Fermions, boundaries, and conformal and chiral anomalies in $d=3$, 4 and 5 dimensions

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In the presence of boundaries, the quantum anomalies acquire additional boundary terms. In odd dimensions, the integrated conformal anomaly, for which the bulk contribution is known to be absent, is nontrivial due to the boundary terms. These terms became a subject of active study in the recent years. In the present paper, we continue our previous study [1,2] and compute explicitly the anomaly for fermions in dimensions $d=3$, 4 and 5. The calculation in dimension 5 is new. It contains both contributions of the gravitational field and the gauge fields to the anomaly. In dimensions $d=3$ and 4, we reproduce and clarify the derivation of the results available in the literature. Imposing the conformal invariant mixed boundary conditions for fermions in odd dimension d, we particularly pay attention to the necessity of choosing the doubling representation for gamma matrices. In this representation, there exists a possibility to define chirality and thus address the question of the chiral anomaly. The anomaly is entirely due to terms defined on the boundary. They are calculated in the present paper in dimensions $d=3$ and 5 due to both gravitational and gauge fields. To complete the picture, we reevaluate the chiral anomaly in $d=4$ dimensions and find a new boundary term that is supplementary to the well-known Pontryagin term.

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I. INTRODUCTION

The role of quantum anomalies in the modern theoretical constructions becomes increasingly important. Conformal anomalies, first discovered by Capper and Duff almost 50 years ago [3], by now serve as excellent example of very rich and mutually useful interplay between the differential geometry, quantum fundamental physics and applications. In the recent years, a new aspect of conformal anomaly came into play. The presence of boundaries changes rather dramatically what we used to think about the anomaly. Indeed, the local geometric invariants from which the anomaly can be constructed have necessarily even dimensionality. So, that the anomaly is conventionally absent in space-time of odd dimension d since no invariant of appropriate odd dimensionality exits. This is no more true in the presence of boundaries. A geometric quantity, extrinsic curvature, characterizes how the boundary is embedded into space-time and it has dimension one. This allows one to construct new invariants of both odd and even dimensionality on the boundary of space-time. As a consequence, the conformal anomaly or, better to say, the integrated conformal anomaly can be now nontrivial even if the dimension of space-time is odd. If dimension is even there, additionally to the bulk terms, appear boundary terms with increasingly reach structure as the dimension d grows. An earlier paper in this direction is [4]. The complete structure or the building blocks from which one can construct the boundary anomaly terms is not yet fully understood for large values of d. In dimensions $d=3$ and $d=4$, the situation is by now quite clear after the works [1,5,6]. In [5], the values of the boundary conformal charges in dimension $d=4$ have been computed for free conformal fields: scalar fields, Dirac fermions, and gauge fields. In dimension $d=5$, a recent progress has been reached after identifying a complete set of boundary conformal invariants in this dimension in [2]. The respective conformal charges for a conformal scalar field in $d=5$ have been computed in [2]. The further developments in this direction include [7–15].

The primary goal of the present paper is to build on our previous work [2] and compute the boundary conformal charges for the fermions in dimension $d=5$. Let us briefly discuss a difficulty that looks technical but whose resolution leads to interesting consequences. Considering the Dirac fermions in space-time with boundaries, one

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TABLE I. Contributions to boundary terms in conformal and chiral anomaly.

Dimensions	$d=3$		$d=4$		$d=5$	
Type of anomaly	Conformal	Chiral	Conformal	Chiral	Conformal	Chiral
Boundary terms due to gravitational field	Yes	No	Yes	Yes	Yes	Yes
Boundary terms due to gauge field	No	Yes	No	No	Yes	Yes

encounters the following problem. One imposes the boundary conditions on the fermion fields of the mixed type: A half of components of the fermion satisfy the Dirichlet condition while the other half a conformal Robin type condition. Thus, one needs to define two projectors $\Pi_+ = \frac{1}{2}(1+\chi)$ and $\Pi_- = \frac{1}{2}(1-\chi)$ such that $\Pi_+ + \Pi_- = 1$. The condition on matrix χ is that it has to anticommute with $\gamma^n = n_k\gamma^k$, where n^k is normal vector to the boundary, and commute with all other gamma matrices $\gamma^a, a = 1, \ldots, d-1$ projected along the boundary. In even dimension d, this matrix can be easily constructed, $\chi = i\gamma^*\gamma^n$. Here γ^* is the chirality matrix, it anticommutes with all gamma matrices. In odd dimension d, provided one uses the usual $2^{\frac{d-1}{2}} \times 2^{\frac{d-1}{2}}$ representation for gamma-matrices, such a matrix γ^* does not exist. This forces us to use other representation for gamma matrices of dimension $2^{\frac{d+1}{2}} \times 2^{\frac{d+1}{2}}$ that is obtained in the so-called doubling procedure. For earlier discussions of this representation, see [16,17]. In this representation, it is known that two chiral matrices exist, what we call below Γ_1^* and Γ_2^*. The notion of chirality is thus naturally defined. That is why in the present paper, we also compute the chiral anomaly in dimensions $d = 3$, 4 and 5.

In dimension $d = 4$, the above mentioned problem does not arise, and one uses the standard representation for gamma matrices. The respective chiral anomaly is due to the Pontryagin term, as is well known. However, the careful analysis presented below reveals a new boundary term in the chiral anomaly that is supplementary to the Pontryagin term.

In order to make our consideration general, we also include the coupling of the fermion field to a background gauge field and compute the contribution of the gauge field to the boundary terms in the conformal and chiral anomalies. Our findings are summarized in Table I.

It should be noted that the key tool in our computations is the heat kernel method. Thus, we heavily rely on the available results for the heat kernel coefficients for manifolds with boundaries given in [18] and [19].

II. THE BASICS

A. Dirac operator in curved space-time

The theory lives on manifold M covered by coordinates x^i and with the metric g_{ij}. This manifold has a boundary ∂M; the study of its contribution to the conformal anomaly is the main topic of this note. We also introduce a basis of orthonormal tangent vectors, the *n-beins* $e_i^p(x)$ at each point on M so that

$$g_{ij} = \eta_{pq} e_i^p e_i^q. \tag{1}$$

We consider a Dirac theory describing a spinor $\psi(x)$ on this manifold. The action reads

$$S = \int d^d x \sqrt{-g}\bar{\psi} i\gamma^k \hat{\nabla}_k \psi, \tag{2}$$

where $\gamma^k = e_p^k \gamma^p$ and γ^p are the Dirac matrices satisfying the Clifford algebra,

$$\gamma^p\gamma^q + \gamma^q\gamma^p = 2\eta^{qp}, \qquad q, p = 0, 1, \ldots d-1, \\ \eta = \mathrm{diag}(-1, +1, \ldots, +1). \tag{3}$$

The covariant derivative is defined as a combination of the purely gravitational covariant derivative and the gauge field $A_i = iB_i + A_i^a \lambda^a$, where λ^a form the algebra of non-Abelian transformations and B_i is the Abelian gauge field,

$$\hat{\nabla}_i = \nabla_i + A_i. \tag{4}$$

The gravitational covariant derivative is defined as

$$\nabla_k \psi = \left(\partial_k + \frac{1}{2}\omega_k^{pq}\Sigma_{pq}\right)\psi, \tag{5}$$

where $\Sigma^{pq} = \frac{1}{4}[\gamma^p, \gamma^q]$ and ω_k^{pq} is the spin connection[1] defined via the relation

$$\nabla_i e_j^p = \partial_i e_j^p - \Gamma_{ij}^k e_k^p + \omega_i{}^p{}_q e_i^q = 0. \tag{6}$$

Defining $R_{ij}{}^{pq} = e_k^p e_\ell^q R_{ij}{}^{k\ell}$, one has[2]

$$R_{ij}{}^{pq} = \partial_i \omega_j{}^{pq} - \partial_j \omega_i^{pq} + [\omega_i, \omega_j]^{pq}. \tag{7}$$

A direct calculation shows that

[1] Also known as coefficients of Fock-Ivanenko [20].

[2] Note that our convention for the Riemann tensor differs by sign from the one used in [18]. On the other hand, our convention for Ricci tensor and Ricci scalar agree with [18].

$$[\nabla_i, \nabla_j] = \frac{1}{4} R_{ij}{}^{pq} \gamma_p \gamma_q. \tag{8}$$

From this, we may define the field strength tensor as

$$\Omega_{ij} = [\hat{\nabla}_i, \hat{\nabla}_j] = F_{ij} + \frac{1}{4} R_{ij}{}^{pq} \gamma_p \gamma_q. \tag{9}$$

The Laplace type operator for Dirac theory is the square of Dirac operator,

$$\Delta^{(\frac{1}{2})} \psi \equiv (i\gamma^k \hat{\nabla}_k)^2 = -(\hat{\nabla}^2 + E)\psi. \tag{10}$$

For E, one finds

$$E = -\frac{1}{4} R + \frac{1}{4} [\gamma^i, \gamma^j] F_{ij}. \tag{11}$$

B. Fermions in even and odd dimensions

First we discuss the Dirac gamma matrices in even and odd dimensions.

C. Even dimension *d*

In even dimensions, there is a unique representation for the Clifford algebra in terms of the $2^{\frac{d}{2}} \times 2^{\frac{d}{2}}$ unitary matrices. In even dimensions, the following unitary matrix anticommutes with all gamma matrices, and thus, it can be used to introduce a chiral representation,

$$\gamma^* = -i^{\frac{d-2}{2}} \gamma^0 \gamma^1 \cdots \gamma^{d-1}, \qquad (\gamma^*)^2 = 1. \tag{12}$$

Explicitly, for $d = 2$,

$$\gamma^0 = -i\sigma^1, \quad \gamma^1 = \sigma^2, \quad \gamma^* = -\gamma^0\gamma^1 = -i\sigma^1\sigma^2 = \sigma^3, \tag{13}$$

where σ^i, $i = 1, 2, 3$ are the 2×2 Pauli matrices.

For $d = 4$, one has

$$\gamma^0 = \begin{pmatrix} 0 & -i\mathbb{I} \\ -i\mathbb{I} & 0 \end{pmatrix}, \qquad \gamma^{1,2,3} = \begin{pmatrix} 0 & -i\sigma^{1,2,3} \\ i\sigma^{1,2,3} & 0 \end{pmatrix},$$
$$\gamma^* = -i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} \mathbb{I} & 0 \\ 0 & -\mathbb{I} \end{pmatrix}. \tag{14}$$

1. Odd dimension d, doubling trick

In odd dimensions, for the standard representation $2^{\frac{d-1}{2}} \times 2^{\frac{d-1}{2}}$ for the gamma matrices, the product

$$\gamma_{\mp}^{d-1} = \mp i^{\frac{d-3}{2}} \gamma^0 \gamma^1 \cdots \gamma^{d-2} \tag{15}$$

belongs to the Clifford algebra by itself. Here, two different signs determine two unequal sets of Dirac matrices. Thus, in odd dimensions, there does not exist a matrix, γ^*, sometimes called the chirality matrix, that would anticommute with all gamma matrices. For the reasons that will be clear shortly, when we will discuss the appropriate mixed boundary conditions for the fermions, we would need such a matrix to exist. This is the primary reason why we shall consider a doubled representation of gamma matrices obtained in a procedure sometimes referred to as a doubling procedure. The physical meaning of this doubling can be understood in this way that, for chirality to be meaningful, we need two distinguished spin states in odd dimensions, which is what the doubling procedure provides [16]. Following the trick, we define

$$\Gamma^k = \gamma^k \otimes \sigma^2, \quad k = 0, 1, \ldots, d-1, \tag{16}$$

where in the last step, we choose $\gamma^{d-1} \equiv \gamma_-^{d-1}$. These new gamma matrices satisfy the Clifford algebra relations,

$$\Gamma^k \Gamma^\ell + \Gamma^\ell \Gamma^k = 2\eta^{k\ell}. \tag{17}$$

The product of first $d-2$ gamma matrices now is not the same as Γ^{d-1}. The respective Dirac fermions have $2^{\frac{d+1}{2}}$ components, twice as the standard Dirac fermions. The other interesting feature is that now there exist two candidates for the chiral matrix,

$$\Gamma_1^* = \mathbb{I} \otimes \sigma_1, \quad \Gamma_2^* = \mathbb{I} \otimes \sigma_3, \quad (\Gamma_1^*)^2 = 1, \quad (\Gamma_2^*)^2 = 1. \tag{18}$$

Both these matrices are Hermitian and anticommute with all gamma matrices (16). Thus, one may define two chiral type transformations,

$$\psi \to e^{i\Gamma_1^* \alpha} \psi \quad \text{and} \quad \psi \to e^{i\Gamma_2^* \beta} \psi. \tag{19}$$

More generally, four matrices, $\mathbb{I}$, Γ_1^*, Γ_2^*, and $[\Gamma_1^*, \Gamma_2^*]$, generate a unitary group of transformations. More on this representation and its applications in physics can be found in [17].

$d = 3$

In 3 dimensions, we start with

$$\gamma^0 = -i\sigma^1, \qquad \gamma^1 = \sigma^2, \qquad \gamma^2 = -\gamma^0\gamma^1 = \sigma^3. \tag{20}$$

Therefore,

$$\Gamma^0 = \begin{pmatrix} 0 & -i\sigma^2 \\ -i\sigma^2 & 0 \end{pmatrix}, \qquad \Gamma^1 = \begin{pmatrix} 0 & -i\sigma^2 \\ i\sigma^2 & 0 \end{pmatrix},$$
$$\Gamma^2 = \begin{pmatrix} \sigma^2 & 0 \\ 0 & -\sigma^2 \end{pmatrix}. \tag{21}$$

These matrices are Hermitian, except Γ^0 which is anti-Hermitian. As explained above, we define two different chirality matrices as

$$\Gamma_1^* = \mathbb{I}_{2\times 2} \otimes \sigma^1 = \begin{pmatrix} \sigma^1 & 0 \\ 0 & \sigma^1 \end{pmatrix}, \tag{22}$$

and

$$\Gamma_2^* = \mathbb{I}_{2\times 2} \otimes \sigma^3 = \begin{pmatrix} \sigma^3 & 0 \\ 0 & \sigma^3 \end{pmatrix}. \tag{23}$$

These two Hermitian matrices anticommute with all of the gamma matrices we introduced above. It can be easily checked that

$$\Gamma_1^*\Gamma_2^*\Gamma^0\Gamma^1\Gamma^2 = -i\mathbb{I}_{4\times 4}, \tag{24}$$

and thus,

$$\mathrm{tr}(\Gamma_1^*\Gamma_2^*\Gamma^i\Gamma^j\Gamma^k) = -4i\epsilon^{ijk}. \tag{25}$$

$d = 5$

In 5 dimensions,

$$\gamma^0 = \begin{pmatrix} 0 & -i\mathbb{I} \\ -i\mathbb{I} & 0 \end{pmatrix}, \qquad \gamma^{1,2,3} = \begin{pmatrix} 0 & -i\sigma^{1,2,3} \\ i\sigma^{1,2,3} & 0 \end{pmatrix}. \tag{26}$$

On the other hand,

$$\gamma^4 = -i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} \mathbb{I} & 0 \\ 0 & -\mathbb{I} \end{pmatrix}. \tag{27}$$

Then, using (16), one can construct $\Gamma^a, a = 0, \ldots, 4$. Two chirality matrices in 5 dimensions are

$$\Gamma_1^* = \mathbb{I}_{4\times 4} \otimes \sigma^1, \tag{28}$$

and

$$\Gamma_2^* = \mathbb{I}_{4\times 4} \otimes \sigma^3. \tag{29}$$

Then one finds that

$$\Gamma_1^*\Gamma_2^*\Gamma^0\Gamma^1\Gamma^2\Gamma^3\Gamma^4 = \mathbb{I}_{8\times 8}, \tag{30}$$

and thus, the trace is

$$\mathrm{tr}(\Gamma_1^*\Gamma_2^*\Gamma^i\Gamma^j\Gamma^k\Gamma^\ell\Gamma^m) = 8\epsilon^{ijk\ell m}. \tag{31}$$

D. Boundary conditions

In the present paper, we are interested in situation when the space-time M has a boundary ∂M. We consider a spacelike boundary so that $n^2 = n_k n^k = 1$ for the normal vector $n = n^k\partial_k$. Respectively, near the boundary, one may separate the normal direction given by vector n and the directions along the boundary given by a basis of tangent vectors $t_a^k\partial_k, a = 1, \ldots, d-1$. The appropriate mixed boundary conditions were first formulated by Gilkey and Branson [21]. In our discussion of the boundary conditions to be imposed on the Dirac fermions, we follow the chapter 3 in the book of Fursaev and Vassilevich [22] and give some necessary clarifications.

For a differential operator of order q, one has to impose q initial conditions, i.e., conditions at the initial constant time hyperurface. If there is a second, final constant time hypersurface, the required boundary conditions are distributed between them so that at each component of the boundary, one has to impose $q/2$ conditions. The Dirac operator $\hat{D} = i\gamma^k\hat{\nabla}_k$ is a first order operator so that in this case, one has to impose the boundary conditions on a half of fermionic components. Suppose that Π_+ is the projector that selects a half of the spinor components. One can define $\Pi_- = 1 - \Pi_+$ the projector on the other half. As projectors, they satisfy the properties: $\Pi_\pm^2 = \Pi_\pm$ and $\Pi_+\Pi_- = \Pi_-\Pi_+$.

A natural physical condition is to require that the normal component of the fermionic current vanishes on the boundary (for simplicity, we use here the Euclidean signature),

$$\psi^\dagger\gamma^n\psi|_{\partial M} = 0, \qquad \gamma^n = n_k\gamma^k. \tag{32}$$

This can be achieved by imposing the Dirichlet boundary condition

$$\Pi_-\psi|_{\partial M} = 0, \tag{33}$$

provided the projector Π_- satisfies certain commutation condition with γ^n. To identify this condition, decomposing $\psi = \Pi_+\psi + \Pi_-\psi$ and using that the projectors are unitary and assuming condition (33), we find

$$\psi^\dagger\gamma^n\psi|_{\partial M} = \psi^\dagger\Pi_+\gamma^n\Pi_+\psi|_{\partial M} = \eta\psi^\dagger\gamma^n\Pi_+\Pi_-\psi|_{\partial M} = 0, \tag{34}$$

where we used the commutativity of the two projectors and assumed a relation $\Pi_+\gamma^n = \eta\gamma^n\Pi_-$. In order to determine the value of η, we apply once again projector Π_+ to both sides of this relation and get equation $\eta^2 = \eta$ that gives us $\eta = 1$.

The square of Dirac operator $\hat{D}^2$ is an operator of second order in derivative, and logically one has to impose more conditions on the spinor components than just (33). These conditions should be valid at least for the eigenvectors of the Dirac operator $\hat{D}\psi = \lambda\psi$. Applying projector Π_- to both sides of this equation and assuming the condition (33), we arrive at

$$\Pi_-\hat{D}\psi|_{\partial M} = 0. \tag{35}$$

In order to simplify our discussion, let us consider the flat space-time and the boundary ∂M to be a plane $x^n = \text{const}$, while $\{x^a\}$ are the coordinates on the boundary. Then we may separate the normal and tangential components in the Dirac operator, $\hat{D} = i\gamma^n\partial_n + i\gamma^a\partial_a$. Then condition (35) leads to

$$\Pi_-(\gamma^n\partial_n + \gamma^a\partial_a)|_{\partial M} = \gamma^n\partial_n\Pi_+\psi|_{\partial M} + \Pi_-\gamma^a\partial_a\psi|_{\partial M} = 0. \tag{36}$$

Since the last term contains derivatives along the boundary, we expect this term to vanish due to (33). This is so provided a commutation relation is valid, $\Pi_-\gamma^a = \gamma^a\Pi_-$ (again the consistency condition requires that a possible numerical prefactor η in this relation to be 1). Thus, we arrive at the Robin type boundary condition on the second half of the spinor components,

$$\partial_n\Pi_+\psi|_{\partial M} = 0. \tag{37}$$

We have also found the commutation relations between the projectors and the gamma matrices,

$$\Pi_+\gamma^n = \gamma^n\Pi_-, \qquad \Pi_-\gamma^a = \gamma^a\Pi_-. \tag{38}$$

Representing $\Pi_\pm = \frac{1}{2}(1 \pm \chi)$, we find that χ has to anticommute with γ^n and commute with γ^a. This matrix γ can be constructed as follows:

$$\chi = i\gamma^*\gamma^n, \tag{39}$$

where γ^* anticommutes with all gamma matrices. This is so called the chirality matrix.

In the case of a curved space-time with a boundary ∂M characterized by nontrivial extrinsic curvature K, the commutations that we performed in (36) are more involved, and they are performed in Appendix C. The respective Robin type boundary condition that generalizes (37) is

$$(\hat{\nabla}_n - S)\Pi_+\psi = 0, \qquad S = -\frac{1}{2}K\Pi_+, \tag{40}$$

where K is the trace of the extrinsic curvature, which we briefly introduce in Sec. II E. We note that the mixed boundary conditions (40) are conformal invariant.

Even dimensions. In even dimensions, the mixed boundary conditions on the boundary ∂M will be realized as

$$\Pi_-\psi|_{\partial M} \oplus (\hat{\nabla}_n - S)\Pi_+\psi|_{\partial M} = 0. \tag{41}$$

We remind again that for Dirac spinors $\Pi_\pm = \frac{1}{2}(1 \pm i\gamma^*\gamma^n)$, where γ^* is the chirality matrix and $\gamma^n = n_k\gamma^k$.

Odd dimensions. In the standard representation for the gamma matrices of dimension $2^{\frac{d-1}{2}} \times 2^{\frac{d-1}{2}}$, there exists no matrix that would anticommute with all gamma matrices. This poses a problem for the formulation of appropriate boundary conditions for the Dirac spinors. This forces us to use the other known representation for the gamma matrices that we discussed in section II C 1. Boundary conditions in odd dimensions are obtained in the same way as in even dimensions and by replacing γ^k with Γ^k and γ^* with one of the two chirality matrices, for instance, Γ^*_1. So, in odd dimensions, we set

$$\chi = \Pi_+ - \Pi_- = i\Gamma^*_1\Gamma^n. \tag{42}$$

One should note that the duplication due to the existence of two nonequivalent representations in odd dimensions is very crucial for properly setting the boundary conditions on fermions. Otherwise, the set of boundary conditions would be ill-defined and over-restricted in such cases. As a result of this duplication, all traces will be doubled when we take the trace of the various terms contributing to the boundary conformal anomalies in odd dimensions. Applying these boundary conditions, we calculate the boundary anomaly for two explicit examples of $d = 3$ and $d = 5$ in the following. In odd dimension d, we thus have

$$\mathrm{tr}\mathbb{I} = 2d_s, \qquad \mathrm{tr}\Pi_\pm = d_s, \tag{43}$$

where $d_s = 2^{\frac{d-1}{2}}$.

Notice that in any (odd or even) dimension, one has that

$$\mathrm{tr}\gamma = 0. \tag{44}$$

Chiral transformations. For a massless Dirac field, one may introduce a chiral transformation $\psi \to e^{i\alpha\gamma^*}\psi$ where γ^* is a chirality matrix. The Dirac action is invariant under such transformation. In even dimension d, the boundary conditions (41) are also invariant. In the case of odd dimension d, there exist two chirality matrices Γ^*_1 and Γ^*_2 and, respectively, two possible choices for the chiral transformations (19). If we choose matrix Γ^*_1 to define the projectors Π_+ and Π_- in the mixed boundary condition, then these conditions are invariant under the chiral transformations generated by the other chirality matrix,[3] Γ^*_2. The invariance under chiral transformations means that the current $j^i_A = \bar{\psi}\gamma^i\gamma^*\psi$ is conserved, $\nabla_i j^i_A = 0$. The conservation is violated in quantum theory that leads to the quantum chiral anomaly. Below we shall compute the anomaly in dimensions $d = 3$, 4 and 5.

[3]The other possibility is to choose Γ^*_1 as the chirality matrix. Then this is similar to what one has in the even dimensions; for instance, $d = 4$. In particular, one has same difficulty (and same resolution) in defying the invariant boundary conditions that we discuss at the end of Sec. III. As far as we can see, the respective chiral anomaly vanishes both in $d = 3$ and $d = 5$ dimensions. That is why we do not consider this case here.

E. Extrinsic geometry

To complete the discussion in this section, let us briefly review the external geometry. External geometry is about how a boundary is embedded in a manifold. The characteristic measure of this geometry is the extrinsic curvature tensor, or as it is often called in mathematical texts, the second fundamental form. This tensor will be defined as

$$K_{ij} = h_i^k h_j^\ell \nabla_{(k} n_{l)}, \tag{45}$$

where n^i is the unit normal vector on the boundary, and $h^i_j = \delta^i_j - n^i n_j$ defines the projection on the boundary. This tensor is symmetric by construction, with no component in the normal direction, $n^i K_{ij} = 0$. If it is preferred to consider the extrinsic curvature as a tensor living on the boundary, ∂M, the following pullback can be calculated

$$K_{ab} = t^i_a t^j_b K_{ij}. \tag{46}$$

Accordingly, one can define the induced metric on, or the first fundamental form of the boundary

$$h_{ab} = t^i_a t^j_b h_{ij} = t^i_a t^j_b g_{ij}. \tag{47}$$

We denote the trace of this tensor, which frequently appears in our equations, by $K = K^i_i = K^a_a$.

The covariant derivative defined with respect to the intrinsic metric h_{ab} is denoted by $\overline{\nabla}_a$ and the respective curvature by $\bar{R}$, $\bar{R}_{ab}$, and $\bar{R}_{abcd}$. The relations between the intrinsic curvature of the boundary and the curvature in the 5 dimensional space-time are given by the Gauss-Codazzi identities presented in Appendix B.

Under the infinitesimal conformal transformations $\delta g_{ij} = 2\sigma g_{ij}$, $\delta n_i = \sigma n_i$, the Weyl tensor transforms as $\delta W_{ijkl} = 2\sigma W_{ijkl}$. The extrinsic curvature transforms as follows:

$$\delta K_{ab} = \sigma K_{ab} + \gamma_{ab} \nabla_n \sigma, \qquad \delta K = -\sigma K + 4\nabla_n \sigma,$$
$$\delta \hat{K}_{ab} = \sigma \hat{K}_{ab}, \tag{48}$$

where $\nabla_n = n^i \nabla_i$ and $\hat{K}_{ab} = K_{ab} - \frac{1}{d-1} h_{ab} K$ is the trace-free part of the extrinsic curvature tensor. The basic conformal tensors are, thus, the bulk Weyl tensor W_{ijkl} and the trace free extrinsic curvature of the boundary $\hat{K}_{ab}$. The intrinsic Weyl tensor of the boundary metric is expressed in terms of the bulk Weyl tensor and the extrinsic curvature by means of the Gauss-Codazzi relations.

III. CONFORMAL AND CHIRAL ANOMALIES

The quantization of the fermionic field in a fixed gravitational and gauge field background leads to the quantum effective action,

$$W_Q = -\frac{1}{2} \ln \det \hat{D}^2, \tag{49}$$

expressed in terms of determinant of the square Dirac operator $\hat{D}^2$. It can be calculated by using the heat kernel $K(x, x') = \langle x | e^{-s\hat{D}^2} | x' \rangle$,

$$W_Q = \frac{1}{2} \int_{\epsilon^2}^{\infty} \frac{ds}{s} \mathrm{Tr} K(\hat{D}^2, x, x, s), \tag{50}$$

where the trace includes also integration over x. The trace of the heat kernel is characterized by its small s expansion,

$$\mathrm{Tr} K(\hat{D}^2, s) = \sum_{p=0} a_p(\hat{D}^2) s^{\frac{(p-d)}{2}}, \qquad s \to 0, \tag{51}$$

where $a_p(\hat{D}^2)$ are the heat kernel coefficients that are represented by the bulk and boundary integrals,

$$a_p(\hat{D}^2) = \int_M \mathrm{tr}\, A_p(x) + \int_{\partial M} \mathrm{tr}\, B_p(x), \tag{52}$$

where the trace is taken over spinor and group indexes, $A_p(x)$ and $B_p(x)$ are local invarinats constructed from the curvature, gauge fields, and the extrinsic curvature; the latter invariants appear in the boundary term $B_p(x)$.

The invariance of the classical theory under local conformal transformations means that the respective classical stress energy tensor is traceless. This property however is violated in the quantum theory that manifests in the conformal anomaly. For a quantum Dirac field, the integrated conformal anomaly in dimension d is determined by the coefficient a_d, see [18,23],

$$\int_{\mathcal{M}_d} \langle T_{ij} \rangle g^{ij} = -a_d(\hat{D}^2). \tag{53}$$

The minus sign here is due to anticommutativity of the fermion fields in the path integral.

Provided there exists a chirality matrix γ^*, the theory classically possesses a conserved axial current $j^i_A = \bar{\psi}\gamma^i\gamma^*\psi$, $\nabla_i j^i_A = 0$. In quantum theory, the conservation is modified by a quantum anomaly. The local form of the anomaly in terms of the heat kernel coefficients was found in earlier works; see, for instance, [18,23]). When there are both bulk and boundary contributions to the anomaly, it can be presented as follows:

$$\nabla_i \langle j^i_A(x) \rangle = -2i(\mathrm{tr}(\gamma^* A_d(x)) + \mathrm{tr}(\gamma^* B_d(x))\delta_{\partial M}). \tag{54}$$

In dimension $d = 4$, one has a difficulty since the boundary conditions (41) are not invariant under the chiral transformations since the chiral matrix γ^* anticommutes rather than commutes with $\chi = \Pi_+ - \Pi_-$. Resolution of this difficulty leads to a generalization of the boundary conditions and the matrix χ. This issue was earlier discussed in [24–26]. One introduces a family of the boundary conditions (known as the chiral bag boundary conditions) with $\chi(\theta)$ defined as

$$\chi(\theta) = i\gamma^* \gamma^n e^{i\gamma^* \theta}, \tag{55}$$

where θ is a parameter. Under the chiral transformations $\psi \to e^{i\gamma^* \phi}\psi$ it transforms as

$$\chi(\theta) \to e^{-i\gamma^* \phi}\chi(\theta)e^{i\gamma^* \phi} = \chi(\theta + 2\phi). \tag{56}$$

The boundary conditions remain invariant if one transforms $\theta \to \theta - 2\phi$. This transformation is even more natural if the parameter of transformation is a local function of coordinates. Then θ is a function of the coordinates and to have the theory invariant one introduces axial gauge field that compensates the gradients of $\phi(x)$. This direction was followed in [26]. We do not consider here this generalization and keep parameter θ to be constant. Then one has to compute the heat kernel of the Dirac operator subject to the boundary conditions parametrized by θ. The heat kernel coefficients then become the rather nontrivial functions of θ. A remarkable fact, however, proved in [24–26] is that the heat kernel coefficient $a_4(\hat{D}^2, \chi(\theta))$ is independent of parameter θ. For the chiral anomaly, this means that

$$\partial_\theta \mathrm{Tr}(\gamma^* a_4(\hat{D}^2, \chi(\theta))) = 0. \tag{57}$$

This property justifies that in what follows; we do the computation of the anomaly for $\theta = 0$.

The other technical but important remark is the following. The computation of the chiral anomaly with the help of (54) requires the asymptotic expansion $\mathrm{tr}F(x)e^{-s\hat{D}^2}$, with $F(x)$ being a matrix, while the results available in [18] and [19] are given for the case when $F(x)$ is unity matrix multiplied by a function. So that these results cannot be directly applicable.[4] The difference is essential; there may appear terms that do not commute with $F(x)$. Some such terms due to different ordering of the operators in the heat kernel expansion have been computed in [27]. It should be noted that this problem does not arise in odd dimensions. Indeed, in this case, $F(x) = \Gamma_2^*$ commutes with all operators that may enter the heat kernel expansion, E, Ω_{ij}, χ, and $\bar{\nabla}_a \chi$ so that the order of operators is not important in this case and one may use the asymptotic expansion obtained in [18] and [19]. In dimension $d = 4$, the situation is rather different. The chirality matrix $\gamma^* = \gamma_5$ commutes with E and Ω_{ij} but anticommutes with χ and $\bar{\nabla}_a \chi$. This means that in the bulk, one may still use the expansion found in [18] and [19]. However, on the boundary, the order of operators becomes important. Luckily, the boundary terms in the asymptotic expansion that contribute to the chiral anomaly in $d = 4$ were previously computed in [27]. We return to this issue in Sec. V B when we compute the chiral anomaly in $d = 4$.

[4]We thank the anonymous referee for rasing this issue.

IV. BOUNDARY ANOMALY FOR DIRAC SPINORS IN $d = 3$ DIMENSIONS

A. Conformal anomaly

In $d = 3$ dimensions, there are two boundary conformal invariants. One invariant is the integrated Euler density

$$E_2 = \int_{\partial M_3} \bar{R} = \int_{\partial M_3} (R - 2R_{nn} - \mathrm{Tr}K^2 + K^2), \tag{58}$$

where in the second equality, we used the Gauss-Coddazzi relations, and the other is

$$I_1 = \int_{\partial M_3} \mathrm{Tr}\hat{K}^2 = \int_{\partial M_3} \left(\mathrm{Tr}K^2 - \frac{1}{2}K^2\right), \tag{59}$$

where $\hat{K}_{ab}$ is the traceless part of the extrinsic curvature tensor. The general form of the conformal anomaly in $d = 3$ dimensions then is

$$\int_{\mathcal{M}_d} \langle T_{ij} \rangle g^{ij} = -\frac{a}{384\pi} E_2 + \frac{c}{256\pi} I_1. \tag{60}$$

For a conformal scalar field, the conformal charges were computed in [1]. In the present normalization, $(a = -1, c = +1)$ for a scalar field satisfying the Robin boundary conditions.

Computationally, we use Eq. (53) that expresses anomaly in terms of the heat kernel coefficient,

$$\int_{\mathcal{M}_d} \langle T_{ij} \rangle g^{ij} = -a_3, \tag{61}$$

of the Dirac operator. The general form of this coefficient is given in (D 1). Notice that in $d = 3$, one defines $\chi = i\Gamma_1^* \Gamma^n$. The various constituent terms of a_3 and their contributions are listed below.

Various terms	Extended forms	Contribution to anomaly
R	R	$\mathrm{tr}(16\chi) = 0$
E	$-\frac{1}{4}R + \frac{1}{2}F_{ij}\Gamma^i\Gamma^j$	$\mathrm{tr}(96\chi) = 0, \mathrm{tr}(\chi\Gamma^i\Gamma^j) = 0$
R_{nn}	R_{nn}	$\mathrm{tr}(-8\chi) = 0$
$\mathrm{Tr}K^2$	$\mathrm{Tr}K^2$	$\mathrm{tr}(2\Pi_+ + 10\Pi_-) = 24$
K^2	K^2	$\mathrm{tr}(13\Pi_+ - 7\Pi_-) = 12$
SK	$-\frac{1}{2}K^2$	$\mathrm{tr}(96\Pi_+) = 192$
S^2	$\frac{1}{4}K^2$	$\mathrm{tr}(192\Pi_+^2) = 384$
$\mathrm{tr}(\nabla_a\chi\nabla^a\chi)$	$4\mathrm{Tr}K^2$	-12

Note that in the third column, we have dropped the overall factor $\frac{1}{384(4\pi)}$.

Collecting all terms, one finds

$$a_3 = \frac{1}{384(4\pi)} \int_{\partial M_3} (-24\mathrm{Tr}K^2 + 12K^2), \quad (62)$$

and this should be rewritten as a combination of the conformal invariants (58) and (59). Matching the coefficients between (60) and (61), we get the following algebraic equations:

$$R: a = 0,$$
$$R_{nn}:\ a = 0,$$
$$\mathrm{Tr}K^2: a + \frac{3}{2}c_1 = 6,$$
$$K^2: -a - \frac{3}{4}c_1 = -3.$$

Solving these equations, we find

$$a = 0, \qquad c = 4. \quad (63)$$

The value of the charge $c = 4$ corresponds to number of components of the spinor in the doubling representation. Notice that in terms of the charges, the $d = 3$ spinor in this representation can be viewed as equal number of conformal scalars satisfying the Dirichlet $(a = 1, c = +1)$ and conformal Robin $(a = -1, c = +1)$ boundary conditions. Note that in [7] the conformal anomaly due to the Dirac field in the standard 2×2 representation was earlier computed. Taking however our notes on the difficulties with imposing consistently the boundary conditions in the standard representation, that calculation was not fully eligible. Notice also that the gauge field A_i does not make a contribution to the conformal anomaly in $d = 3$ dimensions.

B. Chiral anomaly in $d = 3$ dimensions

The chiral transformations that preserve the boundary conditions are defined with respect the chirality matrix Γ_2^*. Computing the chiral anomaly according to (54), we have to compute the trace in the heat kernel coefficient with matrix Γ_2^*. Most of the traces vanish, for instance, $\mathrm{tr}(\Gamma_2^*\chi) = 0$. The only nonvanishing term is $\mathrm{tr}(\Gamma_2^*\chi\Gamma^i\Gamma^j) = -4\epsilon^{nij}$. The chiral anomaly then is due to the gauge fields,

$$\mathrm{tr}(\Gamma_2^*, B_3(x)) = -\frac{1}{8\pi}\epsilon^{nij}F_{ij}. \quad (64)$$

One can identify $\epsilon^{nij} = \epsilon^{ij}$, the intrinsic epsilon tensor defined on the boundary ∂M_3. In this expression, we are supposed to also take trace over the group indexes. Taking that the non-Abelian generators are traceless, only the Abelian component survives. The chiral anomaly then in $d = 3$ is entirely due to Abelian gauge field B^i. The local chiral anomaly is thus due to a boundary term,

$$\nabla_i \langle j_A^i \rangle = -\frac{1}{2\pi}\epsilon^{ij}\partial_i B_j \delta_{\partial M_3}. \quad (65)$$

Notice that the anomaly has only a boundary contribution that is precisely of the form of a chiral anomaly in two dimensions.

V. ANOMALY IN $d = 4$ DIMENSIONS

A. Conformal anomaly in $d = 4$ dimensions

In $d = 4$ dimensions, the integrated conformal anomaly has both bulk and boundary parts as was noticed in [5]. The computation we are about to perform in this section is the one already done by Fursaev [5]. We of course reproduce his result and include it here only for the reasons of completeness of our consideration. In total, there are four possible conformal invariants so that the integrated anomaly reads

$$\int_{\mathcal{M}_d} \langle T_{ij} \rangle g^{ij} = -\frac{a}{180}\chi[M_4] + \frac{b}{1920\pi^2}\int_{M_4} W^2_{ijkl} + \frac{c_1}{240\pi^2}\int_{\partial M_4} \hat{K}^{ab} W_{anbn} + \frac{c_2}{280\pi^2}\int_{\partial M_4} \mathrm{Tr}\hat{K}^3, \quad (66)$$

where we excluded the contribution due to the gauge fields. These contributions will be discussed later separately. The values for the charges a and b for free fields are well known. For a Dirac field, one has $a = 11$ and $b = 6$. We here focus on the boundary terms.

The general expression for heat kernel coefficient a_4, is given by (D 2). The various constituent boundary terms of a_4 and their contributions are listed below.

Various terms	Extended forms	Contribution to anomaly
$\nabla_n R$	$\nabla_n R$	$\mathrm{tr}(12\mathbb{I} - 42\Pi_+ + 18\Pi_-) = 0$
$\nabla_n E$	$-\frac{1}{4}\nabla_n R$	$\mathrm{tr}(60\mathbb{I} - 240\Pi_+ + 120\Pi_-) = 0$
$K^{ab}R_{ab}$	$K^{ab}R_{ab}$	$\mathrm{tr}(-4\mathbb{I}) = -16$
$K^{ab}R_{anbn}$	$K^{ab}R_{anbn}$	$\mathrm{tr}(16\mathbb{I}) = 64$
KR_{nn}	KR_{nn}	$\mathrm{tr}(-4\mathbb{I}) = -16$
KR	KR	$\mathrm{tr}(20\mathbb{I}) = 80$
KE	$-\frac{1}{4}KR$	$\mathrm{tr}(120\mathbb{I}) = 480$
SR	$-\frac{1}{2}KR$	$\mathrm{tr}(120\Pi_+) = 240$
SE	$\frac{1}{8}KR$	$\mathrm{tr}(720\Pi_+) = 1440$
$\mathrm{Tr}K^3$	$\mathrm{Tr}K^3$	$\mathrm{tr}\left(\frac{224\Pi_+ + 320\Pi_-}{21}\right) = \frac{1088}{21}$
$K\mathrm{Tr}K^2$	$K\mathrm{Tr}K^2$	$\mathrm{tr}\left(\frac{168\Pi_+ - 264\Pi_-}{21}\right) = -\frac{64}{7}$
$S\mathrm{Tr}K^2$	$-\frac{1}{2}K\mathrm{Tr}K^2$	$\mathrm{tr}(48\Pi_+) = 96$

(Table continued)

(Continued)

Various terms	Extended forms	Contribution to anomaly
K^3	K^3	$\mathrm{tr}\left(\frac{280\Pi_+ + 40\Pi_-}{21}\right) = \frac{640}{21}$
SK^2	$-\frac{1}{2}K^3$	$\mathrm{tr}(144\Pi_+) = 288$
S^2K	$\frac{1}{4}K^3$	$\mathrm{tr}(480\Pi_+^2) = 960$
S^3	$-\frac{1}{8}K^3$	$\mathrm{tr}(480\Pi_+^3) = 960$
$\mathrm{tr}(\chi\nabla^a\chi\Omega_{an})$	$2K^{ab}R_{anbn}$	-60
$K\mathrm{tr}(\nabla_a\chi\nabla^a\chi)$	$4K\mathrm{Tr}K^2$	-12
$K^{ab}\mathrm{tr}(\nabla_a\chi\nabla_b\chi)$	$4\mathrm{Tr}K^3$	-24
$\mathrm{tr}(\nabla_a\chi\nabla^a\chi S)$	$-K\mathrm{Tr}K^2$	-120

Above in the contribution to anomaly, we drop the overall factor $\frac{1}{360(4\pi)^2}$. So, for the anomaly,

$$\int_{\mathcal{M}_4} \langle T_{ij}\rangle g^{ij} = -a_4, \tag{67}$$

and focusing only on the boundary terms, we find

$$a_4 = \frac{1}{360(4\pi)^2}\int_{\partial M_4}\left(-56K^{ab}R_{anbn} - 16K^{ab}R_{ab} - 16KR_{nn} + 20KR - \frac{928}{21}\mathrm{Tr}K^3 + \frac{104}{7}K\mathrm{Tr}K^2 + \frac{136}{21}K^3\right). \tag{68}$$

Matching the coefficients with the general expression (66), one arrives at the following algebraic equations

$$\begin{aligned}
K^{ab}R_{anbn}&: -8a - 24c_1 = 56,\\
K^{ab}R_{ab}&: 8a - 12c_1 = 16,\\
KR_{nn}&: 8a - 12c_1 = 16,\\
KR&: -4a + 4c_1 = -20,\\
\mathrm{Tr}K^3&: -\frac{16}{3}a + \frac{144}{7}c_2 = \frac{928}{21},\\
K\mathrm{Tr}K^2&: 8a - \frac{144}{7}c_2 = -\frac{104}{7},\\
K^3&: -\frac{8}{3}a + \frac{32}{7}c_2 = -\frac{136}{21},
\end{aligned}$$

solving, which, we find

Conformal charges	Mixed boundary condition
a	11
b	6
c_1	6
c_2	5

The value of b comes from matching the bulk terms and is well known. These values agree with those found in [5].

Let us now discuss the contribution to the bulk and boundary conformal anomaly due to the gauge fields. In the bulk, the possible nontrivial contribution comes from the terms $\mathrm{tr}\Omega_{ij}\Omega^{ij} = 4F_{ij}F^{ij}$ and $\mathrm{tr}E^2 = \frac{1}{4}F_{ij}F_{kl}\mathrm{tr}\gamma^i\gamma^j\gamma^k\gamma^l = -2F_{ij}F^{ij}$. Notice that in these expressions, one also supposes to take trace over the group indexes; we omit it here to avoid additional confusion with traces. Among the boundary terms, there are several terms that depend either on $E = \frac{1}{2}F_{ij}\gamma^i\gamma^j$ or $\Omega_{ij} = F_{ij}$ (we here focus only on dependence on the gauge fields) that potentially may produce a contribution to anomaly due to the gauge fields. However, all these terms identically vanish. For instance, $\mathrm{tr}(\Pi_+E) = 0$ and $\mathrm{tr}(\chi\overline{\nabla}_a\chi\Omega^a_n) = 0$. So, we conclude that the only contribution to the conformal anomaly due to the gauge fields is in the bulk,

$$\int_{\mathcal{M}_4}\langle T_{ij}\rangle g^{ij} = \frac{1}{24\pi^2}\int_{M_4} F_{ij}F^{ij}, \tag{69}$$

where the trace over the group indexes of the gauge fields is assumed to be taken. This anomaly is of course well known.

B. Chiral anomaly in $d = 4$ dimensions

First, we discuss purely gravitational part in the chiral anomaly. We choose present result in the integrated form,

$$\int_{M_4}\nabla_i\langle j^i_A\rangle = -2i\left(\int_{M^4}\mathrm{tr}(\gamma^* A_4(x)) + \int_{\partial M_4}\mathrm{tr}(\gamma^* B_4(p))\right). \tag{70}$$

Analysis shows that only two terms are nonvanishing. One term is in the bulk, $\mathrm{tr}(\gamma^*\Omega_{ij}\Omega^{ij})$ and the other on the boundary $\mathrm{tr}(\gamma^*\overline{\nabla}_a\chi\Omega^a_n)$. Taking that $\Omega_{ij} = \frac{1}{4}R_{ijkl}\gamma^k\gamma^l$ and that $\overline{\nabla}_a\chi = i\gamma^*\gamma^b K_{ab}$ (see Appendix), we find using that $\mathrm{tr}(\gamma^*\gamma^i\gamma^j\gamma^k\gamma^\ell) = -4i\epsilon^{ijk\ell}$,

$$\begin{aligned}
\mathrm{tr}(\gamma^*\Omega_{ij}\Omega^{ij}) &= -\frac{i}{4}\epsilon^{klmn}R_{ijkl}R^{ijmn},\\
\mathrm{tr}(\gamma^*\chi\overline{\nabla}_a\chi\Omega^a_n) &= i\epsilon^{nbcd}R_{nacd}K_{ab}.
\end{aligned} \tag{71}$$

The first term contributes to the anomaly in the bulk and the second term to the anomaly on the boundary. As we discussed earlier in the paper, one cannot directly use the boundary coefficient B_4 since it was obtained in the assumption that it is contracted with a unity operator $F(x)$ that commutes with all other operators that appear in the asymptotic expansion. In (70), matrix $F(x) = \gamma^*$ commutes with E and Ω_{ij} but anticommutes with χ and $\overline{\nabla}_a\chi$ that appear on the boundary. Therefore, one cannot directly use B_4 given in the Appendix since there may appear several terms with different ordering of operators that replace a single term in the commutative case. It is interesting that precisely the terms that we need for computing the chiral anomaly were found in [27],

$$\mathrm{tr}[F(x)(-18\chi\overline{\nabla}^a\chi\Omega_{an} - 12\overline{\nabla}^a\chi\Omega_{an}\chi - 18\Omega_{an}\chi\overline{\nabla}^a\chi + 12\chi\Omega_{an}\overline{\nabla}^a\chi)]. \quad (72)$$

In commutative case [$F(x)$ is proportional to unity matrix], these terms combine to $-60\mathrm{tr}(F(x)\chi\overline{\nabla}^a\chi\Omega_{an})$ that comes from B_4 in appendix. For $F(x) = \gamma^*$, these terms combine to produce

$$-12\mathrm{tr}(\gamma^*\chi\overline{\nabla}^a\chi\Omega_{an}), \quad (73)$$

where we used that χ and $\overline{\nabla}_a\chi$ anticommute.

Combining these elements of the calculation, we conclude that

$$\int_{M_4}\nabla_i\langle j^i_A\rangle = -\frac{1}{384\pi^2}\left[\int_{M_4}\epsilon^{k\ell pq}R_{ijk\ell}R^{ij}{}_{pq} + \frac{8}{5}\int_{\partial M_4}\epsilon^{nabc}K^d_a R_{ndbc}\right]. \quad (74)$$

Using (B2), we can rewrite the second integrand as follows:

$$\epsilon^{nabc}K^d_a R_{ndbc} = -2\epsilon^{nabc}K^d_a\overline{\nabla}_c K_{bd}; \quad (75)$$

therefore,

$$\int_{M_4}\nabla_i\langle j^i_A\rangle = -\frac{1}{384\pi^2}\left[\int_{M_4}\epsilon^{k\ell pq}R_{ijk\ell}R^{ij}{}_{pq} - \frac{16}{5}\int_{\partial M_4}\epsilon^{abc}K^d_a\overline{\nabla}_c K_{bd}\right], \quad (76)$$

where, in the boundary term, we use intrinsic $\epsilon^{abc} = \epsilon^{nabc}$. The first term here is $\frac{1}{12}P$, where P is the Pontryagin number defined as

$$P = \frac{1}{32\pi^2}\int_{M_4}\epsilon^{k\ell pq}R_{ijk\ell}R^{ij}{}_{pq}. \quad (77)$$

The boundary term in the chiral anomaly (74), (76) is new. At the moment, it is not clear whether it should always be combined with the Pontryagin term or it is an independent term.

The other remarks is that the chiral anomaly has a structure similar to that of the parity anomaly, see [28], although the relative coefficients are different. The quantum origin of these two anomalies is however quite different, see discussion in [26]. For other recent works on parity anomaly, see [26,28–33].

It is interesting to note that the boundary term (74) is conformal invariant. Indeed, it can be expressed in terms of the trace-free extrinsic curvature $\hat{K}_{ab}$ and the Weyl tensor, $\epsilon^{nabc}K^d_a R_{ndbc} = \epsilon^{nabc}\hat{K}^d_a W_{ndbc}$.

Our last comment in this section concerns the possible contribution of the gauge fields to the chiral anomaly. Analysis shows that there is only one term in the heat kernel that contributes to the anomaly. This is a bulk term

$$\mathrm{tr}(\gamma^* E^2) = \frac{1}{4}F_{ij}F_{k\ell}\mathrm{tr}(\gamma^*\gamma^i\gamma^j\gamma^k\gamma^\ell) = -i\epsilon^{ijk\ell}F_{ij}F_{k\ell}. \quad (78)$$

We conclude that a contribution due to the gauge fields to the chiral anomaly is given by

$$\int_{M_4}\nabla_i\langle j^i_A\rangle = -\frac{1}{16\pi^2}\int_{M_4}\epsilon^{ijk\ell}F_{ij}F_{k\ell}. \quad (79)$$

This is a known result in the literature. No boundary term due to the gauge fields in the anomaly has been found.

VI. ANOMALY IN $d = 5$ DIMENSIONS

A. Boundary conformal anomaly in $d = 5$ dimensions

The conformal anomaly in 5 dimensions is entirely due to the boundary terms,

$$\int_{\mathcal{M}_4}\langle T_{ij}\rangle g^{ij} = \frac{1}{5760(4\pi)^2}\int_{\partial M_5}\left(aE_4 + \sum_{k=1}^{8}c_k I_k\right). \quad (80)$$

E_4 is the Euler density integrated over boundary ∂M_5, and $\chi[\partial M_5] = \frac{1}{32\pi^2}E_4$ is the Euler number of the boundary. $\{I_k, k = 1, \ldots, 8\}$ is a set of eight conformal invariants. Altogether, they form a complete basis of boundary conformal invariants in five dimensions. For convenience, below we give exact expressions for all these invariants.

1. Boundary invariants in $d = 5$ dimensions

We start our discussion of conformal anomaly in $d = 5$ dimensions by recalling the complete list of boundary conformal invariants in 5 dimensions [2],

$$\begin{aligned} E_4 &= \int_{\partial M_5}(\bar{R}^2_{acbd} - 4\bar{R}^2_{ab} + \bar{R}^2) \\ &= \int_{\partial M_5}(R^2_{acbd} - 4R^2_{ab} + R^2 - 4R^2_{anbn} + 8R^{ab}R_{anbn} \\ &\quad + 4R^2_{nn} - 4RR_{nn} + 4K^{ab}K^{cd}R_{acbd} + 8(K^2)^{ab}R_{ab} \\ &\quad - 8KK^{ab}R_{ab} - 2\mathrm{Tr}K^2R + 2K^2R - 8(K^2)^{ab}R_{anbn} \\ &\quad + 8KK^{ab}R_{anbn} + 4\mathrm{Tr}K^2R_{nn} - 4K^2R_{nn} - 6\mathrm{Tr}K^4 \\ &\quad + 8K\mathrm{Tr}K^3 + 3(\mathrm{Tr}K^2)^2 - 6K^2\mathrm{Tr}K^2 + K^4), \end{aligned} \quad (81)$$

$$I_1 = \int_{\partial M_5}(\mathrm{Tr}\hat{K}^2)^2 = \int_{\partial M_5}\left[(\mathrm{Tr}K^2)^2 - \frac{1}{2}K^2\mathrm{Tr}K^2 + \frac{1}{16}K^4\right], \quad (82)$$

$$I_2 = \int_{\partial M_5}\mathrm{Tr}\hat{K}^4 = \int_{\partial M_5}\left(\mathrm{Tr}K^4 - K\mathrm{Tr}K^3 + \frac{3}{8}K^2\mathrm{Tr}K^2 - \frac{3}{64}K^4\right), \quad (83)$$

$$I_3 = \int_{\partial M_5} W^2_{acbd} = \int_{\partial M_5} \left(R^2_{acbd} - \frac{16}{9} R^2_{ab} + \frac{5}{18} R^2 + \frac{8}{3} R^{ab} R_{anbn} + \frac{4}{9} R^2_{nn} - \frac{8}{9} R R_{nn} \right), \tag{84}$$

$$I_4 = \int_{\partial M_5} W^2_{anbn} = \int_{\partial M_5} \left(\frac{1}{9} R^2_{ab} - \frac{1}{36} R^2 + R^2_{anbn} - \frac{2}{3} R^{ab} R_{anbn} - \frac{4}{9} R^2_{nn} + \frac{2}{9} R R_{nn} \right), \tag{85}$$

$$\begin{aligned} I_5 &= \int_{\partial M_5} \hat{K}^{ab} \hat{K}^{cd} W_{acbd} \\ &= \int_{\partial M_5} \left(K^{ab} K^{cd} R_{acbd} + \frac{2}{3} (K^2)^{ab} R_{ab} - \frac{5}{6} K K^{ab} R_{ab} - \frac{1}{12} \mathrm{Tr} K^2 R + \frac{1}{8} K^2 R \right. \\ &\quad \left. + \frac{1}{2} K K^{ab} R_{anbn} - \frac{1}{6} K^2 R_{nn} \right), \end{aligned} \tag{86}$$

$$\begin{aligned} I_6 &= \int_{\partial M_5} \hat{K}^a_c \hat{K}^{cb} W_{anbn} \\ &= \int_{\partial M_5} \left(-\frac{1}{3} K^a_c K^{cb} R_{ab} + \frac{1}{6} K K^{ab} R_{ab} + \frac{1}{12} \mathrm{Tr} K^2 R - \frac{1}{24} K^2 R \right. \\ &\quad \left. + K^a_c K^{cb} R_{anbn} - \frac{1}{2} K K^{ab} R_{anbn} - \frac{1}{3} \mathrm{Tr} K^2 R_{nn} + \frac{1}{6} K^2 R_{nn} \right), \end{aligned} \tag{87}$$

$$\begin{aligned} I_7 &= \int_{\partial M_5} W^2_{nabc} \\ &= \int_{\partial M_5} \left[2 \bar{\nabla}_c K_{ab} \bar{\nabla}^c K^{ab} - \frac{8}{3} \bar{\nabla}_a K^a_b \bar{\nabla}_c K^{cb} + \frac{4}{3} \bar{\nabla}_a K \bar{\nabla}_b K^{ab} - \frac{2}{3} (\bar{\nabla} K)^2 \right. \\ &\quad \left. - 2 K^{ab} K^{cd} R_{acbd} - 2 (K^2)^{ab} R_{anbn} + 2 (K^2)^{ab} R_{ab} + 2 K \mathrm{Tr} K^3 - 2 (\mathrm{Tr} K^2)^2 \right], \end{aligned} \tag{88}$$

$$\begin{aligned} I_8 &= \int_{\partial M_5} \left(\hat{K}^{ab} \nabla_n W_{anbn} - \frac{1}{2} K \hat{K}^{ab} W_{anbn} - \frac{2}{9} \overline{\nabla}_a \hat{K}^a_b \overline{\nabla}_c \hat{K}^{cb} + 2 (\hat{K}^2)^{ab} \bar{S}_{ab} - \mathrm{Tr} \hat{K}^2 \bar{S}^a_a \right) \\ &= \int_{\partial M_5} \left[\frac{2}{3} K^{ab} \nabla_n R_{anbn} - \frac{1}{12} K \nabla_n R + \frac{2}{3} K^{ab} K^{cd} R_{acbd} - K^a_c K^{bc} R_{ab} \right. \\ &\quad + \frac{1}{3} \mathrm{Tr} K^2 R - \frac{5}{48} K^2 R + \frac{5}{3} K^a_c K^{bc} R_{anbn} - \frac{1}{3} K K^{ab} R_{anbn} - \mathrm{Tr} K^2 R_{nn} + \frac{11}{24} K^2 R_{nn} \\ &\quad + \mathrm{Tr} K^4 - \frac{11}{6} K \mathrm{Tr} K^3 + \frac{47}{48} K^2 \mathrm{Tr} K^2 - \frac{7}{48} K^4 \\ &\quad \left. - \frac{1}{3} \overline{\nabla}_c K_{ab} \overline{\nabla}^c K^{ab} + \frac{8}{9} \overline{\nabla}_a K^a_b \overline{\nabla}_c K^{bc} - \frac{7}{9} \overline{\nabla}_a K^{ab} \overline{\nabla}_b K + \frac{25}{72} (\overline{\nabla} K)^2 \right]. \end{aligned} \tag{89}$$

Here, we have substituted $\bar{S}_{ab} = \frac{1}{2} (\bar{R}_{ab} - \frac{1}{6} \bar{R} h_{ab})$, which is the 4 dimensional Schouten tensor computed with respect to the intrinsic boundary metric h_{ab}.

2. *Heat kernel coefficients in five dimensions*

The general form of the coefficient $B_5(x)$ in the expansion of heat kernel is given in (D 3); see [19,34]. According to the notation of [19],

$$a_5 = \frac{1}{5760 (4\pi)^2} \int_{\partial M_5} \mathrm{tr} (\mathcal{A}^1_5 + \mathcal{A}^2_5 + \mathcal{A}^3_5), \tag{90}$$

where tr is taken on spinor indices. The constituent terms of $\mathrm{tr}\mathcal{A}_5^{\{1,2,3\}}$ and their extended forms in terms of the curvature tensors are listed in the following three tables.

Various terms of $\mathcal{A}_5^1$	Extended forms	Contribution to anomaly
$\nabla_n^2 E$	$-\frac{1}{4}\nabla_n^2 R$	$\mathrm{tr}(360\chi)=0$
$\mathrm{tr}(\nabla_n ES)$	$\frac{1}{2}K\nabla_n R$	-1440
E^2	$\frac{1}{16}R^2$	$\mathrm{tr}(720\chi)=0$
RE	$-\frac{1}{4}R^2$	$\mathrm{tr}(240\chi)=0$
$\Box R$	$\nabla_n^2 R+K\nabla_n R$	$\mathrm{tr}(48\chi)=0$
R^2	R^2	$\mathrm{tr}(20\chi)=0$
R_{ij}^2	$R_{ab}^2+R_{nn}^2+2R_{an}^2$	$\mathrm{tr}(-8\chi)=0$
$R_{ikj\ell}^2$	$R_{acbd}^2+4R_{anbn}^2+4R_{nabc}^2$	$\mathrm{tr}(8\chi)=0$
$R_{nn}E$	$-\frac{1}{4}RR_{nn}$	$\mathrm{tr}(-120\chi)=0$
RR_{nn}	RR_{nn}	$\mathrm{tr}(-20\chi)=0$
$\mathrm{tr}(RS^2)$	K^2R	480
$\nabla_n^2 R$	$\nabla_n^2 R$	$\mathrm{tr}(12\chi)=0$
$\nabla_n^2 R_{nn}$	$\nabla_n^2 R_{nn}$	$\mathrm{tr}(15\chi)=0$
$\mathrm{tr}(\nabla_n RS)$	$-2K\nabla_n R$	-270
$\mathrm{tr}(R_{nn}S^2)$	K^2R_{nn}	120
$\mathrm{tr}(S\bar{\Box}S)$	$-(\bar{\nabla}K)^2-\frac{1}{2}K^2\mathrm{Tr}K^2$	960
$R^{ab}R_{anbn}$	$R^{ab}R_{anbn}$	$\mathrm{tr}(-16\chi)=0$
R_{nn}^2	R_{nn}^2	$\mathrm{tr}(-17\chi)=0$
R_{anbn}^2	R_{anbn}^2	$\mathrm{tr}(-10\chi)=0$
$\mathrm{tr}(ES^2)$	$-\frac{1}{4}K^2R$	2880
$\mathrm{tr}(S^4)$	$\frac{1}{4}K^4$	1440

Various terms of $\mathcal{A}_5^2$	Extended forms	Contribution to anomaly
$K\nabla_n E$	$-\frac{1}{4}K\nabla_n R$	$\mathrm{tr}(-90\Pi_+-450\Pi_-)=-2160$
$K\nabla_n R$	$K\nabla_n R$	$\mathrm{tr}(-\frac{111}{2}\Pi_+-42\Pi_-)=-390$
$K^{ab}\nabla_n R_{anbn}$	$K^{ab}\nabla_n R_{anbn}$	$\mathrm{tr}(-30\Pi_+)=-120$
$\mathrm{tr}(K\bar{\Box}S)$	$2(\bar{\nabla}K)^2$	240
$\mathrm{tr}(K^{ab}\bar{\nabla}_a\bar{\nabla}_b S)$	$2\bar{\nabla}_a K^{ab}\bar{\nabla}_b K$	420
$\mathrm{tr}(\bar{\nabla}_a K\bar{\nabla}^a S)$	$-2(\bar{\nabla}K)^2$	390
$\mathrm{tr}(\bar{\nabla}_a K^{ab}\bar{\nabla}_b S)$	$-2\bar{\nabla}_a K^{ab}\bar{\nabla}_b K$	480
$\mathrm{tr}(\bar{\Box}KS)$	$2(\bar{\nabla}K)^2$	420
$\mathrm{tr}(\bar{\nabla}_a\bar{\nabla}_b K^{ab}S)$	$2\bar{\nabla}_a K^{ab}\bar{\nabla}_b K$	60
$(\bar{\nabla}K)^2$	$(\bar{\nabla}K)^2$	$\mathrm{tr}\left(\frac{487}{16}\Pi_++\frac{413}{16}\Pi_-\right)=225$
$\bar{\nabla}_a K^{ab}\bar{\nabla}_b K$	$\bar{\nabla}_a K^{ab}\bar{\nabla}_b K$	$\mathrm{tr}(238\Pi_+-58\Pi_-)=720$
$\bar{\nabla}_a K_b^a\bar{\nabla}_c K^{bc}$	$\bar{\nabla}_a K_b^a\bar{\nabla}_c K^{bc}$	$\mathrm{tr}\left(\frac{49}{4}\Pi_++\frac{11}{4}\Pi_-\right)=60$
$\bar{\nabla}_c K_{ab}\bar{\nabla}^c K^{ab}$	$\bar{\nabla}_c K_{ab}\bar{\nabla}^c K^{ab}$	$\mathrm{tr}\left(\frac{535}{8}\Pi_+-\frac{355}{8}\Pi_-\right)=90$
$\bar{\nabla}_c K_{ab}\bar{\nabla}^b K^{ac}$	$\bar{\nabla}_c K_{ab}\bar{\nabla}^b K^{ac}$	$\mathrm{tr}\left(\frac{151}{4}\Pi_++\frac{29}{4}\Pi_-\right)=180$
$\bar{\Box}KK$	$-(\bar{\nabla}K)^2$	$\mathrm{tr}(111\Pi_+-6\Pi_-)=420$
$\bar{\nabla}_a\bar{\nabla}_b K^{ab}K$	$-\bar{\nabla}_a K^{ab}\bar{\nabla}_b K$	$\mathrm{tr}(-15\Pi_++30\Pi_-)=60$
$K^{bc}\bar{\nabla}_c\bar{\nabla}^a K_{ab}$	$-\bar{\nabla}_a K_b^a\bar{\nabla}_c K^{bc}$	$\mathrm{tr}\left(-\frac{15}{2}\Pi_+-\frac{75}{2}\Pi_-\right)=120$
$K^{ab}\bar{\nabla}_a\bar{\nabla}_b K$	$-\bar{\nabla}_a K^{ab}\bar{\nabla}_b K$	$\mathrm{tr}\left(\frac{945}{4}\Pi_+-\frac{285}{4}\Pi_-\right)=660$
$K^{ab}\bar{\Box}K_{ab}$	$-\bar{\nabla}_c K_{ab}\bar{\nabla}^c K^{ab}$	$\mathrm{tr}(114\Pi_+-54\Pi_-)=240$
$\mathrm{tr}(KSE)$	$\frac{1}{2}K^2R$	1440
$\mathrm{tr}(KSR_{nn})$	$-2K^2R_{nn}$	30
$\mathrm{tr}(KSR)$	$-2K^2R$	240

(Table continued)

(Continued)

Various terms of $\mathcal{A}_5^2$	Extended forms	Contribution to anomaly
$\mathrm{tr}(K^{ab}R_{ab}S)$	$-2KK^{ab}R_{ab}$	-60
$\mathrm{tr}(K^{ab}SR_{anbn})$	$-2KK^{ab}R_{anbn}$	180
K^2E	$-\frac{1}{4}K^2R$	$\mathrm{tr}(195\Pi_+ - 105\Pi_-) = 360$
$\mathrm{Tr}K^2E$	$-\frac{1}{4}\mathrm{Tr}K^2R$	$\mathrm{tr}(30\Pi_+ + 150\Pi_-) = 720$
K^2R	K^2R	$\mathrm{tr}\left(\frac{195}{6}\Pi_+ - \frac{105}{6}\Pi_-\right) = 60$
$\mathrm{Tr}K^2R$	$\mathrm{Tr}K^2R$	$\mathrm{tr}(5\Pi_+ + 25\Pi_-) = 120$
K^2R_{nn}	K^2R_{nn}	$\mathrm{tr}\left(-\frac{275}{16}\Pi_+ + \frac{215}{16}\Pi_-\right) = -15$
$\mathrm{Tr}K^2R_{nn}$	$\mathrm{Tr}K^2R_{nn}$	$\mathrm{tr}\left(-\frac{275}{8}\Pi_+ + \frac{215}{8}\Pi_-\right) = -30$
$KK^{ab}R_{ab}$	$KK^{ab}R_{ab}$	$\mathrm{tr}(-\Pi_+ - 14\Pi_-) = -60$
$KK^{ab}R_{anbn}$	$KK^{ab}R_{anbn}$	$\mathrm{tr}(\frac{109}{4}\Pi_+ - \frac{49}{4}\Pi_-) = 60$
$K_c^aK^{bc}R_{ab}$	$K_c^aK^{bc}R_{ab}$	$\mathrm{tr}(16\Pi_+ - 16\Pi_-) = 0$
$K_c^aK^{bc}R_{anbn}$	$K_c^aK^{bc}R_{anbn}$	$\mathrm{tr}\left(\frac{133}{2}\Pi_+ + \frac{47}{2}\Pi_-\right) = 360$
$K^{ab}K^{cd}R_{acbd}$	$K^{ab}K^{cd}R_{acbd}$	$\mathrm{tr}(32\Pi_+ - 32\Pi_-) = 0$
$\mathrm{tr}(KS^3)$	$-\frac{1}{2}K^4$	2160
K^2S^2	K^4	1080
$\mathrm{Tr}K^2S^2$	$K^2\mathrm{Tr}K^2$	360
K^3S	$-2K^4$	$\frac{885}{4}$
$K\mathrm{Tr}K^2S$	$-2K^2\mathrm{Tr}K^2$	$\frac{315}{2}$
$\mathrm{Tr}K^3S$	$-2K\mathrm{Tr}K^3$	150
K^4	K^4	$\mathrm{tr}(\frac{2041}{128}\Pi_+ + \frac{65}{128}\Pi_-) = \frac{1053}{16}$
$K^2\mathrm{Tr}K^2$	$K^2\mathrm{Tr}K^2$	$\mathrm{tr}\left(\frac{417}{32}\Pi_+ + \frac{141}{32}\Pi_-\right) = \frac{279}{4}$
$(\mathrm{Tr}K^2)^2$	$(\mathrm{Tr}K^2)^2$	$\mathrm{tr}\left(\frac{375}{32}\Pi_+ - \frac{777}{32}\Pi_-\right) = -\frac{201}{4}$
$K\mathrm{Tr}K^3$	$K\mathrm{Tr}K^3$	$\mathrm{tr}(25\Pi_+ - \frac{17}{2}\Pi_-) = 66$
$\mathrm{Tr}K^4$	$\mathrm{Tr}K^4$	$\mathrm{tr}\left(\frac{231}{8}\Pi_+ + \frac{327}{8}\Pi_-\right) = 279$

Various terms of $\mathcal{A}_5^3$	Extended forms	Contribution to anomaly
E^2	$\frac{1}{16}R^2$	$\mathrm{tr}(-180\mathbb{I}) = -1440$
$\chi E\chi E$	$\frac{1}{16}R^2$	$\mathrm{tr}(180\chi^2) = 1440$
$\mathrm{tr}((\overline{\nabla}S)^2)$	$(\overline{\nabla}K)^2 + \frac{1}{2}K^2\mathrm{Tr}K^2$	-120
$\mathrm{tr}(\chi(\overline{\nabla}S)^2)$	$(\overline{\nabla}K)^2$	720
$\mathrm{tr}(\Omega_{ab}\Omega^{ab})$	$-R_{acbd}^2 - 2R_{nabc}^2$	$-\frac{105}{4}$
$\mathrm{tr}(\chi\Omega_{ab}\Omega^{ab})$	0	120
$\mathrm{tr}(\chi\Omega_{ab}\chi\Omega^{ab})$	$-R_{acbd}^2 + 2R_{nabc}^2$	$\frac{105}{4}$
$\mathrm{tr}(\Omega_{an}\Omega_n^a)$	$-2R_{anbn}^2 - R_{nabc}^2$	-45
$\mathrm{tr}(\chi\Omega_{an}\Omega_n^a)$	0	180
$\mathrm{tr}(\chi\Omega_{an}\chi\Omega_n^a)$	$2R_{anbn}^2 - R_{nabc}^2$	-45
$\mathrm{tr}(\Omega_{an}\chi\overline{\nabla}^aS - \Omega_{an}\overline{\nabla}^aS\chi)$	$-2KK^{ab}R_{anbn}$	-360
$\mathrm{tr}(\chi\overline{\nabla}_a\chi\Omega_n^aK)$	$4KK^{ab}R_{anbn}$	-45
$\mathrm{tr}(\overline{\nabla}_a\chi\overline{\nabla}_b\chi\Omega^{ab})$	$-4K^{ab}K^{cd}R_{acbd}$	-180
$\mathrm{tr}(\chi\overline{\nabla}_a\chi\overline{\nabla}_b\chi\Omega^{ab})$	0	30
$\mathrm{tr}(\chi\overline{\nabla}_a\chi\nabla_n\Omega_n^a)$	$4K^{ab}\nabla_nR_{anbn}$	90
$\mathrm{tr}(\chi\overline{\nabla}^a\chi\overline{\nabla}^b\Omega_{ab})$	$-4K^{ab}K^{cd}R_{acbd} - 4\overline{\nabla}^bK^{ac}R_{abcn}$	120
$\mathrm{tr}(\chi\overline{\nabla}_a\chi\Omega_{bn}K^{ab})$	$4K_c^aK^{bc}R_{anbn}$	-180
$\mathrm{tr}(\overline{\nabla}_a\chi\overline{\nabla}^aE)$	0	300
$\mathrm{tr}(\overline{\nabla}_a\chi\overline{\nabla}^a\chi E)$	$-2\mathrm{Tr}K^2R$	-180

(Table continued)

(Continued)

Various terms of $\mathcal{A}_5^3$	Extended forms	Contribution to anomaly
$\mathrm{tr}(\chi\bar\nabla_a\chi\bar\nabla^a\chi E)$	0	−90
$\mathrm{tr}(\bar\Box\chi E)$	0	240
$\mathrm{tr}(\bar\nabla_a\chi\bar\nabla^a\chi R)$	$8\mathrm{Tr}K^2R$	−30
$\mathrm{tr}(\bar\nabla_a\chi\bar\nabla_b\chi R^{ab})$	$8K^a_cK^{bc}R_{ab}$	−60
$\mathrm{tr}(\bar\nabla_a\chi\bar\nabla_b\chi R_n{}^{ab}{}_n)$	$-8K^a_cK^{bc}R_{anbn}$	−30
$\mathrm{tr}(\bar\nabla_a\chi\bar\nabla^a\chi K^2)$	$8K^2\mathrm{Tr}K^2$	$-\frac{675}{32}$
$\mathrm{tr}(\bar\nabla_a\chi\bar\nabla_b\chi K^a_cK^{bc})$	$8\mathrm{Tr}K^4$	$-\frac{75}{4}$
$\mathrm{tr}(\bar\nabla_a\chi\bar\nabla^a\chi\mathrm{Tr}K^2)$	$8(\mathrm{Tr}K^2)^2$	$-\frac{195}{16}$
$\mathrm{tr}(\bar\nabla_a\chi\bar\nabla_b\chi K^{ab}K)$	$8K\mathrm{Tr}K^3$	$-\frac{675}{8}$
$\mathrm{tr}(\bar\nabla_a\chi\bar\nabla^a SK)$	$-2K^2\mathrm{Tr}K^2$	−330
$\mathrm{tr}(\bar\nabla_a\chi\bar\nabla_b SK^{ab})$	$-2K\mathrm{Tr}K^3$	−300
$\mathrm{tr}(\bar\nabla_a\chi\bar\nabla^a\chi\bar\nabla_b\chi\bar\nabla^b\chi)$	$8(\mathrm{Tr}K^2)^2$	$\frac{15}{4}$
$\mathrm{tr}(\bar\nabla_a\chi\bar\nabla_b\chi\bar\nabla^a\chi\bar\nabla^b\chi)$	$16\mathrm{Tr}K^4-8(\mathrm{Tr}K^2)^2$	$\frac{15}{8}$
$\mathrm{tr}(\bar\Box\chi\bar\Box\chi)$	$8\bar\nabla_aK^a_b\bar\nabla_cK^{bc}+8(\mathrm{Tr}K^2)^2$	$-\frac{15}{4}$
$\mathrm{tr}(\bar\nabla_a\bar\nabla_b\chi\bar\nabla^a\bar\nabla^b\chi)$	$8\bar\nabla_cK_{ab}\bar\nabla^cK^{ab}+8\mathrm{Tr}K^4$	$-\frac{105}{2}$
$\mathrm{tr}(\bar\nabla_a\chi\bar\nabla^a\chi\bar\Box\chi)$	0	−15
$\mathrm{tr}(\bar\nabla_a\bar\Box\chi\bar\nabla^a\chi)$	$-8\bar\nabla_aK^a_b\bar\nabla_cK^{bc}-8(\mathrm{Tr}K^2)^2$	$-\frac{135}{2}$

In derivation of the second columns of the tables, we have extensively used the identities in A and that $\mathrm{tr}\Pi_\pm=4$ in the doubling representation in $d=5$. By adding the similar terms of the second columns, taking into account the coefficients of the third columns and after using the identities of Sec. B, we get the following result:

$$a_5=\frac{1}{5760(4\pi)^2}\int_{\partial\mathcal{M}_5}\Big(240K^{ab}\nabla_nR_{anbn}-30K\nabla_nR \\ -450K^{ab}K^{cd}R_{acbd}+210K^a_cK^{bc}R_{ab}+60KK^{ab}R_{ab}+60\mathrm{Tr}K^2R-15K^2R \\ -810K^a_cK^{bc}R_{anbn}+240KK^{ab}R_{anbn}-30\mathrm{Tr}K^2R_{nn}+45K^2R_{nn} \\ -261\mathrm{Tr}K^4+381K\mathrm{Tr}K^3-\frac{1251}{4}(\mathrm{Tr}K^2)^2+66K^2\mathrm{Tr}K^2-\frac{267}{16}K^4 \\ +300\bar\nabla_cK_{ab}\bar\nabla^cK^{ab}-240\bar\nabla_aK^a_b\bar\nabla_cK^{bc}-15(\bar\nabla K)^2\Big). \tag{91}$$

It is a curious observation that in this expression, the Riemann curvature appears only in combination with the extrinsic curvature. The anomaly thus vanishes for a geodesic boundary. A technical explanation for this fact is that in the heat kernel (D 3), all terms that are expressed only in terms of the Riemann curvature come with matrix $\chi=\Pi_+-\Pi_-$ whose trace vanishes. In the case of a scalar field considered in [2], one has that χ is either $+1$ or -1 depending on the type of the boundary condition. As a result, the curvature terms are present in a_5 and in the anomaly. For exactly same reason, such terms did not appear in $d=3$ dimensions for the Dirac field and appeared for the conformal scalars.

3. Conformal charges in $d=5$

This expression gives us the integrated conformal anomaly

$$\int_{\partial M_5}\langle T_{ij}\rangle g^{ij}=-a_5. \tag{92}$$

Now it has to be presented as a combination of the conformal invariants, E_4 and I_1 to I_8, and compared to the general form of the anomaly (80) and determine the conformal charges $(a,c_1,\ldots,c_8)$. Doing this, we arrive at 27 algebraic equations for nine conformal charges that we present in Appendix E. These equations have unique solutions that give the following values for the conformal charges:

Conformal charges	Mixed boundary condition
a	0
c_1	$-\frac{429}{4}$
c_2	621
c_3	0
c_4	0
c_5	270
c_6	990
c_7	−210
c_8	−360

4. Comparison with conformal scalar field

It is instructive to compare the found charges for the fermions with the charges computed earlier in [2] for a conformal scalar field satisfying either the Dirichlet boundary condition or conformal Robin boundary condition.

Conformal charges	Dirichlet boundary condition	Robin boundary condition
a	$\frac{17}{8}$	$-\frac{17}{8}$
c_1	$-\frac{681}{32}$	$\frac{39}{32}$
c_2	$\frac{609}{8}$	$\frac{309}{8}$
c_3	$-\frac{81}{8}$	$\frac{81}{8}$
c_4	$-\frac{27}{2}$	$\frac{27}{2}$
c_5	$-\frac{9}{8}$	$\frac{189}{8}$
c_6	$\frac{819}{8}$	$\frac{441}{8}$
c_7	$-\frac{615}{16}$	$\frac{195}{16}$
c_8	$-\frac{45}{2}$	$-\frac{45}{2}$

Denoting the charges for Dirichlet, Robin, and mixed boundary conditions with indices, D, R, and M, respectively, the following relations are observed between some charges. For invariants, which are written purely in terms of the Riemann tensor and its contractions, we find a relation

$$a^M = a^D + a^R = 0, \qquad c^M_{3,4} = c^D_{3,4} + c^R_{3,4} = 0. \tag{93}$$

For the charge a, we saw a similar relation in $d = 3$ dimensions. For invariants that include the derivatives, we find a relation

$$c^M_{7,8} = 8(c^D_{7,8} + c^R_{7,8}). \tag{94}$$

No obvious relations were observed for the other charges.

5. Including the gauge fields

Let us now discuss the possible contribution of the gauge fields to the conformal anomaly in 5 dimensions. There are two possible terms in the anomaly that we present in the form,

$$\frac{1}{1536\pi^2}\int_{\partial M_5}(b_1 F_{ab}F^{ab} + b_2 F_{an}F^{a}{}_{n}), \tag{95}$$

that are due to the gauge fields. In the heat kernel coefficient a_5, the gauge field A_i with the field strength F_{ij} may appear either via $E = -\frac{1}{4}R + \frac{1}{2}F_{ij}\gamma^i\gamma^j$ or via $\Omega_{ij} = \frac{1}{4}R_{ijk\ell}\gamma^k\gamma^\ell + F_{ij}$. There are plenty of such terms in the heat kernel coefficient (D 3). Most of them give zero after taking the trace over spinor indexes. The nonvanishing terms are given in the table below.

Various terms including F_{ij}	Extended forms	Contribution to anomaly
$\mathrm{tr}(E^2)$	$-4F_{ab}F^{ab} - 8F_{an}F^{an}$	−180
$\mathrm{tr}(\chi E\chi E)$	$-4F_{ab}F^{ab} + 8F_{an}F^{an}$	180
$\mathrm{tr}(\Omega_{ab}\Omega^{ab})$	$8F_{ab}F^{ab}$	$-\frac{105}{4}$
$\mathrm{tr}(\chi\Omega_{ab}\chi\Omega^{ab})$	$8F_{ab}F^{ab}$	$\frac{105}{4}$
$\mathrm{tr}(\Omega_{an}\Omega^a_n)$	$8F_{an}F^a_n$	−45
$\mathrm{tr}(\chi\Omega_{an}\chi\Omega^a_n)$	$8F_{an}F^a_n$	−45

We have also checked that the possible cross terms that contain both F_{ij} and the Riemann curvature do not appear.

Collecting all terms together, we find that the anomaly

$$\int_{\partial M_5}\langle T_{ij}\rangle g^{ij} = -\frac{3}{128\pi^2}\int_{\partial M_5} F_{an}F^{a}{}_{n}. \tag{96}$$

Comparing with (95), we conclude that in the anomaly for a Dirac fermion the charge $b_1 = 0$, while the only nonvanishing charge is b_2. At the moment, we do not have an explanation for this result.

This result is worth comparing with the gauge field terms in the conformal anomaly for a complex scalar field carrying a representation for the gauge group G and coupled to the gauge fields. In this case, $\chi = -1$ for the Dirichlet boundary conditions, and $\chi = +1$ for the conformal Robin boundary conditions; see [2]. We have computed these terms. Only two terms in the heat kernel coefficient contribute to the anomaly in this case: $\Omega_{ab}\Omega^{ab}$ and $\Omega_{an}\Omega^{an}$. Omitting the details that are quite simple, the result is summarized below.

Conformal charges	Dirichlet boundary condition	Robin boundary condition	Mixed boundary condition
b_1	4	9	0
b_2	−4	−3	36

We find a relation

$$b^M_k = 6(b^D_k + b^R_k), \quad k = 1, 2, \tag{97}$$

between the charges for fermions (M) and scalars with Dirichlet (D) and Robin (R) boundary conditions.

B. Chiral anomaly in $d = 5$ dimensions

1. Parity odd conformal invariants

There are three parity odd conformal invariants in five dimensions, see [13],

$$J_1 = \int_{\partial M_5} \epsilon^{abcd} W_{abef} W_{cd}{}^{ef}, \qquad (98)$$

$$J_2 = \int_{\partial M_5} \epsilon^{abcd} W_{abne} W_{cdn}{}^{e} = 4 \int_{\partial M_4} \epsilon^{abcd} \overline{\nabla}_b \hat{K}_{ae} \overline{\nabla}_d \hat{K}^e_c, \qquad (99)$$

and

$$J_3 = \int_{\partial M_5} \epsilon^{abcd} \hat{K}^e_a \hat{K}^f_b W_{cdef}, \qquad (100)$$

where W_{abcd} is the bulk Weyl tensor with the boundary indices, and $\hat{K}_{ab}$ is the traceless part of the extrinsic curvature tensor. If we include the gauge fields, then there is one more invariant

$$J_4 = \int_{\partial M_5} \epsilon^{\mathrm{nabcd}} F_{ab} F_{cd}. \qquad (101)$$

The chiral anomaly decomposes as follows:

$$\int_{M_4} \nabla_i \langle j^i_A \rangle = -\frac{1}{96(4\pi)^2} (d_1 J_1 + d_2 J_2 + d_3 J_3 + d_4 J_4). \qquad (102)$$

2. Computation using heat kernel coefficients

For the Dirac field, the anomaly is

$$\int_{M_4} \nabla_i \langle j^i_A \rangle = -2 \int_{\partial M_5} \mathrm{tr}(i\Gamma^*_2 B_5(x)). \qquad (103)$$

There are four terms[5] in the heat kernel that produce a nontrivial trace in (103).

Various parity odd terms	Extended forms	Contribution to anomaly
$\mathrm{tr}(i\Gamma^*_2 \chi \Omega_{ab} \Omega^{ab})$	$\frac{1}{2}\epsilon^{\mathrm{nabcd}} R_{abef} R_{cd}{}^{ef}$	120
$\mathrm{tr}(i\Gamma^*_2 \chi \Omega_{an} \Omega^a_n)$	$\frac{1}{2}\epsilon^{\mathrm{nabcd}} R_{abne} R_{cdn}{}^{e} = 2\epsilon^{\mathrm{nabcd}} \overline{\nabla}_b K_{ae} \overline{\nabla}_d K^e_c$	180
$\mathrm{tr}(i\Gamma^*_2 \chi \overline{\nabla}_a \chi \overline{\nabla}_b \chi \Omega^{ab})$	$2\epsilon^{\mathrm{nabcd}} K^e_a K^f_b R_{cdef}$	30
$\mathrm{tr}(i\Gamma^*_2 \chi E^2)$	$2\epsilon^{\mathrm{nabcd}} F_{ab} F_{cd}$	720

It is not difficult to see that in these invariants, the Riemann tensor can be replaced by Weyl tensor and the extrinsic curvature by its trace-free part so that these are precisely invariants that we listed above. Matching the coefficients with the general form (102), we find the following corresponding charges.

Conformal charges	Mixed boundary condition
d_1	2
d_2	3
d_3	2
d_4	48

This completes our consideration of anomaly in $d = 5$ dimensional space-time.

VII. CONCLUSIONS

That the quantum anomalies are modified in the presence of boundaries by the boundary terms is an interesting subject of research that came into light in the recent years. In the present paper, we have developed a systematic calculation for the boundary terms in the conformal anomaly and in the chiral anomaly. The conformal anomaly in even dimensional space-time in this context was studied in [5,6] (see also earlier paper [4]) where the anomaly in $d = 4$ was systematically studied. It is intriguing that the both anomalies, which are usually absent in odd dimensions, can be nontrivial in the presence of boundaries. For the conformal anomaly, this is known already for some time [1]. The complete basis of conformal boundary terms in the anomaly in $d = 5$ was identified in [2], where the respective conformal charges for a scalar field with either Dirichlet or Robin boundary conditions were computed. In the present paper, we continued the previous study in $d = 5$ and have computed the conformal charges for Dirac fermions in $d = 5$ dimensions. A new subject of research that is in the focus of the present paper is the boundary terms in the chiral anomaly. To the best of our knowledge, this issue was not widely discussed before. An earlier paper, known to us, on this subject is [27], where in $d = 4$ dimensions, the boundary terms in chiral anomaly due to the axial gauge fields (not considered in the present paper) were found in a rather restrictive case when the boundary is geodesic. We should also mention here the papers by Vassilevich *et al.* [26,28,32,33] on parity anomaly. This anomaly is different from the chiral anomaly although it has a similar odd structure.

Below, we summarize our findings:

(i) Boundary terms due to gauge field in chiral anomaly in $d = 3$ dimensions.

[5]The coefficient of $\chi \overline{\nabla}_a \chi \overline{\nabla}_b \chi \Omega^{ab}$ was reported as 90 for the first time in [19]. Later in [35], I. Moss corrected this and reported this coefficient as 30. We thank the referee for bringing this to our attention.

(ii) Gravitational boundary term in chiral anomaly in $d = 4$ dimensions.
(iii) Boundary conformal anomaly for fermions in dimension $d = 5$ both due to the gravitational field and the gauge fields.
(iv) The anomaly due to gauge fields for conformal scalars with either Dirichlet or Robin boundary condition in $d = 5$ dimensions that completes our previous study [2] of anomaly for conformal scalars.
(v) Boundary terms in chiral anomaly in $d = 5$ dimensions both due to the gravitational field and the gauge fields.

It would be interesting to develop the holographic aspects for the present calculations of the anomaly, which is the subject of a work in progress [36]. It would also be interesting to find some applications for our findings such as the chiral anomaly in $d = 3$ or the boundary term in the chiral anomaly in $d = 4$. We leave these issues for a further study.

APPENDIX A: IDENTITIES FOR MATRICES AND TRACES

$$\gamma^n\Pi_\pm = \Pi_\mp\gamma^n, \quad \gamma^a\Pi_\pm = \Pi_\pm\gamma^a, \quad \chi\Pi_\pm = \Pi_\pm. \tag{A1}$$

$$\nabla_a\gamma^* = 0, \quad \nabla_a\gamma^n = K_{ab}\gamma^b, \quad \nabla_a\gamma^b = -K_a^b\gamma^n. \tag{A2}$$

$$\begin{aligned}\overline{\nabla}_a\chi &= i\gamma^*\gamma^b K_{ab}\\ \overline{\nabla}_a\overline{\nabla}_b\chi &= i\gamma^*(\gamma^c\overline{\nabla}_a K_{bc} + \gamma^n K^2_{ab}),\end{aligned} \tag{A3}$$

We also need the intrinsic derivatives of $S = -\frac{1}{2}K\Pi_+$, which are calculated below:

$$\begin{aligned}\bar{\nabla}_a S &= -\frac{1}{2}\bar{\nabla}_a K\Pi_+ - \frac{1}{4}K\bar{\nabla}_a\chi,\\ \bar{\nabla}_a\bar{\nabla}_b S &= -\frac{1}{2}(\bar{\nabla}_a\bar{\nabla}_b K)\Pi_+ - \frac{1}{4}K\bar{\nabla}_a\bar{\nabla}_b\chi\\ &\quad - \frac{1}{4}(\bar{\nabla}_a K\bar{\nabla}_b\chi + \bar{\nabla}_b K\bar{\nabla}_a\chi).\end{aligned} \tag{A4}$$

These identities rely only on the commutation relations and do not depend on the choice of the representation for gamma matrices and for the chirality matrix γ^*.

APPENDIX B: IDENTITIES

Gauss-Codazzi relations

$$R_{acbd} = \bar{R}_{acbd} - (K_{ab}K_{cd} - K_{ad}K_{bc}), \tag{B1}$$

$$R_{nabc} = (\overline{\nabla}_c K_{ab} - \overline{\nabla}_b K_{ac}), \tag{B2}$$

where $\bar{R}_{acbd}$ represents the intrinsic Riemann tensor of the boundary. In particular, in 5 dimensions, we need

$$R^2_{nabc} = 2\overline{\nabla}_c K_{ab}\overline{\nabla}^c K^{ab} - 2\overline{\nabla}_c K_{ab}\overline{\nabla}^b K^{ac}, \tag{B3}$$

where the second term can be expanded as (B10).

The contracted equations read

$$R_{an} = R_{na} = (\overline{\nabla}_b K^b_a - \overline{\nabla}_a K). \tag{B4}$$

$$R_{ab} = \bar{R}_{ab} + R_{anbn} + (K^2_{ab} - KK_{ab}), \tag{B5}$$

and a double contraction yields

$$R = \bar{R} + 2R_{nn} + (\mathrm{Tr}K^2 - K^2). \tag{B6}$$

Thus, one finds for the projected Einstein tensor

$$G_{nn} = -\frac{1}{2}\bar{R} - \frac{1}{2}(\mathrm{Tr}K^2 - K^2). \tag{B7}$$

Differential equations

$$\Box R = \overline{\Box} R + \nabla_n^2 R + K\nabla_n R, \tag{B8}$$

$$\nabla_n G_{nn} = K^{ab}R_{ab} - KR_{nn} - \overline{\nabla}_a\overline{\nabla}_b K^{ab} + \overline{\Box} K, \tag{B9}$$

$$\begin{aligned}\overline{\nabla}_c K_{ab}\overline{\nabla}^b K^{ac} &= \overline{\nabla}_a K^a_b\overline{\nabla}_c K^{bc} + K^{ab}K^{cd}R_{acbd}\\ &\quad - K^{ac}K^b_c R_{ab} + K^{ac}K^b_c R_{anbn} - K\mathrm{Tr}K^3\\ &\quad + (\mathrm{Tr}K^2)^2 + \mathrm{T.D.},\end{aligned} \tag{B10}$$

$$\begin{aligned}\nabla^2_n G_{nn} &= -R^{ab}R_{anbn} + R^2_{nn} - K^a_c K^{bc}R_{ab} + \mathrm{Tr}K^2 R_{nn}\\ &\quad + K^{ab}\nabla_n R_{ab} - K\nabla_n R_{nn} - \overline{\nabla}_a K^{ab}\overline{\nabla}_b K\\ &\quad + (\overline{\nabla}K)^2 + \mathrm{T.D.},\end{aligned} \tag{B11}$$

$$\begin{aligned}\nabla_n R_{ab} &= \nabla_n R_{anbn} - 2K^{cd}R_{acbd} - K^c_a R_{bncn} - K^c_b R_{ancn}\\ &\quad + KR_{anbn} + K_{ab}R_{nn} - \mathrm{Tr}K^2 K_{ab} + KK_{ac}K^c_b\\ &\quad + \overline{\nabla}_a\overline{\nabla}_c K^c_b + \overline{\nabla}_b\overline{\nabla}_c K^c_a - \overline{\Box} K_{ab} - \overline{\nabla}_a\overline{\nabla}_b K,\end{aligned} \tag{B12}$$

where we defined $\nabla_n G_{nn} = n^k n^i n^j \nabla_k G_{ij}$, $\Box G_{nn} = n^i n^j \Box G_{ij}$ and $\nabla_n^2 G_{nn} = n^k n^\ell n^i n^j \nabla_k\nabla_\ell G_{ij}$.

APPENDIX C: DERIVATION OF THE ROBIN BOUNDARY CONDITION

Since Dirac operator is a first order operator, one imposes the Dirichlet boundary condition on a half of the Dirac spinor components

$$\Pi_-\psi|_{\partial M} = \frac{1}{2}(1 - i\gamma^*\gamma^n)\psi|_{\partial M} = 0. \tag{C1}$$

Acting the projector on both sides of the eigenvalue equation, $\gamma^k\nabla_k\psi = \lambda\psi$, one gets zero again; thus,

$$\Pi_-(\gamma^k\nabla_k)\psi|_{\partial M} = 0. \tag{C2}$$

Putting $\gamma^n\Pi_-$ on the left-hand side, we get

$$\Pi_+\gamma^n\Pi_-(\gamma^k\nabla_k)\psi|_{\partial M} = 0, \tag{C3}$$

where we have defined $\Pi_+ = \frac{1}{2}(1 + i\gamma^*\gamma^n)$ and have used

$$\gamma^n \Pi_- = \Pi_+ \gamma^n. \tag{C4}$$

By separating the normal and tangential components, we will have

$$\begin{aligned}
&\Pi_+\gamma^n\Pi_-(\gamma^n\nabla_n + \gamma^a\nabla_a)\psi|_{\partial M} = 0 \\
&\to \Pi_+^2\gamma^n\nabla_n\psi|_{\partial M} + \Pi_+\gamma^n\gamma^a\Pi_-\nabla_a\psi|_{\partial M} = 0 \\
&\to \Pi_+\nabla_n\psi|_{\partial M} - \Pi_+\gamma^n\gamma^a(\nabla_a\Pi_-)\psi|_{\partial M} = 0 \\
&\to \Pi_+\left(\nabla_n + \frac{1}{2}\gamma^n\gamma^a\nabla_a\chi\right)\psi|_{\partial M} = 0 \\
&\to \left(\nabla_n + \frac{1}{2}\Pi_+\gamma^n\gamma^a\nabla_a\chi\right)\Pi_+\psi|_{\partial M} = 0,
\end{aligned} \tag{C5}$$

where we have defined $\chi = \Pi_+ - \Pi_- = i\gamma^*\gamma^n$. Comparing with $(\nabla_n - S)\Pi_+\psi|_{\partial M} = 0$, we conclude that

$$S = -\frac{1}{2}\Pi_+\gamma^n\gamma^a\nabla_a\chi\Pi_+. \tag{C6}$$

Now using (A2) and (A3), one deduces

$$S = -\frac{1}{2}K\Pi_+. \tag{C7}$$

APPENDIX D: BULK AND BOUNDARY TERMS IN THE HEAT KERNEL COEFFICIENT a_d IN DIMENSION d [18,19]

1. Heat kernel coefficient in $d=3$

$$\begin{aligned}
B_3(x) = \frac{1}{384(4\pi)}[&96\chi E + 16\chi R - 8\chi R_{nn} + (2\Pi_+ + 10\Pi_-)\mathrm{Tr}K^2 + (13\Pi_+ - 7\Pi_-)K^2 \\
&+ 96SK + 192S^2 - 12\bar{\nabla}_a\chi\bar{\nabla}^a\chi].
\end{aligned} \tag{D1}$$

2. Heat kernel coefficient in $d=4$

$$\begin{aligned}
A_4(x) &= \frac{1}{360(4\pi)^2}(60\Box E + 12\Box R + 2R_{ikj\ell}R^{ikj\ell} - 2R_{ij}R^{ij} + 180E^2 + 60RE + 5R^2 + 30\Omega_{ij}\Omega^{ij}) \\
B_4(x) &= \frac{1}{360(4\pi)^2}\Big[(-240\Pi_+ + 120\Pi_-)\nabla_n E + (-42\Pi_+ + 18\Pi_-)\nabla_n R + 24\bar{\Box}K \\
&+ 120KE + 20KR - 4KR_{nn} + 16K^{ab}R_{anbn} - 4K^{ab}R_{ab} \\
&+ \frac{1}{21}(224\Pi_+ + 320\Pi_-)\mathrm{Tr}K^3 + \frac{1}{21}(168\Pi_+ - 264\Pi_-)K\mathrm{Tr}K^2 + \frac{1}{21}(280\Pi_+ + 40\Pi_-)K^3 \\
&+ 720SE + 120SR + 48S\mathrm{Tr}K^2 + 144SK^2 + 480S^2K + 480S^3 + 120\bar{\Box}S \\
&- 60\chi\bar{\nabla}^a\chi\Omega_{an} - 24K^{ab}\bar{\nabla}_a\chi\bar{\nabla}_b\chi - 12K\bar{\nabla}_a\chi\bar{\nabla}^a\chi - 120S\bar{\nabla}_a\chi\bar{\nabla}^a\chi\Big].
\end{aligned} \tag{D2}$$

3. Heat kernel coefficient in $d=5$

$$\begin{aligned}
B_5(x) = \frac{1}{5760(4\pi)^2}\Big[&360\chi\nabla_n^2E - 1440\nabla_n ES + 720\chi E^2 + 240\chi\bar{\Box}E + 240\chi RE + 48\chi\nabla^2R + 20\chi R^2 \\
&- 8\chi R_{ij}^2 + 8\chi R_{ikj\ell}^2 - 120\chi R_{nn}E - 20\chi RR_{nn} + 480RS^2 + 12\chi\nabla_n^2R \\
&+ 24\chi\bar{\Box}R_{nn} + 15\chi\nabla_n^2R_{nn} - 270S\nabla_n R + 120R_{nn}S^2 + 960S\bar{\Box}S + 16\chi R^{ab}R_{anbn} \\
&- 17\chi R_{nn}^2 - 10\chi R_{anbn}R^a{}_n{}^b{}_n + 2880ES^2 + 1440S^4 \\
&- (90\Pi_+ + 450\Pi_-)K\nabla_n E - \left(\frac{111}{2}\Pi_+ + 42\Pi_-\right)K\nabla_n R - 30\Pi_+K^{ab}\nabla_n R_{anbn} + 240K\bar{\Box}S \\
&+ 420K^{ab}\bar{\nabla}_a\bar{\nabla}_b S + 390\bar{\nabla}_a K\bar{\nabla}^a S + 480\bar{\nabla}_a K^{ab}\bar{\nabla}_b S + 420S\bar{\Box}K + 60\bar{\nabla}_a\bar{\nabla}_b K^{ab}S \\
&+ \left(\frac{487}{16}\Pi_+ + \frac{413}{16}\Pi_-\right)(\bar{\nabla}K)^2 + (238\Pi_+ - 58\Pi_-)\bar{\nabla}_a K^{ab}\bar{\nabla}_b K + \left(\frac{49}{4}\Pi_+ + \frac{11}{4}\Pi_-\right)\bar{\nabla}_a K^{ab}\bar{\nabla}_c K_b^c
\end{aligned}$$

$$
\begin{aligned}
&+\left(\frac{535}{8}\Pi_+ - \frac{355}{8}\Pi_-\right)\bar{\nabla}_c K^{ab}\bar{\nabla}^c K_{ab} + \left(\frac{151}{4}\Pi_+ + \frac{29}{4}\Pi_-\right)\bar{\nabla}_c K_{ab}\bar{\nabla}^b K^{ac} + (111\Pi_+ - 6\Pi_-)K\bar{\Box}K \\
&+ (-15\Pi_+ + 30\Pi_-)K\bar{\nabla}_a\bar{\nabla}_b K^{ab} + \left(-\frac{15}{2}\Pi_+ + \frac{75}{2}\Pi_-\right)K^{bc}\bar{\nabla}_c\bar{\nabla}_a K^a_b + \left(\frac{945}{4}\Pi_+ - \frac{285}{4}\Pi_-\right)K^{ab}\bar{\nabla}_a\bar{\nabla}_b K \\
&+ (114\Pi_+ - 54\Pi_-)K^{ab}\bar{\Box}K_{ab} + 1440KSE + 30KSR_{nn} + 240KSR - 60K_{ab}R^{ab}S + 180K^{ab}R_{anbn}S \\
&+ (195\Pi_+ - 105\Pi_-)K^2E + (30\Pi_+ + 150\Pi_-)\mathrm{Tr}K^2E + \left(\frac{195}{6}\Pi_+ - \frac{105}{6}\Pi_-\right)K^2R \\
&+ (5\Pi_+ + 25\Pi_-)\mathrm{Tr}K^2R + \left(-\frac{275}{16}\Pi_+ + \frac{215}{16}\Pi_-\right)K^2R_{nn} + \left(-\frac{275}{8}\Pi_+ + \frac{215}{8}\Pi_-\right)\mathrm{Tr}K^2R_{nn} \\
&+ (-\Pi_+ - 14\Pi_-)KK^{ab}R_{ab} + \left(\frac{109}{4}\Pi_+ - \frac{49}{4}\Pi_-\right)KK^{ab}R_{anbn} + 16\chi K_{ab}K^b_cR^{ac} \\
&+ \left(\frac{133}{2}\Pi_+ + \frac{47}{2}\Pi_-\right)K^{ac}K^b_cR_{anbn} + 32\chi K^{ab}K^{cd}R_{acbd} + \frac{315}{2}K\mathrm{Tr}K^2S + \left(\frac{2041}{128}\Pi_+ + \frac{65}{128}\Pi_-\right)K^4 \\
&+ 150S\mathrm{Tr}K^3 + \left(\frac{417}{32}\Pi_+ + \frac{141}{32}\Pi_-\right)K^2\mathrm{Tr}K^2 + 1080K^2S^2 + 360\mathrm{Tr}K^2S^2 \\
&+ \left(\frac{375}{32}\Pi_+ - \frac{777}{32}\Pi_-\right)(\mathrm{Tr}K^2)^2 + \frac{885}{4}SK^3 + \left(25\Pi_+ - \frac{17}{2}\Pi_-\right)K\mathrm{Tr}K^3 + 2160KS^3 \\
&+ \left(\frac{231}{8}\Pi_+ + \frac{327}{8}\Pi_-\right)\mathrm{Tr}K^4 - 180E^2 + 180\chi E\chi E - 120(\bar{\nabla}S)^2 + 720\chi(\bar{\nabla}S)^2 \\
&- \frac{105}{4}\Omega_{ab}\Omega^{ab} + 120\chi\Omega_{ab}\Omega^{ab} + \frac{105}{4}\chi\Omega_{ab}\chi\Omega^{ab} - 45\Omega_{an}\Omega^a{}_n + 180\chi\Omega_{an}\Omega^a{}_n - 45\chi\Omega_{an}\chi\Omega^a{}_n \\
&- 360\Omega^a{}_n\chi\bar{\nabla}_aS + 360\Omega^a{}_n\bar{\nabla}_aS\chi - 45\chi\bar{\nabla}_a\chi\Omega^a{}_nK - 180\bar{\nabla}_a\chi\bar{\nabla}_b\chi\Omega^{ab} + 30\chi\bar{\nabla}_a\chi\bar{\nabla}_b\chi\Omega^{ab} \\
&+ 90\chi\bar{\nabla}_a\chi\nabla_n\Omega^a{}_n + 120\chi\bar{\nabla}^a\chi\bar{\nabla}^b\Omega_{ab} - 180\chi\bar{\nabla}_a\chi\Omega_{bn}K^{ab} + 300\bar{\nabla}_a\chi\bar{\nabla}^aE - 180\bar{\nabla}_a\chi\bar{\nabla}^a\chi E \\
&- 90\chi\bar{\nabla}_a\chi\bar{\nabla}^a\chi E + 240\bar{\Box}\chi E - 30\bar{\nabla}_a\chi\bar{\nabla}^a\chi R - 60\bar{\nabla}_a\chi\bar{\nabla}_b\chi R^{ab} - 30\bar{\nabla}_a\chi\bar{\nabla}_b\chi R_n{}^{ab}{}_n \\
&- \frac{675}{32}\bar{\nabla}_a\chi\bar{\nabla}^a\chi K^2 - \frac{75}{4}\bar{\nabla}_a\chi\bar{\nabla}_b\chi K^{ac}K^b_c - \frac{195}{16}\bar{\nabla}_a\chi\bar{\nabla}^a\chi\mathrm{Tr}K^2 - \frac{675}{8}\bar{\nabla}_a\chi\bar{\nabla}_b\chi KK^{ab} \\
&- 330\bar{\nabla}_a\chi\bar{\nabla}^aSK - 300\bar{\nabla}_a\chi\bar{\nabla}_bSK^{ab} + \frac{15}{4}\bar{\nabla}_a\chi\bar{\nabla}^a\chi\bar{\nabla}_b\chi\bar{\nabla}^b\chi + \frac{15}{8}\bar{\nabla}_a\chi\bar{\nabla}_b\chi\bar{\nabla}^a\chi\bar{\nabla}^b\chi - \frac{15}{4}(\bar{\Box}\chi)^2 \\
&- \frac{105}{2}\bar{\nabla}_a\bar{\nabla}_b\chi\bar{\nabla}^a\bar{\nabla}^b\chi - 15\bar{\nabla}_a\chi\bar{\nabla}^a\chi\bar{\Box}\chi - \frac{135}{2}\bar{\nabla}_a\chi\bar{\nabla}^a\bar{\Box}\chi\Bigg]. \qquad \text{(D3)}
\end{aligned}
$$

APPENDIX E: ALGEBRAIC EQUATION FOR CONFORMAL CHARGES IN $d=5$

$$
\begin{aligned}
&\mathrm{Tr}K^4:\ -6a + c_2 + c_8 = 261, \\
&K\mathrm{Tr}K^3: 8a - c_2 + 2c_7 - \frac{11}{6}c_8 = -381, \\
&(\mathrm{Tr}K^2)^2: 3a + c_1 - 2c_7 = \frac{1251}{4}, \\
&K^2\mathrm{Tr}K^2: -6a - \frac{1}{2}c_1 + \frac{3}{8}c_2 + \frac{47}{48}c_8 = -66, \\
&K^4: a + \frac{1}{16}c_1 - \frac{3}{64}c_2 - \frac{7}{48}c_8 = \frac{267}{16}, \\
&R^2_{acbd}: a + c_3 = 0, \\
&R^2_{ab}: -4a - \frac{16}{9}c_3 + \frac{1}{9}c_4 = 0,
\end{aligned}
$$

$$
\begin{aligned}
&R^2{:}\, a+\frac{5}{18}c_3-\frac{1}{36}c_4=0,\\
&R^2_{anbn}{:}\,-4a+c_4=0,\\
&R^{ab}R_{anbn}{:}\,8a+\frac{8}{3}c_3-\frac{2}{3}c_4=0,\\
&R^2_{nn}{:}\,4a+\frac{4}{9}c_3-\frac{4}{9}c_4=0,\\
&RR_{nn}{:}\,-4a-\frac{8}{9}c_3+\frac{2}{9}c_4=0,\\
&K^{ab}K^{cd}R_{acbd}{:}\,4a+c_5-2c_7+\frac{2}{3}c_8=450,\\
&K^a_cK^{bc}R_{ab}{:}\,8a+\frac{2}{3}c_5-\frac{1}{3}c_6+2c_7-c_8=-210,\\
&KK^{ab}R_{ab}{:}\,-8a-\frac{5}{6}c_5+\frac{1}{6}c_6=-60,\\
&\mathrm{Tr}K^2R{:}\,-2a-\frac{1}{12}c_5+\frac{1}{12}c_6+\frac{1}{3}c_8=-60,\\
&K^2R{:}\,2a+\frac{1}{8}c_5-\frac{1}{24}c_6-\frac{5}{48}c_8=30,\\
&K^a_cK^{bc}R_{anbn}{:}\,-8a+c_6-2c_7+\frac{5}{3}c_8=810,\\
&KK^{ab}R_{anbn}{:}\,8a+\frac{1}{2}c_5-\frac{1}{2}c_6-\frac{1}{3}c_8=-240,\\
&\mathrm{Tr}K^2R_{nn}{:}\,4a-\frac{1}{3}c_6-c_8=30,\\
&K^2R_{nn}{:}\,-4a-\frac{1}{6}c_5+\frac{1}{6}c_6+\frac{11}{24}c_8=-45,\\
&\bar{\nabla}_cK_{ab}\bar{\nabla}^cK^{ab}{:}\,2c_7-\frac{1}{3}c_8=-300,\\
&\bar{\nabla}_aK^a_b\bar{\nabla}_cK^{bc}{:}\,-\frac{8}{3}c_7+\frac{8}{9}c_8=240,\\
&\bar{\nabla}_aK^{ab}\bar{\nabla}_bK{:}\,\frac{4}{3}c_7-\frac{7}{9}c_8=0,\\
&(\bar{\nabla}K)^2{:}\,-\frac{2}{3}c_7+\frac{25}{72}c_8=15,\\
&K^{ab}\nabla_nR_{anbn}{:}\,-\frac{2}{3}c_8=240,\\
&K\nabla_nR{:}\,\frac{1}{12}c_8=-30.
\end{aligned}
$$

These equations have a unique solution that determines the values for the conformal charges given in the main text.

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