Isospectral local Hamiltonians for perturbative $\mathcal{PT}$-symmetric Hamiltonians

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A new method to work out the Hermitian correspondence of a $\mathcal{PT}$-symmetric quantum mechanical Hamiltonian is proposed. In contrast to the conventional method, the new method ends with a local Hamiltonian of the form $\frac{1}{2}p^2+\frac{1}{2}m^2x^2+v(x)$ without any higher-derivative terms. This method is available for Hamiltonians with general antilinear symmetries in the perturbative regime. Possible extensions to multivariable quantum mechanics and quantum field theories are discussed.

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I. INTRODUCTION

The discovery of real-spectra $\mathcal{PT}$-symmetric Hamiltonians [1] has inspired a lot of research beyond conventional Hermitian quantum theories [2,3]. Originally, in [1] it was found that Hamiltonians of the form $H=p^2+m^2-(ix)^N(N\geq 2)$ have real spectra. Later, the general framework to describe a $\mathcal{PT}$-symmetric quantum theory was established [4–7]. A nontrivial metric operator $\eta=e^{-Q}$ satisfying $\eta H\eta^{-1}=H^\dagger$ is necessary [5] for the unitary evolution generated by a non-Hermitian $\mathcal{PT}$-symmetric Hamiltonian H, which differs from Hermitian quantum mechanics. With the help of this metric operator, a real-spectra $\mathcal{PT}$-symmetric Hamiltonian H can be recast to an isospectral Hermitian Hamiltonian $h=e^{-Q/2}He^{Q/2}$ equipped with the ordinary Dirac inner product. A remarkable example is the isospectral Hermitian Hamiltonian for $H=p^2-gx^4$, as described in [8,9]. The stability for the $-x^4$ potential is essential to guarantee the stability of the Higgs vacuum [3]. Moreover, a generic method [10] has been developed to calculate the metric operator for a perturbative $\mathcal{PT}$-symmetric Hamiltonian of the form $H=H_0+\epsilon H_1$ where H_0 is Hermitian and H_1 is anti-Hermitian. In this case, Q has the form $Q=\epsilon Q_1+\epsilon Q_3+\cdots$, and each term can be determined perturbatively as follows [3]:

$$[H_0,Q_1]=-2H_1,\quad [H_0,Q_3]=-\frac{1}{6}[[H_1,Q_1],Q_1],\cdots. \tag{1}$$

The isospectral Hermitian Hamiltonian h acquired from this procedure is in general nonlocal in the sense of containing terms in an arbitrarily high order of momentum p, which render the physical meaning of h rather obscure. However, there are vast degrees of freedom in generating h as demonstrated in [11]. In this paper we give an explicit method to calculate the local version of h for perturbative $\mathcal{PT}$-symmetric Hamiltonians whose free parts are nondegenerate. In contrast to the nonlocal h from the above conventional method, we believe a local form has apparent physical meanings and will bring inspirations to the research of $\mathcal{PT}$-symmetric theories. Also, our method applies to Hamiltonians with general antilinear symmetries, because no explicit properties of spatial-inversion and time-reversal symmetries are used in the subsequent derivations. From now on, $\mathcal{PT}$ symmetry stands for any general antilinear symmetry.

Here we summarize the main procedures of our new methods and the structure of this paper. In Sec. II, we start from a single-variable Hamiltonian $H_V=\frac{1}{2}p^2+\frac{1}{2}m^2x^2+V(x,p)$, where $V(x,p)=\sum_{n=1}^{\infty}g^nV_n(x,p)$ is the sum of various polynomial functions $V_n(x,p)$ of x and p with coupling constant g. We assume H_V respects unbroken $\mathcal{PT}$ symmetry. Then we show a similarity transformation of H_V leads to a manifestly diagonal Hermitian Hamiltonian $H_N=m(N+\frac{1}{2})+F(N)$, where

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$F(N)=\sum_{n=1}^{\infty} g^n f_n(N)$ is the sum of various polynomial functions $f_n(N)$ of N and $N=a^\dagger a$ in which $a=\sqrt{\frac{m}{2}}x+i\sqrt{\frac{1}{2m}}p$ is the standard annihilation operator.[1] In Sec. III, we transform H_N to $h_v=\frac{1}{2}p^2+\frac{1}{2}m^2x^2+v(x)$, where $v(x)=\sum_{n=1}^{\infty} g^n v_n(x)$ is the sum of various polynomial functions $v_n(x)$ of x only. The transformation from H_V to H_N is a typical diagonalization procedure. And the key point of the transformation from H_N to h_v is the existence of a one-to-one correspondence between the nth order polynomials of N and x^2. In Sec. IV, we calculate h_v in the ix^3 model as an example. When generalizing to multivariable Hamiltonians, the one-to-one map exists only in the case where the free part of H_V is nondegenerate, and this is discussed in Sec. V together with the generalization to quantum field theories. We conclude in Sec. VI.

II. DIAGONALIZATION OF A HAMILTONIAN WITH THE D-OPERATION

Consider a single-variable Hamiltonian with one[2] real coupling constant g:

$$H_V=\frac{1}{2}p^2+\frac{1}{2}m^2x^2+\sum_{n=1}^{\infty} g^n V_n(x,p). \tag{2}$$

As stated in Sec. I, $V_n(x,p)$ is a polynomial function of x and p respecting $\mathcal{PT}$ symmetry. Creation and annihilation operators can be defined as usual

$$a^\dagger=\sqrt{\frac{m}{2}}x-i\sqrt{\frac{1}{2m}}p, \qquad a=\sqrt{\frac{m}{2}}x+i\sqrt{\frac{1}{2m}}p. \tag{3}$$

In the Fock space defined by $a^\dagger$ and a, diagonal operators are in the form $\sum_{n=0}^{\infty} c_n N^n$ because of the commutation relation $[a,a^\dagger]=1$. We define a linear operation $D()$ on any operator $\mathcal{O}$ that can be expanded by polynomial functions of x and p to take out the diagonal part of $\mathcal{O}$ such that $D(\mathcal{O})=\sum_{n=0}^{\infty} c_n^{\mathcal{O}} N^n$. To be concrete, for an operator $\mathcal{O}$ possessing an expansion of the form $\mathcal{O}=\sum_{n\ell} a_{n\ell}^{\mathcal{O}} x^n p^\ell$, we first recast it into a series of a and $a^\dagger$ using (3), and then we have $\mathcal{O}=\sum_{n\ell} b_{n\ell}^{\mathcal{O}} a^{\dagger n} a^\ell$ where $b_{n\ell}^{\mathcal{O}}$ is determined by $a_{n\ell}^{\mathcal{O}}$ and (3) systematically. Because $a^{\dagger n}a^\ell$ with $n\neq\ell$ has vanishing diagonal components, we simply have $D(\mathcal{O})=\sum_n b_{nn}^{\mathcal{O}} a^{\dagger n}a^n$. Making use of established relations such as $a^{\dagger n}a^n=N(N-1)\cdots(N-n+1)$, we finally have $D(\mathcal{O})=\sum_{n=0}^{\infty} c_n^{\mathcal{O}} N^n$ where $c_n^{\mathcal{O}}$ is determined by $b_{nn}^{\mathcal{O}}$ and is unique by the description of the entire procedure. For example,

$$D(1)=1, \qquad D(x)=D(p)=0,$$
$$D(x^2)=\frac{1}{m^2}D(p^2)=\frac{1}{2m}(2N+1),\cdots. \tag{4}$$

A diagonal operator $\mathcal{O}$ satisfies $\mathcal{O}=D(\mathcal{O})$. If we want to diagonalize H_V with a similarity transformation e^{-R}, it is enough to satisfy the condition

$$e^{-R}H_V e^R=D(e^{-R}H_V e^R). \tag{5}$$

Assume H_V can be evaluated in the perturbative regime, and then R can be written as a perturbation series $R=\sum_{n=1}^{\infty} g^n R_n$. Taking out nth order terms on both sides of (5), we have

$$\begin{aligned}[H_0,R_n]=&D([H_0,R_n])+D(V_n)-V_n\\&+D\left(\sum_{j=2}^{n}\sum_{\substack{\{k_1,\ldots,k_j\}\\k_1+\cdots+k_j=n}}\frac{[[H_0,R_{k_1}],\ldots,R_{k_j}]}{j!}+\sum_{\ell=1}^{n-1}\sum_{j=1}^{n-\ell}\sum_{\substack{\{k_1,\ldots,k_j\}\\k_1+\cdots+k_j=n-\ell}}\frac{[[V_\ell,R_{k_1}],\ldots,R_{k_j}]}{j!}\right)\\&-\left(\sum_{j=2}^{n}\sum_{\substack{\{k_1,\ldots,k_j\}\\k_1+\cdots+k_j=n}}\frac{[[H_0,R_{k_1}],\ldots,R_{k_j}]}{j!}+\sum_{\ell=1}^{n-1}\sum_{j=1}^{n-\ell}\sum_{\substack{\{k_1,\ldots,k_j\}\\k_1+\cdots+k_j=n-\ell}}\frac{[[V_\ell,R_{k_1}],\ldots,R_{k_j}]}{j!}\right),\end{aligned} \tag{6}$$

where $H_0\equiv\frac{1}{2}p^2+\frac{1}{2}m^2x^2=m(N+\frac{1}{2})$.

Because H_0 is diagonal, $[H_0,R_n]$ has vanishing diagonal components and $D([H_0,R_n])=0$. $[H_0,R_n]$ is thus determined completely by lower-order R_ks. It is obvious that $D(\mathcal{O})-\mathcal{O}$ has vanishing diagonal components such that it is in the form $\sum_{k,\ell(k\neq\ell)} c'^{\mathcal{O}}_{k\ell}a^{\dagger k}a^\ell$. We also have the relation $[H_0,a^{\dagger k}a^\ell/(m(k-\ell))+\alpha_{k\ell}(N)]=a^{\dagger k}a^\ell$ where $\alpha_{k\ell}(N)$ is an arbitrary function of N, so that all R_n can be solved iteratively from (6). $H_N\equiv e^{-R}H_V e^R$ is thus in the form $H_N=m(N+\frac{1}{2})+\sum_{n=1}^{\infty} g^n f_n(N)$ as stated in Sec. I, where $f_n(N)$ is given by

[1] $\hbar=1$ is assumed.

[2] Generalization to multi-coupling Hamiltonians is straightforward.

$$f_n(N) = D\left(V_n + \sum_{j=2}^{n} \sum_{\substack{\{k_1,\ldots,k_j\} \\ k_1+\cdots+k_j=n}} \frac{[[H_0,R_{k_1}],\ldots,R_{k_j}]}{j!} + \sum_{\ell=1}^{n-1}\sum_{j=1}^{n-\ell} \sum_{\substack{\{k_1,\ldots,k_j\} \\ k_1+\cdots+k_j=n-\ell}} \frac{[[V_\ell,R_{k_1}],\ldots,R_{k_j}]}{j!}\right). \tag{7}$$

Given a Hermitian H_V with all $V_n(x,p)$ Hermitian, obviously R_1 is anti-Hermitian[3] by $[H_0,R_1] = D(V_1) - V_1$ which is the $n=1$ version of (6). Because the commutator of a Hermitian operator and an anti-Hermitian operator is Hermitian, the right-hand side of (6) is always Hermitian by induction, and thus all R_n are anti-Hermitian and e^{-R} is unitary. However, for $\mathcal{PT}$-symmetric H_V, $V_n(x,p)$ is not necessarily Hermitian, and R_n is not anti-Hermitian, resulting in a nonunitary e^{-R}. This is reasonable: if H_V is $\mathcal{PT}$-unbroken and has a real spectrum, H_N must be Hermitian[4] because it is diagonal and isospectral to H_V by the similarity transformation e^{-R}, and thus Hermitian H_N and non-Hermitian H_V cannot be related by a unitary transformation.

For a $\mathcal{PT}$-symmetric H_V, the Hermiticity of $H_N = e^{-R}H_V e^R$ results from $H_N = D(H_N)$ which is the definition (5) of H_N that means H_N is diagonal. A diagonal operator is not necessarily Hermitian even with entirely real eigenvalues. However, we work in Fock space where number states formed a set of complete and orthonormal basis in the ordinary Dirac inner product, and thus guaranteeing the Hermiticity of H_N. In other bases, concluding the Hermiticity of an operator $\mathcal{O}$ satisfying $\mathcal{O} = D(\mathcal{O})$ must be cautious.[5]

III. THE LOCAL POTENTIAL FROM A DIAGONAL HAMILTONIAN

The diagonalization of H_V makes use of the D-operation, and one may think that H_V can be recovered from the diagonal H_N by some D^{-1}-operation. However, the D-operation is not bijective as shown by (4) such that D^{-1} does not exist. The nonexistence of D^{-1} indicates that there are many different Hamiltonians, which is similar to the same diagonal H_N. As we are going to show, there exists a local Hermitian h_v similar to H_N serving as the Hermitian correspondences of H_V.

To invert the diagonalization procedure, we make use of the fact that $D(x^{2n})$ is a polynomial function of N written as

$$D(x^{2n}) = \sum_{k=0}^{n} X_{nk} N^k, \tag{8}$$

where $X_{nn} \neq 0$. Then we can define a linear operation $L()$ on any operator $\mathcal{O}$ as follows:

$$L(\mathcal{O}) = L(D(\mathcal{O})), \quad L(1) = 1,$$
$$L(N^n) = \frac{1}{X_{nn}}\left(x^{2n} - \sum_{k=0}^{n-1} X_{nk} L(N^k)\right) \quad (n \geq 1), \tag{9}$$

and $L(N^n)$ can be solved iteratively resulting in a $2n$th order polynomial function of x.

The requirement that $h_v \equiv e^{-K}H_N e^K$ is local is simply

$$e^{-K}H_N e^K - H_0 = L(e^{-K}H_N e^K - H_0). \tag{10}$$

Assume K has a perturbative expansion $K = \sum_{n=1}^{\infty} g^n K_n$. Taking out nth order terms on both sides of (10), we have

$$\begin{aligned}[H_0,K_n] = {} & L([H_0,K_n]) + L(f_n(N)) - f_n(N) \\ & + L\left(\sum_{j=2}^{n} \sum_{\substack{\{k_1,\ldots,k_j\} \\ k_1+\cdots+k_j=n}} \frac{[[H_0,K_{k_1}],\ldots,K_{k_j}]}{j!} + \sum_{\ell=1}^{n-1}\sum_{j=1}^{n-\ell} \sum_{\substack{\{k_1,\ldots,k_j\} \\ k_1+\cdots+k_j=n-\ell}} \frac{[[f_\ell(N),K_{k_1}],\ldots,K_{k_j}]}{j!}\right) \\ & - \left(\sum_{j=2}^{n} \sum_{\substack{\{k_1,\ldots,k_j\} \\ k_1+\cdots+k_j=n}} \frac{[[H_0,K_{k_1}],\ldots,K_{k_j}]}{j!} + \sum_{\ell=1}^{n-1}\sum_{j=1}^{n-\ell} \sum_{\substack{\{k_1,\ldots,k_j\} \\ k_1+\cdots+k_j=n-\ell}} \frac{[[f_\ell(N),K_{k_1}],\ldots,K_{k_j}]}{j!}\right). \end{aligned} \tag{11}$$

[3] R_1 can always be taken to be anti-Hermitian by choosing all $\alpha_{k\ell}(N) = 0$ in the analysis about the solution of (6) in the paragraph before (7). $D(V_1) - V_1$ is Hermitian, and any $a^{\dagger k}a^\ell$ must appear as the combination $a^{\dagger k}a^\ell + a^{\dagger \ell}a^k$, R_1 is thus anti-Hermitian by the relation $[H, a^{\dagger k}a^\ell/(m(k-\ell)) - a^{\dagger\ell}a^k/(m(k-\ell))] = a^{\dagger k}a^\ell + a^{\dagger\ell}a^k$.

[4] In this paper we deal with $\mathcal{PT}$-unbroken H_V only, and H_N is no longer Hermitian with a $\mathcal{PT}$-broken H_V. The existence of H_N is rather questionable with a $\mathcal{PT}$-broken H_V with an exceptional point.

[5] We thank the referee for this reminder.

Because $[H_0, K_n]$ has vanishing diagonal components, we have $L([H_0, K_n]) = 0$ by using $D([H_0, K_n]) = 0$ and (9). $[H_0, K_n]$ is thus determined completely by lower-order K_k's. From (8) and (9) it is obvious that $D(L(\mathcal{O})) = D(\mathcal{O})$, which is to say $L(\mathcal{O}) - \mathcal{O}$ has vanishing diagonal components, for any operator $\mathcal{O}$. K_n can thus be solved iteratively by the same reason of R_n's as in Sec. II. From (9), h_v is finally written in the form $h_v = \frac{1}{2}p^2 + \frac{1}{2}m^2x^2 + \sum_{n=1}^{\infty} g^n v_n(x)$ where $v_n(x)$ is given by

$$v_n(x) = L\left(f_n(N) + \sum_{j=2}^{n} \sum_{\substack{\{k_1,\ldots,k_j\} \\ k_1+\cdots+k_j=n}} \frac{[[H_0, K_{k_1}], \ldots, K_{k_j}]}{j!} + \sum_{\ell=1}^{n-1} \sum_{j=1}^{n-\ell} \sum_{\substack{\{k_1,\ldots,k_j\} \\ k_1+\cdots+k_j=n-\ell}} \frac{[[f_\ell(N), K_{k_1}], \ldots, K_{k_j}]}{j!}\right). \tag{12}$$

If H_N is Hermitian, all K_n are anti-Hermitian by the same arguments as those in Sec. II and thus e^{-K} is unitary resulting in a Hermitian h_v.

IV. ix^3 AS AN EXAMPLE

The ix^3 model is a popular toy model for studying $\mathcal{PT}$-symmetric theories [2,3,10,12]. However, a local form of the isospectral Hermitian Hamiltonian has not been given yet. Here we calculate the h_v for $H_V = \frac{1}{2}p^2 + \frac{1}{2}m^2x^2 + igx^3$ up to $\mathcal{O}(g^3)$ and show that h_v is indeed local. Higher-order calculation is systematic as shown by (6), (7), (9), (11), and (12) but rather tedious. Higher-order terms can be calculated whenever needed and will not be presented in this paper.

Various quantities entailed in the calculation of h_v for $H_V = \frac{1}{2}p^2 + \frac{1}{2}m^2x^2 + igx^3$ are as follows, up to $\mathcal{O}(g^3)$, and we take all homogeneous terms when solving for R_n from (6) to be zero:

$$\begin{aligned}
R_1 &= \frac{-i}{m(2m)^{3/2}}\left(\frac{a^{\dagger 3}}{3} + 3a^\dagger + 3a^{\dagger 2}a - 3a^\dagger a^2 - 3a - \frac{a^3}{3}\right), \\
R_2 &= \frac{1}{m(2m^4)}\left(\frac{3}{2}a^{\dagger 4} - 18a^{\dagger 2} - 12a^{\dagger 3}a + 12a^\dagger a^3 + 18a^2 - \frac{3}{2}a^4\right), \\
f_1(N) &= 0, f_2(N) = \frac{1}{8m^4}(30N^2 + 30N + 11), \\
L(N) &= mx^2 - \frac{1}{2}, \quad L(N^2) = \frac{2}{3}m^2x^4 - mx^2, \\
v_1(x) &= 0, \quad v_2(x) = \frac{5}{2m^2}x^4 - \frac{1}{2m^4}.
\end{aligned} \tag{13}$$

The expression for h_v is thus

$$h_v = \frac{1}{2}p^2 + \frac{1}{2}m^2x^2 + \frac{5g^2}{2m^2}x^4 - \frac{g^2}{2m^4} + \mathcal{O}(g^3). \tag{14}$$

A typical result of h using the conventional method proposed in [10] is [12]

$$h = \frac{1}{2}p^2 + \frac{1}{2}m^2x^2 + \frac{3g^2}{2m^4}\left(\{x^2, p^2\} + m^2x^2 + \frac{2}{3}\right) + \mathcal{O}(g^3), \tag{15}$$

where the appearance of $\{x^2, p^2\} = x^2p^2 + p^2x^2$ makes the physical interpretation of h rather complicated.

V. GENERALIZATION TO MULTIVARIABLE QUANTUM MECHANICS AND QUANTUM FIELD THEORIES

Consider a multivariable Hamiltonian with coupling constant g

$$H_V = \sum_i \left(\frac{1}{2}p_i^2 + \frac{1}{2}m_i^2x_i^2\right) + \sum_{n=1} g^n V_n(\{x_j\}, \{p_k\}). \tag{16}$$

If there is no degeneracy in the free part $H_0 = \sum_i (\frac{1}{2}p_i^2 + \frac{1}{2}m_i^2x_i^2)$, which is to say that all linear combinations of the integral multiple of m_i in the form $\sum_i n_i m_i (n_i \in \mathbb{Z}, \exists\, n_j \neq 0)$ is nonzero, there is no obstacle in calculating h_v from H_V. First, the D-operation is generalized trivially resulting in functions of $N_i = a_i^\dagger a_i$, and $D(\mathcal{O}) - \mathcal{O}$ is a linear combination of $\prod_{i,j} a_i^{\dagger n_i} a_j^{\ell_j}$ $(\sum_{i,j}(n_i m_i - \ell_j m_j) \neq 0)$ for any operator $\mathcal{O}$. Second,

R_n is guaranteed to have solutions by the explicit commutation relation $[H_0, (\prod_{i,j} a_i^{\dagger n_i} a_j^{\ell_j})/(\sum_{i,j}(n_i m_i - \ell_j m_j)) + \alpha_{\{n_i,\ell_j\}}(\{N_k\})] = \prod_{i,j} a_i^{\dagger n_i} a_j^{\ell_j}$. Next, the L-operation is also generalized trivially resulting in functions of x_i, and K_n is soluble the same as R_n. Finally, we get a local $h_v = \sum_i (\frac{1}{2} p_i^2 + \frac{1}{2} m_i^2 x_i^2) + \sum_{n=1} g^n v_n(x_i)$ as the isospectral Hermitian Hamiltonian of H_V.

However, if degeneracy does occur in H_0, $D(\mathcal{O}) - \mathcal{O}$ has terms in the form of $\prod_{i,j} a_i^{\dagger n_i} a_j^{\ell_j}$ where $\sum_{i,j}(n_i m_i - \ell_j m_j) = 0$. Consequently, R_n has no solution and the whole procedure breaks down.

A quantum field theory is multivariable, of course. However, Lorentz symmetry requires that all relativistic quantum field theories have the same spectra as free theories. Therefore, any perturbatively well-defined relativistic quantum field theory is equivalent to its corresponding free theory up to a similarity transformation that is constructed explicitly in textbooks such as [13]. A $\mathcal{PT}$-symmetric relativistic quantum field theory is thus isospectral to any local Hermitian theories having the same mass, and restricting conditions other than equivalent spectra are needed to isolate a meaningful one for a $\mathcal{PT}$-symmetric relativistic quantum field theory.

VI. SUMMARY AND OUTLOOK

In this paper we propose a new method to calculate isospectral Hermitian Hamiltonians of $\mathcal{PT}$-symmetric Hamiltonians, and local expressions are acquired for those whose free parts are nondegenerate. Moreover, a general antilinear symmetry rather than explicit $\mathcal{PT}$ symmetry is enough for our method, which is particularly helpful when dealing with higher-dimensional problems where there are many choices for an antilinear symmetry. A real spectrum accompanied by a set of complete eigenvectors, which is guaranteed by an unbroken antilinear symmetry [5–7], is the only thing needed[6] to result in a local and Hermitian Hamiltonian that is isospectral to the ordinary Hamiltonian. In summary, we diagonalize a quantum mechanical Hamiltonian and transform the diagonalized one into a Hermitian Hamiltonian with a local potential making use of a correspondence between nth order polynomials of N and x^2. However, this correspondence, which is denoted as the L-operation, is not unique. There are many polynomials that lead to the same result as x^{2n} under the D-operation because $D(x^{2k+1}) = 0$ is satisfied for any rational number k. Therefore, various definitions of $L(N^n)$ can differ by arbitrary functions of x^{2k+1}, thus resulting in different h_v's that differ from each other by arbitrary functions of x^{2k+1}, too. This nonuniqueness reflects spectral equivalence of different potentials and disappears once we specify the parity property of h_v. Furthermore, our method is only of formal sense, because perturbation theories in infinite-dimensional space are rather tricky.[7]

We have only dealt with $\mathcal{PT}$-unbroken theories in this paper. The Hermiticity of h_v is guaranteed by the Hermiticity of H_N, and the Hermiticity of H_N is guaranteed by the unbroken $\mathcal{PT}$-symmetry of H_V as explained at the ends of Secs. II and III. We do not expect a Hermitian h_v for a $\mathcal{PT}$-broken H_V, because a $\mathcal{PT}$-broken H_V has a complex spectrum and is not even diagonalizable in the exceptional point [3]. As the conventional method is also not valid for $\mathcal{PT}$-broken theories, we left discussions of $\mathcal{PT}$-broken theories for future work.

Our method is also incapable of dealing with theories degenerate in their free parts as discussed in Sec. V. However, the conventional method is also invalid in this case. For example, consider a $\mathcal{PT}$-symmetric Hamiltonian $H = \frac{1}{2} p_1^2 + \frac{1}{2} p_2^2 + \frac{1}{2} m^2 x_1^2 + \frac{1}{2}(2m)^2 x_2^2 + i g x_1^2 x_2$, and the first-order equation needed to calculate the metric operator $\exp(\sum_{n=1}^{\infty} g^{2n+1} Q_{2n+1})$ is $[H_0, Q_1] = -2i x_1^2 x_2$, which has no solution because $\langle 2,0|[H_0, Q_1]|0,1\rangle = 0$ is not consistent with $-2i\langle 2,0|x_1^2 x_2|0,1\rangle = -2i/(2m)^{3/2}$ where $|2,0\rangle$ and $|0,1\rangle$ are bases in the Fock space of $H_0 = \frac{1}{2} p_1^2 + \frac{1}{2} p_2^2 + \frac{1}{2} m^2 x_1^2 + \frac{1}{2}(2m)^2 x_2^2$. We hope more powerful methods can be developed to handle degeneracy problems.

Although degeneracy also occurs in quantum field theories, Lorentz symmetry makes all quantum field theories with the same physical mass equivalent to each other. While the conventional method picks up a Hermitian h for a $\mathcal{PT}$-symmetric Hamiltonian H by its explicit calculation procedure, we point out that there is no special choice of h if we consider only the spectrum of a $\mathcal{PT}$-symmetric Hamiltonian H, and further constraints must be added to select a meaningful Hermitian h. We hope to extract more physical information from $\mathcal{PT}$-symmetric quantum field theories, thus being able to construct a special Hermitian Hamiltonian h for a $\mathcal{PT}$-symmetric Hamiltonian H, which carries the same physical information as H.

[6]We thank the referee for the observation of generality of our method.

[7]We thank the referee for pointing this out.

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