Instantons, analytic continuation, and $\mathcal{PT}$-symmetric field theory

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(Received 23 March 2023; accepted 25 September 2023; published 19 October 2023)

Ordinary Hermitian $\lambda\phi^4$ theory is known to exist in $d < 4$ dimensions when $\lambda > 0$. For negative values of the coupling, it has been suggested that a physical meaningful definition of the interacting theory can be given in terms of $\mathcal{PT}$-symmetric field theory. In this work, we critically reexamine the relation between analytically continued Hermitian field theory with quartic interaction and $\mathcal{PT}$-symmetric field theory, including $O(N)$ models. We find that in general $\mathcal{PT}$-symmetric field theory *does not* correspond to the analytic continuation of the Hermitian theory to negative coupling, except at high temperature where the instanton contribution present in the analytically continued theory can be neglected.

DOI: 10.1103/PhysRevD.108.085013

I. INTRODUCTION

Hermitian field theory is built around the presence of a Hermitian Hamiltonian that is bounded from below. In quantum mechanics, it has long been known that Hermiticity and a lower-bounded potential are sufficient to guarantee a real and lower-bounded spectrum of the Hamiltonian, thus providing the basis for modern quantum field theory. However, it has been found that somewhat weaker conditions than Hermiticity and boundedness, namely symmetry under parity $\mathcal{P}$ and time reversal $\mathcal{T}$, still result in real and semidefinite energy eigenspectra [1]. In fact, it has been proved that $\mathcal{PT}$ symmetry is sufficient to guarantee real spectra in quantum mechanics for the massless theory [2], showing that Hermiticity is not a necessary condition.

A natural generalization of $\mathcal{PT}$-symmetric quantum mechanics is $\mathcal{PT}$-symmetric quantum field theory, which is a fairly recent area of study. In a series of articles, it has been suggested that Hermitian field theory with a quartic interaction and negative coupling constant can be related to $\mathcal{PT}$-symmetric field theory [3–6]. In particular, in [6] it is conjectured that the partition function $Z_{\mathcal{H}}$ of the Hermitian field theory can be related to the partition function of the $\mathcal{PT}$-symmetric field theory $Z_{\mathcal{PT}}$ in $d > 0$ dimensions via

$$\ln Z_{\mathcal{PT}}(g) = \frac{1}{2}\Big[\ln\, Z_{\mathcal{H}}(\lambda = -g + i0^+) + \ln\, Z_{\mathcal{H}}(\lambda = -g - i0^+)\Big], \tag{1}$$

where λ is the coupling constant of the Hermitian theory with $\lambda\phi^4$ interaction and $Z_{\mathcal{H}}$ refers to the analytic continuation of the Hermitian theory's partition function. For the discussion of the structure in the complex coupling plane, and the nature of the discontinuity across the cut on the negative real coupling axis, we refer the interested reader to Ref. [6].

If the Ai, Bender, Sarkar (ABS) conjecture (1) holds for quantum field theory with quartic interaction in general dimensions d, this would provide meaning for quantum field theories in situations where the potential becomes unbounded, in particular scalar quantum field theory in four dimensions; see e.g. Refs. [7–9]. For this reason, it is interesting to study the precise relation between analytically continued Hermitian and $\mathcal{PT}$-symmetric field theory. In particular, we aim to study the ABS conjecture (1) in cases where both sides of the equation can be evaluated. This is particularly easy in $d = 0$, where Ref. [6] already noted that the partition functions fulfill the relation

$$Z_{\mathcal{PT}}(g) = \mathrm{Re}\, Z_{\mathcal{H}}(\lambda = -g), \tag{2}$$

instead of (1).

In this work we examine these two conjectures in the $d = 1$ case, that is, quantum mechanics. Here high-precision numerical calculations are possible, and we find that neither conjecture holds at all values of the dimensionless parameter $\beta^3 g$. However, at high temperatures (equivalently

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at weak coupling), the second conjecture (2) holds to high precision. We provide numerical evidence and, by considering the semiclassical expansion in $\beta^3 g$, an argument from complex analysis indicating that the failure of that conjecture to hold at low temperatures (strong coupling) is due to the presence of nonperturbative bounce[1] contributions to the analytically continued partition function.

The remainder of this paper is structured as follows. We confirm the result (2) for $d = 0$ and consider the extension to multicomponent scalar fields [also known as the $O(N)$ model] in Sec. II. We then continue in Sec. III to study the quantum mechanical ($d = 1$) case, where the partition function for both sides of (1) can be obtained numerically to high precision. We show that there is no correspondence of the form of (1); however, the analog of (2) is true to high precision at low temperatures. The numerical evidence indicates that the difference between the two sides of (2) is due to an extra nonperturbative bounce contribution in the analytically continued partition function. Working in the path integral formalism, we provide an explanation for this fact in Sec. IV. Finally, we discuss the implications of our findings in Sec. V.

II. THE ONE-SITE MODEL

As a warm-up to quantum field theory (QFT), let us first discuss the limiting case of zero dimensions. This section will focus on complex-analytic arguments to reveal the behavior of the partition function without the need to find closed-form expressions; explicit calculations are provided in the Appendix.

A. Warm-up: One component

The partition function for standard Hermitian field theory in $d = 0$ becomes a single integral over the field:

$$Z^{d=0}(\lambda) = \int_{-\infty}^{\infty} \frac{d\phi}{\sqrt{2\pi}} e^{-\lambda\phi^4} = \frac{2\Gamma(\frac{5}{4})}{(4\pi^2\lambda)^{\frac{1}{4}}}. \quad (3)$$

As written this partition function is only defined for $\mathrm{Re}\,\lambda > 0$; however, it may be extended to all $\lambda \neq 0$ (although not uniquely, due to a branch point at the origin) by analytic continuation. This is clearly seen from the right-hand side of (3); however, without access to a closed-form solution for the integral, the analytic continuation is still easily accomplished by deforming the contour of integration to preserve the convergence of the integral as λ is rotated from $\mathbb{R}_+$ to elsewhere on the complex plane. In a slight abuse of notation, for $\lambda = \Lambda e^{i\theta}$, we may write

$$Z^{d=0}(\lambda = \Lambda e^{i\theta}) = \int_{-\infty e^{i\theta/4}}^{\infty e^{i\theta/4}} \frac{d\phi}{\sqrt{2\pi}} e^{-\lambda\phi^4}. \quad (4)$$

This expression also makes clear the nonuniqueness of the analytic continuation. For example, for negative real values of λ, the analytic continuation may involve integrating either along a contour for which ϕ is proportional to $e^{i\pi/4}$ or one proportional to $e^{-i\pi/4}$. However, the integrals along these two contours are related by complex conjugation: The real parts do not differ.

So much for the analytic continuation of the partition function; now we consider the "PT-symmetric" version. This version of the partition function is intended to be real and to correspond to the case $\lambda < 0$, but here the integral no longer converges. To obtain a well-defined partition function, we will deform the domain of integration from the real line to some other contour γ. In general, a contour will yield a convergent integral at $\lambda < 0$ if ϕ^4 approaches $-\infty$ in either direction along the contour. From Cauchy's integral theorem, two such contours will yield the same partition function if one can be smoothly deformed into the other without passing through any regions where $e^{g\phi^4}$ diverges.

We can satisfy all these constraints by defining the $\mathcal{PT}$-symmetric theory as

$$Z^{d=0}_{\mathcal{PT}}(g) = \int_{\gamma_{\mathcal{PT}}} \frac{d\phi}{\sqrt{2\pi}} e^{g\phi^4}, \quad (5)$$

with a contour $\gamma_{\mathcal{PT}}$ defined by

$$\phi(s) = \begin{cases} s e^{i\frac{\pi}{4}} & s < 0 \\ s e^{-i\frac{\pi}{4}} & s \geq 0 \end{cases} \quad (6)$$

with $s \in \mathbb{R}$ parametrizing the contour.

To relate the Hermitian and $\mathcal{PT}$-symmetric partition functions in this $d = 0$, one-component case, it is helpful to define four "partial" integration contours, each connecting the origin to some asymptotic region where $\phi^4 \to -\infty$. Each contour is parametrized by $s \in [0, \infty)$:

$$\gamma_1\colon\ \phi(s) = s e^{i\frac{\pi}{4}}, \quad (7)$$

$$-\gamma_2\colon\ \phi(s) = s e^{i\frac{3\pi}{4}}, \quad (8)$$

$$-\gamma_3\colon\ \phi(s) = s e^{i\frac{5\pi}{4}}, \quad (9)$$

$$\gamma_4\colon\ \phi(s) = s e^{i\frac{7\pi}{4}}. \quad (10)$$

These four contours each lie in a different quadrant of the complex plane and are numbered accordingly. Finally note that γ_2 and γ_3 have reversed orientation, so that the integration is taken from complex infinity to the origin rather than vice versa. As a result, each contour is oriented

[1] We will be dealing with periodic instanton solutions that in the literature are referred to as "bounces"; hence we will use the term bounce in the following.

so that integration is performed from "right to left" on the complex plane.

With these definitions, the contour defining the $\mathcal{PT}$-symmetric theory above is given by $\gamma_{\mathcal{PT}} = \gamma_3 + \gamma_4$. The (clockwise) analytic continuation is defined by integrating instead along $\gamma_{\rm ac} = \gamma_3 + \gamma_1$. Denoting for brevity $I_k = \int_{\gamma_k} e^{\phi^4}$, we see that the various partial integrals are related by

$$I_1 = I_2^* = I_3 = I_4^*. \tag{11}$$

A short calculation therefore relates the (analytically continued) Hermitian and $\mathcal{PT}$-symmetric partition functions in this case: The $\mathcal{PT}$-symmetric partition function $Z = \frac{\Gamma(\frac{5}{4})}{(\pi^2 g)^{\frac14}}$ is simply given by the real part of the analytically continued Hermitian partition function:

$$Z_{\mathcal{PT}}^{d=0}(g) = \mathrm{Re}\Big(Z^{d=0}(\lambda = -g)\Big). \tag{12}$$

As noted in [6], this relation is *different* from the conjecture (1), which involves the logarithm of the partition function.

B. *N*-component scalars

We may now investigate the relation between Hermitian and $\mathcal{PT}$-symmetric field theory for $d = 0$ for N-component scalars $\vec\phi = (\phi_1, \phi_2, \ldots \phi_N)$. In this case, the partition function for the Hermitian field theory is defined as

$$Z_N^{d=0} = \int \frac{d\vec\phi}{(2\pi)^{\frac{N}{2}}} e^{-\frac{\lambda}{N}(\vec\phi^2)^2}. \tag{13}$$

The construction of a $\mathcal{PT}$-symmetric QFT for N-component scalars is not trivial, and we refer the interested reader to Ref. [10] for an in-depth discussion of $\mathcal{PT}$ symmetry in this case. The partition function for $\mathcal{PT}$-symmetric QFT is defined by

$$Z_{\mathcal{PT},N}^{d=0} = \int \frac{d\vec\phi}{(2\pi)^{\frac{N}{2}}} e^{\frac{g}{N}(\vec\phi^2)^2}, \tag{14}$$

where the integration is *not* on the real axis but in the complex plane. For pedagogical reasons, it is useful to first consider the explicit case of $N = 2$ (two component scalar fields) where $\vec\phi = (\phi_0, \phi_1)$. The $\mathcal{PT}$-symmetric field theory is then defined by using the parametrization (6) for both ϕ_0 and ϕ_1, effectively parametrizing a "cone" in the complex four-dimensional parameter space (see Fig. 1). Explicitly, one has

$$\phi_0 = s\Big(e^{\frac{i\pi}{4}}\theta(-s) + e^{-\frac{i\pi}{4}}\theta(s)\Big) \quad \text{and}$$
$$\phi_1 = t\Big(e^{\frac{i\pi}{4}}\theta(-t) + e^{-\frac{i\pi}{4}}\theta(t)\Big), \tag{15}$$

with $s, t \in \mathbb{R}$. The resulting $\mathcal{PT}$-symmetric path integral for $N = 2$ therefore is

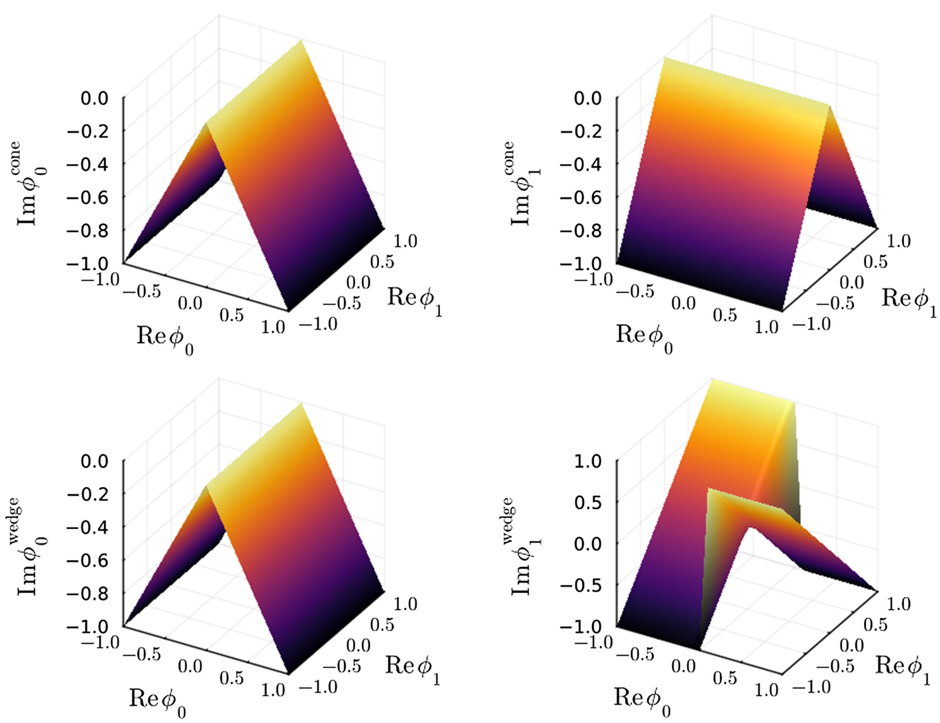

FIG. 1. The cone and wedge contours considered throughout this paper, in the case of a two-component scalar field. The cone contour is shown on the top by plotting $\mathrm{Im}\,\phi_1(\mathrm{Re}\,\phi_1, \mathrm{Re}\,\phi_2)$ and $\mathrm{Im}\,\phi_2(\mathrm{Re}\,\phi_1, \mathrm{Re}\,\phi_2)$. The wedge contour is similarly portrayed on the bottom.

$$Z^{d=0}_{\mathcal{PT},N=2} = \int_0^\infty \frac{dsdt}{(2\pi)}\left[e^{-\frac{g}{2}(s^2+t^2)^2}\cos\left(\frac{\pi}{2}\right) + e^{-\frac{g}{2}(s^2-t^2)^2}\right]. \tag{16}$$

It is straightforward to see that the $N=2$ $\mathcal{PT}$-symmetric partition function diverges, because there is a flat direction $s=t$ in the integrand along with the action is constant. In fact this finding generalizes to any integer $N>1$ when fields are quantized on the cone as a repeated application of (15). As a consequence, we find that for $N>1$, the $\mathcal{PT}$-symmetric partition function obeys neither the ABS conjecture (1) nor the relation (2) proved for $N=1$.

However, it is possible to give a meaningful definition of the path integral with negative coupling constant for the case $N>1$. To this end, consider again the case of $N=2$, but now parametrize fields on a "wedge" in the complex four-dimensional parameter space (see again Fig. 1). Explicitly, one then has the wedge contour defined by

$$\phi_0 = s\left(e^{\frac{i\pi}{4}}\theta(-s) + e^{-\frac{i\pi}{4}}\theta(s)\right) \quad \text{and}$$
$$\phi_1 = t\left(e^{\frac{i\pi}{4}}\theta(-s) + e^{-\frac{i\pi}{4}}\theta(s)\right), \tag{17}$$

where again $s, t \in \mathbb{R}$.

As in the one-site case, we will now show that this contour arises as the real part of the analytic continuation to negative λ of the Hermitian theory. Let us introduce some notation to make working with simple multidimensional integration contours tractable. Given two one-dimensional contours γ_a and γ_b, denote by $\gamma_a \times \gamma_b$ the two-dimensional contour consisting of points $(z_a, z_b) \in \mathbb{C}^2$ with $z_a \in \gamma_a$ and $z_b \in \gamma_b$. In this notation, the wedge contour (17) defined above may be written

$$\gamma_{\text{wedge}} = \gamma_3 \times (\gamma_3 + \gamma_1) + \gamma_4 \times (\gamma_2 + \gamma_4). \tag{18}$$

Meanwhile, the clockwise analytic continuation yields a different contour:

$$\gamma_{\text{ac}} = (\gamma_2 + \gamma_4) \times (\gamma_2 + \gamma_4). \tag{19}$$

For brevity, denote the integral along the contour $\gamma_i \times \gamma_j$ by I_{ij}. The clockwise analytic continuation is equal to $I_{22} + I_{24} + I_{42} + I_{44}$, while the integral along the wedge contour is

$$\int_{\gamma_{\text{wedge}}} \frac{1}{2\pi} e^{-S} = I_{33} + I_{31} + I_{42} + I_{44}. \tag{20}$$

As before, to relate these two note that we have the following relations among the various partial integrals:

$$I_{11} = I_{22}^* = I_{33} = I_{44}^* \quad \text{and} \tag{21}$$

$$I_{13} = I_{24}^*. \tag{22}$$

From this it follows that

$$\text{Re}(I_{22} + I_{23} + I_{32} + I_{33}) = I_{33} + I_{31} + I_{42} + I_{44}, \tag{23}$$

confirming the desired identity. The same proof holds without modification for the case of three or more components; all new components are treated as ϕ_1.

To review: In the N-component, $d=0$ case, we have examined two contours on which we could attempt to define the partition function. The cone contour—arguably the more obvious generalization of the $N=1$ case—results in an undefined partition function. The wedge contour corresponds exactly to the real part of the analytic continuation of the original, Hermitian theory to negative couplings. This establishes an analog of (2) for multicomponent theories in 0 dimensions.

Finally, a brief note on the ABS conjecture itself. Because the analytic continuation of $\log Z^{d=0}$ has a nonzero imaginary part, (2) implies that the ABS conjecture (1) does not hold at any finite N. However, in the large-N limit, both the real and imaginary parts of the free energy—for both the analytically continued and the $\mathcal{PT}$-symmetric theories—may be expanded in powers of N. The leading terms in the real parts are proportional to N, but because of the logarithm the leading term in the imaginary part can at most be $O(N^0)$. As a result, the conjecture (2) directly implies the ABS conjecture in the large-N limit.

Thus at large N and large N only, the Hermitian and wedge-contour parametrized partition functions for the $d=0$ case are related through the ABS conjecture (1), as well as the modified conjecture (2).

III. NUMERICAL COMPARISON

Let us now discuss the case of a single-component quantum field with a quartic interaction in $0+1$ dimensions, with both positive coupling sign (the Hermitian theory) and negative coupling sign (the $\mathcal{PT}$-symmetric theory). First we will define the different theories under consideration—two theories constructed via analytic continuation, and the $\mathcal{PT}$-symmetric theory. Then we detail numerical schemes for computing a high-precision partition function in all three cases, and finally we perform a comparison, the results of which indicate that neither construction via analytic continuation is equivalent to the $\mathcal{PT}$-symmetric theory. One, however, is sufficiently closely related to merit further examination; this is done in the subsequent section.

The Hermitian theory is defined from the Hamiltonian

$$H_{\mathcal{H}} = p^2 + \frac{\lambda}{4}x^4, \tag{24}$$

from which a partition function $Z_{\mathcal{H}}(\beta;\lambda) \equiv \mathrm{Tr}\, e^{-\beta H_{\mathcal{H}}(\lambda)}$ is obtained. As written, this function is defined only on the right half-plane of complex λ; elsewhere the Hamiltonian is unbounded below and the trace diverges. However, noting that solutions to the corresponding time-independent Schrödinger equation are invariant under a certain simultaneous rescaling of x, p, β, and λ, we discover that the spectrum of $H_{\mathcal{H}}$—and therefore the partition function $Z_{\mathcal{H}}$—depends only on the combination $\beta^3\lambda$. As a result we find that $Z_{\mathcal{H}}(\beta,\lambda) = Z_{\mathcal{H}}(\beta\lambda^{1/3}, 1)$. We can use this relation to analytically continue $Z_{\mathcal{H}}$ to values of λ in the left half-plane.

It is important to point out that our Hamiltonian (24) differs from that used in the ABS conjecture [6], which employed an additional mass term. For this reason, the results in this section correspond to the strong-coupling (massless) limit of the theory studied in [6]. The failure of the conjecture in the massless limit is sufficient to establish failure at some sufficiently small finite mass but not at all masses.

The analytically continued Hermitian partition function has a branch point at $\lambda = 0$, and as a result the analytic continuation is not unique. Following the conjecture, we analytically continue to negative values of λ along both clockwise and counterclockwise paths. The two resulting partition functions may be computed as

$$Z_{\mathrm{cw}}(\beta, g) \equiv \mathrm{Tr}\, e^{-\beta e^{i\frac{\pi}{3}} H_{\mathcal{H}}(g)} \quad \text{and} \tag{25}$$

$$Z_{\mathrm{ccw}}(\beta, g) \equiv \mathrm{Tr}\, e^{-\beta e^{-i\frac{\pi}{3}} H_{\mathcal{H}}(g)}, \tag{26}$$

where as in the previous section we have defined $g = -\lambda$ to be the wrong-sign coupling.

From these analytically continued partition functions, we can define a candidate $\mathcal{PT}$-symmetric theory either by averaging either the two partition functions or their logarithms. The former yields a partition function analogous to the one constructed in the $d = 0$ case (2):

$$Z_1 \equiv \mathrm{Re}\,\mathrm{Tr} e^{-\beta e^{i\frac{\pi}{3}} H_{\mathcal{H}}(g)}. \tag{27}$$

The latter approach yields the partition function of the ABS conjecture (1):

$$Z_2 \equiv \left| \mathrm{Tr} e^{-\beta e^{i\frac{\pi}{3}} H_{\mathcal{H}}(g)} \right|. \tag{28}$$

The $\mathcal{PT}$-symmetric theory is defined by quantizing the Hamiltonian (24) at negative coupling $\lambda = -g$, on a contour other than the real line. We will parametrize the contour $x(s) \in \mathbb{C}$ by some $s \in \mathbb{R}$. A wide variety of contours yield the same spectrum; it is sufficient to consider any smooth contour $x(s)$ with $|x| = |s|$ and obeying

$$\exp\left(i \arg \lim_{s\to\pm\infty} x(s)\right) = -ie^{\pm i\frac{\pi}{4}}. \tag{29}$$

A common choice is to take the contour to be the sum of two linear pieces going through the origin:

$$x(s) = \begin{cases} se^{-i\frac{\pi}{4}} & s \geq 0 \\ se^{i\frac{\pi}{4}} & s < 0. \end{cases} \tag{30}$$

From the spectrum of the Hamiltonian (24), the partition function of the $\mathcal{PT}$-symmetric theory is obtained in the usual way:

$$Z_{\mathcal{PT}} = \sum_n e^{-\beta E_n}. \tag{31}$$

All three theories defined above are amenable to high-precision numerical calculation. In the case of the first two, we determine the eigenenergies of $H_{\mathcal{H}}$ by expressing that Hamiltonian in the occupation number basis of the harmonic oscillator and numerically diagonalizing. A truncation of the first 100 states of the harmonic oscillator is found to yield eigenvalues of sufficient precision for this study; all plots and numerical results reported herein come from a truncation of the first 10^3 states. With these eigenenergies determined, it is straightforward to evaluate either (27) or (28) numerically; the sums exhibit exponential convergence even at negative coupling.

In order to obtain (31), we exploit the exact duality demonstrated in [11]: The spectrum of the $\mathcal{PT}$-symmetric Hamiltonian, quantized on a suitable contour, is equal to that of

$$H_{\mathrm{dual}} = p^2 - x + 4x^4. \tag{32}$$

The spectrum of H_{dual} is obtained, as before, by diagonalizing the Hamiltonian expressed in the occupation number basis of the harmonic oscillator. As before, 100 states are sufficient for this study, and 10^3 are used for all results hereafter.

We are now prepared to compute the three different partition functions and compare. The results of this evaluation are shown in Fig. 2. The left panel is a check of the conjecture (2), which is clearly seen to fail at large β where the analytically continued partition function becomes unphysically negative. The right panel checks the ABS conjecture (1), where both partition functions exhibit physical behavior but do not match.

Although the left panel refutes (2), there is still surprising and suggestive agreement at small β (high temperatures or, equivalently, weak couplings). The precise agreement at small β followed by sudden onset of disagreement is suggestive of nonanalytic behavior akin to that of $f(x) = e^{-1/x^2}$ near the origin. The logarithm of the difference between the two partition functions at small β is shown in Fig. 3. To high precision, and across several orders of

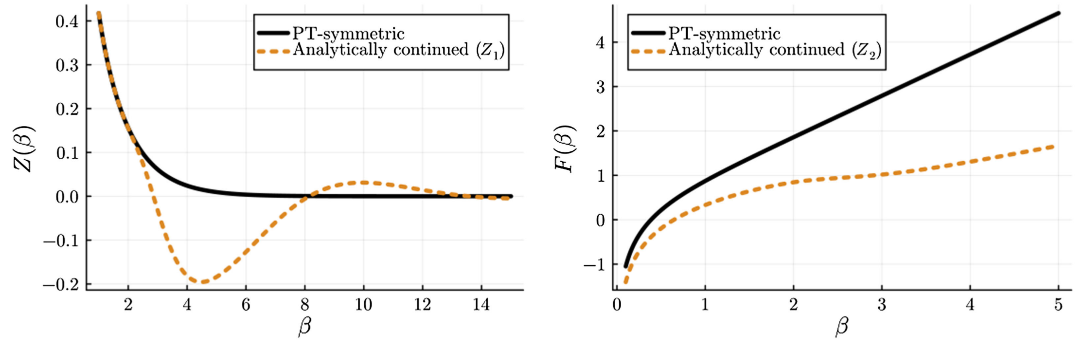

FIG. 2. Numerical checks of the two conjectures. The conjecture (2) is examined in the left panel, where very precise agreement in the partition functions at small β gives way to unphysical behavior in the analytically continued theory at low temperatures. The ABS conjecture itself is checked in the right panel, where the logarithm of the partition functions is plotted.

magnitude of the partition functions, this difference is found to be fit by a function

$$-\log\,(Z_{\mathcal{PT}} - Z_1) \approx \frac{p_1^3}{\beta^3} + \log(p_2 + p_3\beta^3) \qquad (33)$$

with parameters $p_1 \approx 3.963$, $p_2 \approx 0.307$, and $p_3 \approx 0.035$.

The numerically observed form of the failure of (2) at small β provides a clue as to the origin of the difference for high values of β, since it has the same parametric dependence on the coupling—of the form $e^{-1/(\beta^3\lambda)}$—as a bounce contribution [12]. The next section explores this further.

IV. PATH INTEGRALS

To explain the relation between the $\mathcal{PT}$-symmetric theory and the analytically continued partition function Z_1, we switch from the Hamiltonian to the action formalism. The action of either theory is

$$S = \int d\tau \left[\frac{1}{2}\left(\frac{d\phi}{d\tau}\right)^2 + \lambda\phi^4\right], \qquad (34)$$

although making sense of this in either the $\mathcal{PT}$-symmetric or analytically continued cases (where $\lambda < 0$) requires taking the path integral over an appropriate contour.[2] We will see that the choice of contour is what makes the difference observed in the previous section: One contour corresponds to the analytically continued theory and a different contour to the $\mathcal{PT}$-symmetric theory.

The first subsection below traces through the derivation of the path integral starting from the Hamiltonian formalism, showing that if the starting point is the analytically continued theory Z_1, one contour (analogous to the wedge discussed in Sec. II above) is obtained, but if the starting point is the $\mathcal{PT}$-symmetric theory, the path integral must be performed over a different contour (the cone). Next we examine a lattice discretization of the path integral and show numerically that it yields qualitatively similar results to the above. After reviewing some basic facts about Lefschetz thimbles and their intersection numbers, we show that the two contours have the same contribution from the trivial saddle point at the origin and, therefore, must differ in their contribution from some other saddle point. The final subsection examines the saddle points of the action and performs a semiclassical expansion around the nontrivial ones; we find that this expansion matches the functional form (and, to decent precision, the exponent) found by fitting the difference of partition functions above.

A. Two contours

First let us perform a loose derivation of a path integral for the $\mathcal{PT}$-symmetric theory, beginning with the Hamiltonian $H_{\mathcal{PT}} = H_{\mathcal{H}}(\lambda = -g)$. The derivation proceeds in the usual way, but we use the following resolution of the identity:

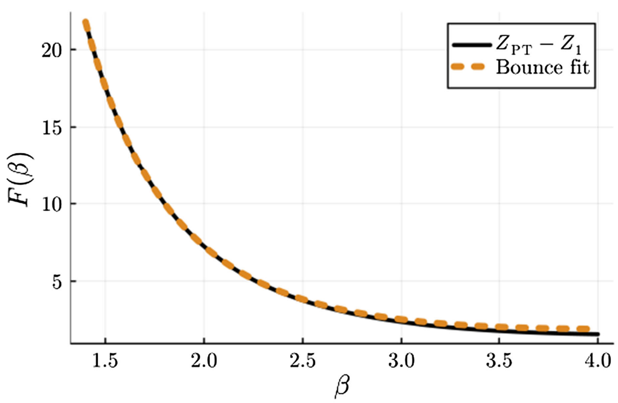

FIG. 3. Detailed study of the failure of (2). The logarithm of the difference between the two partition functions is plotted at small β (equivalent to small coupling λ). A fit to the functional form (33) is performed over the range $\beta \in [1.4, 1.8]$ and compared with the numerical form over $\beta \in [1.4, 4.0]$. The fit agrees to high precision and generalizes well to those smaller temperatures.

[2]Note that this is not the same as the process of "quantizing on a contour" that was used to define the $\mathcal{PT}$-symmetric Hamiltonian theory. For example, in the path integral formulation, $\phi(t)$ and $\phi(t')$ may live on two different contours in $\mathbb{C}$.

$$1 = \int_{\gamma_{\mathcal{PT}}} dx|x\rangle\langle x|. \tag{35}$$

As a result, the partition function $Z_{\mathcal{PT}} \equiv \mathrm{Tr}e^{-\beta H_{\mathcal{PT}}}$ reads

$$Z_{\mathcal{PT}} = \int_{x(t)\in\gamma_{\mathcal{PT}}} \mathcal{D}x(t)e^{-S(x;\lambda=-g)}, \tag{36}$$

where, at every time t, the position $x(t)$ is required to be valued not on the real line but on the deformed contour $\gamma_{\mathcal{PT}}$ used to quantize the $\mathcal{PT}$-symmetric theory.

In the $d=0$ case, we were able to show in Sec. II that an analogous integral corresponded to the real part of the analytic continuation of the original partition function, but critically, this held only for $N=1$. For a multicomponent field, obtaining the real part of the analytic continuation requires the use of the wedge contour, defined by (17) and depicted in Fig. 1. The same derivation holds here without modification.

B. On the lattice

For Hermitian theories, the partition function in quantum mechanics can also be defined as a path integral over real values of the field ϕ:

$$Z_H^{d=1}(\lambda) = \int \mathcal{D}\phi e^{-S}. \tag{37}$$

The path integral may be discretized by dividing the imaginary time interval into K sites [13]:

$$Z^{d=1}(\lambda) = \lim_{K\to\infty} \int \prod_{i=1}^{K} \frac{d\phi_i}{\sqrt{2\pi\varepsilon}} e^{-S_{\mathrm{lat}}}, \tag{38}$$

where the lattice action is defined as

$$S_{\mathrm{lat}} = \varepsilon \sum_{i=1}^{K} \left[\frac{(\phi_i - \phi_{i+1})^2}{2\varepsilon^2} + \lambda\phi_i^4\right], \tag{39}$$

with $\varepsilon = \frac{\beta}{K}$ and periodic boundary conditions $\phi_{K+1} = \phi_1$. In this form, the partition function is amenable to numerical computation for given values of λ and β. The number of sites K must be chosen such that $\varepsilon \ll 1$ in order to be close to the continuum limit of the theory. In practice, we find that in units where $\lambda = 1$, the choice $\varepsilon < 0.5$ gives acceptable quantitative results.

For the $\mathcal{PT}$-symmetric theory, the integration domain is not real. As with the case of $d=0$, one can, however, choose each ϕ_i to be given by (6), such that with $\chi = -\frac{\pi}{4}$

$$Z_{\mathcal{PT}}^{d=1}(g) = \int \prod_{i=1}^{K} \frac{d\phi_i}{\sqrt{2\pi\varepsilon}} e^{-S_{\mathcal{PT}}}, \tag{40}$$

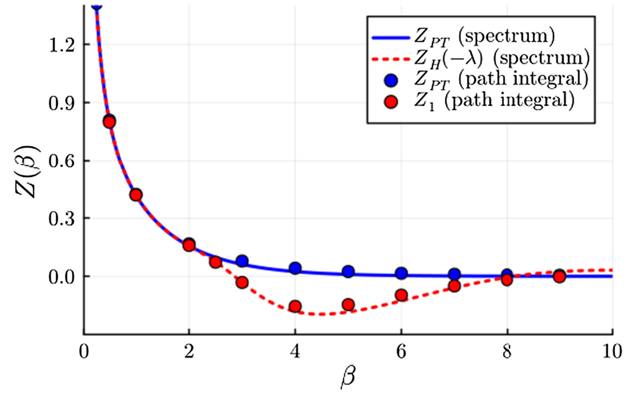

FIG. 4. The same as Fig. 2, but comparing results for $Z^{d=1}$ from the Hamiltonian spectrum to those from the path integral. See text for details.

with the $\mathcal{PT}$-symmetric form of the lattice action $S_{\mathcal{PT}}(g) = S_{\mathrm{lat}}(\lambda = -g)$. This is analogous to the cone contour of previous sections, except now defined for multiple sites rather than multiple components of the field. The resulting path integral is convergent but somewhat unwieldy to implement. Note that it instead of (6) is possible to choose complex integration contours without kinks such that the resulting path integral can be cast in form of a real-integration domain with a real action [14], which is numerically preferable to (40). However, we find that (40) with just four sites ($K=4$) gives qualitatively acceptable results for $Z_{\mathcal{PT}}^{d=1}(g)$ for $\beta < 3$; see Fig. 4.

Recalling the discussion in Sec. II, it was found that in the case of $d=0$, the cone contour of integration did not reproduce the analytically continued Hermitian theory for more than one field $N>1$. However, it was found that a wedge contour (17) faithfully gave the correct analytic continuation. For this reason, we consider a *different* path integral given by choosing the fields to lie on the wedge (17) where instead of $i = 0,1,2,\ldots,N$ the index in (17) now refers to the site location, e.g. $i = 0,1,2,\ldots,K$. With $\chi = -\frac{\pi}{4}$, one finds

$$Z_{\mathrm{wedge}}^{d=1}(g) = \int_0^\infty \frac{ds_0}{\sqrt{2\pi\varepsilon}} \int_{-\infty}^{\infty} \prod_{i=1}^{K-1} \frac{ds_i}{\sqrt{2\pi\varepsilon}} \left(e^{-S^+} + e^{-S^-}\right), \tag{41}$$

$$S^{\pm} = \pm\frac{i\pi K}{4} + \varepsilon \sum_{i=0}^{K-1} \left[\pm i\frac{(s_i - s_{i+1})^2}{2\varepsilon^2} + gs_i^4\right]. \tag{42}$$

The wedge path integral is convergent and can be evaluated numerically using efficient numerical integrators such as VEGAS [15] on modern CPUs for $K \lesssim 20$. Results for $K=10$ for $Z_{\mathrm{wedge}}(g)$ are shown in Fig. 4, suggesting that the wedge path integral indeed corresponds to the analytic continuation of the Hermitian theory to negative coupling.

C. Intersection numbers

To understand the origin of the difference between the integrals on the two contours, we must first review the properties of Lefschetz thimbles (see [16] for a more detailed exposition). We will assume that the model has been defined on a finite number of degrees of freedom, as in the lattice models of the previous section.

Given a holomorphic action $S(z)$ of fields $z \in \mathbb{C}^N$, we define the *upward flow* according to

$$\frac{dz_i}{dt} = \left(\frac{\partial S}{\partial z_i}\right)^*. \tag{43}$$

The upward flow has the important property that along it the imaginary part of the action is constant, while the real part of the action monotonically increases.

The flow vanishes only at solutions to the classical equations of motion—i.e., the saddle points. To each saddle point $z^{(\sigma)}$ is associated a *Lefschetz thimble* $\mathcal{J}_\sigma$: a K-dimensional manifold consisting of the union of all solutions $z(t)$ to (43) obeying $\lim_{t\to-\infty} z(t) = z^{(\sigma)}$. We may similarly define an *antithimble* as the union of all solutions obeying $\lim_{t\to+\infty} z(t) = z^{(\sigma)}$. Note that the integral of e^{-S} along a thimble is finite, while the integral along the antithimble diverges.

Any integration contour that begins and ends at complex infinity is homologous to some linear combination of thimbles, with integer coefficients. In particular, this implies that the integral of any holomorphic function (including of course e^{-S}) along that contour is the sum of the integrals taken along those thimbles:

$$\int_\gamma f(z) = \sum_\sigma n_\sigma \int_{\mathcal{J}_\sigma} f(z). \tag{44}$$

Once an integration contour has been expressed as a linear combination of thimbles, we may perform a saddle point approximation on each thimble. The contribution of the thimble $\mathcal{J}_\sigma$ will be proportional (up to a Jacobian factor) to $e^{-S(z^{(\sigma)})}$.

The integers n_σ in (44) are termed *intersection numbers*. In practice we commonly find $n_\sigma \in \{0, \pm 1\}$. If the integrals on the cone and wedge contour are to differ, then those two contours must have differing intersection numbers. If the difference between those two contours is to be suppressed by a factor of $e^{-1/\lambda\beta^3}$, then the difference in intersection numbers must not be at the origin but at a subdominant saddle point.

In principle, the intersection numbers may be obtained by evolving the integration contour of interest according to (43). In the limit of long-flow times, this will approach a fixed-point manifold exactly equal to some integer combination of the various Lefschetz thimbles. This is typically not a practical procedure, but it provides a useful trick for establishing that an intersection number is 0, as follows. Recall that the upward flow only increases the real part of the action. If, for every point z in the integration contour of interest, the real part of the action is already larger than that at a saddle point $z^{(\sigma)}$, then the associated intersection number is necessarily $n_\sigma = 0$.

We can use this to establish that the cone and wedge contours have the same intersection numbers with the thimble extending from the saddle point $z^{(0)}$ at the origin.[3] In the two-site case, consider the contour defined by the difference between the cone and wedge contours; we will show that this contour has intersection number $n_0 = 0$.

Using the notation of Sec. II, the integration contour that gives the difference between the wedge and cone contours is

$$\gamma_{\text{wedge}} - \gamma_{\text{cone}} = \gamma_3 \times (\gamma_1 - \gamma_4) + \gamma_4 \times (\gamma_2 - \gamma_3). \tag{45}$$

As is, this contour does intersect the origin and, therefore, contains a point on which the action is equal to that at the saddle point. However, the attentive reader may already observe that the contour intersects the origin *twice*, with opposing orientations.

To make clear that the origin has no contribution, we can infinitesimally deform the contours $(\gamma_1 - \gamma_4)$ and $(\gamma_2 - \gamma_3)$ away from the origin. This increases the real part of the action at every point and, therefore, results in a contour where the inequality $S(z) > S(z_0)$ is strict. With respect to this contour, the intersection number with the trivial saddle point must vanish: $n_0 = 0$.

D. Semiclassics

In the previous sections, we found numerically that the partition function for the analytically continued Hermitian theory differs from the partition function of the $\mathcal{PT}$-symmetric theory but that this difference becomes exponentially small for high temperatures; cf. Fig. 2. In this section, we consider the high-temperature limit of the analytically continued theory by performing a semiclassical evaluation of the path integral. Note that at high temperature the semiclassical evaluation is a good approximation because quantum fluctuations are highly suppressed.

To wit, when appropriately rescaling τ and ϕ, the analytically continued partition function is given by

$$Z(\lambda = -g) = \int \mathcal{D}\phi e^{-S}, \quad S = \int_0^1 d\tau \left[\frac{\dot{\phi}^2}{2} - g\beta^3\phi^4\right], \tag{46}$$

[3]Here we are being slightly sloppy. The saddle point at the origin is degenerate, and strictly speaking we ought to break this degeneracy—and any others—by introducing a small perturbation in the action before we can speak of a unique thimble decomposition. However, this does not change the results or any of the reasoning, so we have elided this step to keep the explanation brief and manageable.

subject to periodic boundary conditions $\phi(0) = \phi(1)$. In the high-temperature limit, we may attempt to evaluate this partition function by functional saddle point method. Specifically, we have

$$\phi(\tau) = \phi_{\rm cl}(\tau) + \phi'(\tau), \qquad S = S^{(0)} + S^{(1)} + S^{(2)} + \cdots, \tag{47}$$

where $S^{(0)} = S[\phi = \phi_{\rm cl}]$ and

$$S^{(1)} = \int_0^1 d\tau \phi' \left[-\ddot{\phi}_{\rm cl} - 4g\beta^3 \phi_{\rm cl}^3 \right], \tag{48}$$

$$S^{(2)} = \int_0^1 d\tau \frac{\phi'}{2} \left[-\ddot{\phi}' - 12g\beta^3 \phi_{\rm cl}^2 \phi' \right]. \tag{49}$$

The saddle point condition of vanishing $S^{(1)}$ leads to the classical equations of motion

$$\ddot{\phi}_{\rm cl} = -4g\beta^3 \phi_{\rm cl}^3. \tag{50}$$

The classical solution $\phi_{\rm cl}$ is given by

$$\phi_{\rm cl}(\tau) = \frac{\Omega}{\sqrt{4g\beta^3}} \mathrm{cn}\left(\Omega\tau + B, \frac{1}{2} \right), \tag{51}$$

where cn denotes the Jacobi elliptic cn function and Ω and B are two constants. We may recast the path integral in terms of these constants as follows. Writing

$$Z = \int \frac{d\phi_i d\phi_f}{\sqrt{2\pi}} \delta(\phi_i - \phi_f) \mathcal{D}\phi' e^{-S}, \tag{52}$$

where $\phi_i = \phi_{\rm cl}(0)$, $\phi_f = \phi_{\rm cl}(1)$, and we perform a change of variables $\phi_i, \phi_f \to \Omega, B$ such that

$$Z = \int \frac{d\Omega dB}{\sqrt{2\pi}} \delta(\Omega - \Omega_n) \left| \mathrm{sn}\left(B, \frac{1}{2}\right) \mathrm{dn}\left(B, \frac{1}{2}\right) \right| \int \mathcal{D}\phi' e^{-S}, \tag{53}$$

where sn and dn denote the Jacobi elliptic sn and dn functions, respectively, and $\Omega_n = 4nK(\frac{1}{2})$ with $K(m)$ the complete elliptic integral of the first kind with modulus m. Here Ω_n with $n \in \mathcal{N}$ denotes periodic frequency of the Jacobi elliptic functions that results from the periodicity requirement $\delta(\phi_i - \phi_f)$. Effectively, the integral over the constant Ω turns into a sum over n:

$$Z = \sum_{n=0}^{\infty} \int \frac{dB}{\sqrt{2\pi}} \left| \mathrm{sn}\left(B, \frac{1}{2}\right) \mathrm{dn}\left(B, \frac{1}{2}\right) \right| \int \mathcal{D}\phi' e^{-S}. \tag{54}$$

Restricting the classical solution (51) to $\Omega = \Omega_n$ leads to the classical action

$$S^{(0)} = \frac{\Omega_n^4}{48g\beta^3} \simeq \frac{63.02}{g\beta^3} \times n^4. \tag{55}$$

Note that this corresponds to a bounce contribution proportional to $e^{-3.98/(\beta^3\lambda)}$, consistent with the fit performed in Fig. 3.

The functional integration over the fluctuations ϕ' can be calculated using the Gelfand-Yaglom method; cf. [17]. From $S^{(2)}$ above, the equations of motion for ϕ' are

$$\ddot{\phi}' = -12g\beta^3 \phi_{\rm cl}^2 \phi', \tag{56}$$

with $\phi_{\rm cl}$ given by (51) and Dirichlet boundary conditions $\phi'(0) = \phi'(1) = 0$ because of $\phi = \phi_{\rm cl} + \phi'$ and $\phi_{\rm cl}(0) = \phi(0)$, $\phi_{\rm cl}(1) = \phi(1)$. The Gelfand-Yaglom method implies

$$\int \mathcal{D}\phi' e^{-S^{(2)}} = [u(1)]^{-\frac{1}{2}}, \tag{57}$$

where $u(\tau)$ is a solution to (56) with different boundary conditions $u(0) = 0$, $\dot{u}(0) = 1$. The general solution to (56) can be found by the variation of the classical solution (51), $u(\tau) = \delta\phi_{\rm cl}$ with respect to the parameters Ω and B. The solution fulfilling the boundary conditions can then be constructed straightforwardly, and one finds

$$u(1) = \mathrm{sn}^2\left(B, \frac{1}{2}\right) \mathrm{dn}^2\left(B, \frac{1}{2}\right). \tag{58}$$

Putting everything together, we find in the semiclassical limit

$$Z(\lambda = -g) = \sum_{n=0}^{\infty} (-1)^n e^{-\frac{\Omega_n^4}{48g\beta^3}} \left(\frac{2K(\frac{1}{2})}{\sqrt{2\pi}} + \mathcal{O}(g) \right), \tag{59}$$

where we have taken the integral limits for B to correspond to the points where cn $(B, \frac{1}{2}) = \pm 1$. The origin of the factor $(-1)^n$ can be understood as follows: Regarding $-\partial_\tau^2 - 12g\beta^3\phi_{\rm cl}^2$ as a Schrödinger operator, we see that for $n = 0$ the spectrum of the operator is real and positive, so the square root of the determinant is positive. For $n > 0$, we can identify a zero-energy solution for the special case $B = 0$ that fulfills the boundary conditions $\phi'(0) = \phi'(1) = 0$ with wave function $u(\tau) = \mathrm{sn}(\Omega_n, \frac{1}{2})\mathrm{dn}(\Omega_n\tau, \frac{1}{2})$. For $n = 1$, this wave function has one node. It is well known that the ground-state wave function for the Schrödinger equation has no nodes, so there must be exactly one energy eigenstate with $E < 0$ for $n = 1$ and $B = 0$. If $B \neq 0$, the energy of the first excited state must also be negative; otherwise, the determinant of the operator calculated in (58) would have to be negative. As a result, we find that for $B \neq 0$ and $n = 1$ there must be two negative eigenenergies, and hence the sign of $\det^{-\frac{1}{2}}$ must be negative. For $n = 2$, one can repeat this exercise, now noting that for

$B=0$ has three nodes, and hence there must be four negative energy states for $B>0$, $n=2$. This generalizes to higher n, leading to the factor of $(-1)^n$ shown in (59).

We recognize (59) have the typical form expected for bounces, with $n=0$ the zero-bounce (perturbative) contribution, $n=1$ the one-bounce contribution, and $n>1$ multibounce contributions.

E. N-component scalars in the large-N limit

Finally, let us consider quantum mechanics in N dimensions, for which the Hermitian partition function reads

$$Z^{d=1}_{N,\mathcal{H}} = \int \mathcal{D}\vec{\phi} e^{-\int_0^\beta d\tau \left[\frac{1}{2}\dot{\vec{\phi}}^2 + \frac{\lambda}{N}(\vec{\phi}^2)^2\right]}. \tag{60}$$

Using a Hubbard-Stratonovich transformation introducing the auxiliary field ζ, this can be rewritten as in [18], so that after performing the Gaussian integral over $\vec{\phi}$ one has

$$Z^{d=1}_{N,\mathcal{H}} = \int \mathcal{D}\zeta e^{-\int_0^\beta d\tau \frac{N\zeta^2}{4\lambda} - \frac{N}{2}\mathrm{tr}\,\ln[-\partial_\tau^2 + 2i\zeta]}. \tag{61}$$

At large N, only the zero mode of the field ζ contributes; if in addition we limit our consideration to low temperatures, we have (cf. [19])

$$Z^{d=1}_{N\gg1,\mathcal{H}}(\beta\to\infty) = \int d\zeta_0 e^{-N\beta\left(\frac{\zeta_0^2}{4\lambda} + \sqrt{\frac{i\zeta_0}{2}}\right)}. \tag{62}$$

At large N, the last integral may be calculated exactly using the saddle point method. There is only one saddle on the principal Riemann sheet, located at $i\zeta_0 = 2^{-\frac{1}{3}}\lambda^{\frac{2}{3}}$. One can identify the stable thimble connecting this saddle to the real line by the same technique that was used in the Appendix. Evaluating the action at the saddle, one thus has $\ln Z^{d=1}_{N\gg1,\mathcal{H}}(\beta\to 0) = -\beta E_0^{(N\gg1)}$, where to leading order in large N, $E_0^{N\gg1} = \frac{3(2\lambda)^{\frac{1}{3}}}{8}N$. One can also calculate the contribution of order $\mathcal{O}(N^0)$ to E_0 as follows: Expanding the partition function (61) to second order in fluctuations around the saddle $\zeta=\zeta_0+\zeta'(\tau)$ and performing a Fourier transform on the fields ζ', we obtain the fluctuation action in the small temperature limit as $S_2 = \frac{1}{2}\int\frac{dk}{2\pi}|\zeta'(k)|(\frac{1}{2\lambda}+\Pi(k))$. Here $\Pi(k)=\frac{1}{2}\int\frac{dp}{2\pi}G(p)\times G(p+k)$, with $G^{-1}(p) = p^2+(2\lambda)^{\frac{1}{3}}$ such that

$$\Pi(k) = \frac{1}{(2\lambda)^{\frac{1}{3}}(k^2+4(2\lambda)^{\frac{2}{3}})}. \tag{63}$$

Performing the path integral over ζ' leads to an expression for the spectral gap accurate to next-to-leading order (NLO) in large N:

$$E_0^{N\gg1} = (2\lambda)^{\frac{1}{3}}\left(\frac{3}{8}N + \frac{\sqrt{6}-2}{2}\right) + \mathcal{O}(N^{-1}). \tag{64}$$

A similar calculation can be performed for the "wrong-sign" partition function defined on the wedge contour. Starting with the partition function

$$Z^{d=1}_{\mathrm{wedge}N} = \int_{\mathcal{C}} \mathcal{D}\vec{\phi} e^{-\int_0^\beta d\tau\left[\frac{1}{2}\dot{\vec{\phi}}^2 - \frac{g}{N}(\vec{\phi}^2)^2\right]}, \tag{65}$$

with $\vec{\phi}$ a complex function of real-valued vectors $\vec{s}$ an obvious generalization to (41) to N components. Since $\vec{s}$ is a real-valued vector field, we introduce a Hubbard-Stratonovich transformation just as in the Hermitian theory case. Since the integral over $\vec{s}$ is again Gaussian, we find

$$Z^{d=1}_{\mathrm{wedge},N} = \int \mathcal{D}\zeta\left[\frac{1}{2}e^{-\int_0^\infty d\tau\frac{N\zeta^2}{4g} - \frac{N}{2}\mathrm{tr}\,\ln[-\partial_\tau^2 - 2\zeta]} + \zeta\to-\zeta\right], \tag{66}$$

which is still exact for all N. In the large-N limit, we can again use the fact that the partition function can be evaluated from the saddle points of the action, which is the Fourier zero mode $\zeta(\tau)=\zeta_0$. The calculation then proceeds exactly analogous to the Hermitian case, even though the saddle point locations ζ_0 are complex. One finds

$$E^{N\gg1}_{\mathrm{wedge},0}(g) = \frac{(2g)^{\frac{1}{3}}}{2}\left(\frac{3}{8}N + \frac{\sqrt{6}-2}{2}\right) - \frac{1}{\beta}\ln\left[2\cos\left(\frac{\sqrt{3}N(2g\beta^3)^{\frac{1}{3}}}{16}\right)\right] + \mathcal{O}(N^{-1}). \tag{67}$$

Comparing (67) and (64), we find that, in the zero temperature limit to leading and NLO order in large N,

$$\ln Z_{\mathrm{wedge}}(g) = \mathrm{Re}\,\ln Z_{\mathcal{H}}(\lambda=-g). \tag{68}$$

V. DISCUSSION

In this work, we have examined the relation between interacting quantum theories with quartic interaction. Specifically, we have studied if and how analytically continuing the Hermitian theory to negative coupling can be related to the $\mathcal{PT}$-symmetric theory.

Based on our detailed calculations performed in $d=0$ and $d=1$, our findings are as follows.

(i) We showed that a path-integral formulation on a complex field contour (the wedge) for the wrong-sign Hermitian theory has the property that its partition function equals the real part of the analytically continued Hermitian theory (2).

(ii) We found that this complex integration contour (the wedge) is different from—and yields a different integral than—the complex integration contour used to define the $\mathcal{PT}$-symmetric theory (the cone).

(iii) We found that the difference in integration contours corresponds to a nonperturbative contribution to the partition function (the "bounce"). Evaluating the leading bounce contribution analytically using semiclassics, we find excellent numerical agreement with the difference between the partition functions defined on the two contours.

(iv) We provided evidence from high-precision numerical calculations that the path integrals defined on the wedge and cone, respectively, correspond to the partition function calculated from the known spectrum of the Hamiltonian for the analytically continued Hermitian and $\mathcal{PT}$-symmetric theories.

(v) We found that because the bounce contribution becomes exponentially suppressed at high temperature (equivalently, weak coupling), the partition functions defined on the two integration contours are exponentially close in that limit.

(vi) We found that in the limit of a large number of fields the difference between the relations (1) and (2) becomes large-N suppressed.

Based on these findings, we offer the following interpretations concerning the relation between analytically continued Hermitian and $\mathcal{PT}$-symmetric field theory.

(i) The ABS conjecture (1) is likely incorrect. In all cases we studied, (1) was violated, for reasons we have outlined in this work.

(ii) The relation (2) holds to very good approximation at high temperatures. This is because the nonperturbative corrections to the left-hand side of (2) are exponentially suppressed at high temperature.

(iii) The analytically continued Hermitian partition function does have a consistent formulation as a path integral on a complex integration contour; it is just not the $\mathcal{PT}$-symmetric integration contour. This wedge contour gives the exact analytic continuation of the Hermitian theory for all temperatures and all number of field components.

(iv) In the large volume (zero temperature) limit, we expect the relation

$$\ln Z_{\text{wedge}}(g) = \text{Re}\ln Z_{\mathcal{H}}(\lambda = -g), \qquad (69)$$

which we proved for $d = 0, 1$, to leading and next-to-leading order in $\frac{1}{N}$ to generalize to arbitrary dimension d.

While the original ABS conjecture does not seem to hold, we believe that the existence of the relation (69) puts the analytic continuation of wrong-sign field theories such as those discussed in Refs. [7–9] on firm footing.

ACKNOWLEDGMENTS

We would like to thank Wen-Yuan Ai, Carl Bender, Seth Grable, Sarben Sarkar, and Max Weiner for helpful discussions. This work was supported by the Department of Energy, Grant No. DE-SC0017905.

APPENDIX: ONE-SITE CALCULATIONS FOR N-COMPONENT SCALARS

In this section, we provide some calculational details for the case of N-component scalars in $d = 0$ discussed in Sec. II in the main text. To start, note that (13) can be calculated using spherical coordinates in N dimensions. This leads to

$$Z_N^{d=0} = \frac{2^{1-\frac{N}{2}}}{\Gamma(\frac{N}{2})}\int_0^\infty dr r^{N-1} e^{-\frac{\lambda}{N}r^4} = \left(\frac{4\lambda}{N}\right)^{-\frac{N}{4}} \frac{\Gamma(\frac{N}{4})}{2\Gamma(\frac{N}{2})}. \qquad (A1)$$

In particular, for large $N \gg 1$, the asymptotic expansion for the Γ function then leads to

$$\ln Z_{N\gg 1}^{d=0} = -\frac{N}{4}\ln\frac{4\lambda}{e^1} - \frac{\ln 2}{2} + \mathcal{O}(N^{-1}). \qquad (A2)$$

This large-N behavior may also be obtained directly using the method of steepest descent. To this end, rewrite

$$Z_N^{d=0} = \sqrt{\frac{N}{4\lambda\pi}}\int \frac{d\vec{\phi}}{(2\pi)^{\frac{N}{2}}}\int_{-\infty}^{\infty} d\zeta e^{-i\zeta\vec{\phi}^2 - \frac{\zeta^2 N}{4\lambda}}. \qquad (A3)$$

Now the integral over $\vec{\phi}$ is Gaussian and can be done exactly to give

$$Z^{d=0}N = \sqrt{\frac{N}{4\lambda\pi}} 2^{-\frac{N}{2}}\int_{-\infty}^{\infty} d\zeta e^{-\frac{\zeta^2 N}{4\lambda} - \frac{N}{2}\ln(i\zeta)}. \qquad (A4)$$

For $N \gg 1$, the integral can be evaluated exactly using the method of steepest descent. The saddle point condition then is

$$\frac{\zeta}{2\lambda} + \frac{1}{2\zeta} = 0, \qquad (A5)$$

which is solved by $\zeta = \pm i\sqrt{\lambda}$. To find out which saddle contributes to the path integral, we consider a path parametrized by s in the complex ζ plane, so that $\zeta(s) = a(s) + ib(s)$ with real $a(s)$ and $b(s)$. The special path we are interested in is called a Lefschetz thimble, and it is defined through the solution of the flow equations (43) where here $S = \frac{\zeta^2}{4\lambda} + \frac{1}{2}\ln(i\zeta)$. The thimbles have the special property that the imaginary part of the action is constant along the thimble, which can easily be seen from noting that

$$\frac{dS}{ds} = \frac{\partial S}{\partial \zeta}\frac{d\zeta}{ds} = \left|\frac{\partial S}{\partial \zeta}\right|^2. \qquad (A6)$$

For the saddle located at $\zeta = -i\sqrt{\lambda}$, S is real, which can be used to find the corresponding thimbles passing through this saddle without solving (43). Specifically, one finds that one thimble is given by

$$a(s) = \sqrt{\lambda} x(s) \sqrt{\frac{\operatorname{atan} x(s)}{x(s)}}, \qquad b(s) = -\sqrt{\lambda} \sqrt{\frac{\operatorname{atan} x(s)}{x(s)}}, \tag{A7}$$

with $x(s) = \pm e^s$ on the right and left part of the thimble. There is also an unstable thimble given by $a(s) = 0$ and $b = -s$ with $s > 0$, but this thimble does not connect to the real line at $\zeta = \pm\infty$, so it is dismissed. For the second saddle at $\zeta = +i\sqrt{\lambda}$, one finds that in the complex ζ plane the branch cut of the logarithm implies that these are actually multiple saddles on different Riemann sheets. Not surprisingly, there are no thimbles that connect to the real line on the principal Riemann sheet and go through these saddles, so there is no contribution from the saddle at $\zeta = \pm\infty$ to the path integral.

Using the stable thimble through the saddle $\zeta = -i\sqrt{\lambda}$, expanding $S[\zeta]$ to quadratic order and doing the Gaussian integral then gives

$$\ln Z^{d=0}_{N\gg 1} = -\frac{N}{4} \ln \frac{4\lambda}{e^1} - \frac{1}{2} \ln 2 + \mathcal{O}(N^{-1}), \tag{A8}$$

matching the large-N limit of the exact result (A2).

We close this section by giving detailed results for the $d = 0$ partition function for N components on the wedge contour. Explicitly, in the case of $N = 2$, we have for this choice of contour

$$Z^{d=0}_{\text{wedge},N=2} = \int_0^\infty \frac{ds dt}{2\pi} e^{-\frac{g}{2}(s^2+t^2)} \left(2e^{-\frac{i\pi}{2}} + 2e^{\frac{i\pi}{2}}\right) = 0. \tag{A9}$$

Comparing this result to the Hermitian $O(N)$ result (A1), one finds that the path integral defined on the wedge exactly matches the real part of the analytically continued Hermitian result.

For three or more components, one proceeds in a similar fashion to find

$$Z^{d=0}_{\text{wedge},N} = \frac{2^{1-\frac{N}{2}}}{\Gamma(\frac{N}{2})} \int_0^\infty dr r^{N-1} e^{-\frac{g}{N} r^4} \frac{\left(e^{-\frac{Ni\pi}{4}} + e^{\frac{Ni\pi}{4}}\right)}{2}, \tag{A10}$$

which proves

$$Z^{d=0}_{\text{wedge},N} = \operatorname{Re}\{Z^{d=0}_N(\lambda = -g + i0^+)\}, \tag{A11}$$

for all $N \in \mathbb{N}$. At large $N \gg 1$, one notes that

$$\ln Z^{d=0}_{\text{wedge},N\gg 1} = -\frac{N}{4} \ln \frac{4g}{e^1} - \frac{\ln 2}{2} + \ln \cos \frac{N\pi}{4} + \mathcal{O}(N^{-1}). \tag{A12}$$

Since the logarithm of the cosine is not proportional to N, in addition to (A11), at large N the $d = 0$ theory fulfills the additional relation

$$\ln Z^{d=0}_{\text{wedge},N\gg 1} = \operatorname{Re} \ln Z^{d=0}_{N\gg 1}(\lambda = -g + i0^+) + \mathcal{O}(N^0). \tag{A13}$$

Thus at large N, and large N only, the Hermitian and wedge-contour parametrized partition functions for the $d = 0$ case are related through the original conjecture (2).

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