Vacuum energy with nonideal boundary conditions via an approximate functional equation

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We discuss the vacuum energy of a quantized scalar field in the presence of classical surfaces, defining bounded domains $\Omega \subset \mathbb{R}^d$, where the field satisfies ideal or nonideal boundary conditions. We call it *ideal high-pass Dirichlet* boundary conditions. For the electromagnetic case, this situation describes the conductivity correction to the zero-point energy. Using an analytic regularization procedure, we obtain the vacuum energy for a massless scalar field at zero temperature in the presence of a slab geometry $\Omega = \mathbb{R}^{d-1} \times [0, L]$ with Dirichlet boundary conditions. To discuss the case of nonideal boundary conditions, we employ an asymptotic expansion, based on an approximate functional equation for the Riemann zeta-function, where finite sums outside their original domain of convergence are defined. Finally, to obtain the Casimir energy for a massless scalar field in the presence of a rectangular box, with lengths L_1 and L_2, i.e., $\Omega = [0, L_1] \times [0, L_2]$ with nonideal boundary conditions, we employ an approximate functional equation of the Epstein zeta-function.

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I. INTRODUCTION

Quantum fields are fundamental mathematical objects in the description of natural phenomena. These objects are operator-valued generalized functions with test functions, i.e., distributions in the Schwartz space [1,2]. As a consequence, in Minkowski spacetime, it has been shown that the renormalized vacuum expectation value of a quantum free scalar field stress-energy tensor can exhibit a local negative energy density [3]. In other words, although the energy operator associated with a quantum scalar field is self-adjoint and positive, the (0–0) component of the stress-energy tensor can be negative. The Casimir effect is a measurable macroscopic manifestation of this result [4–11]. It has been measured in different geometric configurations [12–15]. The physical origin of the effect is the changes in the vacuum modes associated with the quantized electromagnetic field by the presence of macroscopic surfaces. The vacuum expectation values of the electric field at distinct points separated by spacelike distances are correlated, like the interaction between atomic dipoles induced by the electromagnetic vacuum field (van der Waals forces). Additionally, any constrained field, such as a massless fermionic field, can be a source of the effect as a consequence of the interaction of quantum field vacuum modes with idealized classical surfaces [16]. Another example is the phononic Casimir effect, where the speed of light is replaced by the speed of sound in the medium in the quasiparticle Landau scenario [17].

In the canonical formalism for bosonic and fermionic fields, vacuum energies are divergent. To obtain finite results, different approaches have been developed. One approach analyzes the local energy densities of quantized fields [18–24]. Another one, known as the global approach, investigates the total energy of the quantized field with idealized boundary conditions [25–27]. This approach uses two natural ways to regularize and renormalize the divergent vacuum energy. The first one is the cutoff method, where an ultraviolet regulator function is introduced in the divergent sum of the eigenfrequencies. On general grounds, the regularized vacuum energy exhibits Weyl's terms with a geometric origin, cutoff independent contributions, and terms that vanish as the cutoff is removed. With these geometric terms in hand, we can implement a renormalization procedure with the introduction of auxiliary boundaries and subtraction of the regularized energies of different configurations. The second one are analytic regularization procedures. One is the spectral zeta-function regularization that was constructed to make sense of functional determinants.Fundamental references in the subject are the Refs. [28–30]. Using this procedure, the

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free energy of Euclidean quantum fields can be calculated [31–33]. Another analytic regularization procedure has also been discussed in [34,35]. Although the cutoff method with the auxiliary configurations and the analytic regularization discussed above are quite different in their grounds, it is possible to compare them and shown to be analytically equivalent in some specific situations [36–39].

On the other hand, on physical grounds, the ideal boundary conditions of the perfect conductivity for all electromagnetic field modes is an idealization. Usually, metallic plates behave as dielectric for high-frequency modes, and as conductors for infrared modes. Following the original formulation, the question of the conductivity correction to the electromagnetic Casimir force arises. To derive this correction, Lifshitz proposed a model treating the electromagnetic field as a classical field, where attractive or repulsive forces arise from the fluctuating charges and currents of the boundaries [40]. Further references include [41–47]. The vacuum energy of scalar quantum fields in the presence of surfaces with general boundary conditions has been discussed in Refs. [48–53]. Additionally the literature has been discussing different situations where the boundary conditions are modeled by some smooth increasing potential functions [54,55]. R. Jaffe and collaborators studied the situation of coupling the quantum field to a background potential. The idealized boundary condition is obtained in some limit [56,57]. Another way of treating idealized semitransparent conductors is to use δ potentials interacting with the quantum field, for, e.g., see Refs. [58–61].

The purpose of this work is to discuss the Casimir energy of a massless scalar field at zero temperature satisfying nonideal boundary conditions. Due to the similarity between the quantized electromagnetic field and massless scalar fields satisfying Dirichlet and Neumann boundary conditions, our problem has formal similarities with the conductivity correction to the Casimir force of the quantized electromagnetic field. A first idea is to describe finite conductivity using microscopic models. A microscopic approach have been extensive studied by G. Barton, for, e.g., see, Refs. [62–64]. Also the case of QED in the dielectric matter background has been analyzed, where various quantization scheme have been proposed. For the nonlinear case see the Refs. [65–67] and, for the dispersive case see Ref. [68].

Instead of discussing the nonlinear problem of the microscopic modeling of finite conductivity, i.e., nonideal boundary conditions, we confine ourselves to make use of spectral theory of elliptic differential operators. The situation of corrections to the Casimir force can be discussed using an analytic regularization procedure and approximate functional equations of spectral zeta-functions. These functional equations can express the Riemann and Epstein zeta-functions as finite sums outside their original domain of convergence. Connection between number theory and quantum field theory is presented in the literature, as can been seem in the known arithmetic quantum theory [69–74].

In our methodology, we use the fact that the total renormalized energy of scalar fields in the presence of bounded domains can always be derived using an analytic regularization procedure, where the Dirichlet and Neumann Laplacian are used. It is known that the vacuum energy in the slab geometry $\mathbb{R}^{d-1} \times [0, L]$ with Dirichlet boundary conditions can be written in terms of the Riemann zeta-function. To calculate its correction due to nonideal boundary conditions, we represent the energy density using an asymptotic expansion derived by Hardy and Littlewood. They obtained an approximate functional equation for the Riemann zeta-function written as finite sums beyond their original domain of convergence [75]. Next, we generalize the previous result in the case of a field in the presence of a rectangular box with lengths L_1 and L_2 with nonideal boundary conditions. For instance, other generalizations of the Riemann functional equation have been presented in the literature. Recently it was discussed the introduction in the integral representation of the zeta-function, different cutoffs that are invariant under the transformation $x \mapsto 1/x$. It has been shown that the Riemann functional equation can be generalized with the same symmetry $s \to (1 - s)$ in the critical strip [76].

This paper is organized as follows. In Sec. II we discuss the asymptotic behavior of the eigenvalues of the Helmholtz wave equation, the Minakshisundaram-Pleijel zeta-function, the spectral decomposition of the heat kernel and classical spectral invariants. In Sec. III we discuss how to obtain the renormalized vacuum energy for a massless scalar field at zero temperature in the presence of perfect mirrors. In Sec. IV we use an approximate functional equation to obtain renormalized vacuum energy, due the ideal high-pass Dirichlet boundary conditions for a slab geometry $\mathbb{R}^{d-1} \times [0, L]$. In Sec. V we employ the same method to obtain the renormalized vacuum energy for the case of a massless scalar field confined in a rectangular box, with lengths L_1 and L_2 with ideal and nonideal boundary conditions. Conclusions are given in Sec. VI. Here we are using that $\hbar = c = k_B = 1$.

II. SPECTRAL PROPERTIES OF THE DIRICHLET LAPLACIAN

In this section, we want to describe briefly spectral methods which are fundamental tools to discuss problems in the definition of the global renormalized zero-point energy of scalar fields with ideal and nonideal boundary conditions. Usually, to discuss vacuum energy issues and one-loop physics, it is necessary to introduce a normalization scale μ. Since we are interested in discussing flat space-time that has boundaries and massless fields, the coefficient c_2 vanishes identically, and therefore the renormalized vacuum energy is independent of the normalization scale μ. Consequently, we do not include the parameter μ in our equations.

Consider the eigenfunctions and eigenvalues of the Laplacian operator $D = (-\Delta)$ on a bounded (open connected) domain Ω in Euclidean space $\mathbb{R}^d$. In this work, we discuss only the Dirichlet Laplacian which has a positive definite real spectrum. Also, the eigenvalues form a countable sequence. Using λ_k for $k = 1, 2\ldots$ they are ordered as

$$0 < \lambda_1 < \lambda_2 \leq \cdots \leq \lambda_k \leq \cdots \tag{1}$$

when $k \to \infty$, with possible multiplicities. The eigenfunctions $\{\phi_k\}_{k=1}^{\infty}$ form a basis in $\mathcal{L}_2(\Omega)$ with Dirichlet boundary conditions. Each ϕ_k has eigenvalues $\lambda_k(\Omega) \equiv \lambda_k$.

In spectral theory, the asymptotic behavior of the Dirichlet Laplacian eigenvalues in the analytic regularization procedure has a fundamental role. This behavior was investigated at first by Weyl [77]. Applying the Fredholm-Hilbert formalism of linear integral equations, it was proved that for $\Omega \subset \mathbb{R}^d$, $(d = 2, 3)$

$$\lim_{k\to\infty} \frac{k}{\lambda_k} = \frac{\mathrm{Vol}_d(\Omega)}{4\pi}, \tag{2}$$

where $\mathrm{Vol}_d(\Omega)$ is the volume of the region Ω.

We start the discussion defining the density of eigenvalues as a sum of delta functions

$$g(\lambda) = \sum_k \delta(\lambda - \lambda_k), \tag{3}$$

and the counting function $N(\lambda) \coloneqq \#\{\lambda_m \colon \lambda_m < \lambda\}$, defined as

$$N(\lambda) = \int_0^{\lambda} \mathrm{d}\lambda' g(\lambda'), \tag{4}$$

which gives the number of elements in the sequence of eigenvalues, smaller than λ. The asymptotic behavior of the counting function is given by

$$N(\lambda) = f(d)\mu_d(\Omega)\lambda^{\frac{d}{2}}, \qquad (\lambda \to \infty), \tag{5}$$

where $f(d)$ is an entire function of d. Furthermore, the other asymptotic terms also give information about the boundary of the domain. As an example, for $\Omega \subset \mathbb{R}^3$, we get a contribution proportional to the surface area of Ω.

Our first observation is about the zero-point energy renormalization. Let us define the Minakshisundaram-Pleijel bilocal zeta-function $\mathcal{Z}(x, y; s)$, for $s \in \mathbb{C}$ as

$$\mathcal{Z}(x, y; s) = \sum_{k=1}^{\infty} \frac{\phi_k(x)\phi_k(y)}{\lambda_k^s}, \tag{6}$$

which converges uniformly in x and y for $\mathrm{Re}(s) > s_0$ and was originally defined in a connected compact Riemannian manifold [78]. From this bilocal zeta-function, is possible to define a spectral zeta-function associated with the eigenvalues of the Laplacian in $\Omega \subset \mathbb{R}^d$. We define $\mathsf{Z}(s) = \mathrm{Tr}(-\Delta)^{-s}$, where

$$\mathsf{Z}(s) = \sum_{k=1}^{\infty} \lambda_k^{-s} = \lim_{m\to\infty} \sum_{k=1}^{m} \lambda_k^{-s}. \tag{7}$$

Using the counting function $N(\lambda)$ and the definition of the Riemann-Stieljes integral we get

$$\sum_{n=1}^{m} \lambda_n^{-s} = \sum_{n=1}^{k-1} \lambda_n^{-s} + \int_a^b \mathrm{d}N(t) t^{-s};$$
$$\lambda_{k-1} \leq a < \lambda_k, \qquad \lambda_m \leq b < \lambda_{m+1}. \tag{8}$$

Therefore the spectral zeta-function can be written as

$$\mathsf{Z}(s) = \sum_{n=1}^{k-1} \lambda_n^{-s} + \int_{\lambda_k}^{\infty} \mathrm{d}N(t) t^{-s}. \tag{9}$$

In principle, this formula is given in the region of the complex plane where the original sum converges. As the sum on the right-hand side is analytic over the entire complex s-plane, the qualitative behavior of its analytic continuation is determined by the Riemann-Stieltjes integral expressed in terms of Weyl's counting function.

To obtain the polar structure of the spectral zeta-function let us consider an evolution equation in $\mathcal{L}_2(\Omega)$ that can be formulated as the following initial-boundary problem in $(0, \infty) \times \Omega$. For $\Omega \subset \mathbb{R}^d$, we get

$$\begin{cases} \frac{\partial u}{\partial t} = \Delta u, \\ u(0, x) = f(x), \\ u(t, x)|_{x \in \partial\Omega} = 0. \end{cases}$$

The weak solution $u(t, x)$, that satisfies the diffusion equation in the sense of distributions is given by

$$u(t, x) = \int \mathrm{d}\mu(y) p_{\Omega}(t, x, y) f(y), \tag{10}$$

where $d\mu(y)$ is the volume element of the domain and $p_{\Omega}(t, x, y)$ is the diffusion kernel, i.e., the positive fundamental solution to the heat equation. For a generic boundary condition, the spectral decomposition of the diffusion kernel can be represented as

$$p_{\Omega}(t, x, y) = \sum_{k=1}^{\infty} e^{-t\lambda_k(\Omega)} \phi_k(x)\phi_k(y). \tag{11}$$

Using a Mellin transform and the definition of the Minakshisundaram-Pleijel zeta-function $\mathcal{Z}(x,y;s)$, we get

$$\Gamma(s)\mathcal{Z}(x,y;s) = \int_0^\infty \mathrm{d}t\, t^{s-1} p_\Omega(t,x,y). \tag{12}$$

For $x \neq y$, $\Gamma(s)\mathcal{Z}(x,y;s)$ is an regular function of s in the entire complex plane. For $x = y$ there is a pole at $s = 1$. From the diffusion kernel, since we are interested in global issues, let us define the trace of the diffusion kernel, written as $\Theta(t) = \mathrm{Tr}(e^{t\Delta})$, where, using the Riemann-Stieljes integral we can write

$$\Theta(t) = \int_0^\infty e^{-\lambda t} dN(\lambda) = \sum_{k=1}^\infty e^{-\lambda_k t} \quad t > 0. \tag{13}$$

The spectral zeta-function can be represented as

$$\mathsf{Z}(s) = \frac{1}{\Gamma(s)} \int_0^\infty \mathrm{d}t\, t^{s-1} \Theta(t). \tag{14}$$

Its polar structure in the extended complex plane is determined by the classical spectral invariants, which are the expansion coefficients at $t \to 0^+$ of the diffusion kernel trace. When $\partial\Omega \neq \varnothing$ the coefficients of the asymptotic expansion of the heat trace have been calculated for a variety of boundary conditions

$$\lim_{t\to 0^+} \Theta(t) = (4\pi t)^{-\frac{d}{2}} \left[\sum_{p=0}^{K} c_p(\Omega) t^{\frac{p}{2}} + o(t^{\frac{K+1}{2}}) \right], \tag{15}$$

where the coefficients $c_p(\Omega)$ are related to the geometric characteristics of the bounded domain. Useful information on the heat kernel coefficients in mathematical and physical literature can be found at Refs. [28–30]. By a Tauberian theorem, we are able to connect the first term of the above asymptotic expansion with Weyl's asymptotic behavior of the Laplace operator spectrum.

For the case of vacuum energy, Fulling has stressed the need to study the cylinder kernel [79,80]. See, for example, [81]. To implement this idea, let us define the zeta-function $\zeta_{\sqrt{D}}(s)$ constructed with the energies ω_k of each normal modes

$$\zeta_{\sqrt{D}}(s) = \sum_{k=1}^\infty \frac{1}{\omega_k^s}; \qquad \mathrm{Re}(s) > s_1. \tag{16}$$

The renormalized vacuum energy is by definition $\langle E\rangle_r = \zeta_{\sqrt{D}}(s)|_{s=-1}$. Using again a Mellin transform we have

$$\sum_{k=1}^\infty \frac{1}{\omega_k^s} = \frac{1}{\Gamma(\frac{s}{2})} \int_0^\infty \mathrm{d}t\, t^{\frac{s}{2}-1} \sum_{k=1}^\infty e^{-\omega_k^2 t}. \tag{17}$$

The zeta-function $\zeta_{\sqrt{D}}(s)$ is a meromorphic function of s with simple poles. In the case where $s = -1$ is a pole, we can obtain a representation in a neighborhood of the pole, including some regular part known as the renormalized vacuum energy. It is important to stress that the measurable Casimir energy is obtained from a mathematical formalism based on analytic continuations, where undesirable polar contributions must be removed through a renormalization procedure.

III. THE VACUUM ENERGY IN THE PRESENCE OF SURFACES WITH IDEAL BOUNDARY CONDITIONS

The aim of this section is to use an analytic regularization procedure to obtain the vacuum energy of a massless scalar field at zero temperature in the presence a slab geometry $\Omega = \mathbb{R}^{d-1} \times [0, L]$ with Dirichlet boundary conditions. Let us assume a free neutral scalar field defined in a $(d+1)$-dimensional flat space-time. Its field equation, the Klein-Gordon equation, reads

$$\left(\frac{\partial^2}{\partial t^2} - \Delta + m_0^2\right)\varphi(t, \mathbf{x}) = 0. \tag{18}$$

To implement the canonical quantization, the field operator and the generalized momentum are expanded in Fourier series enclosed in a finite periodic box. With a defined operator Hamiltonian, $\mathcal{H}$, the energy of the confined field has a pure point spectrum, allowing us to characterize its states in terms of occupation numbers and the Fock representation. For Dirichlet boundary conditions, the situation is similar. The states of the system are described in terms of occupation numbers of elementary excitations, which characterizes the states concerning the ground state, the vacuum state $|0\rangle$.

To proceed, we restrict the field to a d-dimensional box with lengths $(L_1 \times L_2 \times \ldots \times L_{d-1} \times L_d)$. Assuming Dirichlet boundary conditions, the vacuum energy, i.e., the total energy of the quantized field in the box, is $\langle 0|\mathcal{H}|0\rangle = U_d(L_1, \ldots, L_{d-1}, L_d)$. Using the condition $L_d \ll L_i$ for $(i = 1, 2, \ldots, d-1)$, and defining $L_d = L$, the unrenormalized vacuum energy can be written as

$$\begin{aligned} &U_d(L_1, \ldots, L_{d-1}, L) \\ &\quad = \frac{1}{(2\pi)^{d-1}} \left(\prod_{i=1}^{d-1} L_i\right) \\ &\qquad \times \int \prod_{i=1}^{d-1} \mathrm{d}q_i \sum_{n=1}^\infty \left(q_1^2 + \cdots + q_{d-1}^2 + \left(\frac{n\pi}{L}\right)^2 + m_0^2\right)^{\frac{1}{2}}. \end{aligned} \tag{19}$$

To discuss the case similar to the electromagnetic field let us assume $m_0^2 = 0$. The unrenormalized vacuum energy per unit area is defined as

$$\epsilon_d(L) = \frac{U_d(L_1, \ldots, L_{d-1}, L)}{(\prod_{i=1}^{d-1} L_i)}, \tag{20}$$

and this is a divergent expression. It can be written as

$$\epsilon_d(L) = \frac{(4\pi)^{\frac{1-d}{2}}}{\Gamma(\frac{d-1}{2})} \sum_{n=1}^{\infty} \int_0^{\infty} \mathrm{d}r\, r^{d-2} \left[r^2 + \left(\frac{n\pi}{L}\right)^2 \right]^{\frac{1}{2}}. \quad (21)$$

A straightforward calculation led us to

$$\epsilon_d(L) = \frac{(4\pi)^{\frac{1-d}{2}}}{2\Gamma(\frac{d-1}{2})} \left(\frac{\pi}{L}\right)^d \int_0^{\infty} \mathrm{d}x\, x^{\frac{d-3}{2}} (1+x)^{\frac{1}{2}} \sum_{n=1}^{\infty} n^d. \quad (22)$$

In the limit $L \to \infty$ we should obtain the fundamental result that the vacuum is a Lorentz invariant state of zero energy. Using the definition of the Beta function as

$$\mathcal{B}(z,w) = \frac{\Gamma(z)\Gamma(w)}{\Gamma(z+w)}, \quad (23)$$

and an analytic continuation principle, the vacuum energy per unit area is given by

$$\epsilon_d(L) = -\frac{\pi^{\frac{d}{2}}\Gamma(-\frac{d}{2})}{2(2L)^d} \zeta(-d), \quad (24)$$

where $\zeta(s)$ is the Riemann zeta-function, which is a function of the complex variable $s = \sigma + it$, where $\sigma, t \in \mathbb{R}$. It is originally defined in the half-plane $\mathrm{Re}(s) > 1$ through an absolutely convergent Dirichlet series [82,83]. The series is defined by summing over the set of natural numbers $n \in \mathbb{N}$ and can be expressed as

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}. \quad (25)$$

It can be extended to the complex plane as a meromorphic function using the Poisson summation formula with a simple pole at $s = 1$.

It is possible to show that Riemann zeta-function $\zeta(s)$ satisfies a functional equation valid for $s \in \mathbb{C}\backslash\{0,1\}$. This equation connects two functions outside the original domain of convergence. Using the properties of the Gamma function to define $\vartheta(s)$ as

$$\vartheta(s) = \frac{(2\pi)^s \Gamma(1-s)}{\Gamma(1-\frac{s}{2})\Gamma(\frac{s}{2})}, \quad (26)$$

we get a reflection formula for the Riemann zeta-function

$$\zeta(s) = \vartheta(s)\zeta(1-s). \quad (27)$$

The above calculations are an intermediate step crucial to discuss the modifications in the renormalized vacuum energy of a scalar field in the presence of surfaces where the scalar field satisfies nonideal boundary conditions.

IV. RENORMALIZED VACUUM ENERGY WITH NONIDEAL BOUNDARY CONDITIONS

In the Lifshitz approach, the dispersion forces between dissipative media are caused by the fluctuating electromagnetic field defined both within and outside the media. Using the fluctuation-dissipation theorem, the Lifshitz expression for the force between plates depends on the dielectric functions on the surfaces and also on the medium in which they are immersed. The finite conductivity correction to the ideal Casimir calculation is obtained using the frequency dependence of the dielectric function. The imperfect conductivity at high frequencies can be modeled by introducing only the plasma frequency ω_p of the plates. It is important to note that the Casimir result is recovered at distances larger than the plasma wavelength.

In our case, we are discussing the vacuum energy of a quantized scalar field in the presence of classical surfaces, where the field satisfies nonideal boundary conditions. Those can be understood as finite conductivity conditions. We can call it *ideal high-pass Dirichlet boundary condition*. In order to be clear, our boundary condition is over the frequencies, one can think them as the following: for frequencies smaller than some ω_{k_c} we do have the usual Dirichilet boundary conditions, otherwise, the plates are transparent for the field. However, the crucial point is that is not convenient to simply calculate the correction to the renormalized vacuum energy separating the effects of the low-energy vacuum modes from the high-energy modes using a sharp cutoff, once that is a sum of positive terms one always obtain a positive energy density, i.e.,

$$\epsilon_d^{\mathrm{f.c.}}(L) = \sum_{k=1}^{k_c} \omega_k > 0, \quad (28)$$

where ω_{k_c+1} is plasma frequence of the material.

We start using an analytic regularization procedure and the fact that for Dirichlet boundary conditions the eigenvalues vary continuously under a smooth deformation of the domain (spectral stability of elliptic operator under domain deformation) and the minimax principle says that the eigenvalues monotonously decrease when the domain is enlarger,

$$\sigma_m(\Omega_1) \geq \sigma_m(\Omega_2), \qquad \Omega_1 \subset \Omega_2. \quad (29)$$

By the above arguments, we can use approximate functional equation that expresses the Riemann zeta-function as finite sums, outside their original domain of convergence.

Initially, we use a classical result by Hardy and Littlewood following the derivation discussed in Ref. [84]. Let us write the Riemann zeta-function as

$$\begin{aligned}\zeta(s) &= \sum_{n\le n_c} n^{-s} + \sum_{n>n_c} n^{-s}\\ &= \sum_{n\le n_c} n^{-s} + \frac{1}{\Gamma(s)}\int_0^\infty \mathrm{d}x\, x^{s-1}\left(\sum_{n>n_c} e^{-nx}\right)\\ &= \sum_{n\le n_c} n^{-s} + \frac{1}{\Gamma(s)}\int_0^\infty \mathrm{d}x \frac{x^{s-1}e^{-n_c x}}{e^x-1}, \qquad (30)\end{aligned}$$

where the absolute convergence justifies the inversion of the order of summation and integration. To proceed, we analyze the following integral $I(s)$. We have

$$I(s) = \int_C \mathrm{d}z \frac{z^{s-1}e^{-n_c z}}{e^z-1}, \qquad (31)$$

where the contour C starts at infinity on the positive real axis, encircles the origin once in the positive direction excluding the points $\pm 2\pi i, \pm 4\pi i, \ldots$ and returns to infinity. We obtain

$$I(s) = (e^{2\pi i s}-1)\int_0^\infty \mathrm{d}x \frac{x^{s-1}e^{-n_c x}}{e^x-1}. \qquad (32)$$

Using the analytic continuation principle we can write

$$\zeta(s) = \sum_{n\le n_c} n^{-s} + \frac{e^{-\pi i s}\Gamma(1-s)}{2\pi i}\int_C \mathrm{d}z \frac{z^{s-1}e^{-n_c z}}{e^z-1}. \qquad (33)$$

From the above equation, we find an approximate representation of the zeta-function in terms of finite sums. It was proved that

$$\zeta(s) = \sum_{n\le x}\frac{1}{n^s} + \vartheta(s)\sum_{n\le y}\frac{1}{n^{1-s}} + O(x^{-\sigma}) + O(t^{\frac{1}{2}-\sigma}y^{\sigma-1}), \qquad (34)$$

for $0\le\sigma<1$ holds for given $x, y, t > C > 0$ satisfying $2\pi xy = t$ where $t \gg 1$. This is known as an approximate functional equation.

For simplicity, using the approximate functional equation, we discuss the case of a slab geometry $\mathbb{R}^{d-1}\times[0,L]$. Making a parallel with the electromagnetic case, in the scalar field scenario, we define the plasma frequency ω_p and the plasma wavelength $\lambda_p = 2\pi/w_p$. Next, we define a "critical" mode index n_c, which will be related to the plasma wavelength. In order to find an adequate maximum number of states n_c for a single compactified direction, we need to introduce first the notion of density of states $\rho(k)$ in the phase space and the number of states $dN = \rho(k)d^dk$ that lies between k and $k+dk$. In our d-dimensional space, where all the directions are finite and have lengths $L_1, L_2, \ldots, L_{d-1}, L$, then the density of states is simply

$$\rho(k) = \left(\frac{L}{\pi^d}\right)\prod_{i=1}^{d-1} L_i, \qquad (35)$$

we can find the number of states inside a volume that possess the maximum value of moment $k_{\max}$ as

$$N(k_{\max}) = \int_{|k|<k_{\max}} \mathrm{d}^d k\rho(k) = \rho\frac{\pi^{d/2}}{\Gamma(\frac{d}{2}+1)}k_{\max}^d, \qquad (36)$$

where we have used the definitions of the volume of a sphere in d-dimensions. For other side, we are interested in obtaining the maximum number of states in a single compactified direction n_c. We have that

$$N(k_{\max}) = \frac{\pi^{d/2}}{\Gamma(\frac{d}{2}+1)}n_c^d. \qquad (37)$$

Therefore we identified $n_c^d = \rho k_{\max}^d$. Now, we relate the maximum wave number with the plasma frequency of the material in such a manner that $k_{\max} = 2\pi/\lambda_p$. With all this, after some algebra, we conclude that

$$n_c = 2\left(\frac{L^{1/d}}{\lambda_p}\right)\prod_{i=1}^{d-1} L_i^{1/d}, \qquad (38)$$

since all the directions L_i from $i = \{1, 2, \ldots, d-1\}$ are much larger that L. The only dependence of the maximum number of states is of the form

$$n_c(L) \equiv \left(\frac{L}{\lambda_p}\right)^{1/d}. \qquad (39)$$

In the Hardy and Littlewood approximate functional equation, we choose

$$x = y = \left(\frac{L}{\lambda_p}\right)^{1/d} = n_c \Rightarrow t = 2\pi\left(\frac{L}{\lambda_p}\right)^{2/d} = 2\pi n_c^2. \qquad (40)$$

Using the asymptotic expansion, Eq. (34), we get the Casimir energy as

$$\epsilon_d(L) = -\frac{\pi^{\frac{d}{2}}\Gamma(-\frac{d}{2})}{2(2L)^d}[H_{n_c}(-d) + \vartheta(-d)H_{n_c}(d+1)]. \qquad (41)$$

The quantities $H_n(s)$ are the generalized harmonic numbers. Once the Eq. (41) only makes sense as an analytic continuation, those finite sums must be understood as such. Moreover, we stress the fact that the equality holds by analytic continuation outside the strip $0<\sigma<1$. This can be shown using an analytic continuation of the asymptotic expansion.

Each generalized harmonic number has an expression for its domain of interest in the complex plane. Lets us start from the second term in the sum, $H_{n_c}(d+1)$. Formally, this quantity is given by

$$H_{n_c}(d+1) \equiv \sum_{n=1}^{n_c} \frac{1}{n^{d+1}}. \tag{42}$$

However, since we start from Eq. (24), which is an analytic continuation, the finite sum should be taken in the range of interest. In such a situation, we can use a known expression

$$H_{n_c}(d+1) = \zeta(d+1) + \frac{(-1)^d}{d!}\psi_d(n_c+1), \tag{43}$$

which holds for $n_c \in \mathbb{R}\backslash\{-1,-2,-3,\ldots\}$ and $d \in \mathbb{N}$; see, e.g., [85], and $\psi_m(x)$ is the polygamma function. Using a recurrence relation and a expression for large arguments, we can write the polygamma function as

$$\psi_d(n_c+1) = \frac{(-1)^d d!}{n_c^{d+1}} + (-1)^{d+1}\sum_{k=0}^{\infty}\frac{(k+d-1)!}{k!}\frac{B_k}{n_c^{d+k}}, \tag{44}$$

where B_k are the Bernoulli numbers (we take the convention $B_1 = 1/2$). Using the definition of n_c and in the limit of $L/\lambda_p \gg 1$ we can write

$$\psi_d(n_c+1) \approx (-1)^{d+1}\left(\frac{\lambda_p}{L}\right)\left[(d-1)! - \frac{1}{2}d!\left(\frac{\lambda_p}{L}\right)^{\frac{1}{d}}\right], \tag{45}$$

which allow us to write the $H_{n_c}(d+1)$ in powers of λ_p/L.

For the first term of Eq. (41), we formally have

$$H_{n_c}(-d) \equiv \sum_{n=1}^{n_c}\frac{1}{n^{-d}}, \tag{46}$$

and an analytic continuation can be obtained using some elementary operations and the uniqueness of the analytic continuation, is straightforward to see that

$$H_{n_c}(-d) = \zeta(-d) - \zeta_H(-d; n_c+1), \tag{47}$$

where $\zeta_H(-d; n_c+1)$ is the Hurwitz zeta-function, defined by

$$\zeta_H(s; a) \equiv \sum_{n=0}^{\infty}\frac{1}{(n+a)^s}. \tag{48}$$

Let us define the Casimir energy per unit area with nonideal boundary conditions, i.e., finite conductivity (f.c.) by

$$\epsilon_d^{\text{f.c.}}(L) \equiv -\frac{1}{L^d}\frac{\pi^{d/2}}{2^{d+1}}\Gamma\left(-\frac{d}{2}\right)\zeta_H(-d; n_c+1). \tag{49}$$

Once this is performed, we can identify the contribution from the ideal boundary conditions, and the remaining part can be regarded as a correction term. We get

$$\epsilon_d^{\text{f.c.}}(L) = \epsilon_d(L) + \frac{\Gamma(1+d)\lambda_p}{2\Gamma(1+\frac{d}{2})}\left(\frac{1}{4\sqrt{\pi}}\right)^d\left[\frac{1}{L^{d+1}d} - \frac{\lambda_p^{\frac{1}{d}}}{2L^{d+1+\frac{1}{d}}}\right]. \tag{50}$$

As we have observed, in the slab geometry, the Casimir force is a negative quantity ($\epsilon_d(L) < 0$), while the second contribution in the above equation is positive. We have succeeded in deriving the Casimir energy per unit area with nonideal boundary conditions. Note that our first finite conductivity correction to the electromagnetic Casimir energy is the same as the correction obtained using the Lifshitz calculations. In contrast, the second correction is smaller, with the Lifshitz formula giving a second correction as L^{-5}, whereas ours give $L^{-\frac{13}{3}}$. The basic assumption that needs to be carefully investigated is the discussion of vacuum energy in a bounded domain. To proceed, in the next section, we generalize the above result to the $d = 2$ dimensional case for a finite volume box.

V. CASIMIR ENERGY IN A RECTANGULAR BOX WITH NONIDEAL BOUNDARY CONDITIONS

Let us discuss now the eigenvalues of a second-order elliptic self-adjoint partial differential operator on scalar functions on a bounded domain. We consider the eigenvalues of $-\Delta$ on a connected open set Ω in Euclidean space $\mathbb{R}^2$. We assume that the massless scalar field is confined in a rectangular box, with lengths L_1 and L_2 obeying Dirichlet boundary conditions. The eigenfrequencies that we use to expand the field operator are given by

$$\omega_{n_1 n_2} = \left[\left(\frac{n_1\pi}{L_1}\right)^2 + \left(\frac{n_2\pi}{L_2}\right)^2\right]^{\frac{1}{2}};\; n_1, n_2 = 1, 2, \ldots. \tag{51}$$

The unrenormalized vacuum energy in this case is

$$U(L_1, L_2) = \frac{1}{2}\sum_{n_1,n_2=1}^{\infty}\omega_{n_1 n_2}. \tag{52}$$

Making use of an analytic regularization procedure, the divergent expression can be written as

$$E(L_1, L_2, s) = \frac{1}{2}\sum_{n_1,n_2=1}^{\infty}\omega_{n_1 n_2}^{-2s}, \tag{53}$$

for $s \in \mathbb{C}$. Observe that, the vacuum energy is obtained when $s = -\frac{1}{2}$. The above double series converges absolutely and uniformly for $\text{Re}(s) > 1$. An analytic function, which plays an important role in algebraic number theory is the Epstein zeta-functions associated with quadratic forms [86]. Suppose that

$$\phi(a,b,c;x,y) = ax^2 + cxy + by^2, \tag{54}$$

where a, b and $c \in \mathbb{R}$ and $a > 0$ and $\eta = 4ab - c^2 > 0$. Lets us define the function $\mathcal{A}(s)$ by the series

$$\mathcal{A}(a,b,c;s) = \sum_{n_1,n_2=-\infty}^{\infty}{}' \phi^{-s}(a,b,c;n_1,n_2), \tag{55}$$

The above series defines an analytic function for $s = \sigma + it$, ($\sigma \in \mathbb{R}$ and $t \in \mathbb{R}$) and $\sigma > 1$, where we adopt the notation that the prime sign in the summation means that the contribution $n_1 = n_2 = 0$ (the origin of the mode space) must be excluded. This particular case of the Epstein zeta-function can be continued analytically to the whole complex plane, except for a simple pole at $s = 1$ [87]. This double series exhibits a functional equation that can be obtained using properties of the theta-function or the Poisson summation formula. The functional equation reads

$$\mathcal{A}(a,b,c;s) = \left(\frac{2\pi}{\sqrt{\eta}}\right)^{2s-1} \frac{\Gamma(1-s)}{\Gamma(s)} \mathcal{A}\left(\frac{1}{a},\frac{1}{b},\frac{1}{c};1-s\right) \tag{56}$$

We are interested in the case where $c = 0$. Let us define the function $Z(\frac{1}{L_1},\frac{1}{L_2};s)$ by

$$Z\left(\frac{1}{L_1},\frac{1}{L_2};s\right) = \sum_{n_1,n_2=-\infty}^{\infty}{}' \left(\frac{n_1^2}{L_1} + \frac{n_2^2}{L_2}\right)^{-s}, \tag{57}$$

We can find that the vacuum energy is written as

$$\begin{aligned} E(L_1,L_2;s) = {}& \frac{1}{8} Z\left(\frac{\pi^2}{L_1^2},\frac{\pi^2}{L_2^2};s\right) \\ & - \frac{1}{4}\left[\left(\frac{\pi}{L_1}\right)^{-2s} + \left(\frac{\pi}{L_2}\right)^{-2s}\right]\zeta(2s). \end{aligned} \tag{58}$$

As it was discussed, $E(L_1,L_2,s)$ is analytic in $s \in \mathbb{C}\backslash\{\frac{1}{2},1\}$. Using the analytic continuation of the Epstein and the Riemann zeta-function gives the vacuum energy $U(L_1,L_2) = E(L_1,L_2;s=-1/2)$ for the system with Dirichlet boundary conditions. We get

$$\begin{aligned} U(L_1,L_2) = {}& \frac{\pi}{48}\left(\frac{1}{L_1} + \frac{1}{L_2}\right) \\ & - \frac{L_1 L_2}{32\pi} \sum_{n_1,n_2=-\infty}^{\infty}{}' (n_1^2 L_1^2 + n_2^2 L_2^2)^{-\frac{3}{2}}. \end{aligned} \tag{59}$$

The next step involves discussing the scalar case similar to the electromagnetic case of imperfect conductors, where there is a plasma frequency ω_p. Using the same approach discussed in the previous section, we aim to determine the approximate functional equation for the Epstein zeta-function.

Potter [88] has derived the following approximate functional equation:

$$\begin{aligned} \mathcal{A}(a,b,c;s) = {}& \sum_{\phi \le x}{}' \phi^{-s}(a,b,c;n_1,n_2) \\ & + X(s) \sum_{\phi \le y}{}' \phi^{s-1}(a,b,c;n_1,n_2), \end{aligned} \tag{60}$$

for $t \gg 1$, and the condition $4\pi^2 xy = \eta t^2$ must be satisfied, the quantity $X(s)$ is defined by

$$X(s) = \left(\frac{2\pi}{\sqrt{\eta}}\right)^{2s-1} \frac{\Gamma(1-s)}{\Gamma(s)}. \tag{61}$$

Henceforth we take $\mathcal{A}(a,b,0;s) \equiv \mathcal{A}(a,b;s)$ and similar for ϕ.

Of course, to obtain the correction to the Casimir energy via asymptotic series we will need to use the Potter approximate functional equation for the Epstein zeta function, but also the Hatree-Littlewood approximate functional equation for the Riemann zeta-function. Let's start analyzing the Epstein zeta-function. It is convenient to introduce a λ_p term in our expression in order to only have adimensional quantities and establish a parallel with the Casimir energy in a finite conductivity scenario. In this case, we have

$$\mathcal{A}\left(\frac{\pi^2\lambda_p^2}{L_1^2},\frac{\pi^2\lambda_p^2}{L_2^2};s\right) = \sum_{\Phi \le x}{}' \Phi_{12}^{-s} + X(s) \sum_{\Phi \le y}{}' \Phi_{12}^{s-1}, \tag{62}$$

where to the notation be lightened, we defined

$$\begin{aligned} \Phi_{12} \equiv {}& \phi\left(\frac{\pi^2\lambda_p^2}{L_1^2},\frac{\pi^2\lambda_p^2}{L_2^2};n_1,n_2\right) \\ = {}& \frac{\pi^2\lambda_p^2}{L_1^2} n_1^2 + \frac{\pi^2\lambda_p^2}{L_2^2} n_2^2, \end{aligned} \tag{63}$$

once that $4\pi^2 xy = \eta t^2$ with

$$\eta = 4\left(\frac{\pi^2\lambda_p^2}{L_1 L_2}\right)^2 \Rightarrow xy = \left(\frac{\pi\lambda_p^2}{L_1 L_2}\right)^2 t^2. \tag{64}$$

Since

$$X(s) = \left(\frac{L_1 L_2}{\pi\lambda_p^2}\right)^{2s-1} \frac{\Gamma(1-s)}{\Gamma(s)}, \tag{65}$$

using a similar argument that we used before, but now all dimensions remain compact, we can define the quantities

$$n_c^{(1)} \equiv \left(\frac{L_1}{\lambda_p}\right)^{1/2} \quad \text{and} \quad n_c^{(2)} \equiv \left(\frac{L_2}{\lambda_p}\right)^{1/2}$$
$$\Rightarrow xy = \left[\frac{\pi}{(n_c^{(1)} n_c^{(2)})^2}\right]^2 t^2, \tag{66}$$

which, considering the fact that we do not have a preferred direction, indicate to us that the natural choice for t should be

$$t = \frac{1}{\pi}\left(n_c^{(1)} n_c^{(2)}\right)^2 \Rightarrow x = y = n_c^{(1)} n_c^{(2)}. \tag{67}$$

So looking back to the Eq. (62), we see that the sums are over all modes inside the ellipse defined by

$$\frac{n_1^2}{L_1 n_c^{(1)} n_c^{(2)}} + \frac{n_2^2}{L_2 n_c^{(1)} n_c^{(2)}} = \left(\frac{1}{\pi\lambda_p}\right)^2 = \text{constant}, \tag{68}$$

in the (n_1, n_2)-plane with the origin removed.

For the Riemann zeta-function contributions, that are present in Eq. (58), we have

$$\zeta(2s) = \sum_{n\leq u} \frac{1}{n^{2s}} + \vartheta(2s) \sum_{n\leq v} \frac{1}{n^{1-2s}}, \tag{69}$$

for $\alpha \gg 1$ where $2\pi uv = \alpha$. Proceeding exactly as in the slab bag geometry case, we find that

$$u = v \equiv n_c^{(i)} = \left(\frac{L_i}{\lambda_p}\right)^{1/2} \Rightarrow \alpha = 2\pi \frac{L_i}{\lambda_p}; \quad i = 1, 2, \tag{70}$$

continuing from the previous section, we employ an analogous method. Using the same harmonic number definitions, once the range in the complex plane will be the same. Considering the case where $s = -1/2$ and manipulating the equations, is possible to find that.

$$E(L_1, L_2; s) = \frac{\lambda_p^{2s}}{8} \sum_{\Phi \leq n_c^{(1)} n_c^{(2)}}' \Phi_{12}^{-s} + \left(\frac{L_1 L_2}{\pi\lambda_p^2}\right)^{2s-1} \frac{\Gamma(1-s)}{\Gamma(s)} \frac{\lambda_p^{2s}}{8} \sum_{\Phi \leq n_c^{(1)} n_c^{(2)}}' \Phi_{12}^{s-1}$$
$$- \frac{\lambda_p^{2s}}{4} \sum_{i=1}^{2} \left\{ \left(\frac{\lambda_p}{L_i}\right)^{-2s} \left[2\zeta(2s) - \zeta_H(2s; n_c^{(i)} + 1)\right] + (-1)^{-4s+1} \vartheta(2s) \left[\frac{1}{2s}\left(\frac{\lambda_p}{L_i}\right)^{-3s} - \frac{1}{2}\left(\frac{\lambda_p}{L_i}\right)^{\frac{-6s+1}{2}}\right]\right\}. \tag{71}$$

We define the vacuum energy for finite conductivity (f.c.) as

$$E^{\text{f.c.}}\left(L_1, L_2, s = -\frac{1}{2}\right) = U^{\text{f.c.}}(L_1, L_2) \equiv \frac{1}{8\lambda_p} \sum_{\Phi \leq n_c^{(1)} n_c^{(2)}} \Phi_{12}^{\frac{1}{2}} - \frac{1}{4} \sum_{i=1}^{2} \frac{1}{L_i} \left[\zeta_H(-1; n_c^{(i)} + 1) - \frac{1}{6}\right]. \tag{72}$$

Therefore

$$U^{\text{f.c.}}(L_1, L_2) = U(L_1, L_2) - \frac{\pi^2 \lambda_p^3}{32 (L_1 L_2)^2} \sum_{\Phi \leq n_c^{(1)} n_c^{(2)}} \Phi_{12}^{-\frac{3}{2}} + \frac{1}{2\lambda_p (2\pi)^2} \sum_{i=1}^{2} \left[\left(\frac{\lambda_p}{L_i}\right)^{3/2} - \frac{1}{2}\left(\frac{\lambda_p}{L_i}\right)^2\right], \tag{73}$$

is the Casimir energy for a rectangular box with nonideal boundary conditions.

VI. CONCLUSIONS

In this paper, we investigate the total energy of a massless scalar quantized field, which satisfies idealized perfectly boundary conditions, using an analytic regularization procedure. We extend the above result to the case of "imperfect conductor" boundary conditions. The crucial point in this scenario is that it is not convenient to calculate the correction to the renormalized vacuum energy separating the effects of the low-energy vacuum modes from the high-energy modes using a cutoff method without realizing previously a regularization of the zero-point energy. Therefore, to obtain the correction to the Casimir force for imperfect conductors assuming a slab geometry $\mathbb{R}^{d-1} \times [0, L]$, we have to use an approximate functional equation. First we represent the energy density using finite sums outside the original domain of convergence of the Dirichlet series. Next, we demonstrate how it is possible to obtain the correction to the force in the three-dimensional spacetime generated by a massless scalar field in the presence of a rectangular box, with lengths L_1 and L_2.

In the literature, it has been discussed a scenario where classical fluctuations assume the role of the quantum vacuum modes as the original Casimir conceptual framework [89–96]. In a confined system with quenched disorder, a sensitivity to the boundaries may arise, where the distance to the critical situation is given by some nonthermal control parameter. Recently, inspired by the statistical Casimir effect, it was discussed the application of the spectral and distributional zeta-function methods to describe fluctuation-induced forces arising from a quenched disorder field in a continuous Landau-Ginzburg model [97]. A series of representations was employed. From the series representation of the average free energy, it is possible to obtain the force between the boundaries, due to the interaction of the critical fluctuations generated by the moment of the partition function, with the largest correlation length of the fluctuations. In other words, varying continuously the intensity of a nonthermal control parameter, the induced force can be repulsive or attractive between the boundaries [98]. This is the problem of the sign of the Casimir force in the statistical Casimir effect [99]. It is clear that is possible to go beyond the Gaussian approximation, where a perturbative expansion must be implemented with Euclidean Green's functions. In this case, in addition to the traditional bulk counterterms, surface counterterms must be introduced to renormalize the interacting Euclidean field theory in the presence of boundaries [100–103]. Therefore, a natural continuation of this work would involve investigating the statistical Casimir effect in the presence of dirty surfaces [104]. Another possibility is to investigate the analytic expression for the heat kernel coefficients taking into account the plasma frequency, following the lines of Ref. [105]. These subjects are under investigation by the authors.

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