Bianchi cosmologies, magnetic fields, and singularities

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We study the effects of a spatially homogenous magnetic field in Bianchi-I cosmological models. The cases of a pure magnetic field and two models with additional dust and a massless scalar field (stiff matter) are also considered. At the beginning of the cosmological evolution, i.e., in the neighborhood of the singularity, the Universe is described by one of Kasner's solutions, and asymptotically by another Kasner solution when the volume of the Universe tends to infinity. The transition law between these two Kasner regimes is established, and shown to coincide with the analogous law for the empty Bianchi-II universe. The universe filled with dust and a magnetic field undergoes the process of isotropization, while the presence of a massless scalar field induces a modification of the relations between Kasner indices in the two asymptotic regimes. In all of these cases, we analyze the approach to the singularity in some detail and comment on the issue of the possible singularity crossing.

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I. INTRODUCTION

Almost all of modern cosmology is based on the spatially homogeneous and isotropic Friedmann-Lemaître cosmological models. Indeed, the Friedmann-Lemaître cosmology for an expanding or contracting spatially homogeneous and isotropic universe is very successful in describing the global evolution of the Universe from inflation to the present epoch of cosmic acceleration. Moreover, inhomogeneities in our Universe are described by means of the theory of cosmological perturbations on the Friedmann background, which explains the origin of large-scale structures in the contemporary Universe starting from quantum fluctuations in the very early Universe.

However, there are very interesting features in gravity and cosmology beyond the Friedmann models and perturbations on such backgrounds. The study of spatially homogeneous but anisotropic models (see, e.g., Refs. [1–3]) originating in the work of Bianchi, published as early as in the year 1898 [4], presents great interest from both the mathematical and physical points of view. Bianchi elaborated on the complete classification of the three-dimensional homogeneous Riemannian spaces and three-dimensional Lie groups long before Einstein put forward General Relativity in 1915. Bianchi classification was then modernized, simplified, and applied to cosmology in the 1950s and 1960s [5–7].

The simplest spatially homogeneous and anisotropic cosmology is given by the Bianchi-I model. Its isometry group contains three generators (Killing vector fields), which commute and correspond to the metric

$$ds^2 = dt^2 - a^2(t)dx^2 - b^2(t)dy^2 - c^2(t)dz^2. \tag{1}$$

One can see that Eq. (1) reduces to a flat Friedmann metric in the limiting case when the three scale factors $a(t)$, $b(t)$, and $c(t)$ coincide.

The first exact solution for the metric (1) in empty space was found by Kasner [8], in fact before Friedmann's works. The Kasner solution has been rediscovered many times, and its importance was only appreciated later. A particular form of Kasner's solution for the Bianchi-I universe had been discovered earlier in Refs. [9,10]. In 1963, Khalatnikov and Lifshitz began employing the Bianchi universes (Bianchi-I, in particular) for studying the problem of the cosmological singularity [11]. At the end of the 1960s, Belinski, Khalatnikov, and Lifshitz discovered the phenomenon of the oscillatory approach to the cosmological singularity [12,13]. Using the Hamiltonian formalism, Misner referred to this phenomenon as the Mixmaster universe [14]. Later, it was understood that the dynamics of

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the universe becomes chaotic when the singularity is approached [15,16]. Finally, the connection between the chaotic behavior in cosmological models and infinite-dimensional Lie algebras [17] was discovered at the beginning of the new millennium [18–20]. Thus, the study of Bianchi cosmologies, starting from the empty Bianchi-I model, has already led to some unexpected discoveries. In the meantime, exact solutions for the Bianchi-I universe filled with matter were also studied. First of all, we mention the Heckmann-Schucking solution, which describes the Bianchi-I universe filled with dust [21]. The Heckmann-Schucking universe behaves as the Kasner universe at the beginning of the evolution, after which isotropization takes place so that the universe approaches an isotropic Friedmann regime while expanding. The generalization of the Heckmann-Schucking solution to other kinds of isotropic perfect fluids was obtained by Jacobs [22–24], and more generalizations were found recently [25,26].

While exact solutions with a highly anisotropic geometry do exist even in empty space or in a universe filled with isotropic matter, the situation becomes even more riveting in the presence of spatially homogenous but anisotropic matter. A magnetic field is an interesting example of such a source. Indeed, papers devoted to the Bianchi-I universes with a spatially homogeneous magnetic field oriented along one of the coordinate axes in Eq. (1) were published already in the early 1960s, sometimes with additional types of matter [23,27–31].

The interest in solutions involving magnetic fields is not purely academic. The existence of large-scale magnetic fields in our Universe is an important and enigmatic phenomenon (see e.g., Ref. [32]). Their origin is not known and is being widely discussed. Thus, even if it is unrealistic to describe the present-day Universe with the Bianchi-I metric, models where this metric is sustained by a magnetic field could shed some light on processes that occurred in the very early Universe. In spite of the long history of studies of cosmic magnetic fields in general and in the Bianchi-I universe in particular, to our knowledge, there is no detailed description of the qualitative behavior of the corresponding solutions. Besides, it would be interesting to try and bridge the established (and sometimes implicit) knowledge about solutions with magnetic fields with innovative approaches to the problem of the singularities in gravity and theoretical cosmology. In this paper, we shall try to fill some gaps and look for new ways in this direction.

The structure of the paper is the following: in Sec. II, we consider the Bianchi-I model with a spatially homogeneous magnetic field directed along the z axis of Eq. (1). At the beginning and end of the cosmological evolution, the universe is in Kasner regimes. We establish the relationship between the parameters of these two regimes and show that it coincides with the one in the empty Bianchi-II universe. We then consider two other models in Sec. III, one in which dust is added and one with an additional massless scalar field (or, in other words, stiff matter). Section IV is devoted to the possibility of crossing the singularity in these models, and Sec. V contains some concluding remarks.

II. BIANCHI-I MODEL WITH SPATIALLY HOMOGENEOUS MAGNETIC FIELD

Let us consider the Bianchi-I universe with the metric in Eq. (1). The Lagrangian of the electromagnetic field is (in the Gaussian system) [1]

$$L_{\rm em} = -\frac{1}{16\pi} F_{ik} F^{ik}, \tag{2}$$

where Latin indices are four-dimensional, $i = 0, \ldots, 3$, and $x^i = (t, x, y, z)$ in Eq. (1). The energy-momentum tensor of the electromagnetic field with the Lagrangian (2) has the form

$$T^i{}_k = \frac{1}{4\pi}\left(-F^{il}F_{kl} + \frac{1}{4}\delta^i_k F_{lm}F^{lm}\right). \tag{3}$$

In particular, we consider the case with no electric field and a homogeneous magnetic field along z. Hence, the only nonzero component of the electromagnetic field tensor is F_{12}. The sourceless Maxwell equation for the electromagnetic field reads

$$F_{[ij;k]} = 0, \tag{4}$$

where semicolons denote covariant derivatives in the metric (1) and square brackets imply antisymmetrization. Since there is no torsion, the symmetric connection terms cancel out and Eq. (4) simplifies to $F_{[ij,k]} = 0$. Choosing the triplet of indices 0, 1, and 2, we see that $F_{12,0} = 0$, which means that F_{12} is constant. For the diagonal spacetime (1), the only nonvanishing component of the fully contravariant electromagnetic field tensor is thus given by

$$F^{12} = g^{11} g^{22} F_{12} \propto a^{-2} b^{-2}. \tag{5}$$

That means that all the contributions to the mixed components of the energy-momentum tensor in Eq. (3) are proportional to $a^{-2}b^{-2}$. On choosing a convenient parametrization, we can then write

$$T^0{}_0 = -T^1{}_1 = -T^2{}_2 = T^3{}_3 = \frac{B_0^2}{a^2 b^2}, \tag{6}$$

where B_0^2 is a positive constant characterizing the intensity of the magnetic field. It is easy to see that the trace T of the energy-momentum tensor (6) vanishes, as it should.

It is convenient to rewrite the Einstein equations,

$$G^i{}_j \equiv R^i{}_j - \frac{1}{2}\mathcal{R}\delta^i{}_j = T^i{}_j, \tag{7}$$

by parametrizing the scale factors as

$$\begin{aligned} a(t) &= R(t)e^{\alpha(t)+\beta(t)}, \\ b(t) &= R(t)e^{\alpha(t)-\beta(t)}, \\ c(t) &= R(t)e^{-2\alpha(t)}. \end{aligned} \tag{8}$$

The components of the Ricci tensor then read

$$R^0{}_0 = -3\frac{\ddot{R}}{R} - 6\dot{\alpha}^2 - 2\dot{\beta}^2, \tag{9}$$

$$R^1{}_1 = -\frac{\ddot{R}}{R} - 2\frac{\dot{R}^2}{R^2} - \ddot{\alpha} - 3\dot{\alpha}\frac{\dot{R}}{R} - \ddot{\beta} - 3\dot{\beta}\frac{\dot{R}}{R}, \tag{10}$$

$$R^2{}_2 = -\frac{\ddot{R}}{R} - 2\frac{\dot{R}^2}{R^2} - \ddot{\alpha} - 3\dot{\alpha}\frac{\dot{R}}{R} + \ddot{\beta} + 3\dot{\beta}\frac{\dot{R}}{R}, \tag{11}$$

$$R^3{}_3 = -\frac{\ddot{R}}{R} - 2\frac{\dot{R}^2}{R^2} + 2\ddot{\alpha} + 6\dot{\alpha}\frac{\dot{R}}{R}, \tag{12}$$

and Einstein's equations are given by

$$G^0{}_0 = -G^1{}_1 = -G^2{}_2 = G^3{}_3 = \frac{B_0^2}{R^4}e^{-4\alpha(t)}. \tag{13}$$

Note also that the scalar curvature is

$$\mathcal{R} \equiv R^i{}_i = -6\frac{\ddot{R}}{R} - 6\frac{\dot{R}^2}{R^2} - 6\dot{\alpha}^2 - 2\dot{\beta}^2, \tag{14}$$

which must vanish since $T = 0$.

Taking the difference of the mixed ${}^1{}_1$ and ${}^2{}_2$ components of Einstein's equations (13) with the expressions (10) and (11) for the Ricci tensor, we obtain

$$\dot{\beta} = \frac{\beta_0}{R^3}, \tag{15}$$

which is just like in Kasner's and Heckmann-Schucking's solutions Ref. [25]. Likewise, combining Eqs. (10)–(12), viz. $R^1{}_1 + R^2{}_2 - 2R^3{}_3$, yields

$$\ddot{\alpha} + 3\dot{\alpha}\frac{\dot{R}}{R} = \frac{2B_0^2}{3R^4}e^{-4\alpha(t)}. \tag{16}$$

Moreover, the combination $R^1{}_1 + R^2{}_2 + 2R^3{}_3$ provides

$$\ddot{\alpha} + 3\dot{\alpha}\frac{\dot{R}}{R} = 2\frac{\ddot{R}}{R} + 4\frac{\dot{R}^2}{R^2}. \tag{17}$$

By multiplying for the spatial volume, $V(t) \equiv R^3(t)$, the equation above can be rewritten in a more convenient form and then integrated, resulting in

$$\dot{\alpha} = 2\frac{\dot{R}}{R} + \frac{\alpha_0}{R^3}, \tag{18}$$

where α_0 is a constant. Combining Eqs. (16) and (17) leads to

$$\frac{1}{R^3}\frac{d^2R^3}{dt^2} = \frac{B_0^2}{R^4}e^{-4\alpha(t)}. \tag{19}$$

Thus, we can determine $\alpha(t)$ for a given $R(t)$, and we further notice that the second time derivative of the spatial volume must always be positive. Substituting Eqs. (9), (14), (15), (18), and (19) into the ${}^0{}_0$ component of the Einstein Eq. (13), after some manipulation, we obtain an equation for the spatial volume which reads

$$V\ddot{V} + \dot{V}^2 + 4\alpha_0\dot{V} + 3\alpha_0^2 + \beta_0^2 = 0. \tag{20}$$

Remarkably, this equation is integrable, but it is instructive to perform a qualitative analysis first.

Let us point out that not all solutions of Eq. (20) solve the complete system of Einstein's and Maxwell's equations. Since $V(t)$ should always be non-negative and $\ddot{V} > 0$ from Eq. (19), we must have $\alpha_0\dot{V} < 0$ with $\alpha_0 \neq 0$, and

$$-2\alpha_0 - \sqrt{\alpha_0^2 - \beta_0^2} \leq \dot{V} \leq -2\alpha_0 + \sqrt{\alpha_0^2 - \beta_0^2}, \tag{21}$$

which is only possible for $\alpha_0^2 \geq \beta_0^2$.

A. Contracting universe

Let us first consider $\alpha_0 > 0$, corresponding to a contracting universe with $\dot{V} < 0$. One can start at a certain moment in time with a positive value of V and a negative value of $\dot{V}$ satisfying the inequality $\alpha_0^2 \geq \beta_0^2$. Since $\ddot{V} > 0$, the time derivative of $V(t)$ grows, remaining negative, and the absolute value of $\dot{V}$ decreases, always satisfying the constraint

$$|\dot{V}| \geq 2\alpha_0 - \sqrt{\alpha_0^2 - \beta_0^2} \equiv W_1. \tag{22}$$

The universe will therefore reach the singularity characterized by $V = 0$ in a finite period of time.

Now, we can consider two times t_1 and t_2 such that $V(t_1) = 0$ and $\dot{V}(t_2) = -W_1$, and we would like to understand which occurs first. Suppose that $t_1 < t_2$, so that V vanishes while $\dot{V}$ still satisfies the inequality (21) with $|\dot{V}|$ larger than the critical value W_1 from Eq. (22). The time t_1 cannot be infinite because the absolute value of the time derivative is larger than W_1, and the spatial volume $V(t)$ reaches zero in a finite period of time. In particular, one can approximate the volume function for $t \lesssim t_1$ with the expression

$$V \simeq \gamma(t_1 - t)^{\lambda}, \tag{23}$$

where γ and λ are positive constants. For $\lambda > 1$, the velocity becomes

$$\dot{V} \simeq -\lambda\gamma(t_1 - t)^{\lambda-1} \to 0 \quad \text{for } t \to t_1, \tag{24}$$

which contradicts the condition (22). On the other hand, if $\lambda < 1$, the velocity in Eq. (24) diverges for $t \to t_1$, which violates the bound (21). The only choice left is $\lambda = 1$ and we need to include another term in the expansion around t_1, namely

$$V \simeq \gamma(t_1 - t) + \eta_1(t_1 - t)^{\mu}, \tag{25}$$

where η_1 is a positive constant and $\mu > 1$. Substituting the corresponding expressions for V, $\dot{V}$, and $\ddot{V}$ into Eq. (20), we find

$$\begin{aligned} &\mu\eta_1(\mu - 1)[\gamma(t_1 - t) + \eta_1(t_1 - t)^{\mu}](t_1 - t)^{\mu-2} \\ &\quad \simeq -[\gamma + \mu\eta_1(t_1 - t)^{\mu-1} - 2\alpha_0]^2 + \alpha_0^2 - \beta_0^2. \end{aligned} \tag{26}$$

The leading term in the left-hand side above behaves as $(t_1 - t)^{\mu-1}$, which vanishes for $t \to t_1$. The leading term in the right-hand side is instead a constant, which should therefore vanish, giving

$$\gamma^2 - 4\alpha_0\gamma + 3\alpha_0^2 + \beta_0^2 = 0. \tag{27}$$

One of the solutions of this equation is $\gamma = W_1$, which means that $\dot{V}$ reaches the critical value W_1 at the same time when the volume $V(t)$ vanishes, so that $t_1 = t_2$. Next, we equate terms of order $(t_1 - t)^{\mu-1}$ in Eq. (26), which gives

$$\mu = 1 + \frac{2\sqrt{\alpha_0^2 - \beta_0^2}}{2\alpha_0 - \sqrt{\alpha_0^2 - \beta_0^2}}. \tag{28}$$

One can see that $1 < \mu \leq 3$, whereas the constant η_1 takes different values depending on the initial conditions.

We could also consider the case of $\dot{V}$ reaching the critical value $-W_1$ at the moment $t_2 < t_1$ while the volume $V(t_2) > 0$. A simple analysis similar to the above shows that this case is excluded. Thus, we can say that the universe hits the singularity $V(t) = 0$ at some finite time $t = t_1$ when the velocity $\dot{V}$ reaches the critical value $-W_1$ for any contracting evolution.

From the (approximate) evolution law of the volume, we can determine the anisotropy factors $\alpha(t)$ and $\beta(t)$. Using Eq. (18), we get

$$\alpha(t) \simeq \left(\frac{2}{3} - \frac{\alpha_0}{2\alpha_0 - \sqrt{\alpha_0^2 - \beta_0^2}}\right) \ln(t_1 - t). \tag{29}$$

Analogously, from Eq. (15), we immediately find

$$\beta(t) = \beta_0 \int \frac{dt}{V} \simeq -\frac{\beta_0 \ln(t_1 - t)}{2\alpha_0 - \sqrt{\alpha_0^2 - \beta_0^2}}. \tag{30}$$

For definiteness, let us set $\beta_0 \geq 0$. Using the definitions (8), we can write the three scale factors in the Kasner form

$$\begin{aligned} a(t) &\sim (t_1 - t)^{p_1}, \\ b(t) &\sim (t_1 - t)^{p_2}, \\ c(t) &\sim (t_1 - t)^{p_3}, \end{aligned} \tag{31}$$

where

$$\begin{aligned} p_1 &= \frac{\alpha_0 - \beta_0 - \sqrt{\alpha_0^2 - \beta_0^2}}{2\alpha_0 - \sqrt{\alpha_0^2 - \beta_0^2}} < 0, \\ p_2 &= \frac{\alpha_0 + \beta_0 - \sqrt{\alpha_0^2 - \beta_0^2}}{2\alpha_0 - \sqrt{\alpha_0^2 - \beta_0^2}} > 0, \\ p_3 &= \frac{\sqrt{\alpha_0^2 - \beta_0^2}}{2\alpha_0 - \sqrt{\alpha_0^2 - \beta_0^2}} > 0. \end{aligned} \tag{32}$$

It is straightforward to check that the exponents p_1, p_2, and p_3 indeed satisfy the Kasner relations, i.e.,

$$p_1 + p_2 + p_3 = p_1^2 + p_2^2 + p_3^2 = 1, \tag{33}$$

which means that the presence of the magnetic field does not change the character of the singularity. The reason for such a behavior is not hard to guess. Substituting the expressions for V and $\ddot{V}$ into Eq. (19), one obtains that the magnetic field contributes to Einstein's equations as

$$\frac{B_0^2}{a^2b^2} \sim \frac{\mu\eta_1}{\gamma}(\mu - 1)(t_1 - t)^{\mu-3}, \tag{34}$$

where $\mu - 3 > -2$. Therefore, the term (34) is weaker than the anisotropy, which contributes a term of order $(t_1 - t)^{-2}$ and dominates near the singularity. The presence of matter less stiff than stiff matter is well known to leave the Kasner type of singularity unaffected [33].

B. Expanding universe

Let us consider now the expanding universe. In this case, the constant $\alpha_0 < 0$ and the time derivative $\dot{V}$ will correspondingly be positive. The expansion will last for $t \to \infty$ with the time derivative $\dot{V}$ approaching the critical value

$$W_2 \equiv -2\alpha_0 + \sqrt{\alpha_0^2 - \beta_0^2} > 0. \tag{35}$$

Thus, the behavior of $V(t)$ can be approximated in the limit $t \to \infty$ as

$$V \simeq W_2 t - \eta_2 t^{\nu}, \tag{36}$$

where η_2 is a positive constant and $0 < \nu < 1$. On substituting the corresponding expressions for V, $\dot{V}$, and $\ddot{V}$ in Eq. (20), we find again

$$\nu = 1 + \frac{2\sqrt{\alpha_0^2 - \beta_0^2}}{2\alpha_0 - \sqrt{\alpha_0^2 - \beta_0^2}}, \tag{37}$$

with $\frac{1}{3} \leq \nu < 1$. The anisotropy factors then read

$$\alpha(t) = \left(\frac{2}{3} + \frac{\alpha_0}{\sqrt{\alpha_0^2 - \beta_0^2} - 2\alpha_0}\right) \ln t, \tag{38}$$

$$\beta(t) = \frac{\beta_0 \ln t}{\sqrt{\alpha_0^2 - \beta_0^2} - 2\alpha_0}. \tag{39}$$

Correspondingly, the scale factors take again the Kasner form

$$a(t) \sim t^{p_1}, \qquad b(t) \sim t^{p_2}, \qquad c(t) \sim t^{p_3}, \tag{40}$$

where

$$p_1 = \frac{\sqrt{\alpha_0^2 - \beta_0^2} - \alpha_0 + \beta_0}{\sqrt{\alpha_0^2 - \beta_0^2} - 2\alpha_0},$$
$$p_2 = \frac{\sqrt{\alpha_0^2 - \beta_0^2} - \alpha_0 - \beta_0}{\sqrt{\alpha_0^2 - \beta_0^2} - 2\alpha_0},$$
$$p_3 = -\frac{\sqrt{\alpha_0^2 - \beta_0^2}}{\sqrt{\alpha_0^2 - \beta_0^2} - 2\alpha_0}, \tag{41}$$

which also satisfy the Kasner relations (33). The presence of the magnetic field does not influence the asymptotic structure of the metric at $t \to \infty$, which therefore does not isotropize, unlike the Heckmann-Schucking solution with dust [21,25]. The reason for this phenomenon is clear. The energy density of the magnetic field at $t \to \infty$ is given by

$$\frac{B_0^2}{a^2 b^2} \simeq \nu(\nu - 1)\frac{\eta_2}{W_2} t^{\nu - 3}, \tag{42}$$

where $\nu - 3 < -2$. Hence it remains weaker than the anisotropy term.

Now we can try to understand what happens with the expanding universe in its distant past. One can suppose that it was born from the initial singularity at $t = 0$, when its volume $V(0) = 0$, and its time derivative had the smallest critical value

$$\dot{V}(0) = W_3 \equiv -2\alpha_0 - \sqrt{\alpha_0^2 - \beta_0^2} > 0. \tag{43}$$

In this case, the scale factors will be of the form in Eq. (40) with the Kasner exponents

$$p_1' = \frac{\alpha_0 - \beta_0 + \sqrt{\alpha_0^2 - \beta_0^2}}{2\alpha_0 + \sqrt{\alpha_0^2 - \beta_0^2}},$$
$$p_2' = \frac{\alpha_0 + \beta_0 + \sqrt{\alpha_0^2 - \beta_0^2}}{2\alpha_0 + \sqrt{\alpha_0^2 - \beta_0^2}},$$
$$p_3' = -\frac{\sqrt{\alpha_0^2 - \beta_0^2}}{2\alpha_0 + \sqrt{\alpha_0^2 - \beta_0^2}}. \tag{44}$$

They again satisfy the Kasner relations (33). To establish a relation between the set of Kasner indices at the beginning and at the end of the evolution, it is convenient to use the Lifshitz–Khalatnikov parametrization [11]. If the Kasner indices are ordered as

$$p_1 \leq p_2 \leq p_3, \tag{45}$$

they can be represented by means of a real parameter $u \geq 1$ according to

$$p_1 = -\frac{u}{1 + u + u^2},$$
$$p_2 = \frac{1 + u}{1 + u + u^2},$$
$$p_3 = \frac{u(1 + u)}{1 + u + u^2}. \tag{46}$$

The ordering (45) can be obtained, for example, by setting the anisotropy parameters

$$\alpha_0 < 0, \qquad \beta_0 < 0, \qquad |\beta_0| < \frac{3}{5}|\alpha_0|. \tag{47}$$

In particular, this choice implies that the universe expands in the y and z directions, but does so more rapidly along the direction z of the magnetic field, while it contracts along the x axis. Combining Eqs. (44) and (47), we obtain

$$u' = \frac{p_3'}{p_2'} = \frac{\sqrt{\alpha_0^2 - \beta_0^2}}{|\alpha_0| + |\beta_0| - \sqrt{\alpha_0^2 - \beta_0^2}}. \tag{48}$$

It is convenient to introduce the parameter $\xi = \frac{|\beta_0|}{|\alpha_0|}$, which, when plugged into Eq. (48), results in the relation

$$\xi = \frac{(u' + 1)^2 - u'^2}{(u' + 1)^2 + u'^2}. \tag{49}$$

Note that, if ξ satisfies the conditions (47), then $1 < u' < \infty$. Let us now look at the Kasner exponents (41) describing the final stage of cosmological evolution. In this case the order of the exponents is

$$p_3 \le p_1 \le p_2. \tag{50}$$

Correspondingly, we can represent them as

$$p_1 = \frac{1+u}{1+u+u^2},\quad p_2 = \frac{u(1+u)}{1+u+u^2},\quad p_3 = -\frac{u}{1+u+u^2}. \tag{51}$$

Thus,

$$u = \frac{p_2}{p_1} = \frac{\sqrt{\alpha_0^2-\beta_0^2}-\alpha_0-\beta_0}{\sqrt{\alpha_0^2-\beta_0^2}-\alpha_0+\beta_0}. \tag{52}$$

Substituting the formulas (48) and (49) into the equation above, we find

$$u = \frac{1+u'}{u'} < 2. \tag{53}$$

Inversely,

$$u' = \frac{1}{u-1}. \tag{54}$$

For the inverse evolution towards the singularity, as one usually analyzes the oscillating approach towards the cosmological singularity [3,12], one can see that the universe passes through two transformations in the transition from the phase described by the parameter u to the phase described by u', according to Eq. (54). The first transformation is characterized by the shift $u \to u-1$, in which the roles of the x and z axes, corresponding to the exponents p_1 and p_3, are exchanged. This transformation is called "change of the Kasner epoch" [12]. As a result of this transformation, we arrive at a value of the parameter $u-1<1$, see Eq. (53). The next transformation is defined by

$$u-1 \to \frac{1}{u-1}, \tag{55}$$

which exchanges the roles of the axes y and z, and is called "change of Kasner era".

We see that our solution displays a transition between two Kasner regimes chacterized by the same law found in an empty Bianchi-II universe. For a detailed description of the dynamics in the Bianchi-II universe in general relativity and other gravity models see Ref. [34]. In Bianchi-VIII or Bianchi-IX models, the universe passes through an infinite series of changes of the Kasner epochs and eras. In our case, for the choice of parameters in Eq. (47), the law of the transformation for the Kasner exponents (54) includes one change of Kasner epoch and one change of Kasner era.

One can also consider the opposite relation between the anisotropy parameters, giving

$$\alpha_0 < 0, \qquad \beta_0 < 0, \qquad \frac{3}{5}|\alpha_0| < |\beta_0| < 1, \tag{56}$$

or, in other words, $3/5 < \xi < 1$, and we have $u' = p_2'/p_3'$, which corresponds to

$$\xi = \frac{(u'+1)^2-1}{(u'+1)^2+1}, \tag{57}$$

and $u = u'+1$ or, inversely, $u' = u-1$. The last two relations show that we now have only a change of Kasner epoch.

C. Exact evolution

As we already pointed out, in addition to the qualitative analysis given in the previous subsections, Eq. (20) in principle is integrable. We can cast it in the form

$$\frac{d}{dt}(V\dot{V} + 4\alpha_0 V) = -3\alpha_0^2 - \beta_0^2, \tag{58}$$

which can be integrated by defining $V = Xt$, leading to

$$\frac{1}{2}\ln\left(1+\frac{X^2+4\alpha_0 X}{3\alpha_0^2+\beta_0^2}\right) + \frac{\alpha_0}{\sqrt{\alpha_0^2-\beta_0^2}}\left[\ln\left(1+\frac{X}{2\alpha_0+\sqrt{\alpha_0^2-\beta_0^2}}\right) - \ln\left(1+\frac{X}{2\alpha_0-\sqrt{\alpha_0^2-\beta_0^2}}\right)\right] = \ln\left(\frac{t_0}{t}\right), \tag{59}$$

where integration constants are chosen so that $t_0 > 0$ is the time at which the volume $V(t)$, hence $X(t)$, vanishes.

Unfortunately, Eq. (59) cannot be inverted to find $X(t)$ as a function of t. Thus, we cannot use this equation to find exact expressions for the anisotropy factors. Notwithstanding, one can use it to study the approach to the singularity for $t \to t_0$ when $X(t) \to 0$. Expanding the left-hand side of Eq. (59), we see that the leading term is proportional to X^2, and the equation reduces to

$$X^2 \simeq 2(3\alpha_0^2+\beta_0^2)\frac{t_0-t}{t_0}. \tag{60}$$

This implies

$$V \simeq \sqrt{2(3\alpha_0^2+\beta_0^2)t_0(t_0-t)} \tag{61}$$

and $\dot{V} \to -\infty$, which obviously breaks the inequality (21). The approximate solution (60) therefore corresponds to initial conditions which are incompatible with the existence of a homogenous magnetic field.

However, the exact solution (59) admits a different regime corresponding to a volume singularity given by $t \to 0$ with $X(0)$ finite. To describe this regime, one can confront the terms proportional to $\ln t$ in the left-hand side and the right-hand side of Eq. (59). In this case, the function $X(t)$ tends to a positive constant which can be found from a quadratic equation and corresponds to the extremal points of the inequality (21). Thus, the regime of the approach to the singularity described above is compatible with the general solution of Eq. (58) and arises in the vicinity of the point $t = 0$.

D. Anisotropy and relations with the Bianchi-II universe dynamics

The Kasner solution for the Bianchi-I universe is the regime of maximal anisotropy. The degree of this anisotropy does not depend on the particular values of the Kasner indices or, equivalently, on the value of the Lifshitz-Khalatnikov parameter. Indeed, to explain this fact, let us introduce the dispersion of the set of Kasner's indices

$$\sigma^2 = \left(p_1 - \frac{1}{3}\right)^2 + \left(p_2 - \frac{1}{3}\right)^2 + \left(p_3 - \frac{1}{3}\right)^2. \qquad (62)$$

Using the Kasner relations (33), we see that $\sigma^2 = 2/3$ and does not depend on the particular choice of the p_i's. Thus, at the beginning and end of the evolution described in the preceding subsections, the universe is maximally anisotropic (i.e., anisotropy evolves like R^{-6}), and the presence of the magnetic field is not significant in the very early and in the very late Universe, as we have seen. However, during the intermediate stage of evolution, it is the presence of the magnetic field that drives the transition from one Kasner regime to another.

We have already highlighted that the character of this transition exactly coincides with one that takes place in an empty Bianchi-II universe. It is interesting to find the roots of this phenomenon. The laws of change of Kasner epochs and, sometimes, eras are well known and can be worked out in two ways. The first one is to solve exactly the differential equations for the Bianchi-II model (see e.g., [34,35] and references therein). These equations acquire a Liouville-like form if logarithmic time is used. Alternatively, one can study these equations qualitatively, arriving at the same conclusions concerning the asymptotic regimes [1]. Here, we would like to treat the dynamics of the empty Bianchi-II universe using the same method that was used to study the Bianchi-I universe with magnetic field. In this way, we shall manage to understand why the laws of transformation between Kasner regimes coincide in these quite different physical models. Besides, we shall obtain another simple way to derive the law of transformation for the Bianchi-II universe.

All of the Bianchi geometries can be represented in a synchronous reference system by a metric of the form

$$ds^2 = dt^2 - a^2(t)\omega^1 \otimes \omega^1 - b^2(t)\omega^2 \otimes \omega^2 - c^2(t)\omega^3 \otimes \omega^3, \qquad (63)$$

where ω^1, ω^2, and ω^3 are the basis one-forms dual to the basis of the reciprocal vector fields which commute with all the Killing vector fields [2]. The differences between various Bianchi models are encoded by the structure coefficients of the Lie algebra of the three Killing vector fields. Moreover, the components of the Ricci tensor for the metric (63) have the following form:

$$R^i{}_j = R^i{}_j(K) - P^i{}_j, \qquad (64)$$

where $R^i{}_j(K)$ is the part of the Ricci tensor depending on the extrinsic curvature given in Eqs. (9)–(12). The tensor $P^i{}_j$ is constructed instead from the components of the three-dimensional spatial metric and can be expressed in terms of the scale factors $a(t)$, $b(t)$, and $c(t)$ and of the structure constants of the Lie algebra of the Killing vector fields. For the Bianchi-I Lie algebra, the structure constants are zero, hence $P^i{}_j = 0$. For the Bianchi-II Lie algebra, only one structure constant is not zero and can be chosen so that the Lie bracket of the first two Killing vectors be equal to the third Killing vector. The nonvanishing components of the tensor $P^i{}_j$ then read

$$P^1{}_1 = P^2{}_2 = -P^3{}_3 = -\frac{c^2}{2a^2b^2} = -\frac{e^{-8\alpha(t)}}{2R^2}, \qquad (65)$$

It follows that we can treat an empty Bianchi-II universe as a Bianchi-I universe filled with a particular kind of matter characterised by the energy-momentum tensor

$$T^i{}_j = P^i{}_j - \frac{1}{2}P\delta^i_j, \qquad (66)$$

with components

$$T^0{}_0 = -T^1{}_1 = -T^2{}_2 = \frac{1}{3}T^3{}_3 = \frac{e^{-8\alpha(t)}}{4R^2}. \qquad (67)$$

Substituting these expressions into Einstein's equations for the Bianchi-I universe, we can study the dynamics of the Bianchi-II universe. First of all, considering the difference $R^1{}_1 - R^2{}_2$, we again obtain the relation (15). Then, the combination $R^1{}_1 + R^2{}_2 - 2R^3{}_3$ yields a relation similar to Eq. (16), giving

$$\ddot{\alpha} + 3\dot{\alpha}\frac{\dot{R}}{R} = \frac{e^{-8\alpha(t)}}{3R^2}. \qquad (68)$$

Combining $R^1{}_1 + R^2{}_2 + 2R^3{}_3$, we get the relations (17) and (18) again, and a combination of (17) with (68) yields

$$\frac{1}{R^3}\frac{d^2R^3}{dt^2} = \frac{e^{-8\alpha(t)}}{2R^2}, \tag{69}$$

which is similar to Eq. (19). To obtain the counterpart of Eq. (20), we use the ${}^0{}_0$ component of Einstein equations, and get

$$V\ddot{V} + 2(\dot{V}^2 + 4\alpha_0\dot{V} + 3\alpha_0^2 + \beta_0^2) = 0. \tag{70}$$

The only difference between Eqs. (20) and (70) is the additional factor of 2 in front of the round bracket of Eq. (70). As follows from Eq. (69), the second time derivative of the volume should be positive, which implies that the inequality $\alpha_0^2 \geq \beta_0^2$ has to hold.

By repeating all the considerations of the preceding subsections, we can see that the parameters of the initial and final Kasner regimes coincide with those obtained for the Bianchi-I universe with the magnetic field. The only difference consists in the fact that the Bianchi-II universe goes out of the initial Kasner regime rapidly and enters the final Kasner regime early, which is consistent with the Bianchi-I universe filled with a magnetic field. Numerical solutions for the associated equations, namely Eqs. (20) and (70), are compared in Fig. 1 for particular values of the parameters and initial conditions.

III. BIANCHI-I MODELS WITH MAGNETIC FIELD AND ADDITIONAL PERFECT FLUIDS

In the previous section, we analyzed cosmological models just sourced by a magnetic field. Now, we will briefly consider two models in which we add dust and a massless scalar field (stiff matter), respectively.

A. Dust

The energy density of dust is $\rho = \rho_0 R^{-3}$, where ρ_0 is a positive constant. It follows immediately from Einstein's equations that the scalar curvature is then given by $\mathcal{R} = -\rho$.

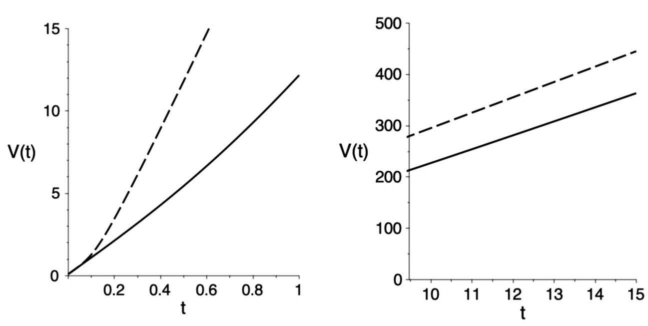

FIG. 1. Numerical solutions of Eqs. (20) (solid line) and (70) (dashed) for $\alpha_0 = -10$, $\beta_0 = 1$ and with initial conditions $V(0) = 10^{-6}$, $\dot{V}(0) = 10.051 \simeq W_3$ from Eq. (43).

It is easy to see that Eq. (15) does not changed in the presence of dust and the anisotropy function β is still the same. Eq. (17) instead changes because the scalar curvature does not vanish and reads

$$\begin{aligned} R^1{}_1 + R^2{}_2 + 2R^3{}_3 &= -4\frac{\ddot{R}}{R} - 8\frac{\dot{R}^2}{R^2} + 6\dot{\alpha}\frac{\dot{R}}{R} + 2\ddot{\alpha} \\ &= 2\mathcal{R} = -\frac{2\rho_0}{R^3}. \end{aligned} \tag{71}$$

This equation can be rewritten in the manner of the beginning of this section, which yields, analogously to Eq. (18),

$$\dot{\alpha} = 2\frac{\dot{R}}{R} - \frac{\rho_0 t}{R^3} + \frac{\alpha_0}{R^3}. \tag{72}$$

Equation (16) is still valid, and we obtain

$$\frac{B_0^2}{R^4}e^{-4\alpha(t)} = \frac{1}{R^3}\frac{d^2R^3}{dt^2} - \frac{\rho_0}{R^3}, \tag{73}$$

which implies $\frac{d^2R^3}{dt^2} > \rho_0$. By performing the analysis as in the case without dust, we obtain the equation for the volume

$$V\ddot{V} = -\dot{V}^2 + 4(\rho_0 t - \alpha_0)\dot{V} - 3\alpha_0^2 - \beta_0^2 + 3\rho_0(2\alpha_0 - \rho_0 t)t, \tag{74}$$

which we cannot solve exactly.

An axisymmetric solution is known in the literature [29–31], corresponding to $\beta_0 = 0$. Since this solution is rather cumbersome and implicit, we shall just undertake a simple qualitative analysis of Eq. (74). One can see that, when the universe approaches the singularity $V(t) = R^3(t) = 0$, the terms proportional to ρ_0 in Eq. (74) become negligible, and we come back to the situation without dust. Thus, we have a Kasner type of singularity with a positive Kasner exponent in the direction of the magnetic field. When the volume of the universe grows towards infinity, we encounter the opposite situation. The term $3\rho_0 t^2$ dominates, since $V(t) \sim t^2$, and we have isotropization just like in the standard Heckmann-Schucking solution.

B. Massless scalar field

Let us consider a Bianchi-I universe with magnetic field as in the preceding sections and a homogeneous massless scalar field, $\phi = \phi(t)$. For a massless scalar field, the pressure equals the energy density, and the perfect fluid with such an equation of state is called stiff matter. The Klein-Gordon equation for $\phi(t)$ in the Bianchi-I universe has the form

$$\ddot{\phi} + 3\frac{\dot{R}}{R}\dot{\phi} = 0, \tag{75}$$

whose general solution is given by $\dot{\phi} = \frac{\tilde{\phi}_0}{R^3} = \frac{\tilde{\phi}_0}{V}$, where $\tilde{\phi}_0$ is a constant. One can see that $\dot{\phi}$ behaves in the same way as the time derivative of the anisotropy factor $\beta(t)$. It is convenient to rescale $\tilde{\phi}_0 = \sqrt{2}\phi_0$, so that the scalar field energy-momentum tensor reads

$$T^0{}_0 = -T^1{}_1 = -T^2{}_2 = -T^3{}_3 = \frac{\phi_0^2}{V^2}, \tag{76}$$

and the scalar curvature $\mathcal{R} = \frac{2\phi_0^2}{V^2}$. Obviously, the presence of ϕ does not change Eq. (16). Moreover, due to the special form of the spatial components in Eq. (76) and $\mathcal{R}$, Eq. (17) also does not change in form. Therefore, the results in Eqs. (18) and (19) still hold. However, the scalar field contributes to the ${}^0{}_0$ component of Einstein's equations (7). From Eq. (76), we see that ϕ_0 enters the equation for the volume (20) on an equal footing with the parameter β_0. Thus, the new equation for $V(t)$ coincides with the formulas in Sec. II with β_0^2 replaced by $\beta_0^2 + \phi_0^2$.

There is no need to rewrite all of the formulas from Sec. II explicitly, but we just focus on the expressions for the scale factors for the expanding universe in the vicinity of the singularity. By expanding around $t = 0$, we obtain the Kasner form (40) with

$$p_1 = \frac{\alpha_0 - \beta_0 + \sqrt{\alpha_0^2 - \beta_0^2 - \phi_0^2}}{2\alpha_0 + \sqrt{\alpha_0^2 - \beta_0^2 - \phi_0^2}},$$
$$p_2 = \frac{\alpha_0 + \beta_0 + \sqrt{\alpha_0^2 - \beta_0^2 - \phi_0^2}}{2\alpha_0 + \sqrt{\alpha_0^2 - \beta_0^2 - \phi_0^2}},$$
$$p_3 = -\frac{\sqrt{\alpha_0^2 - \beta_0^2 - \phi_0^2}}{2\alpha_0 + \sqrt{\alpha_0^2 - \beta_0^2 - \phi_0^2}}, \tag{77}$$

where we have chosen $\alpha_0 < 0$ and $\beta_0 \leq 0$. It is well known [33] that the presence of the massless scalar field changes the relations for the Kasner exponents. While they still satisfy $p_1 + p_2 + p_3 = 1$, the sum of their squares is smaller then one. Namely, using the formulas (77), we obtain $p_1^2 + p_2^2 + p_3^2 = 1 - q^2$, with the parameter

$$q^2 = \frac{2\phi_0^2}{5\alpha_0^2 + 4\alpha_0\sqrt{\alpha_0^2 - \beta_0^2 - \phi_0^2} - \beta_0^2 - \phi_0^2}, \tag{78}$$

satisfying the inequalities

$$0 \leq q^2 < \frac{1}{2}. \tag{79}$$

It is worth noting that, in the case of a universe filled only with the massless scalar field, the parameter q^2 must satisfy the less stringent bounds [33]

$$0 \leq q^2 \leq \frac{2}{3}. \tag{80}$$

One can grasp where the root of the difference lies between Eqs. (79) and (80). The parameter q^2 indicates some kind of isotropization for the universe. The limiting value $q^2 = 2/3$ in Eq. (80) implies that $p_1 = p_2 = p_3 = 1/3$, i.e., that the expansion of the universe is totally isotropic. The presence of a magnetic field in the z direction makes such a high degree of isotropization impossible, which is then reflected in Eq. (79). Finally, the Kasner exponents when the volume of the universe tends to infinity are given by

$$p_1' = \frac{\alpha_0 - \beta_0 - \sqrt{\alpha_0^2 - \beta_0^2 - \phi_0^2}}{2\alpha_0 - \sqrt{\alpha_0^2 - \beta_0^2 - \phi_0^2}},$$
$$p_2' = \frac{\alpha_0 + \beta_0 - \sqrt{\alpha_0^2 - \beta_0^2 - \phi_0^2}}{2\alpha_0 - \sqrt{\alpha_0^2 - \beta_0^2 - \phi_0^2}},$$
$$p_3' = \frac{\sqrt{\alpha_0^2 - \beta_0^2 - \phi_0^2}}{2\alpha_0 - \sqrt{\alpha_0^2 - \beta_0^2 - \phi_0^2}}, \tag{81}$$

so that $(p_1')^2 + (p_2')^2 + (p_3')^2 = 1 - (q')^2$, where

$$(q')^2 = \frac{2\phi_0^2}{5\alpha_0^2 - 4\alpha_0\sqrt{\alpha_0^2 - \beta_0^2 - \phi_0^2} - \beta_0^2 - \phi_0^2}, \tag{82}$$

and $(q')^2 < q^2$. Therefore, in contrast to the isotropization induced by the presence of dust in the Heckmann-Schucking solution, the value of q^2 decreases, going from the singularity towards an infinite expansion, and the degree of anisotropy increases. This effect was noticed in the study of the anisotropic universe with the scalar field only [25,33].

C. Some numbers

While our Bianchi-I universe filled with a spatially homogeneous magnetic field oriented along one of the axes is substantially a mathematical model and cannot represent a realistic description of our Universe, one can hope that it helps to capture some interesting features of cosmology. For example, it could describe the evolution of some local patches of the universe. Thus, it is interesting to try and consider some values for the magnetic field coming from the comparison between models and observations (see e.g., Refs. [36–39]), and check what we can expect in the framework of our solutions.

According to Ref. [36], most constraints on cosmological magnetic fields give an upper bound around 10^{-9} G, assuming coherence on Mpc scales or larger. Let us suppose that, at the end of the evolution, the magnetic field has such a magnitude and estimate the value that the magnetic field could have had close to the initial singularity, or in the very early universe. One can ask; What does it mean "close to the singularity"? For example, we can identify the early time as the beginning of the inflationary expansion. The time interval to consider will correspondingly be the standard number of e-folds, i.e., $N_e \approx 60$ or, in redshift terms, something like $z \sim 10^{50}$. In our solution, the magnetic field $B \propto a^{-1}b^{-1}$, so that the ratio between the "early" and "late" values of the magnetic field is

$$\frac{B_{\rm in}}{B_{\rm fin}} = \frac{a_{\rm fin}}{a_{\rm in}} \frac{b_{\rm fin}}{b_{\rm in}}. \tag{83}$$

If the expansion were isotropic, this ratio would be equal to $z^2 \sim 10^{100}$, and the initial value of the magnetic field would be huge. However, anisotropy can change the situation drastically. Indeed, our evolution represents the transition from one Kasner regime to another. Consider the situation at the beginning of the evolution, when the scale factor $a(t)$ decreases while $b(t)$ grows. In this case,

$$ab \sim t^{p_1+p_2} = t^{1-p_3} \tag{84}$$

and the growth in time of the magnetic field can be very slow if the Kasner index p_3 is close to 1. Subsequently, the scale factors $a(t)$ and $b(t)$ exchange their roles, though that could happen (at high values of the parameter u) such that the index p_3 is still close to 1. One can therefore imagine that, during a very large anisotropic expansion, the value of the magnetic field changes slowly, and the small value of the magnetic field at the end of inflation is quite compatible with reasonable values in the very early universe.

IV. SINGULARITIES AND THEIR CROSSING

The existence of singularities in cosmological models has attracted the attention of researchers working in general relativity and its modifications for a long time. In fact, the question of the initial singularity was already discussed in the seminal paper by Robertson [40], where the early development of Friedmann-type cosmologies was reviewed and generalized. The connection between the presence and sign of the spatial curvature, the value of the cosmological constant, and the dependence of the pressure on the scale factor of the universe were studied there in detail with regard to the appearance of a singularity. It is interesting that Robertson also considered the scenario of a cyclic evolution, described by some trigonometric law, in which the universe emerges from a singularity, expands to the maximum value of its radius, then contracts back, and the process repeats itself indefinitely. It appears that a universe which goes through this singularity did not disturb Robertson too much. Another type of cosmological singularity, which can arise in the future for some finite values of the scale factor of the universe and can be rather soft, was described in Ref. [41]. The interest in such singularities essentially increased during the last few years (see e.g., the review [42] and references therein).

In contrast with the crossing of soft singularities, the idea of crossing the big bang–big crunch singularity appears rather counterintuitive. For many years, the desire to look for models free of such singularities dominated, although the idea of the possible transition from the big crunch to the big bang was studied in some cosmological models. First of all, we would like to mention the string or prebig bang scenario [43–45]. It is worth noting that not only isotropic Friedmann models but also anisotropic Bianchi-I models [46,47] were studied in this framework. In these works, the universe contained a dilaton supplemented by an antisymmetric tensor field, which influenced its dynamics. Another approach to the problem of the singularity, also inspired by superstring theories, was developed in Refs. [48–50]. In Ref. [49], the authors treated the singularity as the transition between a contracting big crunch phase and an expanding big bang phase. A crucial role in their analysis was played by a massless scalar field, a modulus. The theory was reformulated so as to employ variables that remain finite as the scale factor shrinks to zero, which suggests a natural way to match the solutions before and after the singularity. The general features of the approach in Refs. [48–50] are the role played by the scalar field and the construction of variables which are finite at the singularity crossing. These features are also essential for the approach developed in Refs. [51–54].

Recently, other approaches to the problem of the description of such a crossing were elaborated in Refs. [55–62]. Behind these approaches are basically two general ideas. Firstly, to cross the singularity, one must give a prescription matching nonsingular, finite quantities before and after the crossing. Secondly, such a description can be achieved by using a convenient choice of field parametrizations. In Ref. [51], a version of the description of the crossing of singularities in universes filled with scalar fields was elaborated based on the transition between the Jordan and the Einstein frames. The main idea of Ref. [51] is the following. When in the Einstein frame the universe arrives at the big bang–big crunch singularity, from the point of view of the evolution of its counterpart in the Jordan frame, its geometry is regular, but the effective Planck mass is zero. The solution to the equations of motion in the Jordan frame is smooth at this point, and using the relations between the solutions of the cosmological equations in the two frames, one can describe the crossing of the cosmological singularity in a uniquely determined way. In Ref. [63], it was pointed out that in a homogeneous and isotropic universe, one can indeed cross the point where the effective gravitational constant changes sign. However, the presence of anisotropies or inhomogeneities changes the situation drastically because these anisotropies and inhomogeneities grow indefinitely when the value of the effective Planck mass tends to zero, hence the effective gravitational constant diverges. In Ref. [52], the authors investigated this phenomenon, suggesting a simple field reparametrization that allows one to describe the big bang–big crunch singularity crossing in the Bianchi-I model filled with a minimally coupled scalar field. Further studies of the description of the big bang–big crunch singularity crossing in anisotropic models were undertaken in Refs. [53,54]. An attempt to develop a general approach to the problem of the treatment of the singularities in classical and quantum cosmology based on the reparametrizations in field space was also undertaken in Refs. [64–66].

Taking into account all of the above, we shall try to describe what happens with our solution when the Bianchi-I universe with a magnetic field encounters the singularity. Let us tackle the problem by studying the behavior of the differential equations "behind the singularity", where these equations and their solutions remain mathematically well-defined but variables can take on a different meaning from their original one. For example, the variable $V(t)$, defined as a volume, can become negative. Let us come back to Eq. (20), describing the contraction of the universe when the parameter α_0 is positive. We have seen that such a universe reaches $V(t_1) = 0$ when the velocity of the contraction is $\dot{V}(t_1) = -W_1 = -2\alpha_0 + \sqrt{\alpha_0^2 - \beta_0^2}$. If we continue the solution for $t > t_1$, the variable V becomes negative, its time derivative remains negative, and it does not satisfy the inequality (21). It follows from Eq. (20) that the second derivative $\ddot{V}$ remains positive. Thus, $V(t)$ undergoes some kind of "negative expansion" for a period of time during which it remains negative while its absolute value grows. Since $\ddot{V} > 0$, at some moment, say $t = t_3$, the time derivative passes through the value $\dot{V}(t_3) = 0$ and becomes positive. Correspondingly, the function $V(t)$ also starts to increase (its absolute value is decreasing, so we can speak about "negative contraction"). At a later moment, $t = t_4$, the spatial volume $V(t)$ should vanish. However, the velocity $\dot{V} > 0$ and we cannot enter the interval (21) if the parameter α_0 is positive, hence the critical $-W_1 = -2\alpha_0 + \sqrt{\alpha_0^2 - \beta_0^2}$ is negative. This implies that we encounter a singularity of another type when $V(t) \sim \sqrt{t_4 - t}$, which we have described before. And it is impossible to cross this singularity, coming back to the "normal" part of the phase space $(V, \dot{V})$ of our problem, where we were before the first encounter with the singularity and where the magnetic field was well defined.

Nevertheless, a nice way to come back to the "normal universe" after this journey behind the "looking glass" still exists. It consists of a simple prescription; we can change the sign of the constant α_0. The reasonable moment for this change is exactly at $t = t_3$, because the parameter α_0 enters Eq. (20) either squared or multiplied by $\dot{V}$ and the velocity $\dot{V}(t_3) = 0$. To keep the parameters α_0 and β_0 on equal footing, we can change the sign of the parameter β_0 as well. After that, the universe goes towards the singularity, where $V(t_4) = 0$, $\dot{V}(t_4) = W_3 = -2\alpha_0 - \sqrt{\alpha_0^2 - \beta_0^2} > 0$, and crosses into the "normal" part of the phase space expanding.

Now, it is interesting to compare the anisotropy parameters of the universe before it hits the singularity and after it jumps out of it. Using the above solution $\gamma = W_1$ to Eq. (27), we found the expressions (31)–(32) for the scale factors $a(t)$, $b(t)$, and $c(t)$ when the universe is contracting towards the singularity at $t = t_2$. Let us choose $\beta_0 = \xi\alpha_0$ and $0 \leq \xi \leq 3/5$. In this case, $p_3 > p_2$ and

$$u = \frac{\sqrt{\alpha_0^2 - \beta_0^2}}{\alpha_0 + \beta_0 - \sqrt{\alpha_0^2 - \beta_0^2}}, \tag{85}$$

so ξ is just like in Eq. (49). Simple calculations then show that the Kasner indices of the universe that go out from the region behind the singularity coincide with those it had before the encounter with the singularity. It is not clear how we can interpret the period of time that the universe has spent behind the singularity.

One can suggest another way to describe the transition through the singularity. The Einstein equations for our system are invariant under a change in the sign of the scale factors. Indeed, these equations only contain terms like $\ddot{a}/a$, $\dot{a}/a$, or a^2. Thus, we can change the signs of all three scale factors so that the volume becomes positive again. To make the equation (20) for the volume invariant with respect to the change of sign of $V(t)$, we should also change the sign of the parameter α_0. In this way, the positive time derivative of the volume $\dot{V}$ immediately acquires a lower critical value $2|\alpha_0| - \sqrt{\alpha_0^2 - \beta_0^2}$ and an unlimited expansion begins.

Both prescriptions described above imply the same evolution after the singular bounce, but these two bounces have a slightly different origin.

Finally, we would like to make a comment concerning the behavior of the magnetic field as the universe tends toward the singularity. Naturally, the energy density of the magnetic field diverges when the universe enters a Kasner regime approaching the singularity. Namely, this density behaves as $a^{-2}b^{-2} \sim t^{-2(1-p_3)}$. Nevertheless, if the Kasner index p_3 is close to 1, this divergence is much milder than the one in a isotropic universe. In any case, as we have explained before, the presence of the diverging quantities does not prevent us from describing the singularity crossing.

V. CONCLUDING REMARKS

We have studied in detail some features of the Bianchi-I universe filled with a spatially homogeneous magnetic field. Our finding is that the universe is in the Kasner regime at the beginning and end of cosmological evolution. We have established the relationship between the parameters of these two regimes and shown that it coincides with the one in the empty Bianchi-II universe, which goes out of the initial Kasner regime rapidly and enters the final Kasner regime earlier than the Bianchi-I universe filled with a magnetic field. In addition to a spatially homogeneous magnetic field, the universe also containing dust undergoes the process of isotropization, while the presence of a massless scalar field implies a modification of the relations between Kasner indices in two asymptotic regimes.

Here we would also like to briefly mention other recent works in which different aspects of the Bianchi-I geometry were studied. First of all, let us note that the role of the time coordinate and of one of the spatial coordinates in the expression for the metric (1) can be exchanged [1]. This possibility was already considered in the earliest works [8–10] and the study of the spatial Kasner-like geometries (in particular, in the presence of thin or thick walls or slabs) still attracts researchers [67–71]. The geodesics in Kasner's universe were considered in Ref. [72]. Applications of Kasner-type spacetimes to cosmic jets were done in Refs. [73,74], and the behavior of fields and particles with spin in Bianchi-I universes was studied [75,76]. In recent papers [77,78], the comparison of the Bianchi-I geometry with the real very early Universe and our late-time Universe was undertaken, while Ref. [79] was devoted to the study of Horndeski theory and Ref. [80] to loop quantum cosmology in the Bianchi-I universe. Thus, consideration of the simplest nonisotropic cosmological models still flourishes and can bring surprises [81].

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