Realizations of type III stress-energy tensors of the Hawking-Ellis classification in scalar-tensor gravity

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The "ugly duckling" of the Segré-Plebański-Hawking-Ellis classification of stress-energy tensors is believed to be either impossible or extremely difficult to realise in Einstein gravity. *Effective* stress-energy tensors in alternative gravity offer a wider range of possibilities. We report a class of type-III realizations in "first-generation" scalar-tensor and in Horndeski gravity, and their physical interpretation. The ugly duckling may be a freak of nature of limited importance but it is not physically impossible.

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I. INTRODUCTION

In general relativity, the right-hand side of the Einstein field equations

$$G_{ab} \coloneqq R_{ab} - \frac{1}{2} g_{ab} R = T_{ab}, \tag{1.1}$$

(where R_{ab} and $R \coloneqq R^a{}_a$ are the Ricci tensor and the Ricci scalar of the spacetime metric g_{ab}, respectively)[1] contains the energy-momentum tensor of matter $T^{(\mathrm{m})}_{ab}$. The possible forms of $T^{(\mathrm{m})}_{ab}$ expected on physical grounds have been classified by Segre [2], Plebański [3], and Hawking and Ellis [4]. The least-known and least-studied type in this classification is type III, in which the stress-energy tensor has the form

$$T^{(\mathrm{m})}_{ab} = \rho k_a k_b + q_a k_b + q_b k_a, \tag{1.2}$$

where k^a is a null vector field and q^a is spacelike. Because of its unknown nature and unfamiliar properties, the type-III stress-energy tensor has been named the "ugly duckling of the Hawking-Ellis classification" [5,6]. It was believed, although without any firm ground, that type-III stress-energy tensors are unphysical until Podolský, Švarc and Maeda [7], Martin-Moruno and Visser [5,6], and Maeda [8] provided examples in which a gyraton or exotic Lagrangians realize this particular energy-momentum tensor in Einstein gravity.

One could think of obtaining the null vector field k^a of the type-III stress-energy tensor as the gradient of a scalar field ϕ satisfying $\nabla_c\phi\nabla^c\phi = 0$. It is straightforward to see that only a null dust can be obtained in this way in general relativity. However, one can turn to alternative theories of gravity with a built-in scalar.

In alternative gravity, the field equations are often rewritten by moving geometric terms, or terms built out of the extra gravitational degrees of freedom and their derivatives, to the right-hand side of the field equations to make them look like effective Einstein equations (1.1), thus building an *effective* stress-energy tensor T_{ab}, which is formally treated as a mass-energy source of curvature, although it does not describe real matter. These effective stress-energy tensors may provide incarnations of the type-III energy-momentum tensor that are difficult to realize explicitly in Einstein gravity.

The prototype of the alternative theory of gravity is scalar-tensor gravity [9–12], in which a propagating gravitational scalar field ϕ is added to the usual massless spin-two modes of Einstein gravity contained in the metric tensor. Scalar-tensor gravity is the subject of a vast literature and was generalized long ago by Horndeski [13]. His theory went largely unnoticed for many years and was then rediscovered in the quest for the most general scalar-tensor theory of gravity with second order equations of motion. Although this feature was eventually found to be a property of the more general degenerate higher-order scalar-tensor (DHOST) theories ([14–20], see [21,22] for reviews),

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[1] We follow the notation of Ref. [1], in which the metric signature is $(-+++)$. Units are used in which the speed of light and $8\pi G$ (where G is Newton's constant) are unity.

Horndeski gravity has become the subject of a vast literature spanning the last decade (e.g., [23–31]).

Here we analyze "old" scalar-tensor gravity first, and then Horndeski gravity, and we report a class of possible implementations of the ugly duckling type-III *effective* stress-energy tensor in these theories.

Independent motivation for our study comes from a completely different direction. Recently, the analogy between the effective stress-energy tensor of scalar-tensor and viable Horndeski gravity and a dissipative (Eckart) fluid has led to introducing an effective "temperature of gravity" with respect to general relativity and to equations describing the approach of alternative gravity to general relativity (or its departures from it) [32–45]. This formalism, dubbed "first-order thermodynamics of scalar-tensor gravity" is subject to the fundamental limitation that the gradient of the Brans-Dicke-like scalar field be timelike and future oriented. We would like to extend this formalism to situations in which $\nabla_a\phi$ is lightlike instead. As we will see, the goal of introducing an effective temperature of gravity turns out to be impossible but in the process we discover new implementations of the type-III stress-energy tensor, only for effective instead of real fluids.

Let us proceed with the definition of this effective energy-momentum tensor. The (Jordan frame) action of "first-generation" scalar-tensor gravity is [9–12]

$$S = \int d^4x\sqrt{-g}\left[\phi R - \frac{\omega(\phi)}{\phi}\nabla^c\phi\nabla_c\phi - V(\phi)\right] + S^{(\mathrm{m})}, \tag{1.3}$$

where R is the Ricci scalar of the spacetime metric g_{ab} with determinant g and covariant derivative ∇_a, ϕ is the Brans-Dicke-like gravitational scalar with potential $V(\phi)$, $\omega(\phi)$ is the "Brans-Dicke coupling", and $S^{(\mathrm{m})}$ is the matter action. The corresponding field equations are [9–12]

$$G_{ab} = \frac{T_{ab}^{(\mathrm{m})}}{\phi} + \frac{\omega}{\phi^2}\left(\nabla_a\phi\nabla_b\phi - \frac{1}{2}g_{ab}\nabla^c\phi\nabla_c\phi\right) + \frac{1}{\phi}\left(\nabla_a\nabla_b\phi - g_{ab}\Box\phi\right) - \frac{V}{2\phi}g_{ab}, \tag{1.4}$$

$$(2\omega+3)\Box\phi = 8\pi T^{(\mathrm{m})} + \phi V' - 2V - \omega'\nabla^c\phi\nabla_c\phi, \tag{1.5}$$

where $T_{ab}^{(\mathrm{m})}$ is the matter stress-energy tensor, $\Box \equiv \nabla^c\nabla_c$, and a prime denotes differentiation with respect to ϕ. In the next section we restrict ourselves to solutions of these field equations with the property that the scalar field gradient is null, $\nabla^c\phi\nabla_c\phi = 0$. Examples of such solutions are reported in Refs. [46–53].

II. NULL SCALAR FIELD GRADIENT IN FIRST-GENERATION SCALAR-TENSOR GRAVITY

Let ϕ be the Brans-Dicke scalar field and $k_a \coloneqq \nabla_a\phi$. We shall now present a few preliminary results.

Lemma 1. Consider the Brans-Dicke theory (1.3) with $\omega \neq -3/2$, $V=0$, and *in vacuo* $[T_{ab}^{(\mathrm{m})} = 0]$. If k^a is a null vector field (i.e., $k_ak^a = 0$), then $\Box\phi = 0$.

Proof. The field equation for the Brans-Dicke scalar field (1.5) for $\omega \neq -3/2$, $V=0$, and *in vacuo* reduces to

$$(2\omega+3)\Box\phi = -\omega'\nabla^c\phi\nabla_c\phi.$$

If now the scalar field gradient is null, then $\nabla^c\phi\nabla_c\phi = k_ck^c = 0$, that implies $(2\omega+3)\Box\phi = 0$, or simply $\Box\phi = 0$. ■

Proposition 1. Consider the Brans-Dicke theory (1.3) with $\omega \neq -3/2$, $V=0$, and *in vacuo* $[T_{ab}^{(\mathrm{m})} = 0]$. If k^a is a null vector field, then k^a is a geodesic vector field and the corresponding geodesic is affinely parametrized.

Proof. Recalling the definition $k_a \coloneqq \nabla_a\phi$ one has that

$$k^a\nabla_a k_b = k^a\nabla_a\nabla_b\phi = k^a\nabla_b\nabla_a\phi = k^a\nabla_b k_a.$$

Differentiating $k^ak_a = 0$ one gets $k^a\nabla_b k_a = 0$ that, together with the previous equation, allows one to conclude that $k^a\nabla_a k_b = k^a\nabla_b k_a = 0$. In other words, k^a is a null geodesic vector field and the null geodesic curve to which it is tangent is affinely parametrized. ■

Remark. This property holds for both minimally and nonminimally coupled scalar fields ϕ irrespective of whether ϕ is a test field or a gravitating one.

Consider now the congruence of null geodesics with tangent field $k_a = \nabla_a\phi$. Following standard procedure (see, e.g., [54]) and adapting to the special situation in which the null field k^a is a gradient, we define another (nonunique) null vector field n^a normalized so that

$$k_an^a = -1. \tag{2.1}$$

The covariant differentiation of this equation leads to the useful relation

$$n_b\nabla_a k^b = -k_b\nabla_a n^b. \tag{2.2}$$

The 2-metric transverse to both k^a and n^b is defined as

$$h_{ab} \coloneqq g_{ab} + k_an_b + k_bn_a, \tag{2.3}$$

h_{ab} satisfies

$$h_{ab}k^a = h_{ab}k^b = h_{ab}n^a = h_{ab}n^b = 0, \tag{2.4}$$

is 2-dimensional ($h^a{}_a = 2$), and

$$h^a{}_c h^c{}_b = h^a{}_b. \tag{2.5}$$

Defining now

$$B_{ab} := \nabla_b k_a, \tag{2.6}$$

it follows immediately that B_{ab} is symmetric,

$$B_{ab} = \nabla_b \nabla_a \phi = \nabla_a \nabla_b \phi = B_{ba} = B_{(ab)} \tag{2.7}$$

which, in conjunction with the fact that k^a is geodesic, implies that B_{ab} is transverse to k^a, i.e.,

$$B_{ab} k^b = B_{ab} k^a = 0. \tag{2.8}$$

Next, we project B_{ab} onto the 2-space orthogonal to both k^a and n^a, obtaining

$$\tilde{B}_{ab} := h_a{}^c h_b{}^d B_{cd}. \tag{2.9}$$

We can write this quantity explicitly in terms of k^a and n^a using Eq. (2.3), which gives

$$\tilde{B}_{ab} = B_{ab} + k_a n^c B_{cb} + k_b n^c B_{ac} + k_a k_b (n^c n^d B_{cd}). \tag{2.10}$$

As with any rank two tensor, $\tilde{B}_{ab}$ can be decomposed into its symmetric and antisymmetric parts,

$$\tilde{B}_{ab} = \Theta_{ab} + \tilde{\omega}_{ab}, \tag{2.11}$$

$$\Theta_{ab} := \tilde{B}_{(ab)} = \frac{\Theta}{2} h_{ab} + \tilde{\sigma}_{ab}, \qquad \tilde{\omega}_{ab} := \tilde{B}_{[ab]} = 0, \tag{2.12}$$

where $\Theta \equiv \Theta^a{}_a = \nabla_c k^c$ is the expansion scalar while $\tilde{\sigma}_{ab}$ is the shear tensor, which is the symmetric, trace-free part of $\tilde{B}_{ab}$, and the vorticity $\tilde{\omega}_{ab}$ vanishes identically because k_a is a gradient [34].

From this point on let us restrict our attention to the following case.

Assumptions $(\star)$**.** *Vacuum* Brans-Dicke gravity with $V = 0$, $\omega \neq -3/2$, and $k_c = \nabla_c \phi$ a null gradient of the Brans-Dicke scalar field.

In light of *Lemma* **1** we have $\Theta = \nabla_c k^c = \Box \phi = 0$, then it holds that

$$\tilde{B}_{ab} = \tilde{\sigma}_{ab}. \tag{2.13}$$

This particular scenario of Brans-Dicke gravity admits the possibility that k^a is a Killing vector field. In this case, $B_{ab} = 0$ and $T_{ab} = \frac{\omega}{\phi^2} k_a k_b$ describes a pressureless null dust [55]. We do not consider this very special situation further.

In general, the scalar field effective stress-energy tensor for our subclass of Brans-Dicke gravity $(\star)$ reads

$$T_{ab} = \frac{\omega}{\phi^2} \nabla_a \phi \nabla_b \phi + \frac{\nabla_a \nabla_b \phi}{\phi} = \frac{\omega}{\phi^2} k_a k_b + \frac{B_{ab}}{\phi} \tag{2.14}$$

and, using Eq. (2.10), one can write this T_{ab} in the form

$$T_{ab} = \left[\frac{\omega}{\phi^2} - \left(\frac{n^c n^d B_{cd}}{\phi}\right)\right] k_a k_b + \frac{\tilde{B}_{ab}}{\phi} + q_a k_b + q_b k_a, \tag{2.15}$$

where

$$q_a := -\frac{n^c B_{ca}}{\phi} = -\frac{n^c \nabla_a k_c}{\phi} = \frac{k^c \nabla_a n_c}{\phi}. \tag{2.16}$$

Note that

$$q_a k^a = 0 \tag{2.17}$$

because of the k-tranversality of B_{ab}.

Proposition 2. Let u^a be a null vector field and let v^a be a vector field orthogonal to u^a, i.e., $u^a v_a = 0$. Then v^a is either parallel to u^a or it is spacelike.

Proof. If v^a is parallel to u^a, namely $\exists\, \alpha \in \mathbb{R}$ such that $v^a = \alpha u^a$, then $u^a v_a = \alpha u^a u_a = 0$.

If v^a is not parallel to u^a we have to show that $v^a v_a > 0$. Consider a local inertial frame at each spacetime point p where $u^a v_a = 0$ such that $u^\mu = (u, 0, 0, u)$, with $u \in \mathbb{R}$ in this chart. In this local inertial frame the components of v^a will read, in general, as $v^\mu = (v^0, v^1, v^2, v^3)$, with $v^0, \ldots, v^3 \in \mathbb{R}$. The orthogonality condition $u^a v_a = 0$ at p in the local frame reads

$$0 = \eta_{\mu\nu} u^\mu v^\nu = -u v^0 + u v^3,$$

with $\eta_{\mu\nu}$ denoting the Minkowski metric. Therefore in the local frame at each spacetime point where $u^a v_a = 0$ one has that $v^\mu = (v, v^1, v^2, v)$ with $v \in \mathbb{R}$. This implies that $v^\mu v_\mu = (v^1)^2 + (v^2)^2 > 0$ at p. In other words, at each spacetime point p where $u^a v_a = 0$ it holds that $v^a v_a > 0$, if v^a is not parallel to u^a. ∎

Therefore, Eq. (2.17) and *Prop*. **2** tells us that q^a is either parallel to k^a or q^a must be spacelike.

A. Case 1: q^a parallel to k^a

First we observe that:

Lemma 2. Given *Assumption* $(\star)$, if $q_a := -n^c B_{ca}/\phi$ is parallel to k^b then $q^a = -(q_c n^c) k^a$.

Proof. Let q^a be parallel to k^a, this means that $\exists\, \alpha \in \mathbb{R}$ such that $q^a = \alpha k^a$. From the definition of n^a, Eq. (2.1), one has that $n^a q_a = \alpha n^a k_a = -\alpha$. Hence, $q^a = -(q_c n^c) k^a$. ∎

The effective stress-energy tensor (2.15) of the scalar field, given the subclass of Brans-Dicke gravity $(\star)$, becomes

$$T_{ab} = \left[\frac{\omega}{\phi^2} - (q_c n^c)\right] k_a k_b + \frac{\tilde{B}_{ab}}{\phi} \tag{2.18}$$

with energy density

$$\rho = \frac{\omega}{\phi^2} - (q_c n^c), \tag{2.19}$$

and

$$\frac{\tilde{B}_{ab}}{\phi} = \frac{\tilde{\sigma}_{ab}}{\phi}. \tag{2.20}$$

This allows one to introduce an effective, trace-free, anisotropic stress tensor

$$\tilde{\pi}_{ab} \coloneqq \frac{\tilde{\sigma}_{ab}}{\phi} = -2\eta\tilde{\sigma}_{ab}, \tag{2.21}$$

where $\eta = -1/(2\phi)$ is an effective shear viscosity coefficient. The effective energy-momentum tensor (2.18) of ϕ then takes the form of a null fluid,

$$T_{ab} = \rho k_a k_b + \tilde{\pi}_{ab}. \tag{2.22}$$

One can always diagonalize the anisotropic stress tensor $\tilde{\pi}_{ab}$ via a rotation of axes since $\tilde{\sigma}_{ab}$ is a symmetric tensor in a Riemannian 2-dimensional space. There are two spacelike vectors x^a and y^a in this 2-space orthogonal to both k^a and n^a (with metric h_{ab}) such that

$$x_c k^c = y_c k^c = x_c n^c = y_c n^c = x_c y^c = 0 \tag{2.23}$$

and

$$x_c x^c = y_c y^c = 1 \tag{2.24}$$

for which $\tilde{\pi}_{ab}$ is diagonal. In this coordinate system, the effective stress-energy tensor has the null fluid form [1]

$$T_{ab} = \rho k_a k_b + P_1(x_a x_b - y_a y_b). \tag{2.25}$$

In this case the effective stress-energy tensor of ϕ describes a null fluid with anisotropic stresses $\tilde{\pi}_{ab}$ satisfying the constitutive relation of a Newtonian fluid $\tilde{\pi}_{ab} = -2\eta\tilde{\sigma}_{ab}$, with shear viscosity $\eta = -1/(2\phi)$, no heat conduction, and vanishing trace $T^a{}_a$.

We shall now show that the shear can be eliminated, reducing this null fluid to a null dust (type II in the Hawking-Ellis classification [4]).

Consider a congruence of null geodesics with tangent $k_a = \nabla_a\phi$. The proof of this second statement uses the Raychauduri equation for null geodesic congruences [1,4]

$$\frac{d\Theta}{d\lambda} = -\frac{\Theta^2}{2} - \tilde{\sigma}_{ab}\tilde{\sigma}^{ab} + \tilde{\omega}_{ab}\tilde{\omega}^{ab} - R_{ab}k^a k^b, \tag{2.26}$$

where λ is an affine parameter along the null geodesics. In our case, the null geodesic congruence with tangent field k^a has $\Theta = 0$ and $d\Theta/d\lambda = 0$ everywhere, hence any pair of initially parallel geodesics remains parallel. The vorticity $\tilde{\omega}_{ab}$ also vanishes identically. Since we are assuming vacuum, the trace of the matter stress-energy tensor $T^{(\mathrm{m})}_{ab}$ does not contribute to the Ricci scalar, and $\nabla_c\phi\nabla^c\phi = 0$, $V = 0$ and also $\Box\phi = 0$ (*Lemma* **1**). Contracting the vacuum field equation (1.4) yields $R = 0$ and the effective Einstein equation (1.4) contracted twice with k^a gives

$$R_{ab}k^a k^b = T_{ab}k^a k^b = \left(\rho k_a k_b + \frac{\tilde{B}_{ab}}{\phi} + q_a k_b + q_b k_a\right) k^a k^b = 0 \tag{2.27}$$

due to the lightlike nature of k^c and the transversality (2.8) of $\tilde{B}_{ab}$ to k^c. To conclude, the Raychaudhuri equation (2.26) yields $2\tilde{\sigma}^2 \coloneqq \tilde{\sigma}_{ab}\tilde{\sigma}^{ab} = 0$ everywhere, which implies that, since $\tilde{\sigma}^2$ is positive definite, all components of the shear $\tilde{\sigma}_{ab}$ vanish identically [56] and the effective stress-energy tensor (2.18) of ϕ reduces to

$$T_{ab} = \rho k_a k_b, \tag{2.28}$$

which is of type II in the Hawking-Ellis classification system.

These results can therefore be summarized in the following theorem.

Theorem 1. Given *Assumption* ($\star$), if q^a is parallel to k^a then the effective stress-energy tensor (2.15) of the scalar field reduces to the stress-energy tensor of a null dust, which belongs to the type-II family of the Hawking-Ellis classification.

B. Case 2: q^a is spacelike

Let us consider now the second possibility, in which q^a is spacelike. Then, q^a lives in the 2-space orthogonal to both k^a and n^a and $q_a = h_{ab}q^b$. In fact, since $q_c k^c = 0$, the vector field q_a can only have a component parallel to n_a and components in the 2-space orthogonal to both k^a and n^a,

$$\begin{aligned} q_a &= h_{ab}q^b + (q_c n^c) n_a \\ &= q_a + (q_b n^b) k_a + (q_c n^c) n_a \\ &= q_a + (q_c n^c)(k_a + n_a), \end{aligned} \tag{2.29}$$

where we used the definition of h_{ab}. In order for Eq. (2.29) to be satisfied, either $q_c n^c = 0$ (and then q^a lies in the 2-space orthogonal to both k^a and n^a, $q_a = h_{ab}q^b$, $q^\mu = (0, q^1, q^2, 0)$), or else $n_a = -k_a$, which is impossible because n^a is chosen to be independent of k^a and to satisfy $k_c n^c = -1$. Therefore, q^a is orthogonal to both k^b and n^b.

We now have

$$q_c n^c = -\frac{B_{cd} n^c n^d}{\phi} = -\frac{n^c n^d \nabla_c k_d}{\phi} = \frac{n^c k^d \nabla_c n_d}{\phi} = 0 \qquad (2.30)$$

and the effective stress-energy tensor (2.15) reduces to

$$T_{ab} = \frac{\omega}{\phi^2} k_a k_b + \frac{\tilde{B}_{ab}}{\phi} + q_a k_b + q_b k_a, \qquad (2.31)$$

where we used the fact that $q_c n^c = -n^c n^d B_{cd}/\phi = 0$. This energy-momentum tensor can then be written in the form

$$T_{ab} = \rho k_a k_b + \tilde{\pi}_{ab} + q_a k_b + q_b k_a, \qquad (2.32)$$

with $\rho = \omega/\phi^2$. Furthermore, using again the Raychauduri equation for null geodesic congruences we can eliminate the contribution of the shear $\tilde{\pi}_{ab}$, thus reducing the stress-energy tensor for ϕ to

$$T_{ab} = \rho k_a k_b + q_a k_b + q_b k_a, \qquad (2.33)$$

which has vanishing trace (since $k_a k^a = k_a q^a = 0$) and falls into type III of the Hawking-Ellis classification system [4] (see also [2,3]). This type is the least known of this classification and it is largely unknown which physical systems can be described by a type-III tensor. The only known examples, as previously mentioned, are the gyraton and the exotic Lagrangians discussed by Podolský, Švarc and Maeda [7], Martin-Moruno and Visser [5,6,57–59], and Maeda [8]. The stress-energy tensor (2.31) is further reduced to the type III_0 of [5,6] if the Brans-Dicke coupling assumes the special value $\omega = 0$.

In other words, we have shown that:

Theorem 2. Given *Assumption* ($\star$), if q^a spacelike then the effective stress-energy tensor (2.15) of the scalar field reduces to the stress-energy tensor (2.33), which belongs to the type-III family of the Hawking-Ellis classification. Furthermore, if $\omega = 0$ the stress-energy tensor (2.33) further reduces to the type-III_0 class.

III. NULL SCALAR FIELD GRADIENT IN VIABLE HORNDESKI GRAVITY

Let us now move to Horndeski theories of gravity [13], the subject of intense research in the past decade (*e.g.*, [23–31] and references therein), which are much more general than first-generation scalar-tensor theories and contain them as a special case. We restrict ourselves to the so-called viable Horndeski theories in which the coupling functions are constrained by the requirement that gravitational waves propagate at light speed, as shown by the GW170817/GRB170817A multimessenger event detected in gravitational waves and in many electromagnetic bands [60,61].

The action of viable Horndeski gravity is

$$S = \int d^4x \sqrt{-g} (\mathcal{L}_2 + \mathcal{L}_3 + \mathcal{L}_4) + S^{(\mathrm{m})}, \qquad (3.1)$$

where $S^{(\mathrm{m})}$ is the matter action. The Lagrangian densities $\mathcal{L}_i$ ($i = 2, 3, 4$) are

$$\mathcal{L}_2 = G_2(\phi, X), \qquad (3.2)$$

$$\mathcal{L}_3 = -G_3(\phi, X) \Box \phi, \qquad (3.3)$$

$$\mathcal{L}_4 = G_4(\phi) R, \qquad (3.4)$$

where the G_i are regular functions of the scalar field ϕ and of $X \coloneqq -\frac{1}{2} \nabla_c \phi \nabla^c \phi$ (except for G_4 that depends only on ϕ in viable Horndeski theories). The variation of the action (3.1) with respect to the inverse metric g^{ab} yields the effective field equations (see e.g., [39])

$$\begin{aligned} &G_4 G_{ab} - \nabla_a \nabla_b G_4 + \left(\Box G_4 - \frac{G_2}{2} - \frac{1}{2} \nabla_c \phi \nabla^c G_3 \right) g_{ab} \\ &\quad + \left(\frac{G_{3X}}{2} \Box \phi - \frac{G_{2X}}{2} \right) \nabla_a \phi \nabla_b \phi + \nabla_{(a} \phi \nabla_{b)} G_3 = T_{ab}^{(\mathrm{m})}, \end{aligned} \qquad (3.5)$$

and variation with respect to ϕ gives the equation of motion for the scalar field (see e.g., [39]),

$$\begin{aligned} &G_{4\phi} R + G_{2\phi} + G_{2X} \Box \phi + \nabla_c \phi \nabla^c G_{2X} - G_{3X} (\Box \phi)^2 \\ &\quad - (\nabla_c \phi \nabla^c G_{3X}) \Box \phi - G_{3X} \nabla^c \phi \Box \nabla_c \phi \\ &\quad + G_{3X} R_{ab} \nabla^a \phi \nabla^b \phi - \Box G_3 - G_{3\phi} \Box \phi = 0, \end{aligned} \qquad (3.6)$$

where $T_{ab}^{(\mathrm{m})}$ is the matter stress-energy tensor and

$$G_{i\phi} \coloneqq \frac{\partial G_i}{\partial \phi}, \qquad G_{iX} \coloneqq \frac{\partial G_i}{\partial X} \quad (i = 2, 3, 4). \qquad (3.7)$$

The field equations (3.5) can be cast as the effective Einstein equations

$$G_{ab} = T_{ab} + \frac{T_{ab}^{(\mathrm{m})}}{G_4}, \qquad (3.8)$$

where $T_{ab} = T_{ab}^{(2)} + T_{ab}^{(3)} + T_{ab}^{(4)}$ is the scalar field effective stress-energy tensor capturing all deviations from GR, with

$$T_{ab}^{(2)} = \frac{1}{2G_4} \left(G_{2X} \nabla_a \phi \nabla_b \phi + G_2 g_{ab} \right), \qquad (3.9)$$

$$T_{ab}^{(3)} = \frac{1}{2G_4}\left(G_{3X}\nabla_c X\nabla^c\phi - 2XG_{3\phi}\right)g_{ab} - \frac{1}{2G_4}\left(2G_{3\phi} + G_{3X}\Box\phi\right)\nabla_a\phi\nabla_b\phi - \frac{G_{3X}}{G_4}\nabla_{(a}X\nabla_{b)}\phi, \tag{3.10}$$

$$T_{ab}^{(4)} = \frac{G_{4\phi}}{G_4}\left(\nabla_a\nabla_b\phi - g_{ab}\Box\phi\right) + \frac{G_{4\phi\phi}}{G_4}\left(\nabla_a\phi\nabla_b\phi + 2Xg_{ab}\right). \tag{3.11}$$

When the scalar field gradient is null at every spacetime point [48,50,53] and $\Box\phi = 0$, the canonical kinetic term X vanishes with all its derivatives, leading to the much simpler total effective energy-momentum tensor

$$T_{ab} = \left(\frac{G_{2X}}{2G_4} - \frac{G_{3\phi}}{G_4} + \frac{G_{4\phi\phi}}{G_4}\right)\nabla_a\phi\nabla_b\phi + \frac{G_2}{2G_4}g_{ab} + \frac{G_{4\phi}}{G_4}\nabla_a\nabla_b\phi. \tag{3.12}$$

It is shown in the Appendix that the assumptions of this section imply a relation between Horndeski coupling functions at $X = 0, \Box\phi = 0$. It is also possible to show that the scalar field gradient $\nabla^a\phi$ is an eigenvector of the Ricci tensor (the Appendix). For lightlike $k^a = \nabla^a\phi$, we rewrite this effective T_{ab} as

$$T_{ab} = \left(\frac{G_{2X}}{2G_4} - \frac{G_{3\phi}}{G_4} + \frac{G_{4\phi\phi}}{G_4}\right)k_a k_b + \frac{G_2}{2G_4}g_{ab} + \frac{G_{4\phi}}{G_4}\nabla_b k_a. \tag{3.13}$$

Following the logic of the previous section, we define the tensor $B_{ab} \equiv \nabla_b k_a$ and the 2-metric $h_{ab} \equiv g_{ab} + k_a n_b + k_b n_a$ orthogonal to both k^a and the auxiliary null vector n^a, then $\tilde{B}_{ab} = h_a^c h_b^d B_{cd}$ is again given explicitly by Eq. (2.10). The effective stress-energy tensor obtained from substituting these expressions into (3.13) reads

$$T_{ab} = \left(\frac{G_{2X}}{2G_4} - \frac{G_{3\phi}}{G_4} + \frac{G_{4\phi\phi}}{G_4} - \frac{G_{4\phi}n^c n^d B_{cd}}{G_4}\right)k_a k_b + \frac{G_2}{2G_4}h_{ab} + \frac{G_{4\phi}}{G_4}\tilde{\sigma}_{ab} + \left(-\frac{G_{4\phi}}{G_4}n^c B_{ca} - \frac{G_2}{2G_4}n_a\right)k_b + \left(-\frac{G_{4\phi}}{G_4}n^d B_{db} - \frac{G_2}{2G_4}n_b\right)k_a. \tag{3.14}$$

In this form, it is straightforward to identify the relevant fluid quantities:

$$\rho \coloneqq \frac{G_{2X} - 2G_{3\phi} + 2G_{4\phi\phi} - 2G_{4\phi}n^c n^d B_{cd}}{2G_4}, \tag{3.15}$$

$$P \coloneqq \frac{G_2}{2G_4}, \tag{3.16}$$

$$\tilde{\pi}_{ab} \coloneqq \frac{G_{4\phi}}{G_4}\tilde{\sigma}_{ab}, \tag{3.17}$$

$$q_a \coloneqq -\frac{G_{4\phi}}{G_4}n^c B_{ca} - \frac{G_2}{2G_4}n_a, \tag{3.18}$$

i.e., energy density, isotropic pressure, anisotropic stress tensor, and energy current density, respectively. Using the identifications (3.15)–(3.18) the stress-energy tensor (3.14) takes the form

$$T_{ab} = \rho k_a k_b + P h_{ab} + \tilde{\pi}_{ab} + q_a k_b + q_b k_a. \tag{3.19}$$

Note that now there appears the isotropic pressure $P = G_2/(2G_4)$, which was absent in first-generation scalar-tensor gravity because, there, $G_2(\phi, X) = \omega(\phi)X/2$ vanishes for $X = 0$. In viable Horndeski,

$$q_a k^a = \frac{G_2}{2G_4} = P \neq 0 \tag{3.20}$$

in general and the energy flux density q^a no longer lives in the 2-space with metric h_{ab} orthogonal to both k^a and n^a, but has components along the light cone generated by these null vectors since $q_a k^a \neq 0$, $q_a n^a \neq 0$. The trace

$$T^a{}_a = 4P \tag{3.21}$$

now does not vanish. What is more, in viable Horndeski theory q^a does not have a definite causal character because

$$q_a q^a = \frac{G_{4\phi}}{G_4^2}\left(G_{4\phi}B_{ca}B^a{}_d n^c n^d - G_2 n^a n^c B_{ac}\right) \tag{3.22}$$

has indefinite sign.

Following a similar procedure as in Sec. II A it is easy to see that:

Proposition 3. Consider the *vacuum* viable Horndeski gravity with a null scalar field gradient $k_a \coloneqq \nabla_a\phi$ and $\Box\phi = 0$. Then the effective stress-energy tensor (3.19) for the Horndeski scalar field ϕ is shearless.

Proof. The Raychaudhuri equation for null geodesic congruences (2.26) reduces to

$$2\tilde{\sigma}^2 = \tilde{\sigma}_{ab}\tilde{\sigma}^{ab} = -R_{ab}k^a k^b. \tag{3.23}$$

Contracting twice the Horndeski field equation (3.5) for *vacuum* viable Horndeski gravity with k^a, and with the assumption $\Box\phi = 0$, one obtains

$$R_{ab}k^a k^b = T_{ab}k^a k^b.$$

Furthermore,

$$T_{ab}k^ak^b = \frac{1}{G_4}\left[\left(-G_{3\phi} + G_{4\phi\phi} + \frac{G_{2X}}{2}\right)k_ak_b + G_{4\phi}B_{ab} + \frac{G_2}{2G_4}g_{ab}\right]k^ak^b = 0; \tag{3.24}$$

hence, we can conclude that $2\tilde{\sigma}^2 \coloneqq \tilde{\sigma}_{ab}\tilde{\sigma}^{ab} = -R_{ab}k^ak^b = -T_{ab}k^ak^b = 0$. Since $\tilde{\sigma}^2$ is positive-definite, all the components of $\tilde{\sigma}_{ab}$ vanish identically, which completes the proof. ■

The stress-energy tensor of *vacuum* viable Horndeski gravity with a null scalar field gradient $k_a \coloneqq \nabla_a\phi$ and $\Box\phi = 0$ at every spacetime point then reads

$$T_{ab} = \rho k_ak_b + Ph_{ab} + q_ak_b + q_bk_a, \tag{3.25}$$

which is not of type III because q^a is not necessarily spacelike nor orthogonal to the null vector k^a. However, if $G_2(\phi, X = 0) = 0$ one has that the above stress-energy tensor reduces to

$$T_{ab} = \rho k_ak_b + q_ak_b + q_bk_a, \tag{3.26}$$

which instead belongs to the type-III family of the Hawking-Ellis classification.

In other words, we have shown that:

Theorem 3. Given the *vacuum* viable Horndeski gravity with a null scalar field gradient $k_a \coloneqq \nabla_a\phi$, $\Box\phi = 0$, and $G_2(\phi, X = 0) = 0$, the stress-energy tensor (3.19) for the Horndeski scalar field reduces to (3.26), which belongs to the type-III family of the Hawking-Ellis classification.

The stress-energy tensors (2.33) and (3.26) are not the most general type-III stress-energy tensors because the null vector field k^a originates from a gradient and is divergence free, which are not general properties.

IV. DISCUSSION AND CONCLUSIONS

Assuming that the gravitational scalar field ϕ of first-generation scalar-tensor or Horndeski gravity is null everywhere, it is also geodesic and affinely parametrized, and the congruence of null geodesics with tangent $k_a = \nabla_a\phi$ is shear-free, nontwisting, and nonexpanding. We have classified its effective stress-energy tensor, obtained by writing the field equations as effective Einstein equations.

In first-generation scalar-tensor gravity this effective T_{ab} can be of only two types. In the first case, it reduces to the well-known null dust, or type II in the Segré-Plebański-Hawking-Ellis classification. In the second case, it contains an energy flux density q^a which is spacelike, and we have a physical realization of type-III stress-energy tensor (which can even become the simplified type III_0 of [5,6] in $\omega = 0$ Brans-Dicke theory). Therefore, we have an implementation of the ugly duckling type-III stress-energy tensor. This avatar of the ugly duckling is derived directly from the scalar-tensor Lagrangian and not from exotic Lagrangians constructed *ad hoc* (which seems to be the only avenue found as yet in Einstein gravity). Explicit examples are not easy to find and are likely to be contrived. The most likely candidates for type-III effective stress-energy tensors in scalar-tensor gravity are Kundt spacetimes, however the known exact solutions of this kind in "old" scalar-tensor [46,47,49,51,52] and in Horndeski [48,50,53] theories have stress-energy tensors describing pure null dusts (i.e., of type II).

The situation in viable Horndeski gravity is more complicated, as an isotropic pressure appears unless $G_2(\phi, X = 0) = 0$ (in which case the discussion for first-generation scalar-tensor gravity applies again) and the energy flux density q^a is neither orthogonal to k^a nor spacelike.

We suggest a possible physical interpretation of the type-III energy-momentum tensor, with the obvious *caveat* that the null vector n^a and, therefore, the 2-metric h_{ab}, density ρ, and vector q^a are nonunique. The null dust part ρk_ak_b of the effective T_{ab}, with k^a null and geodesic, describes coherent propagation of radiation, which is accompanied by a spacelike (therefore, noncausal) dissipation of energy in the direction transverse to k^a and n^a, as in heat conduction. The interpretation of q^a is essentially the same provided for the dissipative stress-energy tensor $T_{ab} = \rho u_au_b + Ph_{ab} + \pi_{ab} + q_au_b$ when the four-velocity u^a of a dissipative fluid is timelike and q^a is spacelike [62]. It seems counterintuitive that the propagation of a beam at light speed would be compatible with the removal of energy from the beam, but energy is ill-defined for nonasymptotically flat geometries and the simpler pp-waves of general relativity exhibit energy-related features that are difficult to interpret [63], hence this objection may not be substantial after all.

The effective stress-energy tensor of the Brans-Dicke-like scalar field (in the so-called Jordan frame used in the present work) changes type under a conformal transformation, which preserves the lightlike nature of $\nabla^a\phi$. A type-III energy-momentum tensor in the Jordan frame of first-generation scalar-tensor gravity will become type II in the conformally transformed version, the Einstein frame, a property that makes it possible to generate and study Jordan frame geometries sourced by type-III stress-energy tensors using conformal mapping and known Einstein frame solutions associated with type II T_{ab}'s. The reader might have noticed the lack of exact solutions providing explicit examples of type III T_{ab} in the previous section (which, of course, requires one to fix the coupling functions G_i). In a future publication we will search for such solutions using conformal mappings. Incidentally, when the gradient $\nabla_a\phi$ of the gravitational scalar field of scalar-tensor (including Horndeski) gravity is lightlike, the effective "fluid" described by its stress-energy tensor does not lead to a concept of effective "temperature of gravity",[2] as it does when $\nabla_a\phi$ is timelike. Although not unexpected, this conclusion dashes the hope of extending the first-order thermodynamics of scalar-tensor and Horndeski gravity developed in [32–44] to the null case.

We conclude that the ugly duckling of the Segré-Plebański-Hawking-Ellis classification of stress-energy tensors may be a freak of nature of limited importance, but it is not physically impossible.

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APPENDIX: $\nabla^a\phi$ AS AN EIGENVECTOR OF THE RICCI TENSOR AND RELATION BETWEEN HORNDESKI COUPLING FUNCTIONS

In viable Horndeski gravity, assuming $X \equiv 0$, $\Box\phi = 0$, and $G_4 = G_4(\phi)$, the gradient and the d'Alembertian of X also vanish identically and Eq. (3.6) reduces to

$$G_{4\phi}R + G_{2\phi} - G_{3X}\nabla^c\phi\Box\nabla_c\phi + G_{3X}R_{ab}\nabla^a\phi\nabla^b\phi = 0 \tag{A1}$$

at $X = 0$, $\Box\phi = 0$. By applying the commutation relation

$$(\nabla_a\nabla_b - \nabla_b\nabla_c)\omega_c = R_{abc}{}^d\omega_d \tag{A2}$$

to $\omega_a = \nabla_a\phi$ and using $\Box\phi = 0$, one obtains

$$\begin{aligned}\nabla^c\phi\Box\nabla_c\phi &= \nabla^c\phi\nabla^a\nabla_a\nabla_c\phi = \nabla^c\phi\nabla^a\nabla_c\nabla_a\phi \\ &= \nabla^c\phi\left(\nabla_c\Box\phi + R^a{}_{cad}\nabla^d\phi\right) \\ &= \nabla^c\phi\left(\nabla_c\Box\phi + R_{cd}\nabla^d\phi\right) \\ &= R_{ab}\nabla^a\phi\nabla^b\phi \end{aligned} \tag{A3}$$

using the definition of the Ricci tensor $R_{dc} \equiv R_{dac}{}^a$ and the symmetries of the Riemann tensor.

In vacuo, Eq. (3.6) reduces to

$$\begin{aligned}&G_4\left(R_{ab} - \frac{1}{2}g_{ab}R\right) - \nabla_a\nabla_b G_4 \\ &+\left(\Box G_4 - \frac{G_2}{2} - \frac{1}{2}\nabla_c\phi\nabla^c G_3\right)g_{ab} - \frac{G_{2X}}{2}\nabla_a\phi\nabla_b\phi \\ &+\nabla_{(a}\phi\nabla_{b)}G_3 = 0,\end{aligned} \tag{A4}$$

taking the trace of which (and using $\Box G_4 = G_{4\phi\phi}\nabla^a\phi\nabla_a\phi + G_{4\phi}\Box\phi = 0$) produces

$$R = -\frac{2G_2(\phi, X=0)}{G_4(\phi)}. \tag{A5}$$

Now, the contraction of Eq. (A4) with $\nabla^a\phi\nabla^b\phi$ yields

$$R_{ab}\nabla^a\phi\nabla^b\phi = \frac{G_{4\phi}}{G_4}\nabla^a\phi\nabla^b\phi\nabla_a\nabla_b\phi, \tag{A6}$$

but

$$\nabla^a\phi\nabla^b\phi\nabla_a\nabla_b\phi = k^a k^b B_{ab} = 0 \tag{A7}$$

due to the k-transversality of B_{ab}, hence

$$R_{ab}\nabla^a\phi\nabla^b\phi = 0. \tag{A8}$$

Using this result, it follows from Eq. (A3) that the combination

$$G_{3X}\left(-\nabla^c\phi\Box\nabla_c\phi + R_{ab}\nabla^a\phi\nabla^b\phi\right) \tag{A9}$$

appearing in Eq. (A1) vanishes when $\Box\phi = 0$ and $X = 0$, and Eq. (A1) then gives the relation between coupling functions

$$G_{4\phi}R + G_{2\phi} = 0 \tag{A10}$$

at $\Box\phi = 0, X = 0$. Now the comparison of Eqs. (A10) and (A5) yields

$$\frac{G_{2\phi}}{G_2} - \frac{2G_{4\phi}}{G_4} = 0 \tag{A11}$$

at $X = 0, \Box\phi = 0$. Furthermore, substituting Eq. (A5) into Eq. (A4) yields

$$\begin{aligned}&G_4R_{ab} + \frac{G_2}{2}g_{ab} + \left(G_{3\phi} - G_{4\phi\phi} - \frac{G_{2X}}{2}\right)\nabla_a\phi\nabla_b\phi \\ &- G_{4\phi}\nabla_a\nabla_b\phi = 0\end{aligned} \tag{A12}$$

which, contracted with the gradient $\nabla^a\phi$, then gives

[2] This is because q^a does not satisfy a known constitutive relation relating it to a temperature. For timelike $\nabla_a\phi = u_a$, q_a satisfies the Eckart constitutive relation $q_a = -\mathcal{K}h_{ab}(\nabla^b\mathcal{T} + \mathcal{T}\dot{u}^b)$, which allows one to define an effective temperature of gravity.

$$R_{ab}\nabla^a\phi = -\frac{G_2}{2G_4}\nabla_b\phi = \frac{R}{4}\nabla_b\phi; \tag{A13}$$

the scalar field gradient $\nabla^a\phi$ is an eigenvector of the Ricci tensor with eigenvalue $R/4$.

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