Boundary to bound dictionary for generic Kerr orbits

Riccardo Gonzo[*]

Higgs Centre for Theoretical Physics, School of Physics and Astronomy, University of Edinburgh,
Edinburgh EH9 3FD, United Kingdom

Canxin Shi[†]

CAS Key Laboratory of Theoretical Physics, Institute of Theoretical Physics, Chinese Academy of Sciences,
Beijing 100190, China

We establish a new relation between classical observables for scattering and bound orbits of a massive probe particle in a Kerr background. We find an exact representation of the Hamilton-Jacobi action in terms of the conserved charges that admits an analytic continuation, for both the radial and polar contributions, for a general class of geodesics beyond the equatorial case. Remarkably, this allows to extend the boundary to a bound dictionary and provides an efficient method to compute the deflection angles and time delay for scattering orbits, as well as frequency ratios for bound orbits, in the probe limit but at all orders in the perturbative expansion.

I. INTRODUCTION

The existence of gravitational waves was predicted by Einstein's theory of general relativity in 1916, but it took until 2015 for the Laser Interferometer Gravitational-Wave Observatory (LIGO) to detect the first direct evidence of these elusive waves [1]. Since then, LIGO and other gravitational-wave observatories around the world have detected numerous events, opening a new way to study the Universe and test fundamental physics.

To accurately predict the properties of gravitational waves, a theoretical framework is required. One such framework is the post-Minkowskian (PM) expansion, which is a perturbative expansion in powers of the Newton constant G_N. Recent developments in the field of scattering amplitudes [2–19] have pushed our understanding of the PM expansion for the classical two-body problem for spinless [20–31] and spinning bodies [32–50] up to high order for the conservative dynamics. The probe-limit scenario is particularly relevant [51–61], since it provides a concrete example of an exact resummation that makes contact with the self-force expansion [62,63]. For most of the cases, the Hamiltonian extracted from amplitudes can be

directly fed into the effective-one-body machinery [64–68] in order to generate gravitational-wave templates for bound systems [69–71].

Since the classical dynamics is completely captured by differential equations, only the boundary conditions provide the physical distinction between scattering and bound orbits. Building on such intuition, recently Kälin and Porto [72,73] found a way to analytically continue scattering observables like the deflection angle into bound observables like the periastron advance.[1] This "boundary-to-bound" dictionary has been developed for two-body systems of spinless and aligned-spin particles, whose dynamics remain on the equatorial plane at all times. Recently, this was partially extended to radiative observables [31,37,75,76]. In the conservative case, one of the key insights in establishing such correspondence is given by the Hamilton-Jacobi (HJ) action [64,66,77,78], which is related to the solution of the Bethe-Salpeter equation for classical bound states via the "amplitude-action" relation [27,58,78–81].

Interestingly, the Hamilton-Jacobi action can also be used to describe massive probe particles moving in a Kerr metric beyond the equatorial case [82]. This raises the question of whether the boundary-to-bound dictionary can be extended to generic orbits. In this paper, we provide an affirmative answer to this question. We show that there is a natural class of geodesics in a Kerr background that smoothly connect the scattering and the bound dynamics (see Fig. 1), for which an analytic continuation is possible by also taking

---

[*]rgonzo@ed.ac.uk
[†]shicanxin@itp.ac.cn

[1]See also Damour and Deruelle [74] for an interesting earlier proposal.
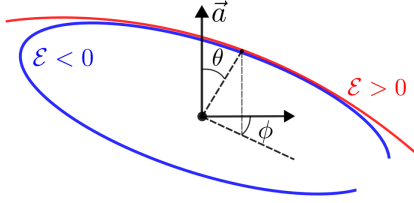

FIG. 1. We consider a class of orbits in a Kerr black hole that smoothly interpolates between scattering ($\mathcal{E} > 0$, in red) and bound ($\mathcal{E} < 0$, in blue) dynamics.

into account the Carter constant Q beyond the energy E and the projection of the angular momentum on the spin axis L. We then derive the scattering angles $(\Delta\phi, \Delta\theta)$ and the time delay ΔT for time-like and null-like geodesics, respectively. Finally, using the new dictionary, we compute the precession of the periastron $K^{\phi r}$ and of the orbital plane $K^{\theta r}$, which are naturally expressed in terms of the fundamental frequencies $(\omega_r, \omega_\phi, \omega_\theta)$ of the motion [83,84].

We use the mostly plus signature convention $(- + ++)$ for the metric and we set $c = 1$.

II. HAMILTON-JACOBI ACTION FOR GENERIC KERR ORBITS

The Kerr metric describes the spacetime of a spinning black hole of mass M and spin $J = Ma$. This can be written in Boyer-Lindquist coordinates (t, r, θ, ϕ) as

$$ds^2 = -\frac{\Delta}{\Sigma}(dt - a\sin^2(\theta)d\phi)^2 + \frac{\Sigma}{\Delta}dr^2 + \Sigma d\theta^2$$
$$+ \frac{\sin^2(\theta)}{\Sigma}[(r^2 + a^2)d\phi - adt]^2,$$
$$\Delta(r) = r^2 - 2Mr + a^2, \qquad \Sigma(r,\theta) = r^2 + a^2\cos^2(\theta), \quad (1)$$

where we have set Newton's constant to unity, $G = 1$, and we have chosen the reference axis to be aligned with the spin direction. The relativistic Hamiltonian for the geodesic motion of a probe particle of mass m and 4-momentum p_μ in this metric is $H(x, p) = 1/2g^{\mu\nu}p_\mu p_\nu$, which guarantees the validity of the geodesic equations

$$p^\mu \nabla_\mu p^\nu = 0, \qquad g^{\mu\nu}p_\mu p_\nu = -m^2. \quad (2)$$

A complete set of constants of motion can be determined for the Kerr metric, as first shown by Carter [82]. First of all, the metric (1) admits two Killing vectors ∂_t and ∂_ϕ as a consequence of time-translation and axial symmetries. Therefore, the total energy E and the angular momentum parallel to the spin axis L as seen by an observer at spatial infinity are conserved,

$$E := -p_\mu \partial_t^\mu = -p_t, \qquad L := p_\mu \partial_\phi^\mu = p_\phi. \quad (3)$$

In addition to the isometries, the Kerr metric also admits an irreducible symmetric Killing tensor $K_{\mu\nu}$, which implies the existence of a new conserved charge Q called the Carter integral,[2]

$$Q = K^{\mu\nu}p_\mu p_\nu - (L - aE)^2$$
$$= p_\theta^2 + a^2(m^2 - p_t^2)\cos^2(\theta) + p_\phi^2\cot^2(\theta). \quad (4)$$

Using a Euclidean flat-space $3d$ notation [85,86], we can write $\vec{r} = r(\sin(\theta)\cos(\phi), \sin(\theta)\sin(\phi), \cos(\theta))$ and $\vec{a} = (0, 0, a)$ so that we can suggestively recast (4) as

$$Q = |\vec{r} \wedge \vec{p}|^2 - (\vec{r} \wedge \vec{p} \cdot \hat{a})^2 - |\vec{p}|^2(a \cdot \hat{r})^2$$
$$= |\vec{L}|^2 - L^2 - |\vec{p}|^2(a \cdot \hat{r})^2, \quad (5)$$

i.e., this is a measure of the motion of the particle off the equatorial plane given by the generalization of the equatorial projection of the orbital angular momentum $|\vec{L}|^2 - L^2$ for a spinning source [87].[3] For the scattering case, the relation between the conserved charges and the incoming kinematics is summarized in Appendix A.

The instantaneous 4-momentum $\mathcal{P} = p_\mu dx^\mu$ of the probe particle can now be expressed [88] in terms of the four constants of motion (m^2, E, L, Q) by directly inverting the Eqs. (2)–(4):

$$\mathcal{P}(x) := -Edt \pm_r \frac{\sqrt{R(r)}}{\Delta(r)}dr \pm_\theta \sqrt{\Theta(\theta)}d\theta + Ld\phi, \quad (6)$$

where we have defined the polar and radial potentials

$$\Theta(\theta) := Q + a^2(E^2 - m^2)\cos^2(\theta) - L^2\cot^2(\theta),$$
$$R(r) := [E(r^2 + a^2) - aL]^2 - \Delta(Q + (L - aE)^2 + m^2r^2), \quad (7)$$

and the signs $\pm_r$ and $\pm_\theta$ depend on the radial and polar direction of the motion, respectively. For convenience, we choose $\pm_r = \pm_\theta = +$. The canonical 1-form (6) provides the transformation to the principal function S, in terms of which the HJ action I is defined as

$$I := S + Et - L\phi = I_r + I_\theta,$$
$$I_r = \frac{1}{2\pi}\int_{\mathcal{C}_r} p_r dr, \qquad I_\theta = \frac{1}{2\pi}\int_{\mathcal{C}_\theta} p_\theta d\theta, \quad (8)$$

where the paths $\mathcal{C}_r$ and $\mathcal{C}_\theta$ correspond to the physical trajectories for the radial and polar motion. Since the

---

[2]The (positive definite) Carter constant is $k = K^{\mu\nu}p_\mu p_\nu$, but Q is more convenient for our purposes.

[3]For $a \to 0$, we recover $Q \overset{a\to 0}{\to} L_x^2 + L_y^2$ while $L \overset{a\to 0}{\to} L_z$.

TABLE I. The table shows the specific class of unbound orbits ($E > m$) with a single turning point r_m (denoted by •) that are smoothly connected with bound orbits ($E < m$) with two turning points r_- (•), r_+ (•̄).

Type	Energy range	Root structure	Radial range
Unbound	$E > m$	$\|+•-•+\rangle$	$r_m \leq r < \infty$
Bound	$E_{\text{ISCO}^+} < E < m$	$\|+•-•+•̄-\rangle$	$r_- \leq r \leq r_+$

TABLE II. A qualitative classification of the polar motion.

Type	Polar range	Conditions
Ordinary	$\theta_- < \pi/2 < \theta_+, \ \theta_- = \pi - \theta_+$	$Q > 0, \mathcal{E} \lessgtr 0$
Vortical	$\theta_- < \theta_+ < \pi/2$ or $\pi/2 < \theta_- < \theta_+$	$Q \leq 0, \mathcal{E} > 0$

dynamics in a Kerr spacetime is separable in Boyer-Lindquist coordinates, the contours $\mathcal{C}_k$ can be localized within the (x^k, p_k) plane on the cotangent bundle.

A. Boundary-to-bound dictionary for generic orbits

We are interested in a class of generic orbits that smoothly connects the scattering and the bound regime. Generic geodesics are such that both end points are either a simple root of the radial potential $R(r)$, the horizon, or infinity. The classification of time-like and null-like orbits in terms of the radial root structure was recently completed in [89,90], respectively. We employ the conventions introduced in [89] for the radial roots, which we review here. We use the symbols $|, +, -$, and $\rangle$ to label, respectively, the Kerr outer horizon, a region where motion is allowed ($R > 0$), a region where motion is disallowed ($R < 0$), and radial infinity, and the • to denote a single root. The radial root structure of the class of geodesics we are interested in is discussed in Table I.[4]

At this point, we can define the cycle of integration for the Hamilton-Jacobi action for unbound and bound geodesics. We introduce the superscript $>$ to denote an expression that is valid for scattering orbits and $<$ to denote an expression that is valid for bound ones. For the radial motion, making manifest the dependence of the radial roots on the conserved charges, the radial integral becomes

$$\int_{\mathcal{C}_r^>} = 2\int_{r_m(\mathcal{E},l,a,l_Q)}^\infty, \qquad \int_{\mathcal{C}_r^<} = 2\int_{r_-(\mathcal{E},l,a,l_Q)}^{r_+(\mathcal{E},l,a,l_Q)}, \quad (9)$$

where we have defined the conserved quantities per unit mass,

$$\mathcal{E} := \frac{E^2 - m^2}{m^2}, \qquad l := \frac{L}{m}, \qquad l_Q := \frac{\sqrt{Q + L^2}}{m}. \quad (10)$$

A direct inspection of the analytic structure of the radial roots shows that we can generalize the boundary-to-bound dictionary from equatorial orbits [72,73] to generic orbits because of the remarkable map

$$r_-(\mathcal{E}, l, a, l_Q) \overset{\mathcal{E} \leq 0}{=} r_m(\mathcal{E}, l, a, l_Q),$$
$$r_+(\mathcal{E}, l, a, l_Q) \overset{\mathcal{E} \leq 0}{=} r_m(\mathcal{E}, -l, -a, -l_Q), \quad (11)$$

where l_Q plays the role of an angular momentum.

For the polar motion, the condition $\Theta \geq 0$ implies that a generic geodesic with $l \neq 0$ is bounded between two turning points $\theta_- < \theta < \theta_+$, which are the solutions of the equation $\Theta = 0$.[5] The polar motion can be of ordinary or vortical type according to the value of $\theta_\pm$, as shown in Table II.

Since we are interested in a class of geodesics that connect the unbound ($\mathcal{E} > 0$) and bound ($\mathcal{E} < 0$) regimes, we are forced to restrict to the case of the oscillatory polar motion with $Q > 0$. After excluding the degenerate case of planar geodesics at fixed polar angle $\theta = \theta_\pm$, the angular integral for the generic configuration reads [88]

$$\int_{\mathcal{C}_\theta} = 2n\left|\int_{\pi/2}^{\theta_\pm}\right| + \eta_{\text{in}}\left|\int_{\pi/2}^{\theta_{\text{in}}}\right| - \eta_{\text{out}}\left|\int_{\pi/2}^{\theta_{\text{out}}}\right|, \quad (12)$$

where θ_{in} (θ_{out}) is the initial (final) polar angle of the trajectory, n is the number of turning points of the polar motion, and we have defined the signs

$$\eta_{\text{in/out}} = \text{sign}(p^\theta_{\text{in/out}})\text{sign}(\cos(\theta_{\text{in/out}})). \quad (13)$$

We now consider the class of geodesics that start on the equatorial plane with $\theta_{\text{in}} = \pi/2$, which is a convenient simplification of our problem and does not affect the validity of the analytic continuation. With this choice, the physical observables we compute depend only on the conserved charges. Therefore, we can effectively use

$$\int_{\mathcal{C}_\theta} \to 2n\left|\int_{\pi/2}^{\theta_\pm}\right| - \eta_{\text{out}}\left|\int_{\pi/2}^{\theta_{\text{out}}}\right|, \quad (14)$$

where θ_{out} is determined explicitly in terms of the conserved charges, as we discuss later.

We are now ready to compute the Hamilton-Jacobi action for our class of geodesics. We start with the radial action, which we can write for scattering orbits as

---

[4]E_{ISCO^+} stands for the energy of the prograde innermost stable circular orbit (ISCO). This will not be of further concern for our work, so we refer to [89] for details.

[5]We exclude the special case $l = 0$ where the north ($\theta = \pi$) or the south ($\theta = 0$) pole can be reached.

$$
\begin{aligned}
I_r^> &:= \frac{1}{2\pi}\int_{\mathcal{C}_r^>} p_r \mathrm{d}r \overset{u=1/r}{=} \frac{1}{\pi}\int_0^{u_m}\frac{\mathrm{d}u}{u^2}\frac{\sqrt{R(u)}}{\Delta(u)} \\
&= \frac{1}{2\pi}\frac{\sqrt{\mathcal{E}}}{\sqrt{M^2-a^2}}\int_0^{u_m}\frac{\mathrm{d}u}{u^2}\prod_{j=1}^4\left(1-\frac{u}{u_j}\right)^{\frac{1}{2}} \\
&\quad \times \left(\frac{1}{u_B-u}-\frac{1}{u_A-u}\right),
\end{aligned}
\tag{15}
$$

where we have defined the radial roots $\{u_j\}_{j=1,\dots,4}$ and

$$
R(u) = -\frac{a^2 Q}{u^4}\prod_{j=1}^4(u-u_j),
$$

$$
u_A = \frac{M+\sqrt{M^2-a^2}}{a^2}, \qquad u_B = \frac{M-\sqrt{M^2-a^2}}{a^2}.
\tag{16}
$$

Having selected the radial root corresponding to the minimum distance according to the pattern identified in Table I, say, $u_4 = u_m$, we can then change variables to $u = u_m\tilde{u}$ so that the radial action reads

$$
\begin{aligned}
I_r^{>,\epsilon} &= \frac{1}{2\pi}\frac{m\sqrt{\mathcal{E}}}{u_m^{1-\epsilon}\sqrt{M^2-a^2}}\int_0^1\frac{\mathrm{d}\tilde{u}}{\tilde{u}^{2-\epsilon}} \\
&\quad \times \prod_{j=1}^4\left(1-\frac{u_m}{u_j}\tilde{u}\right)^{\frac{1}{2}}\left[\frac{1}{u_B-u_m\tilde{u}}-\frac{1}{u_A-u_m\tilde{u}}\right],
\end{aligned}
\tag{17}
$$

where we have introduced an infrared regulator $\epsilon > 0$ to make the integral well defined [4,64]. We can then provide a closed-form expression for the radial action in terms of the Lauricella hypergeometric functions $F_D^{(n)}$,

$$
\begin{aligned}
I_r^{>,\epsilon} &= \frac{1}{2\pi}\frac{m\sqrt{\mathcal{E}}}{u_m^{1-\epsilon}\sqrt{M^2-a^2}}\frac{\Gamma(-1+\epsilon)\Gamma(3/2)}{\Gamma(1/2+\epsilon)} \\
&\quad \times \left[\frac{1}{u_B}F_D^{(4)}\left(\alpha_r,\vec{\beta}_r,\gamma_r;\frac{u_m}{u_B},\frac{u_m}{u_1},\frac{u_m}{u_2},\frac{u_m}{u_3}\right) - (u_B\leftrightarrow u_A)\right] \\
\alpha_r &= -1+\epsilon, \qquad \vec{\beta}_r = \left\{1,-\frac{1}{2},-\frac{1}{2},-\frac{1}{2}\right\}, \\
\gamma_r &= \frac{1}{2}+\epsilon.
\end{aligned}
\tag{18}
$$

For bound orbits, we can use the relation (11) to write the radial contour as

$$
\begin{aligned}
\int_{\mathcal{C}_r^<}\overset{\mathcal{E}<0}{=}&\int_{r_-(\mathcal{E},l,a,l_Q)}^\infty - \int_{r_+(\mathcal{E},l,a,l_Q)}^\infty \\
\overset{\mathcal{E}\le0}{=}&\int_{r_m(\mathcal{E},l,a,l_Q)}^\infty - \int_{r_m(\mathcal{E},-l,-a,-l_Q)}^\infty .
\end{aligned}
\tag{19}
$$

Since p_r is invariant under $(a,l,l_Q)\to(-a,-l,-l_Q)$, we can establish the analytic continuation

$$
I_r^{<,\epsilon}(\mathcal{E},l,a,l_Q)\overset{\mathcal{E}\le0}{=}I_r^{>,\epsilon}(\mathcal{E},l,a,l_Q) - I_r^{>,\epsilon}(\mathcal{E},-l,-a,-l_Q).
\tag{20}
$$

At this point, we focus on the polar action and compute separately the contribution of both terms in (14). Once we choose the initial condition $\mathrm{sign}(p_{\mathrm{in}}^\theta) = 1$, the equations of motion impose (see Appendix B)

$$
\eta_{\mathrm{out}}^> = -1, \qquad n^> = 1, \qquad \eta_{\mathrm{out}}^< = +1, \qquad n^< = 2
\tag{21}
$$

for scattering and bound orbits, respectively. The first contribution for scattering orbits reads

$$
\begin{aligned}
I_\theta^{>(1)} &:= \frac{1}{2\pi}\int_{\mathcal{C}_\theta^>} p_\theta \mathrm{d}\theta\bigg|_1 = \frac{n^>}{\pi}\int_{\pi/2}^{\theta_+}\mathrm{d}\theta\sqrt{\Theta(\theta)} \\
&\overset{\cos^2(\theta)=U_+\tilde{U}}{=}\frac{\sqrt{QU_+}}{2\pi}\int_0^1\frac{\mathrm{d}\tilde{U}(1-\tilde{U})^{\frac{1}{2}}}{\tilde{U}^{\frac{1}{2}}(1-U_+\tilde{U})}\left(1-\frac{U_+}{U_-}\tilde{U}\right)^{\frac{1}{2}},
\end{aligned}
\tag{22}
$$

where the roots $U_\pm$ of the polar potential are[6]

$$
\begin{aligned}
U_\pm &= \frac{\Delta U\pm\sqrt{\Delta U^2+a^2\mathcal{E}(l_Q^2-l^2)}}{a^2\mathcal{E}}, \\
\Delta U &= \frac{a^2\mathcal{E}-l_Q^2}{2},
\end{aligned}
\tag{23}
$$

and it is possible to show that $\theta_\mp = \arccos(\pm\sqrt{U_+})$. Therefore, the first term can be written in terms of the Lauricella hypergeometric function $F_D^{(q)}$,[7]

$$
\begin{aligned}
I_\theta^{>(1)} &= \frac{1}{4}\sqrt{QU_+}F_D^{(2)}\left(\alpha_\theta^{(1)},\vec{\beta}_\theta^{(1)},\gamma_\theta^{(1)};U_+,\frac{U_+}{U_-}\right), \\
\alpha_\theta^{(1)} &= \frac{1}{2}, \qquad \vec{\beta}_\theta^{(1)} = \left\{1,-\frac{1}{2}\right\}, \qquad \gamma_\theta^{(1)} = 2.
\end{aligned}
\tag{24}
$$

Therefore, the second term in (14) can be written as

---

[6]The definition of $U_\pm$ here is slightly different than the conventional one [88], and it allows for a smooth analytic continuation for $\mathcal{E} < 0$.

[7]In the case for $q = 2$, Lauricella function $F_D^{(q)}$ reduces to Appell's F_1 hypergeometric series.

$$I_\theta^{>(2)} = \frac{1}{2\pi} \int_{C_\theta^>} p_\theta d\theta \Big|_2 = -\frac{\eta_{\text{out}}^>}{2\pi} \int_{\pi/2}^{\theta_{\text{out}}^>} d\theta \sqrt{\Theta(\theta)}$$

$$= \frac{1}{2\pi} \sqrt{Q U_{\text{out}}^>} F_D^{(3)} \left(\alpha_\theta^{(2)}, \vec{\beta}_\theta^{(2)}, \gamma_\theta^{(2)}; U_{\text{out}}^>, \frac{U_{\text{out}}^>}{U_+}, \frac{U_{\text{out}}^>}{U_-} \right),$$

$$\alpha_\theta^{(2)} = \frac{1}{2}, \qquad \vec{\beta}_\theta^{(2)} = \left\{ 1, -\frac{1}{2}, -\frac{1}{2} \right\}, \qquad \gamma_\theta^{(2)} = \frac{3}{2}, \qquad (25)$$

where the $U_{\text{out}}^>$ is determined from (see [88])

$$U_{\text{out}}^> = U_+ sn^2 \left(X_0^> \Big| \frac{U_-}{U_+} \right),$$

$$X_0^> = -\sqrt{-a^2 m^2 \mathcal{E} U_-} \int_{C_r^>} \frac{dr}{\sqrt{\mathcal{R}(r)}}. \qquad (26)$$

For bound orbits, the turning points of the polar potential are still $\theta_\mp = \arccos(\pm\sqrt{U_+})$, so we only need to perform the conventional analytical continuation of the radial contour $C_r^> \to C_r^<$ for U_{out} in (26). The polar action for $\mathcal{E} < 0$ will therefore be

$$I_\theta^<(\mathcal{E}, l, a, l_Q; n^>, \eta_{\text{out}}^>) \overset{\mathcal{E}<0}{=} I_\theta^>(\mathcal{E}, l, a, l_Q; n^<, \eta_{\text{out}}^<), \qquad (27)$$

where we emphasize again that $n^> = 1 \to n^< = 2$ and $\eta_{\text{out}}^> = -1 \to \eta_{\text{out}}^< = +1$ as a consequence of the analytic continuation of the equations of motion.

III. PERTURBATIVE EXPANSION OF SCATTERING OBSERVABLES

In this section we derive the PM expansion of the scattering angles $(\Delta\theta, \Delta\phi)$ and the time delay ΔT by using the equations of motion coming from the HJ action (8). Since our main focus is on the weak-field limit, we present our results as a double expansion in M and a, with $a \ll M$. We refer the reader to Appendix C for the exact expression of scattering angles in terms of hypergeometric functions.

A. Polar deflection angle $\Delta\theta$

The polar deflection angle $\Delta\theta = \theta_{\text{out}} - (\pi - \theta_{\text{in}})$ is completely determined by $U_{\text{out}} = \cos^2(\theta_{\text{out}})$, given that we set $\theta_{\text{in}} = \pi/2$. It is straightforward to extend all of our calculations to a generic incoming angle θ_{in}, for example, by using the generic polar contour (12) in the $r - \theta$ Eq. (B1) of Appendix B. A direct perturbative expansion of $\Delta\theta$ from (26) gives, up to $\mathcal{O}(M^3 a^2)$,

$$\frac{\Delta\theta}{\sqrt{l_Q^2 - l^2}} = -\frac{2M(2\mathcal{E}+1)}{\sqrt{\mathcal{E}} l_Q^2} - \frac{3\pi M^2(5\mathcal{E}+4)}{4 l_Q^3} - \frac{2M^3(64\mathcal{E}^3 + 72\mathcal{E}^2 + 12\mathcal{E} - 1)}{3\mathcal{E}^{3/2} l_Q^4} + \frac{4M^3 l^2(2\mathcal{E}+1)^3}{3\mathcal{E}^{3/2} l_Q^6}$$

$$+ a\sqrt{\mathcal{E}+1} \left(\frac{8Ml\sqrt{\mathcal{E}}}{l_Q^4} + \frac{3\pi M^2 l(5\mathcal{E}+2)}{l_Q^5} - \frac{16M^3 l^3(2\mathcal{E}+1)^2}{\sqrt{\mathcal{E}} l_Q^8} + \frac{16M^3 l(16\mathcal{E}^2 + 12\mathcal{E} + 1)}{\sqrt{\mathcal{E}} l_Q^6} \right)$$

$$+ a^2 \left(\frac{2\sqrt{\mathcal{E}} M(l_Q^2 - 4l^2)(2\mathcal{E}+1)}{l_Q^6} + \frac{3\pi M^2(l_Q^2 - 5l^2)(95\mathcal{E}^2 + 88\mathcal{E} + 8)}{32 l_Q^7} \right.$$

$$\left. + \frac{16M^3 l^4(2\mathcal{E}+1)(8\mathcal{E}^2 + 8\mathcal{E} + 1)}{\sqrt{\mathcal{E}} l_Q^{10}} - \frac{12M^3 l^2(88\mathcal{E}^3 + 116\mathcal{E}^2 + 34\mathcal{E} + 1)}{\sqrt{\mathcal{E}} l_Q^8} + \frac{8M^3 \sqrt{\mathcal{E}}(20\mathcal{E}^2 + 26\mathcal{E} + 7)}{l_Q^6} \right). \qquad (28)$$

B. Azimuthal deflection angle $\Delta\phi$

The azimuthal scattering angle $\Delta\phi$ is the conjugate variable to the angular momentum L, i.e.,

$$\frac{\Delta\phi + \pi}{2\pi} = -\frac{\partial I}{\partial L} = -\frac{\partial I_r}{\partial L} - \frac{\partial I_\theta}{\partial L}, \qquad (29)$$

which can be computed from the HJ action in (18), (22), and (24). It is worth stressing that we need to keep $U_{\text{out}}^>$ invariant in taking the derivative over L, since technically it is only fixed dynamically by the equations of motion, whereas the HJ action works at the off-shell level.[8] A direct calculation up to order $\mathcal{O}(M^3 a^2)$ in the PM expansion gives

---

[8]One can also derive $\Delta\phi$ from Hamilton's principal function, which should be understood as a type-2 generating function for a canonical transformation.

$$\Delta\phi = \frac{2Ml(2\mathcal{E}+1)}{\sqrt{\mathcal{E}}l_Q^2} + \frac{3\pi M^2 l(5\mathcal{E}+4)}{4l_Q^3} + \frac{2M^3 l(64\mathcal{E}^3+72\mathcal{E}^2+12\mathcal{E}-1)}{3\mathcal{E}^{3/2}l_Q^4} + a\sqrt{\mathcal{E}+1}\left(\frac{4M\sqrt{\mathcal{E}}(l_Q^2-2l^2)}{l_Q^4}\right.$$

$$+ \frac{\pi M^2(5\mathcal{E}+2)(l_Q^2-3l^2)}{l_Q^5} + \frac{4M^3(16\mathcal{E}^2+12\mathcal{E}+1)(l_Q^2-4l^2)}{\sqrt{\mathcal{E}}l_Q^6}\right) + a^2\left(\frac{2M\sqrt{\mathcal{E}}(2\mathcal{E}+1)(4l^2-3l_Q^2)}{l_Q^6}\right.$$

$$+ \frac{3\pi M^2 l(95\mathcal{E}^2+88\mathcal{E}+8)(5l^2-3l_Q^2)}{32l_Q^7} + \frac{4M^3 l(128\mathcal{E}^3+168\mathcal{E}^2+48\mathcal{E}+1)(2l^2-l_Q^2)}{\sqrt{\mathcal{E}}l_Q^8}\right)$$

$$+ M^3(l_Q^2-l^2)\left[\frac{8l(2\mathcal{E}+1)^3}{3\mathcal{E}^{3/2}l_Q^6} - \frac{32al^2\sqrt{\mathcal{E}+1}(2\mathcal{E}+1)^2}{\sqrt{\mathcal{E}}l_Q^8} + \frac{8a^2 l(2\mathcal{E}+1)(4l^2(8\mathcal{E}^2+8\mathcal{E}+1)-(2\mathcal{E}+1)^2 l_Q^2)}{\sqrt{\mathcal{E}}l_Q^{10}}\right], \quad (30)$$

where with the last square bracket we have isolated the contributions from the polar action, which are proportional to $Q = m^2(l_Q^2 - l^2)$. It is possible to notice a simple relation between the angles $\Delta\phi$ and $\Delta\theta$ as $a \to 0$, i.e., $\Delta\phi^{a\to 0} \sim -(l/\sqrt{l_Q^2-l^2})\Delta\theta$ at the lowest order, which is determined by the fact that the motion happens on an inclined plane and there is always a change of coordinates to bring it to the standard equatorial plane. Moreover, in the limit $l_Q \to l$, we recover the well-known equatorial expression, as expected [32,91].

C. Time delay

The time delay is related to the conjugate variable of the energy E in the HJ action, but it is defined only when we compare the measure to an observer at large distances [56,92–94]. Having defined the impact parameter b for generic null geodesics

$$r_m \overset{M\to 0}{=} b = |\vec{b}| \to b = \frac{\sqrt{L^2+Q}}{E}, \quad (31)$$

and the effective inclination angle [95]

$$\cos(\iota) = \frac{L}{\sqrt{L^2+Q}}, \quad (32)$$

we can then compute the time delay ΔT for generic null geodesics with fixed b relative to an observer with $b' \gg b$ but at the same energy $E' = E$,

$$\Delta T = \frac{\partial I}{\partial E}\Big|_{b,E} - \frac{\partial I}{\partial E}\Big|_{b'\gg b, E'=E}$$

$$= 4M \log\left(\frac{b'}{b}\right) + \frac{15\pi M^2}{2b} + \frac{64M^3}{b^2}$$

$$- \frac{a\cos(\iota)}{b}\left(8M + \frac{15\pi M^2}{b} + \frac{256M^3}{b^2}\right)$$

$$+ \frac{a^2}{b^2}\left(6M\cos(2\iota) + \frac{95\pi M^2}{16b}(1+3\cos(2\iota))\right.$$

$$+ \frac{32M^3}{b^2}(7+13\cos(2\iota))\right), \quad (33)$$

which is accurate up to order $\mathcal{O}(M^3 a^2)$. As expected, ΔT is positive because of causality arguments [92] as long as we impose the physical condition $a \leq M$.

IV. PERTURBATIVE EXPANSION OF BOUND OBSERVABLES

Using the boundary-to-bound dictionary for the Hamilton-Jacobi action developed in (20) and (27), we now proceed to compute the perturbative expansion of bound observables for generic bound orbits that are connected to scattering ones via analytic continuation. We use the same conventions as in Sec. III.

A. Fundamental frequencies $\omega_r, \omega_\phi, \omega_\theta$

The basic properties of Kerr bound orbits are specified by the so-called fundamental frequencies [83,96,97]. Although they are coordinate independent, it is useful to describe them via the conjugate momenta of the action-angle variables in the Boyer-Lindquist representation,

$$J_t := E, \quad J_r := \frac{1}{2\pi}\oint p_r dr = I_r^<,$$

$$J_\theta := \frac{1}{2\pi}\oint p_\theta d\theta = I_\theta^{<(1)}, \quad J_\phi := \frac{1}{2\pi}\oint p_\phi d\phi = L, \quad (34)$$

Since the coordinates are integrated out, the action momenta are constants of the motion $J_\beta(H, E, L, Q)$, where $H = -m^2/2$ is the Hamiltonian for the action-angle variables. The fundamental frequencies are defined as

$$\omega_r = \frac{\partial H}{\partial J_r}, \qquad \omega_\theta = \frac{\partial H}{\partial J_\theta}, \qquad \omega_\phi = \frac{\partial H}{\partial J_\phi}. \quad (35)$$

Note that the partial derivatives are taken with J_β being invariant. Using the results in Appendix D, we can express these frequencies as

$$\omega_r = -\frac{1}{\Omega}\frac{\partial J_\theta}{\partial Q}, \qquad \omega_\theta = \frac{1}{\Omega}\frac{\partial J_r}{\partial Q},$$
$$\omega_\phi = \frac{1}{\Omega}\left(\frac{\partial J_r}{\partial L}\frac{\partial J_\theta}{\partial Q} - \frac{\partial J_r}{\partial Q}\frac{\partial J_\theta}{\partial L}\right), \quad (36)$$

with $\Omega := \frac{\partial J_r}{\partial H}\frac{\partial J_\theta}{\partial Q} - \frac{\partial J_r}{\partial Q}\frac{\partial J_\theta}{\partial H}$. These frequencies have been computed in a closed form in [83,84], but they do not generically admit a weak-field expansion. Indeed, these are considered only as "infinite time average" bound observables because of the dependence on the choice of the time parametrization for each coordinate. Therefore, the natural bound observables are the frequency ratios [98]

$$K^{\phi r} := \frac{\omega_\phi}{\omega_r} = \frac{\partial J_r/\partial Q}{\partial J_\theta/\partial Q}\frac{\partial J_\theta}{\partial L} - \frac{\partial J_r}{\partial L}, \quad (37)$$

$$K^{\theta r} := \frac{\omega_\theta}{\omega_r} = -\frac{\partial J_r/\partial Q}{\partial J_\theta/\partial Q}, \quad (38)$$

which are related to the precession rate of the periastron ($K^{\phi r}$) and of the orbital plane ($K^{\theta r}$).

B. Periastron precession rate ω_ϕ/ω_r

In the weak-field regime, we can identify the precession rate of the orbital ellipse with the so-called periastron advance rate $K^{\phi r}$. A direct calculation of (37) gives, up to order $\mathcal{O}(M^3 a^2)$ in the weak-field expansion,

$$K^{\phi r} = 1 + \frac{3M^2(5\mathcal{E} + 4)}{4l_Q^2}$$
$$+ \frac{aM^2\sqrt{\mathcal{E} + 1}(l_Q - 3l)(5\mathcal{E} + 2)}{l_Q^4}$$
$$+ \frac{3a^2M^2}{32l_Q^6}[l^2(445\mathcal{E}^2 + 416\mathcal{E} + 40)$$
$$- l_Q(l_Q + 2l)(85\mathcal{E}^2 + 80\mathcal{E} + 8)]. \quad (39)$$

In the equatorial limit $l_Q \to l$ we find perfect agreement with the expression in the literature [72,73].

C. Orbital plane precession rate ω_θ/ω_r

The orbital plane precession rate $K^{\theta r}$ can be essentially identified, in the weak-field limit, with the Lense-Thirring effect. Using (38), we obtain up to $\mathcal{O}(M^3 a^2)$

$$K^{\theta r} = 1 + \frac{3M^2(5\mathcal{E} + 4)}{4l_Q^2} - \frac{3aM^2 l\sqrt{\mathcal{E} + 1}(5\mathcal{E} + 2)}{l_Q^4}$$
$$+ \frac{3a^2M^2}{32l_Q^6}[l^2(445\mathcal{E}^2 + 416\mathcal{E} + 40)$$
$$- l_Q^2(85\mathcal{E}^2 + 80\mathcal{E} + 8)]. \quad (40)$$

In the limit $a \to 0$ the azimuthal and polar frequencies in (39) and (40) become degenerate $\omega_\theta = \omega_\phi$, while in the equatorial limit the polar one has no physical interpretation.

V. CONCLUSION

In this paper, we have explored the relationship between scattering and bound observables for generic orbits in a Kerr background. The establishment of a boundary-to-bound dictionary represents a crucial step towards leveraging the computational tools that have been developed for scattering amplitudes in the study of bound systems. Expanding upon previous work in the field, we have extended such a dictionary beyond the equatorial case by considering a smooth class of geodesics that interpolate between scattering and bound dynamics.

Taking advantage of the HJ representation, we have been able to write down a closed-form expression for the radial and polar contributions to the action for such a generic class of scattering and bound orbits. In particular, we have found that in the PM expansion there is one turning point in the scattering case and two turning points in the bound case for both the radial and polar motion. Such an analytic continuation also involves the Carter constant, which plays a crucial role for the dynamics beyond the equatorial plane.

We then computed, in the PM expansion, the azimuthal ($\Delta\phi$) and polar ($\Delta\theta$) deflection angles for time-like geodesics in the Kerr metric and the time delay ΔT for null geodesics. While the azimuthal angle is naturally derived from the action, the polar angle has a more implicit expression since there is no natural conjugate variable. Indeed, in the conventional partial-wave basis an explicit relation has been found only for some degenerate configurations [56,99], but perhaps an alternative basis might help to clarify the general case [58]. Using the new boundary-to-bound dictionary, we then studied the weak-field expansion of the periastron and orbital plane precession rates, $K^{\phi r}$ and $K^{\theta r}$, which are uniquely defined from the ratio of fundamental frequencies [83].

This work offers new promising directions for the analytic continuation of classical scattering and bound observables beyond the equatorial case. First, it would be

important to extend the amplitude-action relation for generic angular momentum orientations, which would include some type of polar action contribution. Furthermore, a natural extension of our work would be to consider a spinning probe in a Kerr background [100], since a generalization of the Carter constant was discovered by Rüdiger in the pole-dipole approximation [101,102] and recently generalized to quadrupolar order [103]. Finally, it would be interesting to see how the extension of the Schwinger-Dyson recursion [81] would allow to compute radiative observables for bound orbits. We hope to come back to these questions in the near future.

ACKNOWLEDGMENTS

APPENDIX A: CONSERVED CHARGES AND KINEMATICS

Here we present the relations between the conserved quantities and the kinematic invariants in the scattering case. Consider a probe particle with incoming momentum $p_{in}^\mu = (E, \vec{p}_{in})$ and impact parameter $b^\mu = (0, \vec{b})$, defined in such a way that $\vec{p}_{in} \cdot \vec{b} = 0$. Then, we have

$$
\begin{aligned}
E^2 &= |\vec{p}_{in}|^2 + m^2, \\
L^2 &= |\vec{p}_{in}|^2(|\vec{b}|^2 - (\hat{a} \cdot \vec{b})^2) - |\vec{b}|^2(\hat{a} \cdot \vec{p}_{in})^2, \\
Q &= (|\vec{b}|^2 - |\vec{a}|^2)(\hat{a} \cdot \vec{p}_{in})^2 + |\vec{p}_{in}|^2(\hat{a} \cdot \vec{b})^2,
\end{aligned} \tag{A1}
$$

where $\hat{a} = \vec{a}/|\vec{a}|$ is the unit vector along the spin direction. The incoming θ angle is determined by

$$
U_{in} = \cos^2(\theta_{in}) = (\hat{a} \cdot \hat{p}_{in})^2. \tag{A2}
$$

In the case considered in this paper, $\theta_{in} = \pi/2$, and therefore we impose $\hat{a} \cdot \hat{p}_{in} = 0$.

APPENDIX B: TURNING POINTS OF THE POLAR MOTION

The (r, θ) components of the geodesic equations in a Kerr black hole imply

$$
\int_{\mathcal{C}_r} \frac{dr}{\sqrt{R(r)}} = \int_{\mathcal{C}_\theta} \frac{d\theta}{\sqrt{\Theta(\theta)}}, \tag{B1}
$$

from which we can find the final polar angle U_{out}, as discussed in Sec. II D of [88]. A direct calculation for our setup shows that (B1) can be reduced to

$$
\sqrt{-a^2 m^2 \mathcal{E} U_-} \int_{\mathcal{C}_r^>} \frac{dr}{\sqrt{R(r)}}
$$
$$
= 2n^> K\left(\frac{U_+}{U_-}\right) - F\left(\arcsin\left(\sqrt{\frac{U_{out}^>}{U_+}}\right) \bigg| \frac{U_+}{U_-}\right) \tag{B2}
$$

for the scattering case $\mathcal{E} > 0$ and to

$$
\sqrt{-a^2 m^2 \mathcal{E} U_-} \int_{\mathcal{C}_r^<} \frac{dr}{\sqrt{R(r)}}
$$
$$
= 2n^< K\left(\frac{U_+}{U_-}\right) + F\left(\arcsin\left(\sqrt{\frac{U_{out}^<}{U_+}}\right) \bigg| \frac{U_+}{U_-}\right) \tag{B3}
$$

in the bound case $\mathcal{E} < 0$. It is worth stressing that the sign flip reflects the fact that $\eta_{out} = -1$ for scattering orbits and $\eta_{out} = +1$ for the corresponding bound orbits. Since U_{out} is independent of n, it turns that a perturbative expansion of (B2) and (B3) completely fixes the number of turning points in the polar motion to (see Fig. 2)

$$
n^> = 1, \qquad n^< = 2. \tag{B4}
$$

APPENDIX C: EXACT EXPRESSIONS FOR SCATTERING OBSERVABLES

We provide here some compact resummed expressions for scattering observables in terms of hypergeometric functions (see also [104] for the equatorial case). These are always functions of the roots of the radial and polar potentials, which need to be explicitly derived for the perturbative calculations.
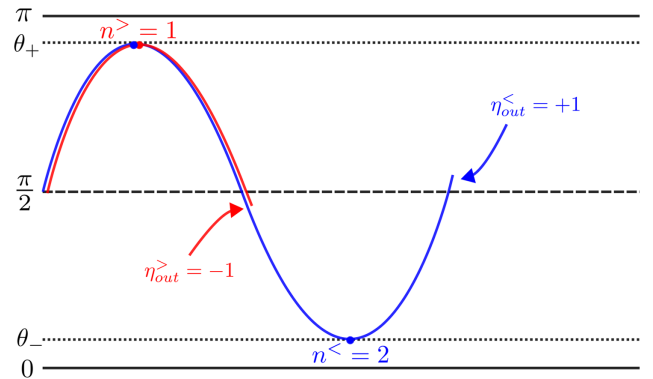

FIG. 2. A pictorial representation of the polar motion for the scattering (short, in red) and bound (long, in blue) classes of geodesics of interest in this paper.

The polar deflection angle is given by (26)

$$\cos^2(\theta_{\text{out}}) = U_+ sn^2\left(X_0^>\Big|\frac{U_-}{U_+}\right), \qquad \mathcal{E} > 0, \qquad Q > 0,$$

$$X_0^> = -4u_m\sqrt{-a^2 U_-}F_D^{(3)}\left(\alpha_{\Delta\theta}, \vec{\beta}_{\Delta\theta}, \gamma_{\Delta\theta}; \frac{u_m}{u_1}, \frac{u_m}{u_2}, \frac{u_m}{u_3}\right),$$

$$\alpha_{\Delta\theta} = 1, \qquad \vec{\beta}_{\Delta\theta} = \left\{\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right\}, \qquad \gamma_{\Delta\theta} = \frac{3}{2}, \tag{C1}$$

while the azimuthal deflection angle is derived from (29),

$$\Delta\phi = u_m\left[\frac{G_A}{u_A}F_D^{(4)}\left(\alpha_{\Delta\phi_1}, \vec{\beta}_{\Delta\phi_1}, \gamma_{\Delta\phi_1}; \frac{u_m}{u_A}, \frac{u_m}{u_1}, \frac{u_m}{u_2}, \frac{u_m}{u_3}\right) - \frac{G_B}{u_B}F_D^{(4)}\left(\alpha_{\Delta\phi_1}, \vec{\beta}_{\Delta\phi_1}, \gamma_{\Delta\phi_1}; \frac{u_m}{u_B}, \frac{u_m}{u_1}, \frac{u_m}{u_2}, \frac{u_m}{u_3}\right)\right]$$

$$+ \frac{\pi l U_2^{\frac{3}{2}}}{2\sqrt{l_Q^2 - l^2}}F_D^{(2)}\left(\alpha_{\Delta\phi_2}, \vec{\beta}_{\Delta\phi_2}, \gamma_{\Delta\phi_2}; U_2, \frac{U_2}{U_1}\right) + l\frac{(U_{\text{out}}^>)^{\frac{3}{2}}}{3\sqrt{l_Q^2 - l^2}}F_D^{(3)}\left(\alpha_{\Delta\phi_3}, \vec{\beta}_{\Delta\phi_3}, \gamma_{\Delta\phi_3}; U_{\text{out}}^>, \frac{U_{\text{out}}^>}{U_1}, \frac{U_{\text{out}}^>}{U_2}\right),$$

$$G_A = \frac{2Mu_A(l - a\mathcal{E}) - l}{\sqrt{\mathcal{E}(M^2 - a^2)}}, \qquad G_B = \frac{2Mu_B(l - a\mathcal{E}) - l}{\sqrt{\mathcal{E}(M^2 - a^2)}},$$

$$\alpha_{\Delta\phi_1} = 1, \qquad \vec{\beta}_{\Delta\phi_1} = \left\{1, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right\}, \qquad \gamma_{\Delta\phi_1} = \frac{3}{2},$$

$$\alpha_{\Delta\phi_2} = \frac{3}{2}, \qquad \vec{\beta}_{\Delta\phi_2} = \left\{1, \frac{1}{2}\right\}, \qquad \gamma_{\Delta\phi_2} = 2,$$

$$\alpha_{\Delta\phi_3} = \frac{3}{2}, \qquad \vec{\beta}_{\Delta\phi_3} = \left\{1, \frac{1}{2}, \frac{1}{2}\right\}, \qquad \gamma_{\Delta\phi_3} = \frac{5}{2}. \tag{C2}$$

$$J_\beta = (J_t, J_r, J_\theta, J_\phi). \tag{D2}$$

APPENDIX D: DERIVATION OF THE FUNDAMENTAL FREQUENCIES

The four integrals of motion,

$$P_\alpha = (H, E, L, Q) = \left(-\frac{1}{2}m^2, E, L, Q\right), \tag{D1}$$

are implicit functions of the action variables $P_\alpha = f(J_\beta)$,

The fundamental frequencies can therefore be computed from the Jacobian of f, i.e.,

$$\frac{\partial P_\alpha}{\partial J_\beta}\frac{\partial J_\beta}{\partial P_\gamma} = \delta_\alpha^\gamma, \tag{D3}$$

which gives directly (36).

[1] B. P. Abbott *et al.* (LIGO Scientific and Virgo Collaborations), Observation of gravitational waves from a binary black hole merger, Phys. Rev. Lett. **116**, 061102 (2016).

[2] T. Damour, High-energy gravitational scattering and the general relativistic two-body problem, Phys. Rev. D **97**, 044038 (2018).

[3] D. Neill and I. Z. Rothstein, Classical space-times from the S matrix, Nucl. Phys. **B877**, 177 (2013).

[4] T. Damour, Classical and quantum scattering in post-Minkowskian gravity, Phys. Rev. D **102**, 024060 (2020).

[5] N. E. J. Bjerrum-Bohr, P. H. Damgaard, G. Festuccia, L. Planté, and P. Vanhove, General relativity from scattering amplitudes, Phys. Rev. Lett. **121**, 171601 (2018).

[6] C. Cheung, I. Z. Rothstein, and M. P. Solon, From scattering amplitudes to classical potentials in the post-Minkowskian expansion, Phys. Rev. Lett. **121**, 251101 (2018).

[7] M.-Z. Chung, Y.-T. Huang, J.-W. Kim, and S. Lee, The simplest massive S-matrix: From minimal coupling to black holes, J. High Energy Phys. 04 (2019) 156.

[8] Z. Bern, C. Cheung, R. Roiban, C.-H. Shen, M. P. Solon, and M. Zeng, Black hole binary dynamics from the double copy and effective theory, J. High Energy Phys. 10 (2019) 206.

[9] Z. Bern, A. Luna, R. Roiban, C.-H. Shen, and M. Zeng, Spinning black hole binary dynamics, scattering amplitudes, and effective field theory, Phys. Rev. D **104,** 065014 (2021).

[10] D. A. Kosower, B. Maybee, and D. O'Connell, Amplitudes, observables, and classical scattering, J. High Energy Phys. 02 (2019) 137.

[11] A. Cristofoli, R. Gonzo, N. Moynihan, D. O'Connell, A. Ross, M. Sergola, and C. D. White, The uncertainty principle and classical amplitudes, arXiv:2112.07556.

[12] B. Maybee, D. O'Connell, and J. Vines, Observables and amplitudes for spinning particles and black holes, J. High Energy Phys. 12 (2019) 156.

[13] G. Kälin and R. A. Porto, Post-Minkowskian effective field theory for conservative binary dynamics, J. High Energy Phys. 11 (2020) 106.

[14] G. Mogull, J. Plefka, and J. Steinhoff, Classical black hole scattering from a worldline quantum field theory, J. High Energy Phys. 02 (2021) 048.

[15] N. E. J. Bjerrum-Bohr, P. H. Damgaard, L. Plante, and P. Vanhove, Chapter 13: Post-Minkowskian expansion from scattering amplitudes, J. Phys. A **55,** 443014 (2022).

[16] J. Vines, Scattering of two spinning black holes in post-Minkowskian gravity, to all orders in spin, and effective-one-body mappings, Classical Quantum Gravity **35,** 084002 (2018).

[17] A. Guevara, A. Ochirov, and J. Vines, Scattering of spinning black holes from exponentiated soft factors, J. High Energy Phys. 09 (2019) 056.

[18] A. Guevara, A. Ochirov, and J. Vines, Black-hole scattering with general spin directions from minimal-coupling amplitudes, Phys. Rev. D **100,** 104024 (2019).

[19] D. A. Kosower, R. Monteiro, and D. O'Connell, The SAGEX review on scattering amplitudes chapter 14: Classical gravity from scattering amplitudes, J. Phys. A **55,** 443015 (2022).

[20] Z. Bern, C. Cheung, R. Roiban, C.-H. Shen, M. P. Solon, and M. Zeng, Scattering amplitudes and the conservative Hamiltonian for binary systems at third post-Minkowskian order, Phys. Rev. Lett. **122,** 201603 (2019).

[21] Z. Bern, J. Parra-Martinez, R. Roiban, M. S. Ruf, C.-H. Shen, M. P. Solon, and M. Zeng, Scattering amplitudes, the tail effect, and conservative binary dynamics at $O(G^4)$. Phys. Rev. Lett. **128,** 161103 (2022).

[22] C. Dlapa, G. Kälin, Z. Liu, and R. A. Porto, Conservative dynamics of binary systems at fourth post-Minkowskian order in the large-eccentricity expansion, Phys. Rev. Lett. **128,** 161104 (2022).

[23] C. Dlapa, G. Kälin, Z. Liu, J. Neef, and R. A. Porto, Radiation reaction and gravitational waves at fourth post-Minkowskian order, Phys. Rev. Lett. **130,** 101401 (2023).

[24] P. Di Vecchia, C. Heissenberg, R. Russo, and G. Veneziano, Universality of ultra-relativistic gravitational scattering, Phys. Lett. B **811,** 135924 (2020).

[25] P. Di Vecchia, C. Heissenberg, R. Russo, and G. Veneziano, The eikonal approach to gravitational scattering and radiation at $\mathcal{O}(G^3)$. J. High Energy Phys. 07 (2021) 169.

[26] N. E. J. Bjerrum-Bohr, P. H. Damgaard, L. Planté, and P. Vanhove, The amplitude for classical gravitational scattering at third post-Minkowskian order, J. High Energy Phys. 08 (2021) 172.

[27] N. E. J. Bjerrum-Bohr, L. Planté, and P. Vanhove, Post-Minkowskian radial action from soft limits and velocity cuts, J. High Energy Phys. 03 (2022) 071.

[28] A. Brandhuber, G. Chen, G. Travaglini, and C. Wen, Classical gravitational scattering from a gauge-invariant double copy, J. High Energy Phys. 10 (2021) 118.

[29] D. Bini, T. Damour, and A. Geralico, Binary dynamics at the fifth and fifth-and-a-half post-Newtonian orders, Phys. Rev. D **102,** 024062 (2020).

[30] D. Bini, T. Damour, and A. Geralico, Sixth post-Newtonian local-in-time dynamics of binary systems, Phys. Rev. D **102,** 024061 (2020).

[31] D. Bini, T. Damour, and A. Geralico, Sixth post-Newtonian nonlocal-in-time dynamics of binary systems, Phys. Rev. D **102,** 084047 (2020).

[32] J. Vines, J. Steinhoff, and A. Buonanno, Spinning-black-hole scattering and the test-black-hole limit at second post-Minkowskian order, Phys. Rev. D **99,** 064054 (2019).

[33] M.-Z. Chung, Y.-T. Huang, and J.-W. Kim, Classical potential for general spinning bodies, J. High Energy Phys. 09 (2020) 074.

[34] N. Arkani-Hamed, Y.-t. Huang, and D. O'Connell, Kerr black holes as elementary particles, J. High Energy Phys. 01 (2020) 046.

[35] D. Kosmopoulos and A. Luna, Quadratic-in-spin Hamiltonian at $\mathcal{O}(G^2)$ from scattering amplitudes, J. High Energy Phys. 07 (2021) 037.

[36] Z. Bern, D. Kosmopoulos, A. Luna, R. Roiban, and F. Teng, Binary dynamics through the fifth power of spin at $\mathcal{O}(G^2)$, Phys. Rev. Lett. **130,** 201402 (2023).

[37] G. U. Jakobsen and G. Mogull, Linear response, Hamiltonian, and radiative spinning two-body dynamics, Phys. Rev. D **107,** 044033 (2023).

[38] G. U. Jakobsen and G. Mogull, Conservative and radiative dynamics of spinning bodies at third post-Minkowskian order using worldline quantum field theory, Phys. Rev. Lett. **128,** 141102 (2022).

[39] F. Febres Cordero, M. Kraus, G. Lin, M. S. Ruf, and M. Zeng, Conservative binary dynamics with a spinning black hole at $\mathcal{O}(G^3)$ from scattering amplitudes, Phys. Rev. Lett. **130,** 021601 (2023).

[40] M. Chiodaroli, H. Johansson, and P. Pichini, Compton black-hole scattering for s ≤ 5/2, J. High Energy Phys. 02 (2022) 156.

[41] L. Cangemi, M. Chiodaroli, H. Johansson, A. Ochirov, P. Pichini, and E. Skvortsov, Kerr black holes enjoy massive higher-spin gauge symmetry, arXiv:2212.06120.

[42] F. Alessio and P. Di Vecchia, Radiation reaction for spinning black-hole scattering, Phys. Lett. B **832**, 137258 (2022).

[43] F. Alessio, Kerr binary dynamics from minimal coupling and double copy, arXiv:2303.12784.

[44] M.-Z. Chung, Y.-t. Huang, J.-W. Kim, and S. Lee, Complete Hamiltonian for spinning binary systems at first post-Minkowskian order, J. High Energy Phys. 05 (2020) 105.

[45] Z. Liu, R. A. Porto, and Z. Yang, Spin effects in the effective field theory approach to post-Minkowskian conservative dynamics, J. High Energy Phys. 06 (2021) 012.

[46] R. Aoude, K. Haddad, and A. Helset, Searching for Kerr in the 2PM amplitude, J. High Energy Phys. 07 (2022) 072.

[47] R. Aoude, K. Haddad, and A. Helset, Classical gravitational spinning-spinless scattering at $O(G^2 S^\infty)$. Phys. Rev. Lett. **129**, 141102 (2022).

[48] W.-M. Chen, M.-Z. Chung, Y.-t. Huang, and J.-W. Kim, The 2PM Hamiltonian for binary Kerr to quartic in spin, J. High Energy Phys. 08 (2022) 148.

[49] G. Menezes and M. Sergola, NLO deflections for spinning particles and Kerr black holes, J. High Energy Phys. 10 (2022) 105.

[50] Y. F. Bautista, Dynamics for super-extremal Kerr binary systems at $\mathcal{O}(G^2)$, arXiv:2304.04287.

[51] C. Cheung, N. Shah, and M. P. Solon, Mining the geodesic equation for scattering data, Phys. Rev. D **103**, 024030 (2021).

[52] C. Cheung and M. P. Solon, Classical gravitational scattering at $\mathcal{O}(G^3)$ from Feynman diagrams, J. High Energy Phys. 06 (2020) 144.

[53] R. Gonzo and C. Shi, Geodesics from classical double copy, Phys. Rev. D **104**, 105012 (2021).

[54] T. Adamo, A. Cristofoli, and A. Ilderton, Classical physics from amplitudes on curved backgrounds, J. High Energy Phys. 08 (2022) 281.

[55] F. Bastianelli, F. Comberiati, and L. de la Cruz, Light bending from eikonal in worldline quantum field theory, J. High Energy Phys. 02 (2022) 209.

[56] Y. F. Bautista, A. Guevara, C. Kavanagh, and J. Vines, Scattering in black hole backgrounds and higher-spin amplitudes: Part I, J. High Energy Phys. 03 (2023) 136.

[57] Y. F. Bautista, A. Guevara, C. Kavanagh, and J. Vinese, Scattering in black hole backgrounds and higher-spin amplitudes: Part II, J. High Energy Phys. 05 (2023) 211.

[58] U. Kol, D. O'connell, and O. Telem, The radial action from probe amplitudes to all orders, J. High Energy Phys. 03 (2022) 141.

[59] Y. Mino, Perturbative approach to an orbital evolution around a supermassive black hole, Phys. Rev. D **67**, 084027 (2003).

[60] E. Poisson, A. Pound, and I. Vega, The motion of point particles in curved spacetime, Living Rev. Relativity **14**, 7 (2011).

[61] A. I. Harte, Mechanics of extended masses in general relativity, Classical Quantum Gravity **29**, 055012 (2012).

[62] S. E. Gralla and R. M. Wald, A rigorous derivation of gravitational self-force, Classical Quantum Gravity **25**, 205009 (2008); **28**, 159501(E) (2011).

[63] L. Barack and A. Pound, Self-force and radiation reaction in general relativity, Rep. Prog. Phys. **82**, 016904 (2019).

[64] T. Damour and G. Schaefer, Higher order relativistic periastron advances and binary pulsars, Nuovo Cimento B **101**, 127 (1988).

[65] A. Buonanno and T. Damour, Effective one-body approach to general relativistic two-body dynamics, Phys. Rev. D **59**, 084006 (1999).

[66] T. Damour, P. Jaranowski, and G. Schaefer, Dynamical invariants for general relativistic two-body systems at the third post-Newtonian approximation, Phys. Rev. D **62**, 044024 (2000).

[67] T. Damour, Introductory lectures on the effective one body formalism, Int. J. Mod. Phys. A **23**, 1130 (2008).

[68] T. Damour, Coalescence of two spinning black holes: An effective one-body approach, Phys. Rev. D **64**, 124013 (2001).

[69] M. Khalil, A. Buonanno, J. Steinhoff, and J. Vines, Energetics and scattering of gravitational two-body systems at fourth post-Minkowskian order, Phys. Rev. D **106**, 024042 (2022).

[70] A. Antonelli, A. Buonanno, J. Steinhoff, M. van de Meent, and J. Vines, Energetics of two-body Hamiltonians in post-Minkowskian gravity, Phys. Rev. D **99**, 104004 (2019).

[71] A. Buonanno, M. Khalil, D. O'Connell, R. Roiban, M. P. Solon, and M. Zeng, Snowmass white paper: Gravitational waves and scattering amplitudes, in *2022 Snowmass Summer Study* (2022), arXiv:2204.05194.

[72] G. Kälin and R. A. Porto, From boundary data to bound states, J. High Energy Phys. 01 (2020) 072.

[73] G. Kälin and R. A. Porto, From boundary data to bound states. Part II. Scattering angle to dynamical invariants (with twist). J. High Energy Phys. 02 (2020) 120.

[74] T. Damour and N. Deruelle, General relativistic celestial mechanics of binary systems. I. The post-Newtonian motion, Ann. Inst. Henri Poincaré Phys. Théor. **43**, 107 (1985).

[75] G. Cho, G. Kälin, and R. A. Porto, From boundary data to bound states. Part III. Radiative effects, J. High Energy Phys. 04 (2022) 154; 07 (2022) 2(E).

[76] M. V. S. Saketh, J. Vines, J. Steinhoff, and A. Buonanno, Conservative and radiative dynamics in classical relativistic scattering and bound systems, Phys. Rev. Res. **4**, 013127 (2022).

[77] R. A. Leacock and M. J. Padgett, Hamilton-Jacobi/action-angle quantum mechanics, Phys. Rev. D **28**, 2491 (1983).

[78] K. W. Ford and J. A. Wheeler, Semiclassical description of scattering, Ann. Phys. (N.Y.) **7**, 259 (1959).

[79] Z. Bern, J. Parra-Martinez, R. Roiban, M. S. Ruf, C.-H. Shen, M. P. Solon, and M. Zeng, Scattering amplitudes and conservative binary dynamics at $\mathcal{O}(G^4)$, Phys. Rev. Lett. **126**, 171601 (2021).

[80] P. H. Damgaard, L. Plante, and P. Vanhove, On an exponential representation of the gravitational S-matrix, J. High Energy Phys. 11 (2021) 213.

[81] T. Adamo and R. Gonzo, Bethe-Salpeter equation for classical gravitational bound states, J. High Energy Phys. 05 (2023) 088.

[82] B. Carter, Global structure of the Kerr family of gravitational fields, Phys. Rev. **174**, 1559 (1968).

[83] W. Schmidt, Celestial mechanics in Kerr space-time, Classical Quantum Gravity **19,** 2743 (2002).

[84] R. Fujita and W. Hikida, Analytical solutions of bound timelike geodesic orbits in Kerr spacetime, Classical Quantum Gravity **26,** 135002 (2009).

[85] S. Balmelli and T. Damour, New effective-one-body Hamiltonian with next-to-leading order spin-spin coupling, Phys. Rev. D **92,** 124022 (2015).

[86] M. Khalil, J. Steinhoff, J. Vines, and A. Buonanno, Fourth post-Newtonian effective-one-body Hamiltonians with generic spins, Phys. Rev. D **101,** 104034 (2020).

[87] K. Rosquist, T. Bylund, and L. Samuelsson, Carter's constant revealed, Int. J. Mod. Phys. D **18,** 429 (2009).

[88] D. Kapec and A. Lupsasca, Particle motion near high-spin black holes, Classical Quantum Gravity **37,** 015006 (2020).

[89] G. Compère, Y. Liu, and J. Long, Classification of radial Kerr geodesic motion, Phys. Rev. D **105,** 024075 (2022).

[90] S. E. Gralla and A. Lupsasca, Null geodesics of the Kerr exterior, Phys. Rev. D **101,** 044032 (2020).

[91] P. H. Damgaard, J. Hoogeveen, A. Luna, and J. Vines, Scattering angles in Kerr metrics, Phys. Rev. D **106,** 124030 (2022).

[92] X. O. Camanho, J. D. Edelstein, J. Maldacena, and A. Zhiboedov, Causality constraints on corrections to the graviton three-point coupling, J. High Energy Phys. 02 (2016) 020.

[93] M. Accettulli Huber, A. Brandhuber, S. De Angelis, and G. Travaglini, Eikonal phase matrix, deflection angle and time delay in effective field theories of gravity, Phys. Rev. D **102,** 046014 (2020).

[94] B. Bellazzini, G. Isabella, and M. M. Riva, Classical vs quantum Eikonal scattering and its causal structure, J. High Energy Phys. 04 (2023) 023.

[95] F. D. Ryan, Effect of gravitational radiation reaction on circular orbits around a spinning black hole, Phys. Rev. D **52,** R3159 (1995).

[96] T. Hinderer and E. E. Flanagan, Two timescale analysis of extreme mass ratio inspirals in Kerr. I. Orbital motion, Phys. Rev. D **78,** 064028 (2008).

[97] M. Kerachian, L. Polcar, V. Skoupý, C. Efthymiopoulos, and G. Lukes-Gerakopoulos, Action-angle formalism for extreme mass ratio inspirals in Kerr spacetime, Phys. Rev. D **108,** 044004 (2023).

[98] A. G. M. Lewis, A. Zimmerman, and H. P. Pfeiffer, Fundamental frequencies and resonances from eccentric and precessing binary black hole inspirals, Classical Quantum Gravity **34,** 124001 (2017).

[99] K. Glampedakis and N. Andersson, Scattering of scalar waves by rotating black holes, Classical Quantum Gravity **18,** 1939 (2001).

[100] V. Witzany, Hamilton-Jacobi equation for spinning particles near black holes, Phys. Rev. D **100,** 104030 (2019).

[101] R. Rüdiger, Conserved quantities of spinning test particles in general relativity. I, Proc. R. Soc. A **375,** 185 (1981).

[102] R. Rüdiger, Conserved quantities of spinning test particles in general relativity. II, Proc. R. Soc. A **385,** 229 (1983).

[103] G. Compère, A. Druart, and J. Vines, Generalized Carter constant for quadrupolar test bodies in Kerr spacetime, arXiv:2302.14549.

[104] G. V. Kraniotis, Frame-dragging and bending of light in Kerr and Kerr-(anti) de Sitter spacetimes, Classical Quantum Gravity **22,** 4391 (2005).