider for example the depressed cubic equation $x^3 + px + q = 0$. Its discriminant is (up to a sign) $\Delta = 4p^3 + 27q^2$. $\Delta = 0$ implies only that two of the roots have collided. One could easily make all three collide by requiring $p = q = 0$, but this demand clearly constitutes two conditions. Incidentally, the vanishing of the discriminant locus is itself a cuspidal cubic in terms of p, q, and it has a cuspidal singularity when $p = q = 0$—precisely at the locus where the three roots coincide.

The case of a general cubic hypersurface $0 = x^3 + a_2x^2 + a_1x + a_0$ can be understood similarly. If two roots coincide and equal u while the third equals v we have $2u + v = -a_2$, $u^2 + 2uv = a_1$ and $u^2v = -a_0$. After eliminating u and v we find

$$a_1^2a_2^2 - 4a_0a_2^3 - 4a_1^3 + 18a_0a_1a_2 - 27a_0^3 = 0. \tag{11}$$

It turns out that this surface has a singularity when three roots coincide. To see this, we set $v = u$ and after eliminating u we find $a_0 = \frac{a_2^3}{27}$ and $a_1 = \frac{a_2^2}{3}$, which is the equation of a rational normal curve. It is now easy to see that if we take the differential of Eq. (11) we find zero, upon replacing the solutions for a_0 and a_1 in terms of a_2 [21]. This means that the surface defined by Eq. (11) is singular along the curve where three roots coincide.

To illustrate the complexities that can arise for higher degrees, let us discuss the quartic equation. To find when the depressed quartic $x^4 + a_2x^2 + a_1x + a_0 = 0$ has multiple solutions we take the roots r_1, r_2, r_3, r_4, which are such that $r_1 + r_2 + r_3 + r_4 = 0$. If we set $r_3 = r_4$ we can solve $r_3 = r_4 = -\frac{1}{2}(r_1 + r_2)$. Plugging back in the expressions for a_0, a_1, a_2 and eliminating r_1 and r_2 we find

$$4a_1^2a_2^3 - 16a_0a_2^4 + 27a_1^4 - 144a_0a_1^2a_2 + 128a_0^2a_2^2 - 256a_0^3 = 0. \tag{12}$$

For the quartic we can also have three roots coinciding or two pairs of roots coinciding. In general, the type of coincident root locus is specified by a partition of the degree of the equation. In the case (3, 1), that is when three roots coincide, we have

$$a_2^2 + 12a_0 = 0, \qquad 9a_1^2 - 32a_0a_2 = 0. \tag{13}$$

In the case (2, 2), that is when two pairs of roots coincide, we have

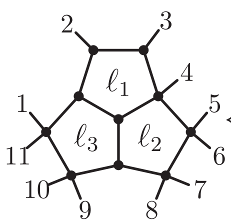
$$a_1 = 0, \qquad a_2^2 - 4a_0 = 0. \tag{14}$$

Again, it can explicitly checked that the differential of Eq. (12) vanishes along the (3, 1) locus and along the (2, 2) locus, which means that the surface defined by Eq. (12) is singular there.

The (3, 1) locus intersects the (2, 2) locus in the (4) locus, where all the roots coincide. In the case of the depressed quartic, this means that all the roots vanish (since their sum vanishes). In the neighborhood where all four roots coincide the vanishing discriminant locus has a swallowtail singularity.

III. *EXEMPLI GRATIA*: A CUBE ROOT AT THREE LOOPS

Consider the scalar three-loop scalar Feynman integral



$$\Leftrightarrow \frac{d^4\ell_1 d^4\ell_2 d^4\ell_3}{(\ell_1|2)(\ell_1|3)(\ell_1|4)(\ell_1|\ell_2)(\ell_2|5)(\ell_2|7)(\ell_2|9)(\ell_2|\ell_3)(\ell_3|9)(\ell_3|11)(\ell_3|2)(\ell_3|\ell_1)},$$

where we have used dual momentum coordinates with $p_a =: x_{a+1} - x_a$ and where we have defined the shorthand $(a|b) \coloneqq (x_a - x_b)^2$. This integral contributes to 11-particle amplitudes in both pure and maximally supersymmetric ($\mathcal{N} = 4$) super Yang-Mills theory.

Here, we will establish that, in order to represent this integral in a reasonable parametrization of the kinematic space, we require cubic roots. We will primarily establish this by examining the on-shell space, and in particular the value of the integral on this space (i.e. at the leading singularity). We will confirm this by parametrizing the integrals in a nonredundant, rational manner, to show that this is not just an artifact of a poor choice of parametrization. However, as suggested by the discussion in Sec. II, we will find that in this nonredundant parametrization it is impossible to approach these roots in a codimension-one limit [22]. Having established what we set out to, we briefly discuss how one might set up the problem of determining the Landau singularities of the diagram, leaving a full derivation for future work.

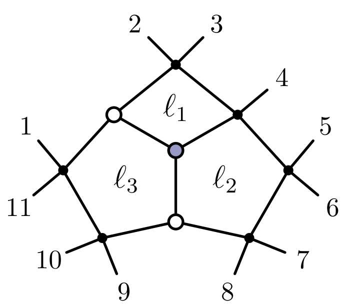
A. On-shell space

To begin, let us discuss the on-shell space. This integral has 16 locations in the (complexified) space of loop momenta which put all its 12 propagators on shell—corresponding to the solutions to its “maximal-cut” equations. These solutions may be organized according to which *particular* solution to the cut equations is involved at each three-particle vertex, which can take one of two forms: with all spinor-helicity variables λ of the participating on-shell momenta being proportional (white) or all their conjugate

$\tilde{\lambda}$'s proportional (blue). When represented in momentum-twistor space [23], these correspond to the cases where the lines (representing the loop momenta ℓ_i and the external dual-momentum points x_a) pass through each other at a single point (white) or when they are coplanar (blue).

As there is a three-particle vertex in the middle of the loop integrand, we can divide the 16 solutions into two groups of eight according to whether the lines representing (ℓ_i) in momentum-twistor space intersect at a point or not (white and blue, respectively). Of these eight, there are three rational solutions, two (a pair) that involve quadratic roots, and three which involve the roots of an irreducible cubic. We would like to analyze and describe these cube-root solutions to the maximal cut equations explicitly.

Let us start with a concrete parametrization of a subset of solutions to the *next-to-maximal* cut [that for which $(\ell_1|3) \neq 0$)—those associated with the parity and coloring of three-particle vertices as indicated in the following contour diagram:



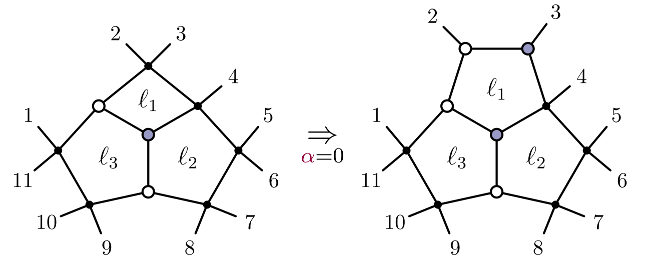
(15)

We may parametrize this one-dimensional family of solutions to the cut equations in momentum-twistor space by writing $(\ell_1) \coloneqq (\hat{2}\,\hat{4})$, $(\ell_2) \coloneqq (\hat{5}\,\hat{9})$, $(\ell_3) \coloneqq (\hat{9}\,\hat{2})$, where

$$\hat{2}(\alpha) \coloneqq z_2 + \alpha z_1, \qquad \hat{5}(\alpha) \coloneqq (5\,4) \cap (6\,7\,\hat{9}),$$
$$\hat{4}(\alpha) \coloneqq (4\,3) \cap (\hat{5}\,\hat{9}\,\hat{2}), \qquad \hat{9}(\alpha) \coloneqq (9\,8) \cap (10\,11\,\hat{2}). \qquad (16)$$

It is not hard to verify that this one-parameter family of $\ell_i(\alpha)$ satisfies the 11 cut equations and that they correspond to the particular branch indicated by the coloring of vertices in (15). Readers interested in the details on how such a parametrization can be constructed may consult Appendix A.

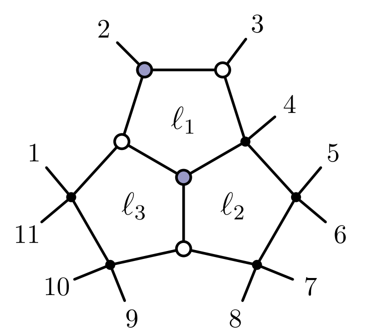
To access the *leading* singularities (that is the singularities arising from the pinch of all the internal lines) of the initial integral from this 11-cut, we must cut the final propagator—that is, take residues about the solutions to $(\ell_1|3) \propto \langle \ell_1 2\,3\rangle = \langle \hat{2}\,\hat{4}\,2\,3\rangle = \alpha\langle 1\,\hat{4}\,2\,3\rangle = 0$. There are four solutions to this final-cut equation: one rational ($\alpha = 0$) and three roots of an irreducible cubic. The $\alpha = 0$ solution corresponds to a rational leading singularity with the following coloring of vertices:



(17)

To see this, notice that when $\alpha = 0$, the line in momentum-twistor space corresponding to $(\ell_1) \coloneqq (\hat{2}\,\hat{4})$ [as parametrized in (16)] passes directly through the point z_2 (as $\hat{2} \to z_2$ when $\alpha \to 0$).

We are interested in the *other* leading singularities—those involving the three solutions to the final-cut equation, where $\langle 1\hat{4}23\rangle = \langle 123(43) \cap (\hat{5}\hat{9}\hat{2})\rangle \propto \langle \hat{2}\hat{5}\hat{9}3\rangle = 0$, which would correspond to leading singularities associated with the following contour diagram:



(18)

It is not hard to check that this final-cut equation is (irreducibly) cubic in the parameter α. In particular, it is given by

$$q(\alpha) \coloneqq \langle \hat{2}\,\hat{5}\,\hat{9}\,3\rangle \eqqcolon c_0 + c_1\alpha + c_2\alpha^2 + c_3\alpha^3 \qquad (19)$$

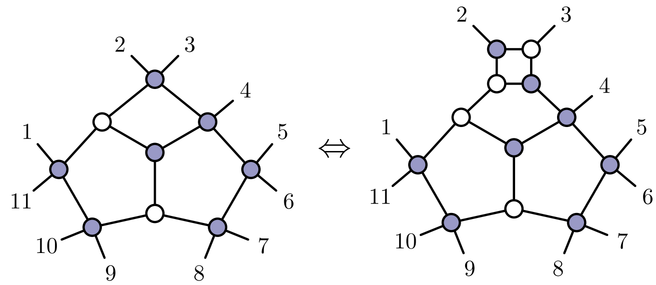
with

$$c_0 \coloneqq \langle 2\,3\,\hat{5}_2\,\hat{9}_2\rangle, \quad c_1 \coloneqq \langle 1\,3\,\hat{5}_2\,\hat{9}_2\rangle + \langle 2\,3\,\hat{5}_1\,\hat{9}_2\rangle + \langle 2\,3\,\hat{5}_2\,\hat{9}_1\rangle,$$
$$c_2 \coloneqq \langle 2\,3\,\hat{5}_1\,\hat{9}_1\rangle + \langle 1\,3\,\hat{5}_2\,\hat{9}_1\rangle + \langle 1\,3\,\hat{5}_1\,\hat{9}_2\rangle, \quad c_3 \coloneqq \langle 1\,3\,\hat{5}_1\,\hat{9}_1\rangle, \qquad (20)$$

where we have introduced $\hat{5}_i \coloneqq (5\,4) \cap (6\,7\,\hat{9}_i)$ and $\hat{9}_i \coloneqq (9\,8) \cap (10\,11\,i)$.

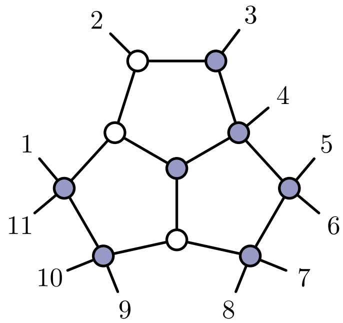
B. Cube-root leading singularities of an amplitude in $\mathcal{N} = 4$ sYM

For maximally supersymmetric Yang-Mills theory (sYM), we can give an explicit formula for the three on-shell functions associated with these cube-root-dependent leading singularities of the original Feynman integral. To do this, we start by first computing the on-shell function for the 11-cut of (15). Representing maximally helicity violating (MHV) and $\overline{\mathrm{MHV}}$ tree amplitudes by blue and white vertices, respectively, the 11-cut on-shell diagram corresponds to



(21)

where the latter is one realization of the former, using one of the (two) Britto-Cachazo-Feng-Witten (BCFW) representations of the four-particle MHV amplitude appearing on the lhs. Thus, we can recognize the on-shell function for the 11-cut as the one-parameter BCFW shift of the following, purely rational leading singularity in sYM:



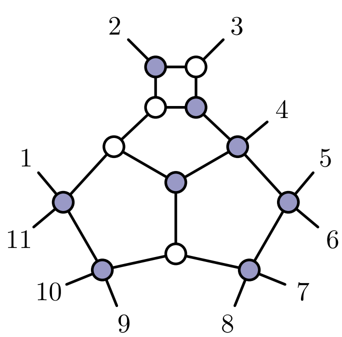
$$= \begin{array}{l} R[8\,9\,10\,11\,2]R[4\,5\,6\,7\,(9\,8)\bigcap(10\,11\,2)] \\ R[3\,4\,(9\,8)\bigcap(10\,11\,2)\,(5\,4)\bigcap(6\,7\,(9\,8)\bigcap(10\,11\,2))] \end{array} \tag{22}$$

This on-shell function has a permutation label given by $\{9,6,7,11,8,3,10,5,1,4,2\}$ [9,10] and has a representation in terms of R invariants given by the expression above—where

$$R[a\,b\,c\,d\,e] \coloneqq \frac{\delta^{1\times4}(\eta_a\langle b\,c\,d\,e\rangle + \eta_b\langle c\,d\,e\,a\rangle + \eta_c\langle d\,e\,a\,b\rangle + \eta_d\langle e\,a\,b\,c\rangle + \eta_e\langle a\,b\,c\,d\rangle}{\langle a\,b\,c\,d\rangle\langle b\,c\,d\,e\rangle\langle c\,d\,e\,a\rangle\langle d\,e\,a\,b\rangle\langle e\,a\,b\,c\rangle}. \tag{23}$$

[The formula given in (22) was found using [10] and recognizing it (via its permutation label) as a series of inverse-soft factors; in general, any leading singularity that could have arisen via BCFW bridges may be constructed recursively by recognizing factorized graphs among its bridge boundaries.]

To implement the BCFW shift as required to represent the 11-cut on-shell function, we merely need to replace $z_2 \mapsto \widehat{z_2} \coloneqq z_2 + \alpha z_1$ and include the prefactor of $1/\alpha$ in the bosonic part of the superfunction. Thus,



$$= \frac{1}{\alpha} R[8\,9\,10\,11\,\widehat{2}]R[4\,5\,6\,7\,\widehat{9}]\,R[3\,4\,\widehat{9}\,\widehat{5}\,\widehat{2}] \tag{24}$$

where $\hat{2}$, $\hat{5}$, and $\hat{9}$ were defined above in (16). This on-shell function corresponds to a 19-dimensional cell in the momentum-space Grassmannian $\mathrm{Gr}_+(11,5)$ labeled by the permutation $\{9,6,7,11,8,2,10,5,1,4,3\}$.

From this 11-cut on-shell function, the 12-cut on-shell functions involving the cube roots can be obtained by taking a simple residue about one of the poles arising from $\langle\hat{2}\,\hat{5}\,\hat{9}\,3\rangle = 0$ (which appears in the denominator of the bosonic part of the third R-invariant factor appearing in the expression above). Notice that this is *precisely* the same cubic as encountered in the final-cut condition above. When a residue is taken on any one of the cube roots, the resulting on-shell function would be labeled by the path permutation $\{9,7,6,11,8,2,10,5,1,4,3\}$ for an $18 = (2\times 11-4)$-dimensional cell in $\mathrm{Gr}_+(11,5)$; using the tools of [9,10], it can be easily confirmed that this cell in the Grassmannian has "intersection number" (with kinematics) equal to 3—meaning that there are *three* solutions to the constraints connecting the auxiliary Grassmannian to kinematic data. These three particular leading singularities correspond to the particular roots of the cubic used to define the cell via (24).

The existence of three leading singularities here leads to an unusual situation. Typically, we expect leading singularities to serve as prefactors of polylogarithmic functions in the integrated expression for an amplitude. Here, we have three distinct possible prefactors, corresponding to distinct residues of Eq. (24) at the roots α_i^* in Eq. (19).

The presence of these distinct possible prefactors raises the question of whether such an integral has differential equations that can be written in canonical form (cf. [24]). This form generally demands pure functions, and there is no normalization under which this integral is pure.

This issue is still present even for a scalar version of the diagram. There the leading singularities are just residues of $\frac{1}{q(\alpha)}$ on its three poles, of the form $\frac{1}{(\alpha_i^*-\alpha_j^*)(\alpha_i^*-\alpha_k^*)}$. As these must sum to zero, we do have one identity:

$$\frac{1}{(\alpha_1^*-\alpha_2^*)(\alpha_1^*-\alpha_3^*)} = -\frac{1}{(\alpha_2^*-\alpha_1^*)(\alpha_2^*-\alpha_3^*)} - \frac{1}{(\alpha_3^*-\alpha_1^*)(\alpha_3^*-\alpha_2^*)}. \tag{25}$$

Thus, we can express one residue in terms of the others, but there are still two independent prefactors.

This situation does not arise for quadratic equations. In that case we have the possible residues $\frac{1}{\alpha_1^*-\alpha_2^*}$ and $\frac{1}{\alpha_2^*-\alpha_1^*}$; but since these can differ only by a sign, we can pull out a global factor $\frac{1}{\alpha_1^*-\alpha_2^*}$, for example.

The appearance of this situation for cubic roots is perhaps not so surprising and is a behavior we expect to continue to higher orders. The final amplitude is, in any event, not generally pure: it has nontrivial leading singularities, which can in fact depend upon arbitrarily high-order algebraic roots involving the kinematics. This behavior is natural from the point of view of unitarity methods, where amplitude integrands are *rational* differential forms on the space of loop momenta; thus, when representing an amplitude using unitarity in terms of any basis of master loop integrands, whatever algebraic normalizations one uses must always conspire with algebraic coefficients (leading singularities) to yield a rational loop integrand.

C. Cluster coordinates for the integral

Following the above, we arrive at an expression for the loop momenta on the leading singularity in terms of momentum-twistor four-brackets. As four-brackets satisfy identities, it is not immediately obvious that the cube root is not reduced to something simpler upon application of these identities. To rule this out, we find expressions on the leading singularity in terms of an explicit twistor chart.

As we are considering an 11-point diagram, one might naively expect to describe it with $3(11)-15 = 18$ variables. However, this diagram is simpler than a generic 11-point dual-conformal diagram because of the presence of pairs of legs at the same corner, forming "masses": (5, 6), (7, 8), (9, 10), and (11, 1). As such, the diagram only depends on the seven dual points x_2, x_3, x_4, x_5, x_7, x_9, and x_{11}. Massless legs contribute three lightlike conditions, bringing the correct number of variables to $4(7)-3-15 = 10$. We would thus ideally want a ten-parameter twistor chart.

In Ref. [25], two of the present authors found twistor charts for a variety of integrals with appropriate numbers of parameters by specializing to particular positroid cells. Unfortunately, this strategy does not suffice here, as there are no ten-dimensional boundaries of the 11-point top cell that preserve dependence on the required dual points. Instead, we will follow a strategy outlined in Ref. [26] based on cluster algebras, finding a subquiver of $G(4,11)$ with ten $\mathcal{X}$-coordinates that spans the correct space of dual points.

To carry out this strategy, we begin with a quiver for $G(4,11)$ and then mutate on all possible nodes. For each mutation, we check to see if there is a subset of its $\mathcal{X}$-coordinates that is independent of the dual points of our integral. We keep only the mutations with the largest sets of $\mathcal{X}$-coordinates that satisfy this condition. We stop when we find at least one quiver which contains eight $\mathcal{X}$-coordinates that are independent from the seven dual points of our diagram, which in this case happens after 14 mutations. Setting these eight $\mathcal{X}$-coordinates to one in a twistor parametrization of the quiver, we find a twistor parametrization with the minimal ten parameters for our diagram.

In terms of momentum-twistor four-brackets, these ten parameters can be written as follows:

$$
\begin{aligned}
e_1 &= \frac{\langle (3\,5)\cap(4\,6\,7)4\,8\,9\rangle}{\langle 3\,4\,8\,9\rangle\langle 4\,5\,6\,7\rangle},\\
e_2 &= \frac{\langle (4\,5)\cap(3\,8\,9)3\,10\,11\rangle\langle (8\,9)\cap(2\,10\,11)4\,6\,7\rangle}{\langle 2\,3\,10\,11\rangle\langle 8\,9\,10\,11\rangle\langle (3\,5)\cap(4\,6\,7)4\,8\,9\rangle},\\
e_3 &= \frac{\langle 3\,4\,6\,7\rangle\langle (4\,5)\cap(3\,8\,9)3\,10\,11\rangle}{\langle 3\,4\,10\,11\rangle\langle (4\,5)\cap(3\,6\,7)3\,8\,9\rangle},\\
e_4 &= \frac{\langle 3\,4\,10\,11\rangle\langle 8\,9\,10\,11\rangle\langle (3\,5)\cap(4\,6\,7)4\,8\,9\rangle\langle (4\,5)\cap(3\,6\,7)3\,8\,9\rangle}{\langle 6\,7(3\,4\,5)\cap(8\,9(10\,11)\cap(3\,4\,5))\rangle\langle 3\,4\,8\,9\rangle\langle (8\,9)\cap(3\,10\,11)4\,6\,7\rangle},\\
e_5 &= \frac{\langle 3\,4\,8\,9\rangle\langle (1\,2)\cap(3\,4\,5)3\,10\,11\rangle}{\langle 1\,2\,3\,4\rangle\langle (4\,5)\cap(3\,8\,9)3\,10\,11\rangle},\\
e_6 &= -\frac{\langle 1\,2\,10\,11\rangle\langle 2\,3\,4\,5\rangle\langle (8\,9)\cap(3\,10\,11)4\,6\,7\rangle}{\langle (1\,2)\cap(3\,4\,5)3\,10\,11\rangle\langle (8\,9)\cap(2\,10\,11)4\,6\,7\rangle},\\
e_7 &= \frac{\langle 6\,7(3\,4\,5)\cap(8\,9(10\,11)\cap(3\,4\,5))\rangle\langle 1\,2\,3\,4\rangle\langle 2\,3\,10\,11\rangle}{\langle 6\,7(3\,4\,5)\cap(8\,9(10\,11)\cap(1\,2\,3))\rangle\langle 2\,3\,4\,5\rangle\langle 3\,4\,10\,11\rangle},\\
e_8 &= \frac{\langle 8\,9(3\,10\,11)\cap(4\,6\,7)\rangle\langle (4\,5)\cap(3\,6\,7)3\,8\,9\rangle}{\langle 3\,4\,6\,7\rangle\langle 6\,7\,8\,9\rangle\langle (4\,5)\cap(3\,8\,9)3\,10\,11\rangle},\\
e_9 &= -\frac{\langle 6\,7(3\,4\,5)\cap(8\,9(10\,11)\cap(1\,2\,3))\rangle\langle (4\,5)\cap(3\,8\,9)3\,10\,11\rangle}{\langle 8\,9\,10\,11\rangle\langle (1\,2)\cap(3\,4\,5)3\,10\,11\rangle\langle (4\,5)\cap(3\,6\,7)3\,8\,9\rangle},\\
e_{10} &= -\frac{\langle 6\,7(3\,4\,5)\cap(8\,9(10\,11)\cap(3\,4\,5))\rangle\langle 3\,4\,8\,9\rangle}{\langle (3\,5)\cap(4\,6\,7)4\,8\,9\rangle\langle (4\,5)\cap(3\,8\,9)3\,10\,11\rangle}.
\end{aligned}
\tag{26}
$$

We give an explicit parametrization of the momentum twistors in these coordinates in Appendix B and also include them in Supplemental Material [27].

1. Structure of the cubic root

Out of 18 cluster $\mathcal{X}$-coordinates, we retain ten coordinates e_i in our ten-parameter chart. In this chart, the cubic equation for α takes the following form:

$$
\frac{9}{4}e_2^4e_3e_4^4e_5^2e_7^2e_8e_9^2e_{10}^2(e_8e_2+e_2+5)^4[c_0+c_1\alpha + c_2\alpha^2+c_3\alpha^3]=0. \tag{27}
$$

The coefficients in this expression are quite long, so we omit them from the main text. They are presented in Appendix B and in Supplemental Material [27].

For a general cubic equation $c_0+c_1x+c_2x^2+c_3x^3=0$, one can write the three solutions as

$$
x_k=-\frac{1}{3c_3}\left(c_2+\zeta^kC+\frac{\Delta_0}{\zeta^kC}\right),\quad \text{where } k\in\{0,1,2\}. \tag{28}
$$

Here we have ζ a third root of unity and define

$$
C\coloneqq\left(\frac{\Delta_1\pm\sqrt{\Delta_1^2-4\Delta_0^3}}{2}\right)^{\frac{1}{3}}, \tag{29}
$$

with

$$
\begin{aligned}
\Delta_0 &\coloneqq c_2^2-4c_3c_1,\\
\Delta_1 &\coloneqq 2c_2^3-9c_3c_2c_1+27c_3^2c_0.
\end{aligned}\tag{30}
$$

It is clear that, in order for the leading singularity of this diagram to have a singularity that goes as $\rho^{1/3}$ as some kinematic parameter $\rho\to 0$, we must have $C\to 0$ in this limit. This in turn demands that $\sqrt{\Delta_1^2-4\Delta_0^3}\to\Delta_1$ and thus that $\Delta_0\to 0$. Generically, in a limit where $\Delta_0\to 0$, C behaves as

$$
\begin{aligned}
C&\sim\left(\frac{\Delta_1\pm(\Delta_1-2\frac{\Delta_0^3}{\Delta_1}+\mathcal{O}(\Delta_0^6))}{2}\right)^{\frac{1}{3}}\sim\left(\frac{\Delta_0^3}{\Delta_1}+\mathcal{O}(\Delta_0^6)\right)^{\frac{1}{3}}\\
&\sim\frac{\Delta_0}{\Delta_1^{1/3}}+\mathcal{O}(\Delta_0^4).
\end{aligned}\tag{31}
$$

Thus, C will generically vanish linearly, not as a third power, in such limits. C only vanishes as a third power if both Δ_0 and Δ_1 simultaneously vanish, as suggested by the form of Eq. (31). This is a codimension-two limit and thus not forbidden by Landau's analysis.

There are two potential loopholes in this general behavior. If Δ_0 vanishes identically for generic kinematics, then we only need a codimension-one limit to uncover the cubic root. If Δ_0 and Δ_1 both vanish in the same kinematic limit, then it might also be possible to achieve a cubic-root singularity.

In our case, these loopholes can be addressed by the explicit forms of Δ_0 and Δ_1. These are too complicated to print in full here, but their relevant structure is easy to display:

$$\begin{aligned} \Delta_0 &= e_6^2 P(e_i), \\ \Delta_1 &= e_6^3 Q(e_i). \end{aligned} \tag{32}$$

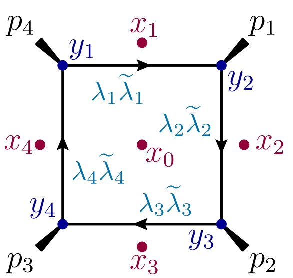
$P(e_i)$ and $Q(e_i)$ are both complicated polynomials, but crucially for our purposes they have no common factors. This means that we can only have a cube-root singularity when a common factor of Δ_0 and Δ_1 vanishes, namely when $e_6 \to 0$. As $e_6 \to 0$, we have $\sqrt{\Delta_1^2 - 4\Delta_0^3} \sim \sqrt{e_6^6}$, and thus $C \sim e_6$. Thus, there exists no codimension-one limit of this integral that behaves like $\rho^{1/3}$, as expected [28].

D. Landau equations

The behavior established in the previous section should also be able to be made manifest at the level of the Landau equations. Here we will describe how this calculation may be initiated, though at this time we have found it too computationally intractable to be worth pursuing.

Examining the leading Landau singularity (with all propagators on shell), one must in addition impose the Landau loop equations.

For the box integral below



$$\begin{aligned} x_a - x_0 &=: \lambda_a \tilde{\lambda}_a \\ \alpha_a \lambda_a \tilde{\lambda}_a &=: y_{a+1} - y_a \end{aligned} \tag{33}$$

the Landau loop equations in dual coordinate language read

$$\alpha_1(x_1 - x_0) + \cdots + \alpha_4(x_4 - x_0) = 0. \tag{34}$$

This equation is not manifestly dual-conformal invariant. As such, it is not straightforward to represent it in terms of a nonredundant parametrization of the kinematics. To make the equation dual-conformal invariant, we may upgrade it to an equation in embedding space, where each dual point is represented as a six-dimensional null vector $X_i =: (x_i^\mu, X_i^+, X_i^-)$ with metric such that

$$X_i \cdot X_j \coloneqq X_i^+ X_j^- + X_i^- X_j^+ - x_i \cdot x_j, \tag{35}$$

where for any nonzero real constant c we take $X_i^+ = c$ for all i and $X_i^- = \frac{1}{2c} x_i^2$. Indeed, using the Landau equations in Eq. (34) and the on-shell conditions it can be shown that in a gauge where X_i^+ is a constant independent on i,

$$\alpha_1(X_1 - X_0) + \cdots + \alpha_4(X_4 - X_0) = 0. \tag{36}$$

For example, suppose we gauge $X_i^+ = 1$ for all X_i. Then the equations for the x_i^μ components are identical to those in Eq. (34), while the X_i^+ equation is trivial, so we only need to check X_i^-. In this gauge $X_i^- = \frac{x_i^2}{2}$, so the equation for this component becomes

$$\alpha_1(x_1^2 - x_0^2) + \cdots \alpha_4(x_4^2 - x_0^2) = 0. \tag{37}$$

Since we have imposed that the propagators are on shell, we have $(x_i - x_0)^2 = 0$ for all i. Thus,

$$\begin{aligned} x_i^2 - x_0^2 &= x_i^2 - x_0^2 - (x_i - x_0)^2 \\ &= 2x_i \cdot x_0 - 2x_0^2 \\ &= 2x_0 \cdot (x_i - x_0), \end{aligned} \tag{38}$$

which shows that the equation for the X_i^- components is implied by our original equation in x space.

Using this type of manifestly dual-conformal invariant form of the Landau equations for the three-loop integral we have discussed in this section, and writing the dual points of the loop momenta as X_A, X_B, X_C, we find

$$\begin{aligned} 0 = {}&\alpha_1(X_A - X_3) + \alpha_2(X_A - X_4) + \alpha_3(X_A - X_B) \\ &+ \alpha_4(X_A - X_C) + \alpha_5(X_A - X_2), \end{aligned} \tag{39}$$

$$\begin{aligned} 0 = {}&\alpha_6(X_B - X_5) + \alpha_7(X_B - X_7) + \alpha_8(X_B - X_9) \\ &+ \alpha_9(X_B - X_C) + \alpha_3(X_B - X_A), \end{aligned} \tag{40}$$

$$\begin{aligned} 0 = {}&\alpha_{10}(X_C - X_9) + \alpha_{11}(X_C - X_{11}) + \alpha_{12}(X_C - X_2) \\ &+ \alpha_4(X_C - X_A) + \alpha_9(X_C - X_B). \end{aligned} \tag{41}$$

Separating these component by component gives 18 equations in the 12 α_i, with coefficients that are algebraic functions in the cluster $\mathcal{X}$-coordinates. Some of these equations are redundant, but 12 are independent, giving a 12-by-12 matrix of coefficients. The leading Landau singularity occurs when the determinant of this matrix vanishes.

It would be interesting to find and factorize the determinant of this matrix. One would expect that it would have a factor in common with Δ_0 and thus that it vanishing would lead a pair of roots of the cubic to coincide. Unfortunately, we are unable to confirm this due to the complexity of the coefficients present [29].

Before presenting a less computationally intensive approach in the next section, we make a few brief comments about the geometry of the problem. Much as the conservation of loop momentum is solved by the introduction of dual coordinates, one can formally solve the Landau loop equations by introducing new auxiliary coordinates associated to the vertices of a diagram that incorporate the variables α_i. Then the solution to the Landau equations can be characterized as a set of geometrical constraints on both the momentum twistors and a set of twistors corresponding to these new auxiliary coordinates.

We can briefly illustrate this approach with the example of the one-loop box integral [see Eq. (33)]. Here in analogy with the dual coordinates $x_a - x_0 =: \lambda_a\tilde{\lambda}_a$, we have introduced auxiliary coordinates y_a such that $\alpha_a\lambda_a\tilde{\lambda}_a =: y_{a+1} - y_a$. One can then think of the problem as a set of geometric constraints linking the momentum twistors and dual momentum twistors

$$Z = (\lambda, i\lambda x), \qquad \tilde{Z} = (-ix\tilde{\lambda}, \tilde{\lambda}), \tag{42}$$

with similar objects introduced for the auxiliary variables, $Y_i = (\lambda, i\lambda y_i)$ and $\tilde{Y}_i = (-iy_i\tilde{\lambda}, \tilde{\lambda})$. This approach appears promising, and we will pursue more applications of it in future work.

IV. *EXEMPLI GRATIA*: A PAIR OF OCTICS

In this section we analyze the singularity arising from the Landau diagram in Fig. 2. We do not take $\langle A_iB_iA_{i+1}B_{i+1}\rangle = 0$ so this is a more general case of the kinematics analyzed before, but it has the advantage of being more symmetric. In this case, we can describe the Landau locus in a particularly compact manner. We analyze only the case where the three-point amplitude in the center is $\overline{\text{MHV}}$, so the λ spinors of the internal lines are proportional. This means we can set the corresponding twistors equal; as a result, they are all labeled as C in the diagram. The parity-conjugate case can be obtained by projective duality. Projective duality exchanges points and planes and sends lines to lines so the dual of three lines intersecting in one point is a configuration of three lines belonging to the same plane.

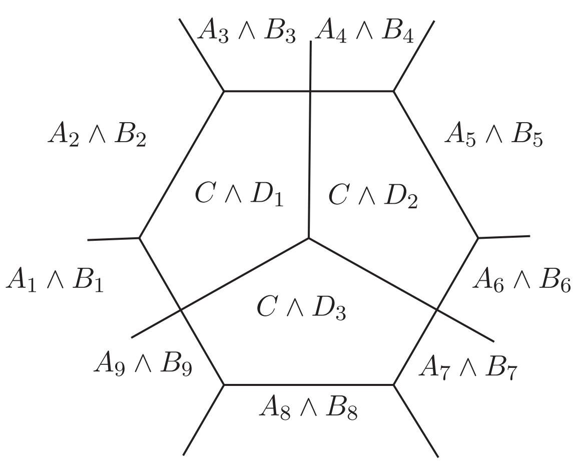


FIG. 2. Three-pentagon cubic-root singularity. We have described the dual points of the diagram in terms of lines in twistor space.

A. On-shell space

We begin by discussing the on-shell space, using some notions from projective geometry. In order for all propagators of the diagram in Fig. 2 to be on shell, we must have that the lines in twistor space $C \wedge D_1$, $C \wedge D_2$, and $C \wedge D_3$ intersect with all of the lines defining their adjacent dual points.

To begin, we focus on solving the on-shell conditions for the propagators on the outside of the diagram. The skew lines $A_1 \wedge B_1$, $A_2 \wedge B_2$ and $A_3 \wedge B_3$ are contained in a unique quadric surface Q_1. Similarly for $A_4 \wedge B_4$, $A_5 \wedge B_5$, $A_6 \wedge B_6$ which determine a unique quadric Q_2 and for $A_7 \wedge B_7$, $A_8 \wedge B_8$, $A_9 \wedge B_9$ which determine a quadric Q_3. Through any point of Q_1 passes a line which intersects $A_1 \wedge B_1$, $A_2 \wedge B_2$ and $A_3 \wedge B_3$. Conversely, any line which intersects these three lines is completely contained in the quadric Q_1. Thus, in order for the propagators surrounding this loop to be on shell, we must have that the line parametrizing the loop momentum is contained in Q_1. In order to parametrize this line, we may write a generic point $P_{A_1B_1}$ on the line $A_1 \wedge B_1$ with

$$P_{A_1B_1} = A_1 + \nu_1 B_1. \tag{43}$$

Then we can write the space of lines I_1 contained in the quadric Q_1 as

$$\begin{aligned} I_1 &= (P_{A_1B_1}A_2B_2) \cap (P_{A_1B_1}A_3B_3) \\ &= (P_{A_1B_1} \wedge A_2)\langle B_2, P_{A_1B_1}, A_3, B_3\rangle \\ &\quad - (P_{A_1B_1} \wedge B_2)\langle A_2, P_{A_1B_1}, A_3, B_3\rangle. \end{aligned} \tag{44}$$

Following this same procedure for the quadrics Q_2 and Q_3, one obtains three lines I_1, I_2, and I_3 parametrizing the three-loop momenta in terms of three parameters ν_1, ν_2, ν_3.

Finally, we must impose the on-shell conditions of the internal lines. These are enforced by demanding that the lines parametrizing the loop momenta intersect. We need $\langle I_1I_2\rangle = \langle I_1I_3\rangle = \langle I_2I_3\rangle = 0$. We have evaluated these equations for generic external kinematics using SageMath [31]. Two of the variables can be eliminated rationally, resulting in a degree-16 polynomial in the final ν_3. This polynomial is reducible: it factors into two irreducible degree-eight polynomials. Thus, the on-shell space for this polynomial involves roots of irreducible octics.

B. Landau equations

We now proceed to describe the Landau equations for this integral. We introduce dual embedding space coordinates

$$X_i = \frac{A_i \wedge B_i}{\langle IA_iB_i\rangle}, \qquad W_a = \frac{C \wedge D_a}{\langle ICD_a\rangle}. \tag{45}$$

The components $(\alpha\dot{\alpha})$ of these embedding space dual coordinates are the usual four-dimensional dual coordinates (which we denote by lowercase letters).

In terms of these four-dimensional dual coordinates the Landau loop equations read

$$\begin{aligned}&\alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 + \alpha'_2 w_2 - \alpha'_3 w_3\\ &\quad = (\alpha_1 + \alpha_2 + \alpha_3 + \alpha'_2 - \alpha'_3) w_1,\end{aligned} \tag{46}$$

$$\begin{aligned}&\alpha_4 x_4 + \alpha_5 x_5 + \alpha_6 x_6 + \alpha'_3 w_3 - \alpha'_1 w_1\\ &\quad = (\alpha_4 + \alpha_5 + \alpha_6 + \alpha'_3 - \alpha'_1) w_2,\end{aligned} \tag{47}$$

$$\begin{aligned}&\alpha_7 x_7 + \alpha_8 x_8 + \alpha_9 x_9 + \alpha'_1 w_1 - \alpha'_2 w_2\\ &\quad = (\alpha_7 + \alpha_8 + \alpha_9 + \alpha'_1 - \alpha'_2) w_3.\end{aligned} \tag{48}$$

When upgraded to twistor language we get

$$\begin{aligned}&\alpha_1 \frac{A_1 \wedge B_1}{\langle IA_1B_1\rangle} + \alpha_2 \frac{A_2 \wedge B_2}{\langle IA_2B_2\rangle} + \alpha_3 \frac{A_3 \wedge B_3}{\langle IA_3B_3\rangle}\\ &\quad + \alpha'_2 \frac{C \wedge D_2}{\langle ICD_2\rangle} - \alpha'_3 \frac{C \wedge D_3}{\langle ICD_3\rangle}\\ &\quad = (\alpha_1 + \alpha_2 + \alpha_3 + \alpha'_2 - \alpha'_3) \frac{C \wedge D_1}{\langle ICD_1\rangle}.\end{aligned} \tag{49}$$

In the following it will be convenient to redefine

$$\alpha_i \to \alpha_i \langle IA_iB_i\rangle, \qquad \alpha'_a \to \alpha'_a \langle ICD_a\rangle. \tag{50}$$

The three-loop Landau equations are

$$\begin{aligned}&\alpha_1 A_1 \wedge B_1 + \alpha_2 A_2 \wedge B_2 + \alpha_3 A_3 \wedge B_3 + \alpha'_2 C \wedge D_2 - \alpha'_3 C \wedge D_3\\ &\quad = \frac{\alpha_1\langle IA_1B_1\rangle + \alpha_2\langle IA_2B_2\rangle + \alpha_3\langle IA_3B_3\rangle + \alpha'_2\langle ICD_2\rangle - \alpha'_3\langle ICD_3\rangle}{\langle ICD_1\rangle} C \wedge D_1,\end{aligned} \tag{51}$$

$$\begin{aligned}&\alpha_4 A_4 \wedge B_4 + \alpha_5 A_5 \wedge B_5 + \alpha_6 A_6 \wedge B_6 + \alpha'_3 C \wedge D_3 - \alpha'_1 C \wedge D_1\\ &\quad = \frac{\alpha_4\langle IA_4B_4\rangle + \alpha_5\langle IA_5B_5\rangle + \alpha_6\langle IA_6B_6\rangle + \alpha'_3\langle ICD_3\rangle - \alpha'_1\langle ICD_1\rangle}{\langle ICD_2\rangle} C \wedge D_2,\end{aligned} \tag{52}$$

$$\begin{aligned}&\alpha_7 A_7 \wedge B_7 + \alpha_8 A_8 \wedge B_8 + \alpha_9 A_9 \wedge B_9 + \alpha'_1 C \wedge D_1 - \alpha'_2 C \wedge D_2\\ &\quad = \frac{\alpha_7\langle IA_7B_7\rangle + \alpha_8\langle IA_8B_8\rangle + \alpha_9\langle IA_9B_9\rangle + \alpha'_1\langle ICD_1\rangle - \alpha'_2\langle ICD_2\rangle}{\langle ICD_3\rangle} C \wedge D_3.\end{aligned} \tag{53}$$

By wedging with C it follows that

$$\alpha_1 A_1 \wedge B_1 \wedge C + \alpha_2 A_2 \wedge B_2 \wedge C + \alpha_3 A_3 \wedge B_3 \wedge C = 0, \tag{54}$$

$$\alpha_4 A_4 \wedge B_4 \wedge C + \alpha_5 A_5 \wedge B_5 \wedge C + \alpha_6 A_6 \wedge B_6 \wedge C = 0, \tag{55}$$

$$\alpha_7 A_7 \wedge B_7 \wedge C + \alpha_8 A_8 \wedge B_8 \wedge C + \alpha_9 A_9 \wedge B_9 \wedge C = 0. \tag{56}$$

From the first equation we obtain that

$$\alpha_1\langle A_1B_1CA_3\rangle + \alpha_2\langle A_2B_2CA_3\rangle = 0, \tag{57}$$

$$\alpha_1\langle A_1B_1CB_3\rangle + \alpha_2\langle A_2B_2CB_3\rangle = 0. \tag{58}$$

If we impose the condition that these two equations are compatible for nonvanishing α_1 and α_2, we obtain that

$$\langle A_1B_1CA_3\rangle\langle A_2B_2CB_3\rangle - \langle A_1B_1CB_3\rangle\langle A_2B_2CA_3\rangle = 0. \tag{59}$$

This is just the condition that C belongs to the quadric containing the lines $A_1 \wedge B_1$, $A_2 \wedge B_2$ and $A_3 \wedge B_3$.

There are three such quadrics. Recall that we have denoted by Q_1 the quadric generated by the lines $A_1 \wedge B_1$, $A_2 \wedge B_2$ and $A_3 \wedge B_3$, etc. The three quadrics Q_1, Q_2 and Q_3 in $\mathbb{P}^3$ intersect in $2 \times 2 \times 2 = 8$ points. We will discuss the surprisingly rich geometry of these intersection points in Sec. IV C.

We can use the equations above to solve for α_2 and α_3 in terms of α_1, etc. We obtain

$$\alpha_2 = -\alpha_1 \frac{\langle A_1B_1CA_3\rangle}{\langle A_2B_2CA_3\rangle}, \qquad \alpha_3 = -\alpha_1 \frac{\langle A_1B_1CA_2\rangle}{\langle A_3B_3CA_2\rangle}. \tag{60}$$

We can also contract the first Landau loop equation (51) with a line which is transversal to A_1B_1, A_2B_2, A_3B_4 and CD_2. Recall that four skew lines always have at least two (possibly coinciding) transversals. One of the transversals is CD_1. To obtain the other, we use that the quadric which contains the lines A_1B_1, A_2B_2 and A_3B_3 has equation

$$\langle A_1B_1A_3X\rangle\langle A_2B_2B_3X\rangle - (A_3 \leftrightarrow B_3) = 0. \tag{61}$$

Even though this expression looks like it is not symmetric under permutations of the lines A_iB_i for $i = 1, 2, 3$, it can be shown (via Plücker identities) that it is symmetric. The line CD_2 intersects this quadric in two points, which are given by the solutions of the following quadratic equation in μ:

$$\langle A_1B_1A_3(C+\mu D_2)\rangle\langle A_2B_2B_3(C+\mu D_2)\rangle - (A_3 \leftrightarrow B_3) = 0. \tag{62}$$

One of the solutions is $\mu = 0$ since C belongs to the quadric (see above). Then the other solution can be found straightforwardly:

$$\mu = -\frac{\langle A_1B_1A_3D_2\rangle\langle A_2B_2B_3C\rangle + \langle A_1B_1A_3C\rangle\langle A_2B_2B_3D_2\rangle - (A_3 \leftrightarrow B_3)}{\langle A_1B_1A_3D_2\rangle\langle A_2B_2B_3D_2\rangle - (A_3 \leftrightarrow B_3)}. \tag{63}$$

If we denote this intersection point by $P_{12} = C + \mu D_2$, then the second transversal line can be taken to be $L_{12} = (P_{12}A_1B_1) \cap (P_{12}A_2B_2)$. Then, using the transversality property we have

$$\langle A_1B_1L_{12}\rangle = \langle A_2B_2L_{12}\rangle = \langle A_3B_3L_{12}\rangle = \langle CD_2L_{12}\rangle = 0, \tag{64}$$

so upon contraction with the Landau loop Eq. (51) with L_{12} we find

$$-\alpha_3'\langle CD_3L_{12}\rangle = \frac{\alpha_1\langle IA_1B_1\rangle + \alpha_2\langle IA_2B_2\rangle + \alpha_3\langle IA_3B_3\rangle + \alpha_2'\langle ICD_2\rangle - \alpha_3'\langle ICD_3\rangle}{\langle ICD_1\rangle}\langle CD_1L_{12}\rangle. \tag{65}$$

We can similarly build a transversal L_{13} to A_1B_1, A_2B_2, A_3B_3 and CD_3 (which is different from CD_1). Contracting the Landau loop Eq. (51) with L_{13} we find

$$\alpha_2'\langle CD_2L_{13}\rangle = \frac{\alpha_1\langle IA_1B_1\rangle + \alpha_2\langle IA_2B_2\rangle + \alpha_3\langle IA_3B_3\rangle + \alpha_2'\langle ICD_2\rangle - \alpha_3'\langle ICD_3\rangle}{\langle ICD_1\rangle}\langle CD_1L_{13}\rangle. \tag{66}$$

Hence,

$$\frac{\alpha_2'}{\alpha_3'} = -\frac{\langle CD_1L_{13}\rangle\langle CD_3L_{12}\rangle}{\langle CD_2L_{13}\rangle\langle CD_1L_{12}\rangle}. \tag{67}$$

Similarly, we find

$$\begin{aligned}\frac{\alpha_3'}{\alpha_1'} &= -\frac{\langle CD_2L_{21}\rangle\langle CD_1L_{23}\rangle}{\langle CD_3L_{21}\rangle\langle CD_2L_{23}\rangle},\\ \frac{\alpha_1'}{\alpha_2'} &= -\frac{\langle CD_3L_{32}\rangle\langle CD_2L_{31}\rangle}{\langle CD_1L_{32}\rangle\langle CD_3L_{31}\rangle}.\end{aligned} \tag{68}$$

Taking the product of these ratios we find

$$\begin{aligned}&\langle CD_1L_{13}\rangle\langle CD_3L_{12}\rangle\langle CD_2L_{21}\rangle\langle CD_1L_{23}\rangle\\ &\quad\times\langle CD_3L_{32}\rangle\langle CD_2L_{31}\rangle + \langle CD_2L_{13}\rangle\langle CD_1L_{12}\rangle\\ &\quad\times\langle CD_3L_{21}\rangle\langle CD_2L_{23}\rangle\langle CD_1L_{32}\rangle\langle CD_3L_{31}\rangle = 0.\end{aligned} \tag{69}$$

This equation expresses a codimension-one constraint on the external kinematics, written in terms of lines in twistor space parametrizing the loop momenta in the on-shell space and transversals to those lines. It is the leading Landau singularity for this diagram, defining points at which the integral can develop a branch cut.

C. Intersection of three quadrics

In practice, even when an algebraic root is needed to describe the on-shell space for an initial choice of kinematic variables, it is often possible to find other variables which rationalize the root. One fruitful way to find such variables is to consider the problem geometrically and find variables that naturally parametrize the on-shell space. These variables will typically be complicated to express in terms of the external kinematics, as they are in some sense derived "outside in," starting from the solution to the on-shell conditions and deriving a parametrization of the external kinematics from that. As such, one should not think of the existence of these parametrizations as evidence that a diagram lacks a given root: rather, they move the complexity of the root to the complexity of the external kinematics.

A familiar example of this kind is that of the four-mass box integral (with massless internal lines). It is known that this integral can be computed in terms of two quantities z and $\bar{z}$ which are the roots of a quadratic equation in the Mandelstam invariants of external lines (see Ref. [32] for a discussion). Unlike the case of the two-loop six-point planar MHV amplitude in $\mathcal{N} = 4$ super Yang-Mills (see Ref. [33]), these square roots do not disappear even if we use momentum twistors to parametrize the external kinematics. However, we can choose to parametrize the kinematics in a different way instead. It turns out (see Ref. [32]) that the quantities z and $\bar{z}$ have the following geometric interpretation: in momentum twistor space the external kinematics of a four-mass box is represented by four skew lines in $\mathbb{P}^3$. Such four lines generically have two transversals (lines which intersect all four of them). On each of these transversals there are four intersection points. Taking the cross-ratio of these four points on one of the transversals one obtains z and on the other transversal $\bar{z}$ [34]. Using this result, we can parametrize the on-shell space of the massive box integral as follows. We first pick the two transversals; then we pick four pairs of points, with the first member of the pair on the first transversal and the second on the second transversal. These four lines are the lines corresponding to the dual external points in twistor space and therefore we have specified the kinematics completely. However, in this parametrization the variables z and $\bar{z}$ arise naturally and do not involve any square roots.

We will now attempt a similar exercise for the three-loop diagram in Fig. 2. Consider three quadrics in $\mathbb{P}^3$ which are in general position. Then, by Bezout's theorem they intersect in eight points. However, the converse does not hold. That is, given eight points in general position (that is no four of them coplanar and therefore no three of them collinear) it is not possible to find three quadrics containing all of them. Indeed, suppose it was possible to find such three quadrics Q_1, Q_2, Q_3. Then we have a two-parameter (parametrized by points in $\mathbb{P}^2$) quadric $\lambda_1 Q_1 + \lambda_2 Q_2 + \lambda_3 Q_3$ which also passes through all eight points. However, given seven points in general position (meaning that they impose independent conditions on the coefficients of a quadric containing them) they determine seven of the nine independent coefficients (up to scaling) of a generic quadric. In other words, two coefficients remain undetermined. But through eight points in generic position passes only a one-parameter family of quadrics. The resolution of this paradox is that specifying seven of the points uniquely determines the eighth.

This is an instance of what is sometimes called *Cramer's paradox*. This paradox first arose in the study of the intersection of cubic curves in the projective plane. The polynomial defining a cubic curve in $\mathbb{P}^2$ has ten terms. Considered up to rescaling by a nonvanishing quantity, such a cubic is parametrized by a point in $\mathbb{P}^9$. Imposing that a point is contained in the curve imposes one constraint. If we impose that the curve contains nine points (and if these conditions are independent), then the cubic is completely determined. Now consider the intersection of two such cubic curves. According to Bezout's theorem, they intersect in $3 \times 3 = 9$ points. However, we have shown that through nine points in generic position passes a single cubic. The conclusion is that the nine intersection points are not independent. Indeed, one can show that if two cubic curves pass through eight points, then they pass through a ninth point as well and this ninth point is uniquely determined by the other eight.

If we denote the eight intersection points of three quadrics by $1, \ldots, 8$, we should be able to express the constraints on the eighth point in terms of four-brackets. Indeed, Turnbull (see Ref. [35]) has given such constraints. A necessary condition is

$$\det\begin{pmatrix} \langle 1256\rangle\langle 3476\rangle & \langle 1258\rangle\langle 3478\rangle \\ \langle 1276\rangle\langle 3456\rangle & \langle 1276\rangle\langle 3458\rangle \end{pmatrix} = 0. \quad (70)$$

If points $1, \ldots, 7$ are kept fixed, this is the equation of a quadric in the coordinates of the eighth point. It can be easily checked that the determinant in Eq. (70) vanishes when point 8 coincides with any of $1, \ldots, 7$. See also Ref. [36], p. 153.

Eight points which are the intersection of three quadrics are sometimes called a *Cayley octad*. Their geometry has been studied more recently by Dolgachev and Ortland (see Ref. [37]). Interestingly, their description of the space of Cayley octads is in terms of a concept called Gale duality. The general form of this duality can be described very concretely as follows (see Ref. [38] for an introduction). Given m points in $\mathbb{P}^n$ (with $m > n + 2$), we can represent their configuration by a $(n + 1) \times m$ matrix, with a left action of $PGL(n + 1)$. If this configuration is generic, then we can, by this left action, make the left $(n + 1) \times (n + 1)$ minor the identity. Then the configuration is parametrized by a $(n + 1) \times (m - n - 1)$ matrix A. This matrix A fits in $(m - n - 1) \times m$ matrix $(\mathbf{1}_{m-n-1}, A^T)$. The matrix A^T, in turn, corresponds to a configuration of m points in $\mathbb{P}^{m-n-2}$. It is advantageous to apply this duality when the dimension of the embedding projective space decreases. This has also been described, in a related geometric context, in Ref. [39], p. 299 (see also Refs. [33,40] for applications). The constraints linking these eight points can be described as follows: the configuration of the eight intersection points of three quadrics in $\mathbb{P}^3$ is Gale self-dual.

The eighth point can be parametrized rationally in terms of the other seven, as described in Ref. [41], Proposition 7.1. Indeed, we can take the first seven points to have homogeneous coordinates

$$(1{:}0{:}0{:}0),\quad (0{:}1{:}0{:}0),\quad (0{:}0{:}1{:}0),\quad (0{:}0{:}0{:}1), \tag{71}$$

$$(1{:}1{:}1{:}1),\quad (\alpha_6{:}\,\beta_6{:}\gamma_6{:}\delta_6),\quad (\alpha_7{:}\beta_7{:}\gamma_7{:}\delta_7). \tag{72}$$

Then, the eighth point has coordinates $(\alpha_8{:}\beta_8{:}\gamma_8{:}\delta_8)$ given by

$$\alpha_8 = \frac{-\gamma_6\beta_7 + \delta_6\beta_7 + \beta_6\gamma_7 - \delta_6\gamma_7 - \beta_6\delta_7 + \gamma_6\delta_7}{-\beta_6\delta_6\beta_7\gamma_7 + \gamma_6\delta_6\beta_7\gamma_7 + \beta_6\gamma_6\beta_7\delta_7 - \gamma_6\delta_6\beta_7\delta_7 - \beta_6\gamma_6\gamma_7\delta_7 + \beta_6\delta_6\gamma_7\delta_7}, \tag{73}$$

$$\beta_8 = \frac{-\gamma_6\alpha_7 + \delta_6\alpha_7 + \alpha_6\gamma_7 - \delta_6\gamma_7 - \alpha_6\delta_7 + \gamma_6\delta_7}{-\alpha_6\delta_6\alpha_7\gamma_7 + \gamma_6\delta_6\alpha_7\gamma_7 + \alpha_6\gamma_6\alpha_7\delta_7 - \gamma_6\delta_6\alpha_7\delta_7 - \alpha_6\gamma_6\gamma_7\delta_7 + \alpha_6\delta_6\gamma_7\delta_7}, \tag{74}$$

$$\gamma_8 = \frac{-\beta_6\alpha_7 + \delta_6\alpha_7 + \alpha_6\beta_7 - \delta_6\beta_7 - \alpha_6\delta_7 + \beta_6\delta_7}{-\alpha_6\delta_6\alpha_7\beta_7 + \beta_6\delta_6\alpha_7\beta_7 + \alpha_6\beta_6\alpha_7\delta_7 - \beta_6\delta_6\alpha_7\delta_7 - \alpha_6\beta_6\beta_7\delta_7 + \alpha_6\delta_6\beta_7\delta_7}, \tag{75}$$

$$\delta_8 = \frac{-\beta_6\alpha_7 + \gamma_6\alpha_7 + \alpha_6\beta_7 - \gamma_6\beta_7 - \alpha_6\gamma_7 + \beta_6\gamma_7}{-\alpha_6\gamma_6\alpha_7\beta_7 + \beta_6\gamma_6\alpha_7\beta_7 + \alpha_6\beta_6\alpha_7\gamma_7 - \beta_6\gamma_6\alpha_7\gamma_7 - \alpha_6\beta_6\beta_7\gamma_7 + \alpha_6\gamma_6\beta_7\gamma_7}. \tag{76}$$

To check that these eight points belong to three independent quadrics we proceed as follows. We start with a quadric

$$q(x) = \sum_{i,j=0}^{3} q_{ij}x_i x_j, \tag{77}$$

with $q_{ij} = q_{ji}$. Imposing that the first five points belong to this quadric implies that $q_{00} = q_{11} = q_{22} = q_{33} = 0$ and $q_{01} + q_{02} + q_{03} + q_{12} + q_{13} + q_{23} = 0$. The conditions that the last four points belong to the quadric are linear constraints on the $(q_{01}, q_{02}, q_{03}, q_{12}, q_{13}, q_{23})$ which can be put in the form of a 4×6 matrix:

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ \alpha_6\beta_6 & \alpha_6\gamma_6 & \alpha_6\delta_6 & \beta_6\gamma_6 & \beta_6\delta_6 & \gamma_6\delta_6 \\ \alpha_7\beta_7 & \alpha_7\gamma_7 & \alpha_7\delta_7 & \beta_7\gamma_7 & \beta_7\delta_7 & \gamma_6\delta_7 \\ \alpha_8\beta_8 & \alpha_8\gamma_8 & \alpha_8\delta_8 & \beta_8\gamma_8 & \beta_8\delta_8 & \gamma_8\delta_8 \end{pmatrix}. \tag{78}$$

Then we can check that with the values in Eq. (73) all the 4×4 minors of this matrix vanish. This means that the eighth point is automatically contained in any quadric that contains the other seven.

Hence, it is possible to parametrize the on-shell kinematics rationally, in a sense from the "inside out." That is, we first pick seven points as above and an eighth point whose coordinates are given by Eq. (73). Then we find three quadrics such that these eight points are their intersection. Then, in each quadric we pick three lines, member of the same ruling and not containing any of the eight intersection points. Finally, we take these nine lines as the external lines A_iB_i for $i = 1, \ldots, n$. The ability to rationally parametrize the on-shell spaces is connected with the possibility of computing the integral in terms of polylogarithms (see Ref. [42]). It would be interesting to understand better the relation between this parametrization and those in terms of cluster variables (see Appendix B).

Finally, let us discuss possible degenerations of these configurations of eight points (which are the on-shell space of the diagram in Fig. 2). This has been studied in Dolgachev and Ortland (see Ref. [37], p. 176). They found three classes of possible degenerations:

(1) Two of the eight points become coincident.
(2) Four points become coplanar (when this happens the other four points also become coplanar as follows from Gale duality).
(3) The eight points lie on a twisted cubic. This is a codimension-two condition.

The physical interpretation of the second and third possibilities is mysterious, with third in particular able to correspond to a codimension-two Landau singularity. The first, on the other hand, is exactly the Landau locus of Eq. (69), as we confirm below.

When two points P_7 and P_8 of a Cayley octad become coincident, we have $\langle m, n, 7, 8\rangle = 0$. Therefore by Gale duality, the four-bracket of the complementary points $\langle i\,j\,k\,l\rangle = 0$ where the points $\{P_m, P_n, P_7, P_8\} \cup \{P_i, P_j, P_k, P_l\} = \{P_1, \ldots, P_8\}$. Since this is true for all groups of four points in $\{P_1, \ldots, P_6\}$, this means that the points $P_1, \ldots, P_6$ belong to the same plane. In fact, they also belong to the same conic, since if they did not then they would not belong to a quadric (since the intersection of a quadric with a plane is a conic). Six points do not generically belong to a conic, a conic being determined by five points (in general position in the same plane).

Since the points $P_1, \ldots, P_6$ belong to the same plane we can pick them without loss of generality to have zero fourth component. We take

$$P_1 = (1{:}0{:}0{:}0), \qquad P_2 = (0{:}1{:}0{:}0), \tag{79}$$

$$P_3 = (0{:}0{:}1{:}0), \qquad P_4 = (1{:}1{:}1{:}0), \tag{80}$$

$$P_5 = (a_1{:}\ b_1{:}c_1{:}0), \qquad P_6 = (a_2{:}b_2{:}c_2{:}0). \tag{81}$$

If the equation of the conic is

$$c_{00}x_0^2 + 2c_{01}x_0x_1 + 2c_{02}x_0x_2 + c_{11}x_1^2 + 2c_{12}x_1x_2 + c_{22}x_2^2 = 0, \tag{82}$$

then imposing that the points $P_1, \ldots, P_6$ belong to it implies that $c_{00} = c_{11} = c_{22} = 0$, $a_{12} = -a_{01} - a_{02}$. The conditions for the last two points to belong to this conic are

$$\frac{c_{01}}{c_{02}} = -\frac{c_1(a_1 - b_1)}{b_1(a_1 - c_1)} = -\frac{c_2(a_2 - b_2)}{b_2(a_2 - c_2)}. \tag{83}$$

This first equality can be used to determine the coefficients c_{ij} (up to a multiplicative factor) while the last equality can be used to determine c_2.

Without loss of generality we take the remaining (coinciding) two points to be

$$P_7 = P_8 = (0{:}0{:}0{:}1). \tag{84}$$

Note that this parametrization cannot be obtained from the one in Eq. (73) for finite values of α, β, γ, δ. This not unexpected when one considers that a configuration where two points coincide lies on the boundary of the space of those parameters.

A quadric $Q \subset \mathbb{P}^3$ is defined by a general equation

$$\sum_{0\le i\le j\le 3} q_{ij}x_ix_j. \tag{85}$$

Imposing the conditions that the points $P_1, \ldots, P_6 \in Q$ we obtain that $q_{ij} = c_{ij}$ if $i, j \in \{0, 1, 2\}$. If we impose the condition that $P_7, P_8 \in Q$, we obtain $q_{33} = 0$. The coefficients q_{03}, q_{13} and q_{23} remain undetermined.

We therefore obtain three natural quadrics which contain all eight points:

$$Q_1(x) = \sum_{0\le i\le j\le 2} c_{ij}x_ix_j + q_{03}x_0x_3, \tag{86}$$

$$Q_2(x) = \sum_{0\le i\le j\le 2} c_{ij}x_ix_j + q_{13}x_1x_3, \tag{87}$$

$$Q_3(x) = \sum_{0\le i\le j\le 2} c_{ij}x_ix_j + q_{23}x_2x_3. \tag{88}$$

It can be checked that these quadrics are generically smooth, by computing the determinants of the associated symmetric matrices. We can set the coefficients q_{03}, q_{13} and q_{23} to one without loss of generality.

Next, we need to choose three skew lines in each of these quadrics. To find a line in a smooth quadric $Q = \sum_{i,j=0}^{3} q_{ij}x_ix_j$ we first pick a point on Q with coordinates $(\xi_0{:}\ \xi_1{:}\xi_2{:}\xi_3)$. We can find this point by, for example, picking ξ_0, ξ_1, ξ_2 and solving for ξ_3, which is a linear equation.

The (projective) tangent plane to Q at ξ is given by the equation $\sum_{ij} q_{ij}x_i\xi_j = 0$. The intersection of this tangent plane with the quadric consists of two lines. Doing this three times we obtain six lines and we can choose three which are skew. Then we repeat this construction for Q_1, Q_2 and Q_3 to obtain the lines A_iB_i for $i = 1, \ldots, 9$.

In practice, we can build these lines using the Segre map. For example, we have

$$Q_1(x) = x_0(c_{01}x_1 + c_{02}x_2 + x_3) + c_{12}x_1x_2. \tag{89}$$

Then, the condition imposed by this quadric can be solved by

$$x_0 = \alpha_0\beta_0, \qquad c_{01}x_1 + c_{02}x_2 + x_3 = \alpha_1\beta_1, \tag{90}$$

$$-c_{12}x_1 = \alpha_0\beta_1, \qquad x_2 = \alpha_1\beta_0. \tag{91}$$

Then for every point C on the surface defined by Q_1 we can find coordinates $(\bar\alpha_0{:}\ \bar\alpha_1)$ and $(\bar\beta_0{:}\bar\beta_1)$. The lines through C are given by α being constant or β being constant. The points where Q_1 vanishes can then be parametrized by

$$x_0 = \alpha_0\beta_0, \qquad x_1 = -\frac{\alpha_0\beta_1}{c_{12}}, \tag{92}$$

$$x_2 = \alpha_1\beta_0, \qquad x_3 = \alpha_1\beta_1 + \frac{c_{01}}{c_{12}}\alpha_0\beta_1 - c_{02}\alpha_1\beta_0. \tag{93}$$

For Q_2 we have

$$Q_2(x) = x_1(c_{01}x_0 + c_{12}x_2 + x_3) + c_{02}x_0x_2. \tag{94}$$

We can then solve this constraint by

$$x_0 = \alpha_0\beta_0, \qquad x_1 = \alpha_0\beta_1, \tag{95}$$

$$x_2 = -\frac{\alpha_1\beta_1}{c_{02}}, \qquad x_3 = \alpha_1\beta_0 - c_{01}\alpha_0\beta_0 + c_{12}\frac{\alpha_1\beta_1}{c_{02}}. \tag{96}$$

Finally, for Q_3 we have

$$Q_3(x) = x_2(c_{02}x_0 + c_{12}x_1 + x_3) + c_{01}x_0x_1, \tag{97}$$

which can be solved by

$$x_0 = \alpha_0\beta_0, \qquad x_1 = -\frac{\alpha_1\beta_1}{c_{01}}, \tag{98}$$

$$x_2 = \alpha_0\beta_1, \qquad x_3 = \alpha_1\beta_0 - c_{02}\alpha_0\beta_0 + c_{12}\frac{\alpha_1\beta_1}{c_{01}}. \tag{99}$$

Next, we pick C to be one of the points $P_1, \ldots, P_8$ and find the points D_1, D_2, D_3. The point C can be either one of the two coinciding points $P_7 = P_8$ or one of the six conconic points $P_1, \ldots, P_6$. We consider one example of each kind.

If we pick $C = P_7 = (0{:}0{:}0{:}1)$, then we can take

$$D_1^{\pm} = \begin{cases} (0{:}0{:}\beta_0{:}\ \beta_1 - c_{02}\beta_0), \\ (0{:}-\alpha_0{:}0{:}c_{12}\alpha_1 + c_{01}\alpha_0), \end{cases} \tag{100}$$

$$D_2^{\pm} = \begin{cases} (0{:}0{:}-\beta_1{:}c_{02}\beta_0 + c_{12}\beta_1), \\ (\alpha_0{:}0{:}0{:}\alpha_1 - c_{01}\alpha_0), \end{cases} \tag{101}$$

$$D_3^{\pm} = \begin{cases} (0{:}-\beta_1{:}0{:}c_{01}\beta_0 + c_{12}\beta_1), \\ (\alpha_0{:}0{:}0{:}\alpha_1 - c_{02}\alpha_0). \end{cases} \tag{102}$$

We can in principle analyze all of these possibilities, but to illustrate let us just pick D_1^+, D_2^+ and D_3^+. Then, we can show that Eq. (69) holds as follows. We first notice that $\langle CD_2L_{21}\rangle = \langle CD_1L_{21}\rangle$ (up to a multiplicative factor) since C, D_1, D_2 are collinear. Then, $\langle CD_1L_{21}\rangle = 0$ since L_{21} is a line transversal to CD_1 in Q_2. Similarly, we have $\langle CD_1L_{12}\rangle = \langle CD_2L_{12}\rangle = 0$.

We can instead pick $C = P_1 = (1{:}0{:}0{:}0)$. Then we find

$$D_1^{\pm} = \begin{cases} (\alpha_0{:}\ 0{:}\alpha_1{:}-c_{02}\alpha_1), \\ \left(\beta_0{:}-\frac{\beta_1}{c_{12}}{:}0{:}\frac{c_{01}\beta_1}{c_{12}}\right), \end{cases} \tag{103}$$

$$D_2^{\pm} = \begin{cases} (\alpha_0{:}0{:}0{:}\alpha_1 - c_{01}\alpha_0), \\ (\beta_0{:}\beta_1{:}0{:}-c_{02}\beta_0), \end{cases} \tag{104}$$

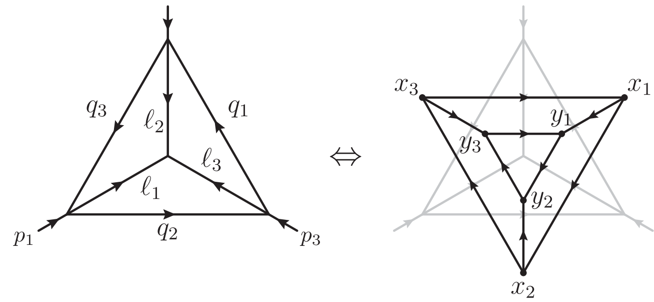
$$D_3^{\pm} = \begin{cases} (\alpha_0{:}0{:}0{:}\alpha_1 - c_{02}\alpha_0), \\ (\beta_0{:}0{:}\beta_1{:}-c_{02}\beta_0). \end{cases} \tag{105}$$

As before, let us pick the case D_1^+, D_2^+ and D_3^+ to analyze in detail. Here we have that $D_2^+ = D_3^+$. Then, we have $\langle CD_2L_{13}\rangle = \langle CD_3L_{13}\rangle = 0$ and $\langle CD_3L_{12}\rangle = \langle CD_2L_{12}\rangle = 0$. Plugging this in Eq. (69) we find that it is satisfied.

Checking for the other cases, we find that, in general, making two of the points $P_1 \ldots P_8$ coincident lands us on the Landau locus, as expected.

V. *EXEMPLI GRATIA*: A SEXTIC IN TWO DIMENSIONS

Let us now consider the following integral for two-dimensional kinematics:



(106)

This integral involves six propagators and three, two-dimensional loop momenta; as such, its leading singularities would correspond to residues around which all propagators become on-shell. How many solutions to the cut equations do we get?

Let us consider the case where all external momenta are massive, with $p_i^2 =: M_i^2$, and for the "all-mass" case of internal propagators: where the q_i propagators have poles at $q_i^2 =: m_i^2$ and the ℓ_i propagators have poles at $\ell_i^2 =: \mu_i^2$.

To determine the leading singularities of this Feynman integral, it is useful to introduce dual coordinates $p_1 =: x_3 - x_2$, etc., and $\ell_1 =: y_3 - y_2$, etc. [43]; in terms of these, we have $q_i = y_i - x_i$. Without loss of generality, we may translate the x_i's and express them in light-cone coordinates so that

$$x_1 := (0,0), \quad x_2 = (M_3/M_2, M_3M_2), \quad x_3 =: (x_3^+, M_2^2/x_3^+), \tag{107}$$

where

$$x_3^+ + \frac{1}{x_3^+} := \frac{M_2^2 + M_3^2 - M_1^2}{M_2M_3}. \tag{108}$$

Next, we may parametrize the solutions to the on-shell conditions for q_1, ℓ_2 and ℓ_3 by expressing

$$\begin{aligned} &y_1 := (y_1^+, m_1^2/y_1^+), \qquad y_2 =: (\psi_2^+, \mu_3^2/\psi_2^+) + y_1, \\ &y_3 =: (\psi_3^+, \mu_2^2/\psi_3^+) + y_1. \end{aligned} \tag{109}$$

In terms of these variables, the on-shell condition for ℓ_1 ($\ell_1^2 = \mu_1^2$) now reads

$$\ell_1^2 = (y_3 - y_2)^2 = (\psi_3^+ - \psi_2^+)\left(\frac{\mu_3^2}{\psi_2^+} - \frac{\mu_2^2}{\psi_2^+}\right), \tag{110}$$

which can be reexpressed as the condition that

$$\frac{\mu_3\psi_3^+}{\mu_2\psi_2^+}+\left(\frac{\mu_3\psi_3^+}{\mu_2\psi_2^+}\right)^{-1}=\frac{\mu_2^2+\mu_3^2-\mu_1^2}{\mu_2\mu_3}. \tag{111}$$

This leaves only the final two on-shell conditions

$$q_2^2=m_2^2=(x_2-y_2)^2 \quad \text{and} \quad q_3^2=m_3^2=(x_3-y_3)^2 \tag{112}$$

to solve.

Let us define

$$a:=x_3^+,\qquad b:=\frac{\mu_3\psi_3^+}{\mu_2\psi_2^+},\quad \text{and let } x:=\frac{\mu_2}{M_2\psi_2^+},$$
$$y:=\frac{M_2y_1^+}{m_1}. \tag{113}$$

We may think of a, b as being fixed by external kinematics since

$$a+a^{-1}=\frac{M_2^2+M_3^2-M_1^2}{M_2M_3}\quad \text{and}\quad b+b^{-1}=\frac{\mu_2^2+\mu_3^2-\mu_1^2}{\mu_2\mu_1}, \tag{114}$$

leaving us only with x and y to determine using the final equations (112).

In terms of x, y, the final-cut conditions (112) are given by

$$\begin{aligned}m_2^2&=M_3^2+\mu_3^2+m_1^2+m_1\mu_3\left(xy+\frac{1}{xy}\right)-M_3\mu_3\left(x+\frac{1}{x}\right)\\&\quad-m_1M_3\left(y+\frac{1}{y}\right),\\m_3^2&=M_2^2+\mu_2^2+m_1^2+m_1\mu_2\left(\frac{xy}{b}+\frac{b}{xy}\right)-M_2\mu_2\left(\frac{b}{ax}+\frac{ax}{b}\right)\\&\quad-m_1M_2\left(\frac{y}{a}+\frac{a}{y}\right).\end{aligned} \tag{115}$$

To count the number of solutions to these cut equations, we may proceed as follows. Notice that the equations (115) take the following form in y:

$$\begin{aligned}0&=A_1(x)+B_1(x)y+C_1(x)y^{-1},\\0&=A_2(x)+B_2(x)y+C_2(x)y^{-1},\end{aligned} \tag{116}$$

where A_i, B_i, C_i are some rational functions of x. Such a system which is linear in y and y^{-1} may be solved with the compatibility condition

$$\begin{vmatrix}A_1 & C_1\\ A_2 & C_2\end{vmatrix}\begin{vmatrix}B_1 & A_1\\ B_2 & A_2\end{vmatrix}=\begin{vmatrix}B_1 & C_1\\ B_2 & C_2\end{vmatrix}^2 \tag{117}$$

which takes the general form (in terms of x) of

$$c_3x^3+c_2x^2+c_1x+c_0+c_{-1}x^{-1}+c_{-2}x^{-2}+c_{-3}x^{-3}=0, \tag{118}$$

where the coefficients $c_{-3},\ldots,c_3$ are rational functions of the external kinematic variables a, b, μ_i, m_i, M_i. It is clear that there are generally six solutions in x to (118), and for each of them we may uniquely specify the corresponding point y. Therefore, there are six roots in (x,y) to the final-cut conditions.

Explicit expressions for the coefficients c_i in (118) can be given but turn out to be rather cumbersome; for example,

$$\begin{aligned}c_3=&-\frac{M_2^2M_3a^2m_1^2\mu_2\mu_3^2}{b}+\frac{M_2^2M_3a^2m_1^2\mu_2^2\mu_3}{b^2}\\&+\frac{M_2M_3^2am_1^2\mu_2\mu_3^2}{b}-\frac{M_2M_3^2am_1^2\mu_2^2\mu_3}{b^2}.\end{aligned}$$

In all, the coefficients $c_{-3},\ldots,c_3$ involve sums of 4, 29, 86, 124, 86, 29, and 4 monomials, respectively. We have checked (using generic numerical values for the masses) that the roots of the sextic (118) are not expressible in terms of radicals.

The Landau equations imply that singularities of the diagram occur when some of these six roots coincide, and this happens when the discriminant of the degree-six equation in Eq. (118) vanishes. The discriminant of a polynomial of degree n is homogeneous in the coefficients with degree $2n-2$, so for a degree-six polynomial we obtain a discriminant of degree ten. For our case the discriminant is a degree-ten polynomial in $c_{-3},\ldots,c_3$ with 246 terms [44].

Let us now discuss the Landau loop equations. The Landau equations read

$$\begin{aligned}0&=\alpha_2q_2+\beta_3\ell_3-\beta_1\ell_1,\\0&=\alpha_1q_1+\beta_2\ell_2-\beta_3\ell_3,\\0&=\alpha_3q_3+\beta_1\ell_1-\beta_2\ell_2.\end{aligned} \tag{119}$$

Okun and Rudik (see Ref. [46]) take the cross-products [47] with q_i, which removes the terms dependent on α. Doing this we obtain

$$\begin{aligned}\beta_3[\ell_3q_2]&=\beta_1[\ell_1q_2],\\\beta_2[\ell_2q_1]&=\beta_3[\ell_3q_1],\\\beta_1[\ell_1q_3]&=\beta_2[\ell_2q_3].\end{aligned} \tag{120}$$

Taking the product and simplifying we obtain

$$[\ell_1q_3][\ell_2q_1][\ell_3q_2]=[\ell_1q_2][\ell_2q_3][\ell_3q_1]. \tag{121}$$

This has the geometrical interpretation that the lines through x_i and y_i intersect in a single point. Let us show that this is indeed the case.

The lines through the points (x_i, y_i) intersect in a single point if there exist t_1, t_2 and t_3 such that

$$y_1 + t_1(x_1 - y_1) = y_2 + t_2(x_2 - y_2) = y_3 + t_3(x_3 - y_3). \tag{122}$$

Using the fact that $q_i = y_i - x_i$, $\ell_1 = y_3 - y_2$, $\ell_2 = y_1 - y_3$ and $\ell_3 = y_2 - y_1$ we find

$$\begin{aligned} &y_1 - y_2 + t_1(x_1 - y_1) - t_2(x_2 - y_2) = 0, \\ &y_2 - y_3 + t_2(x_2 - y_2) - t_3(x_3 - y_3) = 0, \\ &y_3 - y_1 + t_3(x_3 - y_3) - t_1(x_1 - y_1) = 0, \end{aligned} \tag{123}$$

which implies

$$\begin{aligned} &-\ell_3 - t_1 q_1 + t_2 q_2 = 0, \\ &-\ell_1 - t_2 q_2 + t_3 q_3 = 0, \\ &-\ell_2 - t_3 q_3 + t_1 q_1 = 0. \end{aligned} \tag{124}$$

Taking the cross-product with the l_i, separating the remaining terms in the left-hand side and right-hand side and taking their product we obtain

$$[q_1\ell_3][q_2\ell_1][q_3\ell_2] = [q_2\ell_3][q_3\ell_1][q_1\ell_2]. \tag{125}$$

This equation and its derivation look similar to Eq. (69) arising in the $\mathcal{N} = 4$ super Yang-Mills theory.

We have presented this example as two-dimensional (massive) cousin of our four-dimensional (massless) examples; indeed, the pentagon loops in four dimensions are similar to triangle loops in two dimensions. Due to the fact that it arises already at three points, it is possible to analyze this example in full detail, with fewer algebraic complexities. The geometry of the problem is also much easier to understand thanks to the two-dimensional nature of the problem.

This example also serves to show that higher-order polynomials arise quite generically once the equations enforcing the on-shell condition are sufficiently coupled. As such, we expect such examples to become more common in the literature as the community investigates more complicated diagrams.

VI. CONCLUSIONS AND DISCUSSION

As we have shown, leading singularities in Feynman diagrams can indeed involve cubic or higher roots, contrary to the naive expectation one might have from Landau's analysis. However, Landau's analysis is still correct: as we have argued, these cubic or higher roots do not lead to forbidden behavior on codimension-one singularities, because any such singularity will only make two roots coincide. To make more roots coincide requires a singularity of higher codimension. We have illustrated this behavior in three concrete examples, a cubic root in a diagram in planar $\mathcal{N} = 4$ super Yang-Mills, roots of a pair of octic polynomials for a more general diagram in the same theory, and roots of a sextic polynomial for massive scalars in two dimensions.

The existence of cubic-root and more unusual singularities in higher codimension limits may have several implications. It would be interesting to see how their existence interacts with approaches that attempt to derive, not just leading, but iterative singularities of general Feynman diagrams [17,48,49], in which one would expect to need to take into account these singularities of higher codimension in some way. They should also be relevant for series expansions of Feynman diagrams. In particular, there should be special kinematic limits in which these codimension-two limits are uncovered as leading behavior. It would be interesting to see if there is a kinematic limit of physical interest in which these singularities are especially relevant.

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APPENDIX A: EXTENDED DISCUSSION OF THE LEADING SINGULARITY

Due to our choice of a blue vertex in the center of the on-shell diagram in Eq. (15), we have $\ell_1 = A \wedge B$, $\ell_2 = B \wedge C$ and $\ell_3 = A \wedge C$. This solves the on-shell conditions for the internal lines. We are left with the following on-shell conditions:

$$\langle AB12 \rangle = \langle AB34 \rangle = 0, \tag{A1}$$

$$\langle BC45 \rangle = \langle BC67 \rangle = \langle BC89 \rangle = 0, \tag{A2}$$

$$\langle AC12 \rangle = \langle AC89 \rangle = \langle AC, 10, 11 \rangle = 0. \tag{A3}$$

Consider the following two equalities: $\langle BC89 \rangle = \langle AC89 \rangle = 0$. If C and z_8 and z_9 are not collinear, then we have that A and B belong to the plane spanned by C and z_8 and z_9. Equivalently, we can say that z_8 and z_9 belong to the plane spanned by A, B and C.

Similarly, from $\langle AC12\rangle = \langle AB12\rangle = 0$ if A and z_1 and z_2 are not collinear, then we have that B and C belong to the plane spanned by A and z_1 and z_2. Equivalently, we have that z_1, z_2 belong to the plane spanned by A, B and C. This means that the lines $z_8 \wedge z_9$ and $z_1 \wedge z_2$ belong to the same plane $A \wedge B \wedge C$, which means that they must intersect. But this is only possible if $\langle 1289\rangle = 0$.

So instead, let us take A on the line $z_1 \wedge z_2$ and C on the line $z_8 \wedge z_9$. This means that the two three-point vertices neighboring the middle three-point vertex are white, which implies that the lines $A \wedge B$ and $A \wedge C$ and $z_1 \wedge z_2$ intersect in a point and similarly, the lines $A \wedge C$, $B \wedge C$ and $z_8 \wedge z_9$ intersect in a point.

We have the following constraints left to satisfy:

$$\langle BC45\rangle = \langle BC67\rangle = \langle AC, 10, 11\rangle = \langle AB34\rangle = 0. \quad \text{(A4)}$$

Since A belongs to the line $z_1 \wedge z_2$ we have $A = z_2 + \alpha z_1 = \hat{2}$ for some complex α. Next, the constraint $\langle AC, 10, 11\rangle = 0$ can be solved by taking C to belong to the plane $A \wedge z_{10} \wedge z_{11}$. Since C also belongs to the line $z_8 \wedge z_9$, we have $C = (89) \cap (\hat{2}, 10, 11) = -\hat{9}$.

The three remaining constraints involve the point B: $\langle BC45\rangle = \langle BC67\rangle = \langle AB34\rangle = 0$. Geometrically this means that B belongs to the planes $C \wedge z_4 \wedge z_5$, $C \wedge z_6 \wedge z_7$ and $A \wedge z_3 \wedge z_4$, so it must belong to their intersection

$$B = (C45) \cap (C67) \cap (A34) = (\hat{9}45) \cap (\hat{9}67) \cap (\hat{2}34).$$

If desired, the intersection $(\hat{9}45) \cap (\hat{9}67)$ can be expanded as

$$(\hat{9}45) \cap (\hat{9}67) = z_{\hat{9}} \wedge z_4 \langle 5\hat{9}67\rangle - z_{\hat{9}} \wedge z_5 \langle 4\hat{9}67\rangle = -z_{\hat{9}} \wedge z_{\hat{5}},$$

where we have introduced $z_{\hat{5}} = (54) \cap (67\hat{9}) = z_5 \langle 467\hat{9}\rangle - z_4 \langle 567\hat{9}\rangle$.

Each hatted variable is linear in α so B is cubic in α. The final on-shell condition we impose is $\langle AB23\rangle = 0$ which is $\alpha\langle 123B\rangle = 0$. If we take $\alpha \neq 0$, we obtain a cubic polynomial in α from $\langle 123B\rangle = 0$.

APPENDIX B: EXPLICIT PARAMETRIZATION OF THE CUBIC ROOT

In this appendix we present a few expressions that were too long for the main text, regarding the cluster parametrization in Sec. III C.

First, we give our parametrization for the momentum twistors in terms of our cluster chart, as these were too long for the main text:

$$Z_1 = (2e_6 + 1, e_7(e_2(e_8 + 1) + 5), 0, 0), \quad \text{(B1)}$$

$$Z_2 = (1, 0, 0, 0), \quad \text{(B2)}$$

$$Z_3 = (0, 0, 0, 1), \quad \text{(B3)}$$

$$Z_4 = \left(0, 0, \frac{3}{2} e_2 e_4 e_{10}, (e_2(2e_7 e_6 + e_6 + 1)e_4 + e_4 + 1)e_{10} + 1\right), \quad \text{(B4)}$$

$$\begin{aligned} Z_5 = \Big(&0, -e_1 e_2^2 e_3 e_4 e_8 (e_2(e_8 + 1) + 5)e_{10}, \\ &\frac{3}{2} e_2 e_4((e_3(e_5 + 1)e_1 + e_1 + 1)e_8 e_2 + e_2 + 5)e_{10}, \\ &e_2 + ((e_2 + 5)(e_2(2e_7 e_6 + e_6 + 1)e_4 + e_4 + 1) \\ &+ e_2(e_4(e_3 + e_2((e_3 + 1)(e_6 + 1) + 2(e_3(e_5 + 1) + 1)e_6 e_7) + 1)e_1 \\ &+ e_1 + e_4 + e_2 e_4(2e_7 e_6 + e_6 + 1) + 1)e_8)e_{10} + 5\Big), \end{aligned} \quad \text{(B5)}$$

$$\begin{aligned}Z_6 = \Big(&-e_2e_3e_4^2e_8e_9^2e_{10},\\
&-e_2e_3e_4(e_2(e_8+1)+5)((e_8((e_7+1)e_9e_4+e_4+1)+1)e_{10}e_9+e_9+5),\\
&\frac{3}{2}e_2e_4(e_9+e_3((e_5+1)(e_9+5)\\
&+e_9(e_8(e_4(e_9+1)+1)e_5+e_5+e_8+1)e_{10})+5),\\
&e_4((e_3+e_2((e_3+1)(e_6+1)+2(e_3(e_5+1)+1)e_6e_7)+1)(e_9+5)\\
&+e_3e_9(e_2(e_8+e_6(e_8+e_7(2(e_8+1)\\
&+e_5(e_8(e_4(e_9+2)+2)+2))+1)+1)+1)e_{10})+5\Big),\end{aligned} \tag{B6}$$

$$\begin{aligned}Z_7 = \Big(&-e_2e_3e_4^2e_8e_9^2e_{10},\\
&-e_2e_3e_4(e_2(e_8+1)+5)((e_8((e_7+1)e_9e_4+e_4+1)+1)e_{10}e_9+e_9+4),\\
&\frac{3}{2}e_2e_4(e_9+e_3((e_5+1)(e_9+4)\\
&+e_9(e_8(e_4(e_9+1)+1)e_5+e_5+e_8+1)e_{10})+4),\\
&e_4((e_3+e_2((e_3+1)(e_6+1)+2(e_3(e_5+1)+1)e_6e_7)+1)(e_9+4)\\
&+e_3e_9(e_2(e_8+e_6(e_8+e_7(2(e_8+1)\\
&+e_5(e_8(e_4(e_9+2)+2)+2))+1)+1)+1)e_{10})+4\Big),\end{aligned} \tag{B7}$$

$$\begin{aligned}Z_8 = \Big(&-(e_2+4)e_4e_9, -(e_2(e_8+1)+5)((e_2+4)e_4((e_7+1)e_9+1)+4),\\
&\frac{3}{2}e_5((e_2+4)e_4(e_9+1)+4)+6,\\
&e_6(e_7(e_5((e_2+4)e_4(e_9+2)+8)+8)+4)+3\Big),\end{aligned} \tag{B8}$$

$$\begin{aligned}Z_9 = \Big(&-(e_2+3)e_4e_9, -(e_2(e_8+1)+5)((e_2+3)e_4((e_7+1)e_9+1)+3),\\
&\frac{3}{2}(e_5((e_2+3)e_4(e_9+1)+3)+3),\\
&e_6(e_7(e_5((e_2+3)e_4(e_9+2)+6)+6)+3)+2\Big),\end{aligned} \tag{B9}$$

$$Z_{10} = \left(-1, -\frac{1}{3}(3e_7+2)(e_2(e_8+1)+5), e_5, \frac{1}{3}e_5e_6e_7\right), \tag{B10}$$

$$Z_{11} = (-4, -2(2e_7+1)(e_2(e_8+1)+5), 3e_5, 0). \tag{B11}$$

Second, we presented our cubic polynomial in the cluster coordinates in Eq. (27) in terms of coefficients c_i. These were also too long for the main text and are presented here:

$$\begin{aligned}
c_0 = {} & e_4e_2^2+e_1e_4e_8e_2^2+e_1e_3e_4e_8e_2^2+e_4e_8e_2^2+e_4e_9e_2^2+e_4e_7e_9e_2^2+e_1e_4e_8e_9e_2^2\\
&+e_1e_3e_4e_8e_9e_2^2+e_4e_8e_9e_2^2+e_1e_4e_7e_8e_9e_2^2+e_1e_3e_4e_7e_8e_9e_2^2+e_4e_7e_8e_9e_2^2\\
&+e_1e_3e_4e_5e_7e_8e_9e_2^2+2e_4e_2+e_1e_8e_2+e_1e_4e_8e_2+e_1e_3e_4e_8e_2+e_4e_8e_2+e_8e_2\\
&+2e_4e_9e_2+2e_4e_7e_9e_2+e_1e_4e_8e_9e_2+e_1e_3e_4e_8e_9e_2+e_4e_8e_9e_2+e_1e_4e_7e_8e_9e_2\\
&+e_1e_3e_4e_7e_8e_9e_2+e_4e_7e_8e_9e_2+e_1e_3e_4e_5e_7e_8e_9e_2+e_2+e_4+e_4e_9+e_4e_7e_9+1,
\end{aligned} \tag{B12}$$

$$\begin{aligned}
c_1 = {} & e_4e_2^2+4e_4e_6e_2^2+e_1e_4e_8e_2^2+e_1e_3e_4e_8e_2^2+e_4e_8e_2^2+4e_1e_4e_6e_8e_2^2+4e_1e_3e_4e_6e_8e_2^2\\
&+4e_4e_6e_8e_2^2+e_4e_9e_2^2+4e_4e_6e_9e_2^2+4e_4e_6e_7e_9e_2^2+e_1e_4e_8e_9e_2^2+e_1e_3e_4e_8e_9e_2^2\\
&+e_4e_8e_9e_2^2+4e_1e_4e_6e_8e_9e_2^2+4e_1e_3e_4e_6e_8e_9e_2^2+4e_4e_6e_8e_9e_2^2+4e_1e_4e_6e_7e_8e_9e_2^2\\
&+4e_1e_3e_4e_6e_7e_8e_9e_2^2+4e_4e_6e_7e_8e_9e_2^2+4e_1e_3e_4e_5e_6e_7e_8e_9e_2^2+2e_4e_2+10e_4e_6e_2\\
&+5e_6e_2+e_1e_8e_2+e_1e_4e_8e_2+e_1e_3e_4e_8e_2+e_4e_8e_2+5e_1e_6e_8e_2+5e_1e_4e_6e_8e_2\\
&+5e_1e_3e_4e_6e_8e_2+5e_4e_6e_8e_2+5e_6e_8e_2+e_8e_2+2e_4e_9e_2+10e_4e_6e_9e_2\\
&+10e_4e_6e_7e_9e_2+e_1e_4e_8e_9e_2+e_1e_3e_4e_8e_9e_2+e_4e_8e_9e_2+5e_1e_4e_6e_8e_9e_2\\
&+5e_1e_3e_4e_6e_8e_9e_2+5e_4e_6e_8e_9e_2+5e_1e_4e_6e_7e_8e_9e_2+5e_1e_3e_4e_6e_7e_8e_9e_2\\
&+5e_4e_6e_7e_8e_9e_2+5e_1e_3e_4e_5e_6e_7e_8e_9e_2+e_2+e_4+6e_4e_6+6e_6+e_4e_9\\
&+6e_4e_6e_9+6e_4e_6e_7e_9+1,
\end{aligned} \tag{B13}$$

$$\begin{aligned}
c_2 = {} & e_6(2e_4e_6^2e_2^2+e_4e_2^2+4e_4e_6e_2^2+2e_1e_4e_6^2e_8e_2^2+2e_1e_3e_4e_6^2e_8e_2^2+2e_4e_6^2e_8e_2^2\\
&+e_1e_4e_8e_2^2+e_1e_3e_4e_8e_2^2+e_4e_8e_2^2+4e_1e_4e_6e_8e_2^2+4e_1e_3e_4e_6e_8e_2^2+4e_4e_6e_8e_2^2\\
&+2e_4e_6^2e_9e_2^2+e_4e_9e_2^2+4e_4e_6e_9e_2^2+2e_4e_6^2e_7e_9e_2^2+3e_4e_6e_7e_9e_2^2+2e_1e_4e_6^2e_8e_9e_2^2\\
&+2e_1e_3e_4e_6^2e_8e_9e_2^2+2e_4e_6^2e_8e_9e_2^2+e_1e_4e_8e_9e_2^2+e_1e_3e_4e_8e_9e_2^2+e_4e_8e_9e_2^2\\
&+4e_1e_4e_6e_8e_9e_2^2+4e_1e_3e_4e_6e_8e_9e_2^2+4e_4e_6e_8e_9e_2^2+2e_1e_4e_6^2e_7e_8e_9e_2^2\\
&+2e_1e_3e_4e_6^2e_7e_8e_9e_2^2+2e_4e_6^2e_7e_8e_9e_2^2+2e_1e_3e_4e_5e_6^2e_7e_8e_9e_2^2+3e_1e_4e_6e_7e_8e_9e_2^2\\
&+3e_1e_3e_4e_6e_7e_8e_9e_2^2+3e_4e_6e_7e_8e_9e_2^2+3e_1e_3e_4e_5e_6e_7e_8e_9e_2^2+8e_4e_6^2e_2+4e_6^2e_2\\
&+3e_4e_2+14e_4e_6e_2+6e_6e_2+4e_1e_6^2e_8e_2+4e_1e_4e_6^2e_8e_2+4e_1e_3e_4e_6^2e_8e_2\\
&+4e_4e_6^2e_8e_2+4e_6^2e_8e_2+e_1e_8e_2+e_1e_4e_8e_2+e_1e_3e_4e_8e_2+e_4e_8e_2+6e_1e_6e_8e_2\\
&+6e_1e_4e_6e_8e_2+6e_1e_3e_4e_6e_8e_2+6e_4e_6e_8e_2+6e_6e_8e_2+e_8e_2+8e_4e_6^2e_9e_2\\
&+3e_4e_9e_2+14e_4e_6e_9e_2+8e_4e_6^2e_7e_9e_2+10e_4e_6e_7e_9e_2+4e_1e_4e_6^2e_8e_9e_2\\
&+4e_1e_3e_4e_6^2e_8e_9e_2+4e_4e_6^2e_8e_9e_2+e_1e_4e_8e_9e_2+e_1e_3e_4e_8e_9e_2+e_4e_8e_9e_2\\
&+6e_1e_4e_6e_8e_9e_2+6e_1e_3e_4e_6e_8e_9e_2+6e_4e_6e_8e_9e_2+4e_1e_4e_6^2e_7e_8e_9e_2\\
&+4e_1e_3e_4e_6^2e_7e_8e_9e_2+4e_4e_6^2e_7e_8e_9e_2+4e_1e_3e_4e_5e_6^2e_7e_8e_9e_2+4e_1e_4e_6e_7e_8e_9e_2\\
&+4e_1e_3e_4e_6e_7e_8e_9e_2+4e_4e_6e_7e_8e_9e_2+4e_1e_3e_4e_5e_6e_7e_8e_9e_2+e_2+8e_4e_6^2+8e_6^2\\
&+2e_4+12e_4e_6+12e_6+8e_4e_6^2e_9+2e_4e_9+12e_4e_6e_9+8e_4e_6^2e_7e_9+8e_4e_6e_7e_9+2),
\end{aligned} \tag{B14}$$

$$
\begin{aligned}
c_3 = e_6^2(&e_4e_2^2 + 2e_4e_6e_2^2 + e_1e_4e_8e_2^2 + e_1e_3e_4e_8e_2^2 + e_4e_8e_2^2 + 2e_1e_4e_6e_8e_2^2 \\
&+ 2e_1e_3e_4e_6e_8e_2^2 + 2e_4e_6e_8e_2^2 + e_4e_9e_2^2 + 2e_4e_6e_9e_2^2 + 2e_4e_6e_7e_9e_2^2 + e_1e_4e_8e_9e_2^2 \\
&+ e_1e_3e_4e_8e_9e_2^2 + e_4e_8e_9e_2^2 + 2e_1e_4e_6e_8e_9e_2^2 + 2e_1e_3e_4e_6e_8e_9e_2^2 + 2e_4e_6e_8e_9e_2^2 \\
&+ 2e_1e_4e_6e_7e_8e_9e_2^2 + 2e_1e_3e_4e_6e_7e_8e_9e_2^2 + 2e_4e_6e_7e_8e_9e_2^2 + 2e_1e_3e_4e_5e_6e_7e_8e_9e_2^2 \\
&+ 4e_4e_2 + 8e_4e_6e_2 + 4e_6e_2 + 2e_1e_8e_2 + 2e_1e_4e_8e_2 + 2e_1e_3e_4e_8e_2 + 2e_4e_8e_2 \\
&+ 4e_1e_6e_8e_2 + 4e_1e_4e_6e_8e_2 + 4e_1e_3e_4e_6e_8e_2 + 4e_4e_6e_8e_2 + 4e_6e_8e_2 + 2e_8e_2 \\
&+ 4e_4e_9e_2 + 8e_4e_6e_9e_2 + 8e_4e_6e_7e_9e_2 + 2e_1e_4e_8e_9e_2 + 2e_1e_3e_4e_8e_9e_2 + 2e_4e_8e_9e_2 \\
&+ 4e_1e_4e_6e_8e_9e_2 + 4e_1e_3e_4e_6e_8e_9e_2 + 4e_4e_6e_8e_9e_2 + 4e_1e_4e_6e_7e_8e_9e_2 \\
&+ 4e_1e_3e_4e_6e_7e_8e_9e_2 + 4e_4e_6e_7e_8e_9e_2 + 4e_1e_3e_4e_5e_6e_7e_8e_9e_2 + 2e_2 + 4e_4 + 8e_4e_6 \\
&+ 8e_6 + 4e_4e_9 + 8e_4e_6e_9 + 8e_4e_6e_7e_9 + 4).
\end{aligned} \tag{B15}
$$

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[21] This means that the cotangent space along the locus where three roots coincide is ill defined. In turn this implies that the tangent space is ill defined and therefore the surface has a singularity.
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[28] One might also wonder about singularities in which C diverges rather than vanishes. A similar calculation shows that these also cannot contribute a cubic-root singularity.

[29] Implementing the problem in Magma [30] in terms of a matrix in the appropriate quotient ring ran for six days on a computer with a 2.3 GHz 18-core processor and used approximately 30 GB of memory without finishing.

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