Extended combinatorial algebraic approach for the second-generation time-delay interferometry

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This work elaborates on an algebraic approach to second-generation time-delay interferometry (TDI). The proposed method is closely related to the algorithm first developed by Dhurandhar *et al.* and its recent generalizations. While the relevant equation is derived from a geometric TDI perspective, the resulting TDI solutions are primarily generated by a basis consisting of four-tuples. Unlike the original study, the present scheme is not subject to any constraint equation and spans the underlying solution space much further. Moreover, the algorithm does not rely on specific subscript permutations regarding the two elements of the commutator that furnishes the TDI solution. Employing the proposed method, we explicitly show that all the existing second-generation TDI combinations, most established via the geometric TDI approach, can be derived. It is argued that the current approach provides an alternative perspective on the algebraic structure of the second-generation TDI solutions.

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I. INTRODUCTION

To date, most space-based gravitational wave detection projects, namely, LISA [1], TianQin [2], Taiji [3], and DECIGO [4], are based on a layout of an approximately equilateral triangle formed by three spacecraft. As the gravitational waves pass through the detector, the data stream might capture the encoded information on spacetime distortion in terms of the Doppler frequency shifts. One of the primary challenges of spaceborne detectors resides in the variety of noises in the data stream, which consist of laser frequency noise, test mass noise, optical bench motion noise, and clock-jitter noise, among others. In particular, laser frequency noise is overwhelmingly more significant than others, as its strength is typically seven or eight orders of magnitude above that of the inevitable ones [5]. Moreover, in the context of space-based detectors, the laser frequency noise embedded in the beat notes cannot be straightforwardly canceled out using the strategy of their ground-based counterpart. This is because the detectors' armlengths are essentially governed by the spacecraft's orbital motions, which are not only of different sizes but also time-varying. In order to suppress the laser frequency noise to the desired level, the time-delay interferometry (TDI) technique was proposed and has become a crucial method to be employed in the data postprocessing stage.

The main idea behind the TDI algorithm is to construct an effective equal-arm interferometer by linearly combining the time-delayed data streams. Ever since it was first proposed by Tinto *et al.* in 1999 [6], the algorithm has been developed extensively over the next two decades. Regarding the specific orders where the truncations are taken in the optical paths expanded in terms of the rate of change of the armlengths, the TDI combinations can be classified as first-generation, modified first-generation, second-generation, and modified second-generation ones [7]. The first-generation TDI [8] treats the spacecraft constellation as static. Modified first-generation TDI [9] considers the Sagnac effect caused by the rigid rotation of the entire constellation, and therefore, distinctions between the clockwise and counterclockwise light propagations are made. Algebraically, the difference between the first-generation and modified first-generation TDI resides in whether the number of distinct time-delay operations is

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three [10] or six [11]. The relevant solutions can be viewed as the elements of the *first module of syzygies* of an ideal in a (commutative) polynomial ring $\mathscr{R}$ [10]. The module's generators can be obtained using the Groebner basis [12]. In practice, as discussed above, the armlengths do not remain constant in time due to the orbital dynamics, and the rate of change of armlengths is estimated to be up to ~ 10 m/s [13]. In this regard, the second-generation TDI was proposed to take into consideration the contributions that are first-order in time, $\dot{L}$ [7,14]. Similar to the extension to the modified first-generation TDI, it was also meaningful to distinguish the different cyclic directions in the rate of change of armlengths. Specifically, one may consider explicitly six different rates of change of the armlengths instead of three. Nonetheless, the resulting modified second-generation TDI solutions, by definition, constitute a subset of the second-generation TDI combinations [7]. Alternatively, one may view the second-generation TDI combinations as solutions of algebraic equations furnished by noncommutative time-delay operators. The latter cannot be solved straightforwardly owing to the difficulties in the underlying algebraic geometry. From somewhat different perspectives, many efforts have been made, notably the geometric TDI [15], matrix-based approaches [16–18], and combinatorial algebraic methods [19–22]. The geometric TDI method is a method of exhaustion that focuses on constructing virtual equal-arm optical paths. It can be essentially implemented by a ternary search algorithm to enumerate the solution space [7]. The resulting TDI solutions are typically presented for a given number of links n, in the form of spacetime diagrams of the virtual light propagation trajectory. On the other hand, the matrix-based TDI methods are more concerned with data sampled at discrete time instants. The specific configuration of the spaceborne detector, consisting of laser setup and spacecraft orbits, gives rise to a *design matrix* typically of significant rank. Subsequently, a feasible TDI solution is furnished by the basis of the null space of the transpose of the design matrix. This is because the TDI equation corresponds to the vanishing condition when the design matrix is multiplied by a row matrix from the left [16].

The combinatorial algebra method was first proposed by Dhurandhar *et al.* [19]. The algorithm derives feasible modified second-generation TDI solutions by essentially enumerating commutators formed by the time-delay operators in a particular order. In the original study, the approach was primarily applied to the particular scenario with one arm dysfunctional. In other words, the derived solutions are of the Michelson type. The approach was then generalized [20] to construct other types of modified second-generation TDI combinations by taking into account different constraint equations while introducing inverse operators. Besides, an iterative procedure was proposed to "lift" first-generation TDI solutions into modified second-generation ones [22], for which the residual noise in the form of a commutator vanishes. More recently, the approach was extended to consider second-order commutators [21]. The present study involves an attempt to generalize the combinatorial algebraic approach further. While following a similar strategy, it has two notable features relevant for a significant span in the underlying solution space. First, the present scheme is not subject to any constraint equation. Second, the algorithm does not rely on some specific subscript permutation associated with the two elements of the commutator that furnishes the TDI solution in the original approach. As a result, while the relevant equations are derived from a geometric TDI perspective, the resulting TDI solutions are primarily generated by a basis consisting of four tuples. By employing the proposed method, we explicitly show that all the existing second-generation TDI combinations, most established via the geometric TDI approach, can be readily derived. We argue that the present scheme provides an alternative perspective on the algebraic structure of the second-generation TDI solutions.

The remainder of the paper is organized as follows. In Sec. II, we introduce the notations and conventions used in this paper. The definitions for different generations of TDI solutions are briefly revisited. In Sec. III, the original combinatorial algebraic algorithm is reviewed. We discuss the basic assumptions and some of the main results relevant to the present study. In Sec. IV, we derive a system of equations in terms of the time-displacement operators from the original TDI equation, which defines, on a more general ground, the solution space from an algebraic viewpoint. We show that the resulting TDI solutions can be generated mainly by a basis consisting of four-tuples. Subsequently, we present an enumeration algorithm to derive the second-generation TDI solutions. In Sec. V, we illustrate how the proposed scheme can be utilized to enumerate second-generation TDI solutions systematically. In particular, it is shown that all known second-generation geometric TDI combinations can be derived using the approach. The last section is devoted to further discussions and concluding remarks. Specific technical details are relegated to Appendix.

II. TDI EQUATIONS, NOTATIONS, AND CONVENTIONS

For space-based gravitational wave detectors such as LISA and TianQin, three spacecraft form an approximately equilateral triangle. As shown in Fig. 1, the spacecraft are indicated by SCi ($i = 1, 2, 3$), and each spacecraft has two optical benches labeled by i and i'. The armlengths facing the spacecraft SCi are denoted as L_i and $L_{i'}$ ($i' = 1', 2', 3'$) for light propagation in counterclockwise and clockwise directions. The schematic diagram of optical bench 1 on SC1 is shown in Fig. 2. Three types of data streams are primarily involved in the measurements, namely, the science data stream, test mass data stream, and reference data stream, namely,

$$
\begin{aligned}
s_i(t) &= D_{i-1}p_{(i+1)'}(t) - p_i(t) + \nu_{(i+1)'}[\vec{n}_{i-1}\cdot D_{i-1}\dot{\vec{\Delta}}_{(i+1)'}(t) + \vec{n}_{(i-1)'}\cdot\dot{\vec{\Delta}}_i(t)] + H_i(t) + N_i^{\rm opt}(t),\\
\epsilon_i(t) &= p_{i'}(t) - p_i(t) - 2\nu_{i'}[\vec{n}_{(i-1)'}\cdot\dot{\vec{\delta}}_i(t) - \vec{n}_{(i-1)'}\cdot\dot{\vec{\Delta}}_i(t)] + \mu_{i'}(t),\\
\tau_i(t) &= p_{i'}(t) - p_i(t) + \mu_{i'}(t),
\end{aligned}
\tag{1}
$$

and

$$
\begin{aligned}
s_{i'}(t) &= D_{(i+1)'}p_{i-1}(t) - p_{i'}(t) + \nu_{i-1}[\vec{n}_{i+1}\cdot\dot{\vec{\Delta}}_{i'}(t) + \vec{n}_{(i+1)'}\cdot D_{(i+1)'}\dot{\vec{\Delta}}_{i-1}(t)] + H_{i'}(t) + N_{i'}^{\rm opt}(t),\\
\epsilon_{i'}(t) &= p_i(t) - p_{i'}(t) - 2\nu_i[\vec{n}_{i+1}\cdot\dot{\vec{\delta}}_{i'}(t) - \vec{n}_{i+1}\cdot\dot{\vec{\Delta}}_{i'}(t)] + \mu_i(t),\\
\tau_{i'}(t) &= p_i(t) - p_{i'}(t) + \mu_i(t),
\end{aligned}
\tag{2}
$$

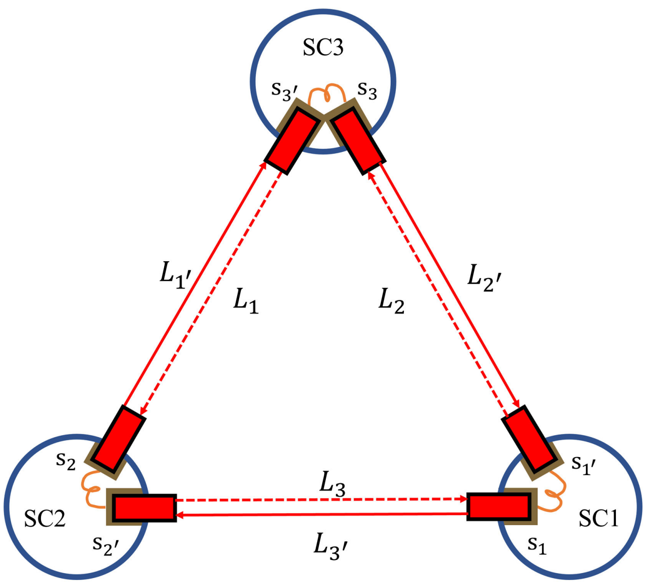

FIG. 1. Schematic diagram of the three-spacecraft constellation for the space-based gravitational wave detector.

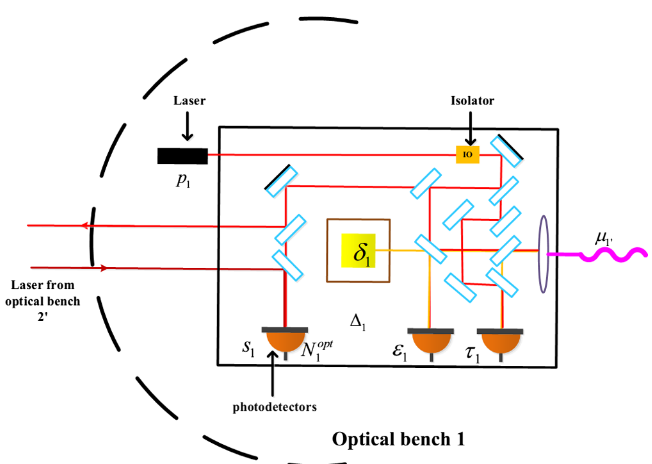

FIG. 2. Schematic diagram of one-way measurement on optical bench 1.

where s_i and $s_{i'}$ are science data streams obtained from the interference between two lasers of the local and distant spacecraft; ϵ_i and $\epsilon_{i'}$ are the test mass data streams obtained from the interference between the two lasers, bounced off from the test masses. τ_i and $\tau_{i'}$ are the reference data streams obtained from the interference between the two local lasers. D_i and $D_{i'}$ are the time-delay operators along the armlengths L_i and $L_{i'}$ which, in the time domain, satisfy

$$
\begin{aligned}
D_i f(t) &= f(t - L_i(t)),\\
D_j D_i f(t) &= f(t - L_j(t) - L_i(t - L_j(t))),
\end{aligned}
\tag{3}
$$

where the speed of light in vacuum c is assumed to be unity. ν_i and $\nu_{i'}$ indicate the laser frequencies $\approx 3\times 10^{14}$ Hz. $\vec{n}_i$ and $\vec{n}_{i'}$ represent the unit vectors along the armlengths. The information on the gravitational wave H_i is expected to be embedded in the science data stream. The laser frequency noise is represented by p_i and $p_{i'}$, typically seven to eight orders of magnitude larger than the gravitational wave signals. $\dot{\vec{\delta}}_{i(i')}(t)$ and $N^{\rm opt}_{i(i')}(t)$ denote the test mass noise and optical path noise, respectively. $\dot{\vec{\Delta}}_{i(i')}(t)$ is the noise due to the vibrations of optical benches. $\mu_{i(i')}(t)$ denote the optical fiber noise. The indices $i = 1, 2, 3$ are understood to be the remainder of the modulo operation by 3. More specific details on the notations utilized by Eqs. (1) and (2) can be found in [5,23], whereas we have ignored the shot noise in $\epsilon_{i(i')}, \tau_{i(i')}$ and the clock jitter noise in $s_{i(i')}, \epsilon_{i(i')}, \tau_{i(i')}$. The measurement of $\epsilon_{i(i')}, \tau_{i(i')}$ has high signal-to-noise ratios so that the shot noise can be ignored [5] and the cancellation of clock jitter noise can be made by introducing the sideband modulation [23,24].

According to the standard procedure [23], the optical bench noise and primed laser frequency noise can be eliminated by using the nine equations regarding the test mass and reference data streams. As a result, three independent laser frequency noise remains, while the test mass noise and optical path noise are considered inevitable. This gives rise to six observables that read

$$\eta_i(t) = H_i(t) + D_{i-1}p_{i+1}(t) - p_i(t) + \nu_{(i+1)'}\vec{n}_{i-1}[D_{i-1}\dot{\vec{\delta}}_{(i+1)'}(t) - \dot{\vec{\delta}}_i(t)] + N_i^{\rm opt}(t),$$
$$\eta_{i'}(t) = H_{i'}(t) + D_{(i+1)'}p_{i-1}(t) - p_i(t) + \nu_{i-1}\vec{n}_{i+1}\cdot[\dot{\vec{\delta}}_{i'}(t) - D_{(i+1)'}\dot{\vec{\delta}}_{i-1}(t)] + N_{i'}^{\rm opt}(t), \quad (4)$$

where we assume the fiber can be reciprocal, the fiber noise terms have been canceled.

By the linear combination of time-delayed observables η_i and $\eta_{i'}$, one aims to construct a TDI solution that eliminates the laser frequency noise [5]. A TDI solution is generally written as

$$\mathrm{TDI} = \sum_{i=1,2,3}(q_i\eta_i + q_{i'}\eta_{i'}), \quad (5)$$

where the unknown coefficients q_i and $q_{i'}$ are the polynomials in D_i and $D_{i'}$. The coefficients before individual laser frequency noise p_i must vanish for a valid solution, which implies

$$q_1 + q_{1'} - q_{2'}D_{3'} - q_3D_2 = 0, \quad (6a)$$

$$q_2 + q_{2'} - q_{3'}D_{1'} - q_1D_3 = 0, \quad (6b)$$

$$q_3 + q_{3'} - q_{1'}D_{2'} - q_2D_1 = 0. \quad (6c)$$

The system of equations Eq. (6) can be viewed as the starting point for most TDI algorithms in the literature. If one ignores the armlength's time dependence, the operators D_i and $D_{i'}$ are commutative. As a result, the solution of Eq. (6) can be significantly simplified [10] and give rise to the first-generation TDI solution. In practice, owing to Eq. (3), the commutators of the time-delay operators do not vanish identically. Subsequently, one has to resort to the expansions of optical paths in the rate of change of the armlengths. The condition for the cancellation up to velocity terms leads to the second-generation TDI solutions. For the latter, the residual laser frequency noise, whose specific form is related to the fact that Eq. (6) are solved under an appropriate approximation, is understood to be suppressed below the noise floor consisting of the inevitable ones.

III. COMBINATORIAL ALGEBRAIC APPROACHES FOR SECOND-GENERATION TDI

A. Expansion of armlengths and TDI's generations

As time-dependent functions, the armlengths $L_i(t)$ can be expanded as

$$L_i(t) = L_i + t\dot{L}_i + \frac{1}{2}t^2\ddot{L}_i + \cdots, \quad (7)$$

where $\dot{L}_i$ and $\ddot{L}_i$ are first-order and second-order time derivatives.

Formally, a TDI solution can be derived by substituting Eq. (7) into the arguments on the right-hand side of Eq. (3) and requiring Eq. (5), or equivalently Eq. (6), to vanish when truncating the relevant equations up to a given order. Different generations of TDI explored in the literature are distinct regarding specific truncations and assumptions utilized in solving Eq. (6).

As discussed above, to start with, one ignores the contributions of $\dot{L}$ and other higher-order terms. To be specific, for the first-generation TDI combinations, one assumes:

$$L_i(t) = L_{i'}(t) = L_i. \quad (8)$$

In other words, D_i and $D_{i'}$ are treated as the same operator, and different operators commute.

For modified first-generation TDI, one takes into account the Sagnac effect in rigid rotation and assumes:

$$L_i(t) = L_i, \qquad L_{i'}(t) = L_{i'}, \qquad L_i \neq L_{i'}. \quad (9)$$

Therefore, $D_i \neq D_{i'}$, but the time-delay operators are still commutative. Subsequently, the residuals of laser frequency noise of the first- and modified first-generation TDI combinations are primarily governed by the rate of change of armlengths terms $\dot{L}$.

The second and modified second-generation TDI combinations further consider the cancelation of first-order derivative terms. Specifically, for second-generation TDI, we have

$$\begin{aligned} L_i(t) &= L_i + t\dot{L}_i, \\ L_{i'}(t) &= L_{i'} + t\dot{L}_{i'}, \\ L_i &\neq L_{i'}, \\ \dot{L}_i &= \dot{L}_{i'}, \end{aligned} \quad (10)$$

where the rates of change of armlengths in different cyclic directions, namely, $\dot{L}_i$ and $\dot{L}_{i'}$, are not distinguished.

On the other hand, the modified second-generation TDI assumes:

$$\begin{aligned} L_i(t) &= L_i + t\dot{L}_i, \\ L_{i'}(t) &= L_{i'} + t\dot{L}_{i'}, \\ L_i &\neq L_{i'}, \\ \dot{L}_i &\neq \dot{L}_{i'}. \end{aligned} \quad (11)$$

For second- and modified second-generation TDI combinations, the residuals of laser frequency noise are largely determined by the second contributions, such as $\dot{L}^2$ and $\ddot{L}$. The fact that the time-delay operators are not commutative in the context of second-generation TDI leads to the complexity of solving the underlying TDI equations.

B. Existing combinatorial algebraic algorithms

In this subsection, we briefly revisit the original combinatorial algebraic approaches and their generalizations developed in the literature. These algorithms aim to systematically construct second-generation TDI solutions by employing an enumeration scheme. It is worth mentioning that they typically make use of a specific constraint equation, and therefore the derived solutions pertain to the corresponding TDI class. As a result, the obtained solutions are, by and large, not exhaustive. We will focus on some of the relevant results, which will also be utilized by the generalized approach elaborated below in Sec. IV.

A central piece of the algorithm proposed by Dhurandhar *et al.* [19] resides in the following equality, which was shown to be valid up to first-order contributions

$$[D_{u_1u_2\ldots u_n}, D_{v_1v_2\ldots v_n}]f(t) = \left(\sum_{k=1}^{n}\delta_{u_k}L_{u_k}\sum_{k'=1}^{n}\delta_{v_{k'}}\dot{L}_{v_{k'}} - \sum_{k'=1}^{n}\delta_{v_{k'}}L_{v_{k'}}\sum_{k=1}^{n}\delta_{u_k}\dot{L}_{u_k}\right) \times \dot{f}\left(t - \sum_{k=1}^{n}\delta_{u_k}L_{u_k} - \sum_{k'=1}^{n}\delta_{v_{k'}}L_{v_{k'}}\right), \tag{12}$$

for an arbitrary function $f(t)$, where δu_k and $\delta v_{k'}$ are defined as

$$\delta u_k = \begin{cases} -1 & \text{if } u_k = \lambda_m \text{ for any } m = 1, \ldots, r \\ +1 & \text{otherwise} \end{cases},$$
$$\delta v_{k'} = \begin{cases} -1 & \text{if } v_{k'} = \gamma_j \text{ for any } j = 1, \ldots, s \\ +1 & \text{otherwise} \end{cases}. \tag{13}$$

On the left-hand side of Eq. (12), one considers a commutator composed of two monomials $[D_{u_1u_2\cdots u_n}, D_{v_1v_2\cdots v_n}]$. We assume that there are r instances of inverse operators in the first monomial $D_{u_1u_2\cdots u_n}$, which are denoted as D_{λ_m}, where $m = 1, \ldots, r$ and λ_m $(= \bar{i}$ or $\bar{i'})$ are r distinct elements chosen from the subscripts u_k with $k = 1, \ldots, n$. Similarly, one assumes that the second monimial $D_{v_1v_2\cdots v_n}$ contains s inverse operators, denoted by D_{γ_j} with $j = 1, \ldots, s$. In the remainder of the paper, we will refer to $[D_{u_1u_2\ldots u_n}, D_{v_1v_2\ldots v_n}]$ as "the commutator", $D_{u_1u_2\ldots u_n}$ as the first monomial of the commutator, and $D_{v_1v_2\ldots v_n}$ as the second monomial of the commutator. The relation was first introduced in Ref. [19] for time-delay operators and then generalized to include inverse time-delay operators [20].

In what follows, we briefly outline an informal but intuitive derivation while referring the interested reader to the appendix of [20]. For simplicity, we will use the shorthand notation

$$D_jD_if(t) \equiv D_{ji}f(t). \tag{14}$$

By substituting the expansion Eq. (7) into Eq. (3) and ignoring second and higher order terms, we have

$$\begin{aligned} D_if(t) &\simeq f(t - L_i) - \dot{f}(t - L_i)t\dot{L}_i, \\ D_{ji}f(t) &\simeq f(t - L_j(t) - L_i(t) + \dot{L}_iL_j(t)) \\ &\simeq f(t - L_i - L_j) + \dot{f}(t - L_i - L_j)\dot{L}_iL_j \\ &\quad - \dot{f}(t - L_i - L_j)(\dot{L}_i + \dot{L}_j)t. \end{aligned} \tag{15}$$

Observing Eq. (15), the latter can be generalized to read

$$D_{u_n}\ldots D_{u_1}f(t) = f\left(t - \sum_{i=1}^{n}L_{u_i}\right) + \dot{f}\left(t - \sum_{i=1}^{n}L_{u_i}\right) \times \left(\sum_{j=1}^{n-1}\dot{L}_{u_j}\sum_{k=j+1}^{n}L_{u_k}\right) - \dot{f}\left(t - \sum_{i=1}^{n}L_{u_i}\right)\left(\sum_{i=1}^{n}\dot{L}_{u_i}\right)t, \tag{16}$$

where D_{u_i} is limited to the time-delay operators. It is noted that the inverse operators $D_{\bar{i}}$

$$D_{\bar{i}}f(t) = f(t + L_i(t + L_i)) \tag{17}$$

can be introduced to formalism as the inverse of its time-delay counterpart. The latter can be shown straightforwardly by noticing, up to the order $\dot{L}$,

$$\begin{aligned} D_{\bar{i}}f(t) &\simeq f(t + L_i) + \dot{f}(t + L_i)(t + L_i)\dot{L}_i \\ &= f(t + L_i) + \dot{f}(t + L_i)t\dot{L}_i + \dot{f}(t + L_i)L_i\dot{L}_i. \end{aligned} \tag{18}$$

By further generalizing Eq. (16) to include inverse operators and evaluating the commutator on the left-hand side (lhs) of Eq. (12), the equality can be readily established.

Now, the algorithm claims that the second-generation TDI solutions can be obtained for specific commutators of the form Eq. (12). In particular, the lhs of Eq. (12) is related to the solution in terms of the coefficients given in Eq. (5), while the rhs of Eq. (12) gives rise to the residual laser frequency noise.

For the rhs of Eq. (12) to vanish, a rather straightforward but intriguing example is when $u_1u_2\cdots u_n$ is a permutation of $v_1v_2\cdots v_n$, namely,

$$v_i = u_{\pi(i)}, \tag{19}$$

where $\pi \in \mathcal{S}_n$ is an arbitrary element of the permutation group of degree n. This is precisely the case explored in Refs. [19,20,22].

To establish the lhs of Eq. (12) to feasible TDI coefficients $q_{i(i')}$ defined in Eq. (5), a few different strategies have been employed. In Refs. [19–21], two constraint equations are introduced so that one falls to the solution space of a specific TDI class, such as the one-arm dysfunctional Michelson combinations. In Ref. [19], the constraint equations read

$$q_2 = q_{3'} = 0, \tag{20}$$

which corresponds to the scenario when the communications through the armlength connecting SC2 and SC3 are interrupted. By substituting Eq. (20) into Eq. (6), one finds

$$q_1(1-a) + q_{1'}(1-b) = 0, \tag{21}$$

where $a = D_{33'}$ and $b = D_{2'2}$.

To accomplish our goal, on the one hand, one needs to write down a tentative TDI solution, for which the subscripts of the two elements on the lhs of Eq. (12) is an arbitrary permutation. On the other hand, one needs to extract the coefficients q_1 and $q_{1'}$ in Eq. (21). As a matter of fact, it was shown that it could be achieved when the subscripts are exclusively expressed in terms of a and b. More detailed and comprehensive discussions can be found in Refs. [19,20], which will be further generalized in the next section. Here, we will merely illustrate the algorithm with a simple example.

Let us consider the commutator

$$[ab, ba] = [D_{33'2'2}, D_{2'233'}]. \tag{22}$$

On the one hand, due to Eq. (12), it vanishes when applied to the laser noise of an arbitrary form. On the other, we have

$$\begin{aligned} abba &= -abb(1-a) - ab(1-b) - a(1-b) - (1-a) + 1, \\ baab &= -baa(1-b) - ba(1-a) - b(1-a) - (1-b) + 1, \end{aligned} \tag{23}$$

and therefore

$$\begin{aligned} [ab, ba] = &(-1 + b + ba - abb)(1-a) \\ &+ (1 - a - ab + baa)(1-b). \end{aligned} \tag{24}$$

When compared against Eq. (21), one readily finds

$$\begin{aligned} q_1 &= -1 + b + ba - ab^2, \\ q_{1'} &= 1 - a - ab + ba^2, \end{aligned} \tag{25}$$

Before closing this section, we comment on the recent developments of the original algorithm. In Ref. [20], various constraint equations corresponding to different TDI classes were considered. This was feasible, thanks to the introduction of the inverse operators Eq. (12). Ref. [21] explores the rhs of Eq. (12), instead of considering commutators whose constituent monomials' subscripts are related by permutation, one generalizes the context to the second-order commutators and polynomials. Reference [22], on the other hand, revises the construction of the TDI solution on the lhs of Eq. (12). In particular, second-generation solutions are manifestly derived by *lifting* up the first-generation ones.

IV. A GENERALIZED COMBINATORIAL ALGEBRAIC APPROACH

In this section, we propose a generalized version of the combinatorial algorithm discussed in [19,20]. It has two pertinent features. First, the present scheme is not subject to any constraint equation. Second, similar to Eq. [21], the algorithm does not rely on some specific subscript permutation associated with the two elements of the commutator in question. It will become clear that these two features give rise to a significant span in the underlying solution space. The novelty of the approach can be intuitively explained regarding Eq. (12). For the rhs of Eq. (12), a system of relevant equations is derived from a more general perspective instead of extracting TDI solutions by enumerating different permutations. For the lhs of Eq. (12), as the constraints are entirely lifted, the solutions are no longer restricted to any specific TDI class and, in principle, generically distinct from the existing ones. As shown below, the resulting TDI solutions are generated mainly by a basis consisting of four-tuples. It is apparent that the procedure is rather different from other approaches in the literature.

In Sec. IV A, we rewrite the TDI equation Eq. (6) regarding the algebraic approach but do not introduce any constraint equation. In Sec. IV B, from the viewpoint of the rhs of Eq. (12), we derive a system of equations that guarantees the laser frequency noise residual vanishes up to first-order terms. Subsequently, in Sec. IV C, from the perspective of the lhs of Eq. (12), we elaborate on the scheme to derive the coefficients of the TDI combinations. The proposed extended version of combinatorial algebraic approach is summarized in Sec. IV D.

A. The TDI equation without constraint

To proceed, we first substitute the forms of q_2 and q_3 from Eqs. (6b) and (6c), namely,

$$\begin{aligned} q_2 &= q_{3'}D_{1'} + q_1 D_3 - q_{2'}, \\ q_3 &= q_{1'}D_{2'} + q_2 D_1 - q_{3'}, \end{aligned} \tag{26}$$

into Eq. (6a), to find

$$\begin{aligned} &q_1(1 - D_{312}) + q_{1'}(1 - D_{2'2}) + q_{2'}(D_{12} - D_{3'}) \\ &\quad + q_{3'}(D_2 - D_{1'12}) = 0, \end{aligned} \tag{27}$$

which is an equation in four variables $q_1, q_{1'}, q_{2'}$, and $q_{3'}$ When compared against Eq. (21), the number of independent variables increases because we have not utilized any constraint equation. Also, regarding the four given variables, we note that Eq. (27) is not unique. For instance, if one substitutes the forms of q_2 and q_3 derived from Eqs. (6a) and (6c) into Eq. (6b), we have

$$q_1(D_{\bar{2}\bar{1}} - D_3) + q_{1'}(D_{\bar{2}\bar{1}} - D_{2'\bar{1}}) + q_{2'}(1 - D_{3'\bar{2}\bar{1}}) + q_{3'}(D_{\bar{1}} - D_{1'}) = 0. \quad (28)$$

However, Eq. (28) can be obtained by right-multiplying $D_{\bar{2}\bar{1}}$ on both sides of Eq. (27). Similarly, by substituting q_2 and q_3 from Eqs. (6a)–(6b), one encounters an equation that can be obtained by right-multiplying D_2 on both sides of Eq. (27). As a result, a TDI solution of Eq. (27) naturally gives rise to a solution of Eq. (28), and the residual of the two solutions are not identical but are of the same order. Therefore, it suffices to only focus on Eq. (27).

By defining $R_{2'} = q_{2'}D_{12}$ and $R_{3'} = q_{3'}D_2$, Eq. (27) can be rewritten in a form reminiscent of Eq. (21)

$$q_1(1-a) + q_{1'}(1-b) + R_{2'}(1-c) + R_{3'}(1-d) = 0, \quad (29)$$

where $a = D_{312}$, $b = D_{2'2}$, $c = D_{\bar{2}\bar{1}3'}$, and $d = D_{\bar{2}1'12}$. A TDI solution is characterized by the values of the four variables $q_1, q_{1'}, R_{2'}$, and $R_{3'}$. Regarding the coefficients defined in Eq. (5), $q_{2'} = R_{2'}D_{\bar{2}\bar{1}}$, $q_{3'} = R_{3'}D_{\bar{2}}$, and q_2 and q_3 can be obtained using Eq. (26).

B. The conditon for vanishing laser-noise residual

By observing the process illustrated in the last section, it is attempting to solve Eq. (29) by considering commutators of monomials whose subscripts are related by some permutations as given by Eq. (19). Although such a procedure is viable, unfortunately, it leads to somewhat restrictive solution space. In this regard, we will directly resort to Eq. (12). Similar to the strategy employed in Refs. [19,20], we also assume that a TDI solution is given in the form of a commutator. The latter consists of two monomials solely formed by a, b, c, d and $\bar{a}$, $\bar{b}$, $\bar{c}$, $\bar{d}$, where $\bar{a} = D_{\bar{2}\bar{1}\bar{3}}$ is the inverse of a, and similarly, $\bar{b} = D_{\bar{2}\bar{2'}}$, $\bar{c} = D_{\bar{3}'12}$, and $\bar{d} = D_{\bar{2}\bar{1}\bar{1}'2}$.

We introduce two tuples with six and four components to facilitate the following discussions. The six-tuple (l, m, n, l', m', n') is defined for any given monomial such as a, b, c, or d. Each component of the tuple is determined by the number of instances for a given time-displacement operator that appears in the monomial. The order of the tuple's components follows the sequence 1, 2, 3, $1'$, $2'$, and $3'$. Like the conventions given by Eq. (13), each time-delay operator will count "+1" for the corresponding component, while the inverse operator contributes a count of "−1".

As an example, for $D_{312'}$, the corresponding six-tuple reads (1,0,1,0,1,0). Similaly, the six-tuple of $D_{\bar{2}\bar{1}1'2}$ is $C = (-1, 0, 0, 1, 0, 0)$. Following these examples, the tuples for the four monomials a, b, c, d, and the inverses are found to be

$$\begin{aligned}
&a = D_{312}, && A = (1,1,1,0,0,0);\\
&b = D_{2'2}, && B = (0,1,0,0,1,0);\\
&c = D_{\bar{2}\bar{1}3'}, && C = (-1,-1,0,0,0,1);\\
&d = D_{\bar{2}1'12}, && D = (1,0,0,1,0,0);\\
&\bar{a} = D_{\bar{2}\bar{1}\bar{3}}, && \bar{A} = (-1,-1,-1,0,0,0) = -A;\\
&\bar{b} = D_{\bar{2}\bar{2'}}, && \bar{B} = (0,-1,0,0,-1,0) = -B;\\
&\bar{c} = D_{\bar{3}'12}, && \bar{C} = (1,1,0,0,0,-1) = -C;\\
&\bar{d} = D_{\bar{2}\bar{1}\bar{1}'2}, && \bar{D} = (-1,0,0,-1,0,0) = -D.
\end{aligned} \quad (30)$$

It is straightforward to observe that the tuple of a monomial consisting of products of a, b, c, d and their inverses is nothing but a "component-wise" summation of the corresponding tuples of individual factors, given by Eq. (30). For example, the six-tuple of $ab\bar{d}c\bar{b}$ is $A + B - D + C - B = (-1, 0, 1, -1, 0, 1)$.

As discussed above, the monomials we utilize to construct the TDI solutions involve only a, b, c, and d and their inverses. Moreover, as it turns out, the corresponding counts are directly associated with valid TDI solutions, where the contribution of $\bar{a}$ will be counted as "−1". In particular, the integer numbers n_a, n_b, n_c, and n_d will be used to constitute the components of a four-tuple. Unless specified, the four-tuple defined above will be used exclusively to indicate the first monomial of the commutator on the lhs of Eq. (12).

For a given monomial, the components of the four-tuple (n_a, n_b, n_c, n_d) and six-tuple (l, m, n, l', m', n') are related by

$$n_aA + n_bB + n_cC + n_dD = (l, m, n, l', m', n'), \quad (31)$$

which implies that

$$\begin{aligned}
&n_a - n_c + n_d = l,\\
&n_a + n_b - n_c = m,\\
&n_a = n, n_d = l', \quad n_b = m', \quad n_c = n'.
\end{aligned} \quad (32)$$

Since there are only four free variables in (l, m, n, l', m', n'), l, m can be determined in terms of n, l', m', n'. It is noting that even though for a given four-tuple (n_a, n_b, n_c, n_d) the six-tuple (l, m, n, l', m', n') is well defined, the underlying monomial cannot be uniquely determined. This is because we can always insert additional factors of a and $\bar{a}$ in pairs into the monomial at arbitrary positions. This will modify the monomial and, subsequently, the TDI solution but the tuple (l, m, n, l', m', n') remains unchanged. Nonetheless, the lack of uniqueness gives rise to more distinct solutions of Eq. (29). In other words, for those who are still wondering, the main reason to introduce the notion of tuples defined by counting resides in the fact that, for the present context, the validity of a TDI solution only depends on the composition of the monomials but not the specific order of its factors. Explicit examples will be given below, but it does not come as a surprise when compared against the combinatorial algorithms proposed in Refs. [19,20].

To proceed, one writes down a system of equations that ensures that the rhs of Eq. (12) vanishes. We denote the tuple of the first monomial of the commutator $[D_{u_1u_2\ldots u_n}, D_{v_1v_2\ldots v_n}]$ as $(x_1, x_2, x_3, x_{1'}, x_{2'}, x_{3'})$ and that of the second monomial as $(y_1, y_2, y_3, y_{1'}, y_{2'}, y_{3'})$. All the coefficients governed by the first factor of the rhs of Eq. (12) correspond to first-order contributions to the residual laser noise. For the rhs of Eq. (12) to vanish, all independent contributions must vanish identically.

Without loss of generality, let us pick out the terms associated with the coefficients governed by the first factor of the rhs of Eq. (12) that involve $\dot{L}_1$, which read

$$
\begin{aligned}
&(x_1L_1 + x_2L_2 + x_3L_3 + x_{1'}L_{1'} + x_{2'}L_{2'} + x_{3'}L_{3'})y_1\dot{L}_1\\
&\quad - (y_1L_1 + y_2L_2 + y_3L_3 + y_{1'}L_{1'} + y_{2'}L_{2'} + y_{3'}L_{3'})x_1\dot{L}_1.
\end{aligned} \tag{33}
$$

As we consider second-generation TDI solutions, according to the last line of Eq. (10), the above coefficients should be dealt together with the terms involving $\dot{L}_{1'}$, which is found to be

$$
\begin{aligned}
&(x_1L_1 + x_2L_2 + x_3L_3 + x_{1'}L_{1'} + x_{2'}L_{2'} + x_{3'}L_{3'})y_{1'}\dot{L}_{1'}\\
&\quad - (y_1L_1 + y_2L_2 + y_3L_3 + y_{1'}L_{1'} + y_{2'}L_{2'} + y_{3'}L_{3'})x_{1'}\dot{L}_{1'}.
\end{aligned} \tag{34}
$$

Now, one can argue that terms of the form $\dot{L}_i\Delta L p(t)$, where ΔL is the difference between two armlengths, is numerically negligible. Taking typical parameters for the LISA mission, the rate of change of armlengths $\dot{L} \sim 10$ m/s, the mismatch of armlengths $\Delta L \sim 1\% \times L \sim 2.5 \times 10^4$ km, and the laser frequency noise $p_i \sim 10^{-13}/\sqrt{\mathrm{Hz}}$ [13]. Therefore, $\dot{L}_i\Delta L p_i \sim 1.5 \times 10^{-24}/\sqrt{\mathrm{Hz}}$@3 mHz, which is insignificant when compared with the typical gravitational wave signal $H(t) \simeq 10^{-20}/\sqrt{\mathrm{Hz}}$. This implies that one can safely make the replacement

$$
\dot{L}_1 L_i \to \dot{L}_1 L, \tag{35}
$$

where L refers to any of the armlengths. The approximation Eq. (35) is a common practice in geometric TDI implementations.

By adding Eq. (33) to Eq. (34) while taking into account of Eq. (35), one finds the coefficient relevant to $\dot{L}_1$

$$
\begin{aligned}
&\dot{L}_1[(y_{1'}x_1 - x_{1'}y_1) + (y_1x_{1'} - x_1y_{1'}) + x_2(y_1 + y_{1'}) - y_2(x_1 + x_{1'}) + x_{2'}(y_{1'} + y_1) - y_{2'}(x_{1'} + x_1)\\
&\quad + x_3(y_1 + y_{1'}) - y_3(x_1 + x_{1'}) + x_{3'}(y_{1'} + y_1) - y_{3'}(x_{1'} + x_1)]L\\
&= \dot{L}_1[(x_2 + x_{2'} + x_3 + x_{3'})(y_1 + y_{1'}) - (y_2 + y_{2'} + y_3 + y_{3'})(x_1 + x_{1'})]L.
\end{aligned} \tag{36}
$$

The terms associated with $\dot{L}_2$ or $\dot{L}_3$ can be obtained by cycling the indices, and the vanishing condition gives rise to the following equations,

$$
\begin{aligned}
&(x_2 + x_{2'} + x_3 + x_{3'})(y_1 + y_{1'}) - (y_2 + y_{2'} + y_3 + y_{3'})(x_1 + x_{1'}) = 0\\
&(x_1 + x_{1'} + x_3 + x_{3'})(y_2 + y_{2'}) - (y_1 + y_{1'} + y_3 + y_{3'})(x_2 + x_{2'}) = 0\\
&(x_2 + x_{2'} + x_1 + x_{1'})(y_3 + y_{3'}) - (y_2 + y_{2'} + y_1 + y_{1'})(x_3 + x_{3'}) = 0
\end{aligned} \tag{37}
$$

The above equations can be rewritten as

$$
Y_1(X_2 + X_3) = (Y_2 + Y_3)X_1, \tag{38a}
$$

$$
Y_2(X_3 + X_1) = (Y_3 + Y_1)X_2, \tag{38b}
$$

$$
Y_3(X_1 + X_2) = (Y_1 + Y_2)X_3, \tag{38c}
$$

where

$$
X_i = x_i + x_{i'}, \tag{39a}
$$

$$
Y_i = y_i + y_{i'}. \tag{39b}
$$

It is observed that only two of the above three equations are independent, as any two of the equations can be used to derive the remaining one. Making use of the relations Eq. (32), one finds the first two equations give

$$(Y_2+Y_3-2Y_1)n_a-2Y_1n_b-(Y_2+Y_3)n_c+2(Y_2+Y_3)n_d=0, \tag{40a}$$

$$(Y_3+Y_1-2Y_2)n_a+2(Y_1+Y_3)n_b-(Y_3+Y_1)n_c-2Y_2n_d=0. \tag{40b}$$

A possible way to view Eq. (38) is to consider that the form of the second monomial of Eq. (12) is already given in terms of the variables Y's and try to solve for the first monomial in terms of the variables X's. Since the TDI solutions are assumed to be expressed in terms of products of the monomials a, b, c, d and their inverses, it is thus meaningful to replace X's by the components of the four-tuple. Therefore, for given Y's, we proceed to solve Eq. (40) for all possible four-tuples (n_a,n_b,n_c,n_d). As derived in appendix, the solution reads

$$(n_a,n_b,n_c,n_d)=\begin{cases} k_1\left(\frac{-2Y_3}{\mathrm{GCD}(2Y_3,Y_3-Y_1,Y_3-Y_2)},\frac{Y_3-Y_2}{\mathrm{GCD}(2Y_3,Y_3-Y_1,Y_3-Y_2)},0,\frac{Y_3-Y_1}{\mathrm{GCD}(2Y_3,Y_3-Y_1,Y_3-Y_2)}\right) & \\ +k_2(-1,1,1,1) & Y_1+Y_2+Y_3\neq 0, Y_1\neq 0 \\ k_1\left(\frac{Y_2+Y_3}{\mathrm{GCD}(Y_2,Y_2+Y_3)},0,\frac{Y_3-Y_2}{\mathrm{GCD}(Y_2,Y_2+Y_3)},-\frac{Y_2}{\mathrm{GCD}(Y_2,Y_2+Y_3)}\right)+k_2(-1,1,1,1) & Y_1+Y_2+Y_3\neq 0, Y_1=0, \\ k_1(1,0,3,0)+k_2(0,1,2,0)+k_3(0,0,2,1) & Y_1+Y_2+Y_3=0 \\ & \text{and } Y_1^2+Y_2^2+Y_3^2\neq 0 \\ (k_1,k_2,k_3,k_4)\text{ arbitrary four-tuple} & Y_1=Y_2=Y_3=0 \end{cases} \tag{41}$$

where k_1, k_2 and k_3 are arbitrary integers, $\mathrm{GCD}(Y_2,Y_2+Y_3)$ is the greatest common divisor of Y_2 and Y_2+Y_3.

To summarize thus far, for a given second monomial of the commutator, we have derived the general form of the four-tuple (n_a,n_b,n_c,n_d) associated with the first monomial of the commutator, which leads to a vanishing laser-noise residual. As discussed above, a given four-tuple (n_a,n_b,n_c,n_d) dictates the value of the six-tuple (l,m,n,l',m',n'). However, the degree of the monomial is not uniquely defined, and neither do the specific elements nor their order. Such freedom implies a broader solution space than those furnished merely by index permutations as explored in preceding studies [19,20]. In particular, it is worth pointing out that TDI solutions generated by index permutation between the first and second monomials correspond to nothing but the following trivial solution of Eq. (38),

$$\begin{aligned} Y_1&=X_1,\\ Y_2&=X_2,\\ Y_3&=X_3. \end{aligned} \tag{42}$$

For the scheme to derive the TDI coefficients to be discussed in the following subsection, we make use of the relation between the variables Y's and the four-tuple (n'_a,n'_b,n'_c,n'_d) associated with the second monomial

$$\begin{aligned} Y_1&=n'_a-n'_c+2n'_d,\\ Y_2&=n'_a+2n'_b-n'_c,\\ Y_3&=n'_a+n'_c, \end{aligned} \tag{43}$$

to give

$$(n'_a,n'_b,n'_c,n'_d)=\left(\frac{-2k_5+Y_1+Y_3}{2},\frac{2k_5-Y_1+Y_2}{2},\frac{2k_5-Y_1+Y_3}{2},k_5\right), \tag{44}$$

where k_5 is an arbitrary integer.

Until now, the discussions have primarily focused on TDI solutions associated with the commutators. Similar to the strategy in Ref. [20], the solution space can be further expanded by attaching arbitrary monomials to both sides of a valid commutator $[s_l,s_r]$ as follows

$$m_l[s_l,s_r]m_r=m_ls_ls_rm_r-m_ls_rs_lm_r, \tag{45}$$

where m_l and m_r are arbitrary monomials. When considering the procedure discussed in the following subsection, they consist of a, b, c, d and their inverses. The proof can be carried out straightforwardly by showing that the residual remains of the same order.

In the following subsection, we elaborate on how to encounter the corresponding TDI coefficients defined in Eq. (5) for a commutator. However, before proceeding further, let us explicitly show that Eq. (41) implies novel solutions not included in the existing combinatorial algebraic algorithms [19,20]. Consider the particular example that the second monomial of the commutator is trivially given by (0,0,0,0,0,0), and the laser residual will vanish regardless of the form of the first monomial. Moreover, as will be shown later, this leads to nontrivial TDI solutions.

C. Derivation of TDI coefficients

The laser frequency noise is guaranteed to be adequately suppressed for those four-tuples acquired according to Eq. (41). In this subsection, we elaborate a scheme to derive the corresponding coefficients in Eq. (29) and thus establish the TDI solution. The process is a generalization of what has been utilized in Refs. [19–21]. The main difference is that now we are dealing with four monomials a, b, c, d and their inverse rather than two, as discussed in Eq. (21).

Before presenting the detailed algorithm, we illustrate the main strategy with two simple examples. Reminiscent of Eq. (23), we have

$$abcd=-abc(1-d)-ab(1-c)-a(1-b)-(1-a)+1, \tag{46}$$

and

$$\bar{b}cd\bar{a}=\bar{b}cd\bar{a}(1-a)-\bar{b}c(1-d)-\bar{b}(1-c)+\bar{b}(1-b)+1. \tag{47}$$

The above two examples demonstrate that given any monomial h exclusively consisting of a, b, c, d and their inverses, $h-1$ can be decomposed into summations of terms proportional to either of $(1-a), (1-b), (1-c), (1-d)$. In what follows, we elaborate a scheme to rewrite an arbitrary monomial h_n of degree n into the form

$$h_n = h_a(1-a) + h_b(1-b) + h_c(1-c) + h_d(1-d) + 1 \tag{48}$$

Subsequently, the difference between two monomials is an element of the left ideal $I = \langle(1-a), (1-b), (1-c), (1-d)\rangle$, which is readily applied to any commutator given by the lhs of Eq. (29).

The general procedure to decompose an arbitrary monomial h_n of degree n into Eq. (48) is as follows:

(1) Initiate $h_a = 0$, $h_b = 0$, $h_c = 0$, and $h_d = 0$.

(2) It is noted that h_n ends in either $a, b, c, d, \bar{a}, \bar{b}, \bar{c}$, or $\bar{d}$, namely, $h_n = h_{n-1}a$, $h_n = h_{n-1}b$, $h_n = h_{n-1}c$, $h_n = h_{n-1}d$, $h_n = h_{n-1}\bar{a}$, $h_n = h_{n-1}\bar{b}$, $h_n = h_{n-1}\bar{c}$ or $h_n = h_{n-1}\bar{d}$.
   - (a) If $h_n = h_{n-1}a$, let $h_a = h_a - h_{n-1}$;
   - (b) If $h_n = h_{n-1}b$, let $h_b = h_b - h_{n-1}$;
   - (c) If $h_n = h_{n-1}c$, let $h_c = h_c - h_{n-1}$;
   - (d) If $h_n = h_{n-1}d$, let $h_d = h_d - h_{n-1}$;
   - (e) If $h_n = h_{n-1}\bar{a}$, let $h_a = h_a + h_{n-1}\bar{a}$;
   - (f) If $h_n = h_{n-1}\bar{b}$, let $h_b = h_b + h_{n-1}\bar{b}$;
   - (g) If $h_n = h_{n-1}\bar{c}$, let $h_c = h_c + h_{n-1}\bar{c}$;
   - (h) If $h_n = h_{n-1}\bar{d}$, let $h_d = h_d + h_{n-1}\bar{d}$.

(3) Repeat step 2 for h_{n-1} until the degree of the monomial vanishes, namely, one encounters $h_0 = 1$.

(4) Finally, we will have $h_n - h_0 = h_a(1-a) + h_b(1-b) + h_c(1-c) + h_d(1-d)$.

The corresponding flow chart of the algorithm is given in Fig. 3. As an example, for the case of Eq. (47), we have $h_4 = \bar{b}cd\bar{a}$. Subsequently, we have $h_3 = \bar{b}cd$, $h_2 = \bar{b}c$, $h_1 = \bar{b}$ and $h_0 = 1$, so that $h_a = \bar{b}cd\bar{a}$, $h_b = \bar{b}$, $h_c = -\bar{b}$ and $h_d = -\bar{b}c$, consistent with Eq. (47).

D. Outline of the proposed approach

We summarize the proposed combinatorial algebraic scheme as follows.

(i) Choose a three-integer-tuple (Y_1, Y_2, Y_3) and determine the composition of the second monomial on the lhs of Eq. (12) using Eq. (44) while including all possible variations. The latter essentially brings in another four arbitrary integers that account for the numbers of $a\bar{a}$ type pairs.

(ii) Find the composition of the first monomial on the lhs of Eq. (12) based on the four-tuple obtained using Eq. (41) while including all possible

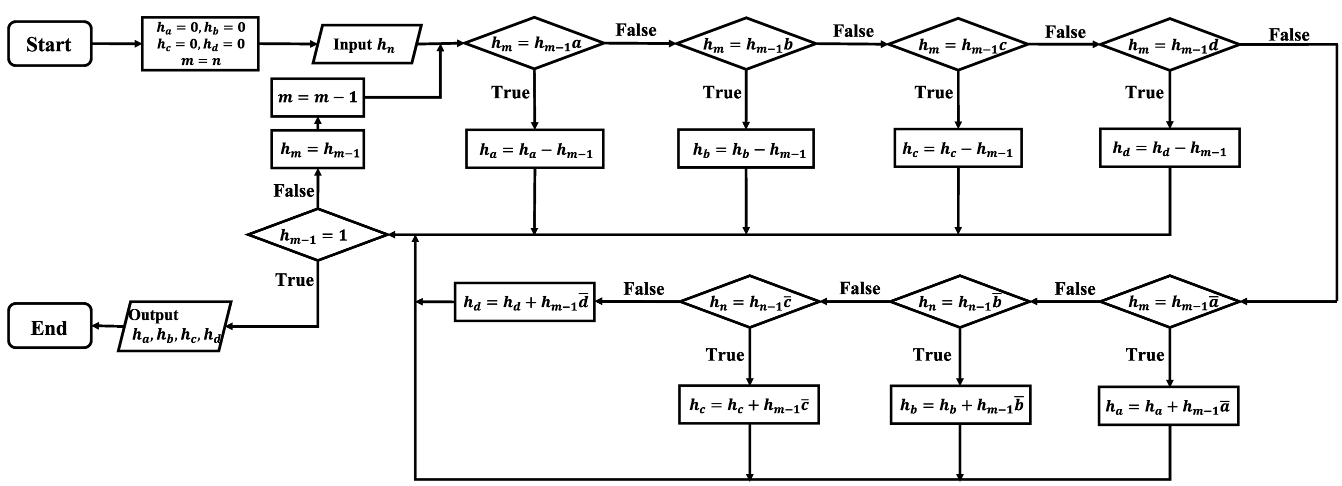

FIG. 3. The algorithm's flow chart for the decomposition of h_n.

variations. The latter essentially brings in another four arbitrary integers that account for the numbers of $a\bar{a}$ type pairs.

(iii) Consider all possible permutations of the time-shift operators, as explored in Refs. [19,20].

(iv) Multiply some arbitrary monomial to the commutator's left and/or right sides.

(v) Derive the corresponding TDI coefficients using the algorithm discussed in Sec. IV C.

V. APPLICATIONS OF THE PROPOSED APPROACH

A. Illustration of the algorithm by a few examples

In Secs. IV A, Sec. IV B, and Sec. IV C, we derive a generalized version of the combinatorial algebraic method for the second-generation TDI combinations. In order to illustrate the use of the proposed algorithm, we give a few particular examples regarding some restrictive choices in the solution space. The first four examples below correspond to the four rows in Eq. (41). We then elaborate on a more sophisticated example to illustrate how the approach can be employed to exhaust possible (restrictive) parameter space. The last example concerns the scenarios where the laser noise residual is not a commutator.

For the first example, let us derive the first monomial by considering the second monomial in the form of a six-tuple (0,0,0,1,1,1). According to Eq. (32), the latter is governed by its last four components (0,1,1,1). We have, $Y_1 = 1$, $Y_2 = 1$, $Y_3 = 1$, $k_5 = 1$, and $Y_3 + Y_2 + Y_1 \neq 0$. Using the first row of Eq. (41), one has the freedom to choose different values for the integers k_1 and k_2. To enumerate distinct TDI solutions, we can choose different degrees of the monomial and consider different permutations. For instance, for $k_1 = -1$ and $k_2 = 0$, the four-tuple associated with the first monomial reads

$$(n_a, n_b, n_c, n_d) = (1, 0, 0, 0). \tag{49}$$

As discussed above, for the four-tuple given by Eq. (49), monomials such as "a", "$ca\bar{c}$", "$\bar{b}ab$" are all valid options. Specifically, let us choose the left monomial as "a" and the right monomial as "bdc". The resultant TDI combination reads

$$[a, bdc], \tag{50}$$

which, by employing the algorithm discussed in Sec. IV C, can be decomposed as

$$\begin{aligned}[a, bdc] = (-1 + bdc)(1 - a) + (1 - a)(1 - b) \\ + (bd - abd)(1 - c) + (b - ab)(1 - d).\end{aligned} \tag{51}$$

Therefore, the coefficients q_i and $q_{i'}$ are

$$\begin{aligned} q_1 &= -1 + D_{2'1'3'}; \\ q_2 &= -D_3 + D_{2'1'3'3}; \\ q_3 &= -D_{31} + D_{2'1'3'31}; \\ q_{1'} &= 1 - D_{312}; \\ q_{2'} &= D_{2'1'} - D_{3122'1'}; \\ q_{3'} &= D_{2'} - D_{3122'}. \end{aligned} \tag{52}$$

Eq. (52) is readily recognized as a twelve-link second-generation Sagnac combination $[\alpha]_1^{12}$ [7].

As for the second example, we choose the second monomial's six-tuple to be (0,0,1,0,0,1). We have, $Y_1 = Y_2 = 0$, $Y_3 \neq 0$, $k_5 = 0$ and subsequently, $Y_1 + Y_2 + Y_3 \neq 0$. Due to the second row of Eq. (41), there are two integers k_1 and k_2 to be determined. By taking $k_1 = 1$, $k_2 = 1$, the four-tuple associated with the first monomial reads

$$(n_a, n_b, n_c, n_d) = (0, 1, 2, 1). \tag{53}$$

As in the first example, let us specifically choose the left monomial as "ac" and the right monomial as "$bdcc$". The resulting commutator reads

$$[ac, bdcc] \tag{54}$$

can be decomposed as

$$\begin{aligned}[ac, bdcc] = (bdcc - 1)(1 - a) + (1 - ac)(1 - b) \\ + (bdcca + bdc + bd - acbdc - acbd - a) \\ \times (1 - c) + (b - acb)(1 - d)\end{aligned} \tag{55}$$

Subsequently, the coefficients q_i and $q_{i'}$ are

$$\begin{aligned} q_1 &= D_{2'1'3'\bar{2}\bar{1}3'} - 1; \\ q_2 &= D_{33'2'1'3'\bar{2}\bar{1}} - D_{2'1'3'\bar{2}\bar{1}}; \\ q_3 &= D_{33'2'1'3'\bar{2}} - D_{2'1'3'\bar{2}}; \\ q_{1'} &= 1 - D_{33'}; \\ q_{2'} &= D_{2'1'3'\bar{2}\bar{1}3'3} + D_{2'1'3'\bar{2}\bar{1}} + D_{2'1'} - D_{33'2'1'3'\bar{2}\bar{1}} \\ &\quad - D_{33'2'1'} - D_3; \\ q_{3'} &= D_{2'} - D_{33'2'}. \end{aligned} \tag{56}$$

For the third example, we choose the second monomial's six-tuple to be $(0, 1, -1, 0, 1, -1)$. We have, $Y_1 = 0$, $Y_2 = -Y_3 = 2, k_5 = 0$ and subsequently, $Y_1 + Y_2 + Y_3 = 0$. Due to the third row of Eq. (41), there are three integers k_1, k_2 and k_3 to be determined. By taking $k_1 = 0$, $k_2 = 1$ and $k_3 = -1$, the four-tuple associated with the first monomial reads

$$(n_a, n_b, n_c, n_d) = (0, 1, 0, -1). \tag{57}$$

As in the first example, let us specifically choose the left monomial to be "$b\bar{d}$" and the right monomial as "$\bar{a}b\bar{c}$", the resulting commutator

$$[b\bar{d}, \bar{a}b\bar{c}] \tag{58}$$

can be decomposed as

$$[b\bar{d}, \bar{a}b\bar{c}] = (b\bar{d}\,\bar{a} - \bar{a})(1-a) + (\bar{a} + \bar{a}b\bar{c} - 1 - b\bar{d}\,\bar{a})(1-b) + (b\bar{d}\,\bar{a}\,b\bar{c} - \bar{a}b\bar{c})(1-c) + (b\bar{d} - \bar{a}b\bar{c}b\bar{d})(1-d) \tag{59}$$

Subsequently, the coefficients q_i and $q_{i'}$ are

$$\begin{aligned}
q_1 &= D_{2'\bar{1}\bar{1}'\bar{1}\bar{3}} - D_{\bar{2}\bar{1}\bar{3}};\\
q_2 &= D_{2'\bar{1}} - D_{\bar{2}\bar{1}\bar{3}2'2\bar{3}'122'\bar{1}} + D_{2'\bar{1}\bar{1}'\bar{1}} - D_{\bar{2}\bar{1}} - D_{2'\bar{1}\bar{1}'\bar{1}\bar{3}2'2\bar{3}'} + D_{\bar{2}\bar{1}\bar{3}2'2\bar{3}'};\\
q_3 &= D_{\bar{2}\bar{1}\bar{3}2'} - D_{2'\bar{1}\bar{1}'\bar{1}\bar{3}2'} - D_{\bar{2}} - D_{2'\bar{1}\bar{1}'\bar{1}\bar{3}2'2\bar{3}'1} + D_{\bar{2}\bar{1}\bar{3}2'2\bar{3}'1} + D_{\bar{2}\bar{1}\bar{3}2'2\bar{3}'122'\bar{1}\bar{1}'};\\
q_{1'} &= -1 + D_{\bar{2}\bar{1}\bar{3}} + D_{\bar{2}\bar{1}\bar{3}2'2\bar{3}'12} - D_{2'\bar{1}\bar{1}'\bar{1}\bar{3}};\\
q_{2'} &= D_{2'\bar{1}\bar{1}'\bar{1}\bar{3}2'2\bar{3}'} - D_{\bar{2}\bar{1}\bar{3}2'2\bar{3}'};\\
q_{3'} &= D_{2'\bar{1}\bar{1}'} - D_{\bar{2}\bar{1}\bar{3}2'2\bar{3}'122'\bar{1}\bar{1}'}.
\end{aligned} \tag{60}$$

For the fourth example, we choose the second monomial's six-tuple to be $(-1, -1, -1, 1, 1, 1)$, which is, again, governed by the last four components. We have, $Y_1 = 0$, $Y_2 = 0$, $Y_3 = 0$, $k_5 = 1$ and subsequently, $Y_1 + Y_2 + Y_3 = 0$. According to the last row of Eq. (41), the first monomial's four-tuple comprises arbitrary integers. Specifically, we choose

$$(n_a, n_b, n_c, n_d) = (0, 0, 0, 1). \tag{61}$$

We consider the following specific commutator $[d, \bar{a}bdc]$

$$[d, \bar{a}bdc] \tag{62}$$

that meets the above conditions. It can be decomposed as

$$\begin{aligned}
[d, \bar{a}bdc] = {} & (d\bar{a} - \bar{a})(1-a) + (\bar{a} - d\bar{a})(1-b)\\
& + (\bar{a}bd - d\bar{a}bd)(1-c)\\
& + (\bar{a}b + \bar{a}bdc - d\bar{a}b - 1)(1-d).
\end{aligned} \tag{63}$$

The above commutator will furnish a valid second-generation TDI combination where the coefficients q_i and $q_{i'}$ are

$$\begin{aligned}
q_1 &= D_{\bar{2}1'\bar{3}} - D_{\bar{2}\bar{1}\bar{3}};\\
q_2 &= D_{\bar{2}\bar{1}\bar{3}2'1'3'\bar{2}1'} - D_{\bar{2}\bar{1}};\\
q_3 &= D_{\bar{2}\bar{1}\bar{3}2'1'3'\bar{2}1'1} - D_{\bar{2}\bar{1}\bar{3}2'1'3'\bar{2}};\\
q_{1'} &= D_{\bar{2}\bar{1}\bar{3}} - D_{\bar{2}1'\bar{3}};\\
q_{2'} &= D_{\bar{2}\bar{1}\bar{3}2'1'} - D_{\bar{2}1'\bar{3}2'1'};\\
q_{3'} &= -D_{\bar{2}} + D_{\bar{2}\bar{1}\bar{3}2'} - D_{\bar{2}1'\bar{3}2'} + D_{\bar{2}\bar{1}\bar{3}2'1'3'\bar{2}}.
\end{aligned} \tag{64}$$

Now, we illustrate how one enumerates all possible commutators using the strategy outlined in Sec. IV D. For the sake of simplicity, we will content ourselves by dealing with a somewhat restrictive solution space. In particular, we choose

$$\begin{aligned}
&Y_1 = 1, \quad Y_2 = 1, \quad Y_3 = 1, \quad k_1 = -1,\\
&k_2 = 1, \quad k_5 = 0.
\end{aligned} \tag{65}$$

As a result, the first monomial's four-tuple reads

$$(n_a, n_b, n_c, n_d) = (0, 1, 1, 1), \tag{66}$$

and that of the second monomial is

$$(n'_a, n'_b, n'_c, n'_d) = (1, 0, 0, 0). \tag{67}$$

Under the above conditions, in Table I, we show all possible monomials by exclusively inserting b and $\bar{b}$ up to a given length and the corresponding commutators. It is straightforward to show that we have $24 \times 3 = 72$ different

TABLE I. Possible choices of the monomials and the corresponding commutators for the four-tuples given by Eqs. (67)–(66).

	Length	Possible variations	Count
First monomial	3	dbc, dcb, bdc, bcd, cbd, cdb	6
	5	$dbbc\bar{b}$, $d\bar{b}cbb$, $bdbc\bar{b}$, $bd\bar{b}cb$, $bcbd\bar{b}$, $bc\bar{b}db$ $bbdc\bar{b}$, $bbd\bar{b}c$, $bbcd\bar{b}$, $bbc\bar{b}d$, $cbbd\bar{b}$, $c\bar{b}dbb$ $\bar{b}dbcb$, $\bar{b}dbbc$, $\bar{b}dcbb$, $\bar{b}cdbb$, $\bar{b}cbdb$, $\bar{b}cbbd$	18
Second monomial	1	a	1
	3	$\bar{b}ab$, $ba\bar{b}$	2

Lengths of the monomials	Commutator
3 + 1	$[dbc,a]$, $[dcb,a]$, $[bdc,a]$, $[bcd,a]$, $[cbd,a]$, $[cdb,a]$
3 + 3	$[dbc,\bar{b}ab]$, $[dcb,\bar{b}ab]$, $[bdc,\bar{b}ab]$, $[bcd,\bar{b}ab]$, $[cbd,\bar{b}ab]$, $[cdb,\bar{b}ab]$ $[dbc,ba\bar{b}]$, $[dcb,ba\bar{b}]$, $[bdc,ba\bar{b}]$, $[bcd,ba\bar{b}]$, $[cbd,ba\bar{b}]$, $[cdb,ba\bar{b}]$
5 + 1	$[dbbc\bar{b},a]$, $[d\bar{b}cbb,a]$, $[bdbc\bar{b},a]$, $[bd\bar{b}cb,a]$, $[bcbd\bar{b},a]$, $[bc\bar{b}db,a]$ $[bbdc\bar{b},a]$, $[bbd\bar{b}c,a]$, $[bbcd\bar{b},a]$, $[bbc\bar{b}d,a]$, $[cbbd\bar{b},a]$, $[c\bar{b}dbb,a]$ $[\bar{b}dbcb,a]$, $[\bar{b}dbbc,a]$, $[\bar{b}dcbb,a]$, $[\bar{b}cdbb,a]$, $[\bar{b}cbdb,a]$, $[\bar{b}cbbd,a]$
5 + 3	$[dbbc\bar{b},\bar{b}ab]$, $[d\bar{b}cbb,\bar{b}ab]$, $[bdbc\bar{b},\bar{b}ab]$, $[bd\bar{b}cb,\bar{b}ab]$, $[bcbd\bar{b},\bar{b}ab]$, $[bc\bar{b}db,\bar{b}ab]$ $[bbdc\bar{b},\bar{b}ab]$, $[bbd\bar{b}c,\bar{b}ab]$, $[bbcd\bar{b},\bar{b}ab]$, $[bbc\bar{b}d,\bar{b}ab]$, $[cbbd\bar{b},\bar{b}ab]$, $[c\bar{b}dbb,\bar{b}ab]$ $[\bar{b}dbcb,\bar{b}ab]$, $[\bar{b}dbbc,\bar{b}ab]$, $[\bar{b}dcbb,\bar{b}ab]$, $[\bar{b}cdbb,\bar{b}ab]$, $[\bar{b}cbdb,\bar{b}ab]$, $[\bar{b}cbbd,\bar{b}ab]$ $[dbbc\bar{b},ba\bar{b}]$, $[d\bar{b}cbb,ba\bar{b}]$, $[bdbc\bar{b},ba\bar{b}]$, $[bd\bar{b}cb,ba\bar{b}]$, $[bcbd\bar{b},ba\bar{b}]$, $[bc\bar{b}db,ba\bar{b}]$ $[bbdc\bar{b},ba\bar{b}]$, $[bbd\bar{b}c,ba\bar{b}]$, $[bbcd\bar{b},ba\bar{b}]$, $[bbc\bar{b}d,ba\bar{b}]$, $[cbbd\bar{b},ba\bar{b}]$, $[c\bar{b}dbb,ba\bar{b}]$ $[\bar{b}dbcb,ba\bar{b}]$, $[\bar{b}dbbc,ba\bar{b}]$, $[\bar{b}dcbb,ba\bar{b}]$, $[\bar{b}cdbb,ba\bar{b}]$, $[\bar{b}cbdb,ba\bar{b}]$, $[\bar{b}cbbd,ba\bar{b}]$

commutators consisting of the monomials enumerated in Table I. They are one-to-one mapped to valid second-generation TDI combinations.

Lastly, as pointed out, the solution space can be further expanded by attaching monomials to a valid commutator. For example, $b[a,bdc]$, $[a,bdc]cd$, and $\bar{d}b[a,bdc]cd$ are all feasible solutions. Subsequently, the TDI coefficients $q_{i(i')}$ can be derived using the algorithm mentioned above. It is not difficult to show that the commutator

$$a[cbd\bar{a},\bar{a}b]\bar{b}, \tag{68}$$

gives rise to the following TDI coefficients

$$\begin{aligned}
q_1 &= D_{33'2'1'\bar{3}\bar{2}\bar{1}\bar{3}} + D_{33'2'1'\bar{3}} - D_{2'\bar{1}3'2'1'\bar{3}} - 1;\\
q_2 &= D_{33'2'1'\bar{3}\bar{2}\bar{1}} - D_{2'\bar{1}};\\
q_3 &= D_{33'2'1'\bar{3}\bar{2}} - D_{2'\bar{1}3'2'1'\bar{3}\bar{2}};\\
q_{1'} &= 1 + D_{2'\bar{1}3'} - D_{33'} - D_{2'\bar{1}3'2'1'\bar{3}\bar{2}2'};\\
q_{2'} &= D_{2'\bar{1}} - D_3;\\
q_{3'} &= D_{2'\bar{1}3'2'} - D_{33'2'},
\end{aligned} \tag{69}$$

which is a fourteen-link geometric TDI combination $[T]^{16}_{11}$ [7].

B. Comparison with Tinto *et al.*'s algebraic method

Recently, Tinto *et al.* proposed an algebraic method to "lift" the first-generation TDI combinations to form the modified second-generation solutions by canceling the laser noise up to the velocity terms [22]. Four first-generation TDI solutions, namely, α, β, γ, and X, constitute the bases. The lifting is achieved by an iterative process to combine the lower-order synthesized beam. The resultant TDI solutions belong to the scenario where the subscripts of the two monomials are related by a permutation Eq. (19). It is important to note that the proposed lifting procedure does not change the sensitivity functions. In this subsection, we explore the connection between such an algorithm and the method proposed in the present study by deriving the resultant combinations in [22] using the present approach. We consider the following two examples [25].

For the Michelson-type combinations X_2 (c.f. Eq. (4.5) of [22]), we choose the following four-tuple for the second monomial

$$(n_a, n_b, n_c, n_d) = (1,1,1,0). \tag{70}$$

The corresponding six-tuple is (0,1,1,0,1,1). We therefore have $Y_1 = 0$, $Y_2 = 2$, $Y_3 = 2$ and $Y_1 + Y_2 + Y_3 \neq 0$. By taking $k_1 = 1$, $k_2 = 1$ on the last row of Eq. (41), one obtains the commutator $[acb, bac]$, which can be decomposed as

$$
\begin{aligned}
[acb, bac] = (bac + b - acbb - 1)(1 - a) \\
+ (1 + bacac - acb - ac)(1 - b) \\
+ (ba + baca - acbba - a)(1 - c).
\end{aligned} \tag{71}
$$

Therefore, the coefficients q_i and $q_{i'}$ are

$$
\begin{aligned}
q_1 &= -1 + D_{2'2} + D_{2'233'} - D_{33'2'22'2}; \\
q_2 &= 0; \\
q_3 &= D_{2'} - D_{33'2'} - D_{33'2'22'} + D_{2'233'33'2'}; \\
q_{1'} &= 1 - D_{33'} - D_{33'2'2} + D_{2'233'33'}; \\
q_{2'} &= -D_3 + D_{2'23} + D_{2'233'3} - D_{33'2'22'23}; \\
q_{3'} &= 0.
\end{aligned} \tag{72}
$$

For the Sagnac-type combinations α_2, β_2, γ_2, it suffices to consider α_2 (cf. Eq. (4.11) of [22]) as the other two solutions can be derived by permuting the indices. It is noted that the solution is different from Eq. (52), as it is obtained by lifting twice. We choose the second monomial's four-tuple to be

$$
(n_a, n_b, n_c, n_d) = (1, 1, 1, 1). \tag{73}
$$

One takes $k_1 = -2$, $k_2 = 1$, according to the first row of Eq. (41). The relevant commutator is chosen to be $[bdca, abdc]$, which gives

$$
\begin{aligned}
&= (1 - bdc - bdca + abdcbdc)(1 - a) \\
&+ (a + abdc - bdcaa - 1)(1 - b) \\
&+ (abd + abdcbd - bdcaabd - bd)(1 - c) \\
&+ (ab + abdcb - b - bdcaab)(1 - d).
\end{aligned} \tag{74}
$$

The corresponding TDI coefficients q_i and $q_{i'}$ read

$$
\begin{aligned}
q_1 &= 1 - D_{2'1'3'} - D_{2'1'3'312} + D_{3122'1'3'2'1'3'}; \\
q_2 &= D_3 - D_{2'1'3'3} - D_{2'1'3'3123} + D_{3122'1'3'2'1'3'3}; \\
q_3 &= D_{31} - D_{2'1'3'31} - D_{2'1'3'31231} + D_{3122'1'3'2'1'3'31}; \\
q_{1'} &= -1 + D_{312} + D_{3122'1'3'} - D_{2'1'3'312312}; \\
q_{2'} &= -D_{2'1'} + D_{3122'1'} + D_{3122'1'3'2'1'} - D_{2'1'3'3123122'1'}; \\
q_{3'} &= -D_{2'} + D_{3122'} + D_{3122'1'3'2'} - D_{2'1'3'3123122'}.
\end{aligned} \tag{75}
$$

C. Comparison with geometric TDI method

The geometric TDI is a method of exhaustion that has been employed extensively for studying TDI solutions. In the literature, the approach has been utilized to derive most second-generation TDI solutions. In a recent study [7], up to sixteen links, thirty-eight distinct second-generation TDI combinations were reported. The derived solutions include a specific class that does not belong to the conventional TDI types, namely, Michelson, relay, beacon, and monitor.

Based on the discussions in Ref. [20], a given TDI class is essentially governed by the related constraint equations, such as Eq. (20). The current approach, however, does not depend on such specific constraints. As a result, the algorithm is expected to reproduce all the geometric TDI solutions, inclusively the ones that do not belong to any specific class. In this subsection, we explicitly show that it is indeed the case.

In Table II, we list all the second-generation TDI combinations encountered using the geometric method in the literature, which consists of three twelve-link, four fourteen-link, and thirty-one sixteen-link combinations. The corresponding laser trajectories and the residual laser noise in commutators are also presented. The latter manifestly shows that all these geometric TDI solutions can be obtained using the algorithm proposed in the present study.

A few comments are in order regarding the decompositions of the geometric TDI solutions shown in Table II. One first writes down the residual of laser frequency noise expressions for a geometric TDI combination. The geometric approach dictates that the laser-noise residual can always be expressed as the difference between two monomials. It is noted that the two monomials are of the same degree but might be composed of different armlengths, as shown in Table II. In order to transform the laser-noise residual into a linear summation of commutators, one might have to introduce a few intermediate monomials by inserting and removing some particular links. The choice of the replacement links follows the principle that (1) it must lead to some valid commutators, and (2) some links will be "annihilated" so that all the monomials will maintain their original degree. Repeating such a procedure allows the laser-noise residual to be rewritten into the desired form. As discussed in Sec. IV B, the residual's validity associated with a commutator that satisfies Eq. (38) remains unchanged by multiplying some monomials on both sides. These monomials may take arbitrary forms composed of a, b, c, d and their inverses, such as b and bdc. Two exceptions are that one may also right-multiply $D_{\bar{2}}$ and $D_{\bar{2}\,\bar{1}}$. This is because the TDI coefficients of the latter two options can be derived by considering the two variations of the TDI equation discussed above near Eq. (28) and their relation with Eq. (27). As mentioned earlier, the difference in the monomials' compositions can be handled by adequately multiplying specific monomials to a valid commutator. As a result, we show that all geometric TDI combinations can be rewritten as a summation of such algebraic solutions.

D. Specific solution with distinct sensitivity

In this subsection, we address possible implications of the present method. To this end, we present a combination obtained using the algorithm whose sensitivity curve and

TABLE II. Enumeration of the second-generation TDI combinations encountered using the geometric TDI approach. We show the solutions' laser link trajectories and the laser-noise residual regarding commutators. The TDI combinations are labeled using the convention in Ref. [7], where the superscript indicates the number of links. For laser link trajectory, $1 \rightarrow 2$ indicates that the laser beam emitted from SC1 propagates to SC2.

TDI combination	Laser link trajectory and residual noise
$[\alpha]_1^{12}$	$1 \leftarrow 2 \leftarrow 3 \leftarrow 1 \leftarrow 3 \leftarrow 2 \leftarrow 1 \rightarrow 3 \rightarrow 2 \rightarrow 1 \rightarrow 2 \rightarrow 3 \rightarrow 1$ $D_{3122'1'3'} - D_{2'1'3'312} = [a, bdc]$
$[\alpha]_2^{12}$	$1 \leftarrow 2 \leftarrow 3 \leftarrow 2 \leftarrow 1 \rightarrow 3 \rightarrow 2 \rightarrow 1 \leftarrow 3 \leftarrow 1 \rightarrow 2 \rightarrow 3 \rightarrow 1$ $D_{311'3'\bar{2}\bar{1}} - D_{2'1'3'\bar{2}\bar{2}'3} = b[\bar{b}a, dc]D_{\bar{2}\bar{1}}$
$[\alpha]_3^{12}$	$1 \leftarrow 2 \leftarrow 1 \rightarrow 3 \leftarrow 2 \rightarrow 1 \leftarrow 3 \rightarrow 2 \rightarrow 3 \leftarrow 1 \rightarrow 2 \leftarrow 3 \rightarrow 1$ $D_{33'\bar{2}1'\bar{3}2'} - D_{2'\bar{1}3'\bar{2}1'1} = a[cd, \bar{a}b]D_{\bar{2}}$
$[U]_1^{14}$	$1 \leftarrow 2 \leftarrow 1 \leftarrow 3 \leftarrow 2 \leftarrow 1 \rightarrow 3 \rightarrow 2 \rightarrow 1 \rightarrow 2 \leftarrow 3 \leftarrow 1 \rightarrow 2 \rightarrow 3 \rightarrow 1$ $D_{33'2'1'3'\bar{2}\bar{1}} - D_{2'1'3'\bar{2}\bar{1}3'3} = [ac, bdcc]\bar{c}D_{\bar{2}\bar{1}}$
$[U]_2^{14}$	$1 \leftarrow 2 \leftarrow 3 \leftarrow 1 \leftarrow 3 \rightarrow 2 \rightarrow 1 \leftarrow 3 \leftarrow 2 \leftarrow 1 \rightarrow 3 \rightarrow 1 \rightarrow 2 \rightarrow 3 \rightarrow 1$ $D_{3122'\bar{1}\bar{3}2'} - D_{2'1'3'2'2\bar{3}'\bar{1}'} = ([a, bdc]\bar{c}\,\bar{d}\,\bar{a} + bdca[\bar{c}\,\bar{d}, \bar{a}b]\bar{b})bD_{\bar{2}}$
$[EP]_1^{14}$	$1 \leftarrow 2 \leftarrow 3 \leftarrow 2 \rightarrow 1 \leftarrow 3 \leftarrow 2 \leftarrow 1 \rightarrow 3 \rightarrow 2 \rightarrow 3 \leftarrow 1 \rightarrow 2 \rightarrow 3 \rightarrow 1$ $D_{311'\bar{3}2'1'3'} - D_{2'1'3'\bar{2}1'12} = a[d, \bar{a}bdc]$
$[EP]_2^{14}$	$1 \leftarrow 2 \leftarrow 1 \leftarrow 3 \rightarrow 2 \leftarrow 1 \rightarrow 3 \leftarrow 2 \rightarrow 1 \rightarrow 2 \rightarrow 3 \leftarrow 1 \rightarrow 2 \leftarrow 3 \rightarrow 1$ $D_{33'2'\bar{1}3'\bar{2}1'} - D_{2'\bar{1}3'\bar{2}1'3'3} = [ac, bcd\bar{a}]aD_{\bar{2}\bar{1}}$
$[U]_4^{16}$	$1 \leftarrow 2 \leftarrow 1 \leftarrow 3 \leftarrow 2 \leftarrow 3 \leftarrow 1 \leftarrow 2 \leftarrow 1 \rightarrow 3 \rightarrow 2 \rightarrow 1 \rightarrow 2 \rightarrow 1 \rightarrow 2 \rightarrow 3 \rightarrow 1$ $D_{33'2'1'1233'} - D_{2'1'3'33'312} = a([\bar{a}bdc, aca] + [a, cbd]c)$
$[U]_5^{16}$	$1 \leftarrow 2 \leftarrow 1 \leftarrow 3 \leftarrow 2 \rightarrow 1 \rightarrow 2 \leftarrow 3 \leftarrow 1 \leftarrow 2 \leftarrow 1 \rightarrow 3 \rightarrow 2 \rightarrow 1 \rightarrow 2 \rightarrow 3 \rightarrow 1$ $D_{33'2'1'\bar{3}\bar{3}'12} - D_{2'1'3'312\bar{3}'\bar{3}} = [a, bdc]\bar{c}\,\bar{a} - a[\bar{c}, cbd\bar{a}]$
$[U]_6^{16}$	$1 \leftarrow 2 \leftarrow 1 \leftarrow 3 \leftarrow 1 \leftarrow 2 \leftarrow 1 \rightarrow 3 \rightarrow 2 \rightarrow 1 \rightarrow 2 \leftarrow 3 \leftarrow 2 \rightarrow 1 \rightarrow 2 \rightarrow 3 \rightarrow 1$ $D_{33'2'233'\bar{2}\bar{1}} - D_{2'1'3'3\bar{1}'\bar{1}3'3} = ([ac, bdcc]\bar{c}\,\bar{c}\,\bar{d}\,ac + bdc([ca\bar{c}, a]\bar{a}\,\bar{d}\,ac + a([ca\bar{c}\,\bar{a}, \bar{d}ac] + \bar{d}[ac, ca\bar{c}\,\bar{a}])))D_{\bar{2}\bar{1}}$
$[PE]_1^{16}$	$1 \leftarrow 2 \leftarrow 3 \leftarrow 2 \leftarrow 1 \rightarrow 3 \rightarrow 2 \rightarrow 3 \leftarrow 1 \rightarrow 2 \leftarrow 3 \leftarrow 2 \rightarrow 1 \leftarrow 3 \rightarrow 2 \rightarrow 3 \rightarrow 1$ $D_{311'3'\bar{2}\bar{1}\bar{1}'2} - D_{2'1'1\bar{2}'3\bar{1}'\bar{1}3'} = a([dc\bar{d}, \bar{c}] + \bar{a}b[\bar{b}a\bar{c}, d]c\bar{d})c$
$[PE]_2^{16}$	$1 \leftarrow 2 \leftarrow 3 \leftarrow 2 \rightarrow 1 \rightarrow 2 \leftarrow 3 \leftarrow 2 \leftarrow 1 \rightarrow 3 \rightarrow 2 \rightarrow 3 \leftarrow 1 \leftarrow 3 \rightarrow 2 \rightarrow 3 \rightarrow 1$ $D_{311'\bar{3}\bar{3}'11'3'} - D_{2'1'1\bar{2}'\bar{2}1'12} = [d, bd\bar{b}] + dbca\bar{c}[\bar{b}\,\bar{c}, c\bar{a}\,\bar{c}\,d]c + [a, dbc]\bar{c}\,\bar{b}\,\bar{a}\,\bar{c}\,dc$
$[PE]_3^{16}$	$1 \leftarrow 2 \leftarrow 3 \leftarrow 2 \leftarrow 3 \leftarrow 2 \leftarrow 1 \rightarrow 3 \rightarrow 2 \rightarrow 3 \leftarrow 1 \rightarrow 2 \rightarrow 1 \leftarrow 3 \rightarrow 2 \rightarrow 3 \rightarrow 1$ $D_{311'11'3'\bar{2}\bar{1}} - D_{2'1'1\bar{2}'33'\bar{2}1'} = b([\bar{b}a, dc]\bar{c}dc + d([cd, \bar{b}a] + c[\bar{b}a\bar{c}, d]c))D_{\bar{2}\bar{1}}$
$[PE]_4^{16}$	$1 \leftarrow 2 \leftarrow 1 \leftarrow 3 \leftarrow 1 \leftarrow 2 \leftarrow 1 \rightarrow 3 \leftarrow 2 \rightarrow 1 \rightarrow 2 \rightarrow 3 \rightarrow 2 \rightarrow 1 \rightarrow 2 \leftarrow 3 \rightarrow 1$ $D_{33'2'233'\bar{2}1'} - D_{2'\bar{1}3'311'3'3} = ([ac, bcdc]\bar{c}\,\bar{d}\,\bar{c}\,acd + bc[dca\bar{d}\,\bar{c}, a]cd)D_{\bar{2}\bar{1}}$
$[PE]_5^{16}$	$1 \leftarrow 2 \leftarrow 1 \leftarrow 3 \leftarrow 2 \rightarrow 1 \rightarrow 2 \rightarrow 1 \rightarrow 2 \leftarrow 3 \leftarrow 1 \leftarrow 2 \leftarrow 1 \rightarrow 3 \rightarrow 2 \rightarrow 3 \rightarrow 1$ $D_{33'2'1'\bar{3}\bar{3}'\bar{3}\bar{3}'} - D_{2'1'12\bar{3}'\bar{3}\bar{2}\bar{1}} = ([ac, bdcc]\bar{c}\,\bar{c}\,\bar{a}\,\bar{c}\,\bar{a} + bd[cca, \bar{c}\,\bar{a}\,\bar{c}]\bar{a})\bar{c}D_{\bar{2}\bar{1}}$
$[PE]_6^{16}$	$1 \leftarrow 2 \leftarrow 1 \leftarrow 3 \leftarrow 2 \rightarrow 1 \rightarrow 2 \rightarrow 3 \leftarrow 1 \leftarrow 2 \leftarrow 1 \rightarrow 3 \rightarrow 2 \rightarrow 1 \rightarrow 2 \leftarrow 3 \rightarrow 1$ $D_{33'2'1'\bar{3}\bar{3}'\bar{1}'2} - D_{2'\bar{1}3'312\bar{3}'\bar{3}} = acbd\bar{a}\,\bar{c}\,\bar{d} - bca\bar{c}\,\bar{a} = a([cbd\bar{a}, \bar{c}]\bar{d} + \bar{a}([a, bdc]\bar{c}\,\bar{a}\,\bar{d} + b[d, ca\bar{c}\,\bar{a}]\bar{d}))$
$[PE]_7^{16}$	$1 \leftarrow 2 \leftarrow 1 \leftarrow 3 \rightarrow 2 \rightarrow 1 \rightarrow 2 \leftarrow 3 \leftarrow 1 \leftarrow 2 \leftarrow 1 \rightarrow 3 \leftarrow 2 \rightarrow 1 \rightarrow 2 \rightarrow 3 \rightarrow 1$ $D_{33'2'\bar{1}\bar{3}\bar{3}'12} - D_{2'1'3'3\bar{1}'2\bar{3}'\bar{3}} = [ac, bdcc]\bar{c}\,\bar{c}\,\bar{d}\,\bar{a}\,\bar{c} + bdca[\bar{a}cad, \bar{d}\,\bar{c}]\bar{d}\,\bar{a}\,\bar{c}$
$[PE]_8^{16}$	$1 \leftarrow 2 \leftarrow 1 \leftarrow 3 \rightarrow 2 \rightarrow 3 \leftarrow 1 \leftarrow 2 \leftarrow 1 \rightarrow 3 \leftarrow 2 \rightarrow 1 \rightarrow 2 \rightarrow 1 \rightarrow 2 \leftarrow 3 \rightarrow 1$ $D_{33'2'\bar{1}\bar{1}'233'} - D_{2'\bar{1}3'33'3\bar{1}'2} = [ac, bcdc]\bar{c}\,\bar{d}\,\bar{c}\,\bar{d}\,ac + bcdc[a\bar{d}, \bar{c}\,\bar{d}\,ac]$
$[PE]_9^{16}$	$1 \leftarrow 2 \leftarrow 1 \leftarrow 2 \leftarrow 1 \rightarrow 3 \rightarrow 2 \rightarrow 3 \leftarrow 1 \leftarrow 3 \leftarrow 2 \rightarrow 1 \rightarrow 2 \rightarrow 1 \rightarrow 2 \leftarrow 3 \rightarrow 1$ $D_{33'33'\bar{2}\bar{1}\bar{1}'2} - D_{2'\bar{1}3'33'3\bar{1}'2} = ([ac, bcdc]\bar{c}\,\bar{d}\,\bar{c}\,\bar{b}\,ac\bar{d}b + bcdc([a\bar{d}, \bar{c}\,\bar{d}\,ac] + a\bar{d}\,\bar{c}[\bar{b}ac, \bar{d}b]))\bar{b}$
$[PE]_{10}^{16}$	$1 \leftarrow 2 \leftarrow 1 \leftarrow 2 \leftarrow 1 \rightarrow 3 \leftarrow 2 \rightarrow 1 \rightarrow 2 \rightarrow 1 \rightarrow 2 \leftarrow 3 \leftarrow 1 \leftarrow 3 \rightarrow 2 \rightarrow 3 \rightarrow 1$ $D_{33'33'\bar{2}1'\bar{3}\bar{3}'} - D_{2'1'1\bar{2}'\bar{2}\bar{1}3'3} = (([ac, bcdc]\bar{c}\,\bar{d}\,\bar{c}\,\bar{b}\,ac\bar{d}b + bcdc([a\bar{d}, \bar{c}\,\bar{d}\,ac] + a\bar{d}\,\bar{c}[\bar{b}ac, \bar{d}b]))\bar{b}d - bd[\bar{b}d, \bar{d}caca\bar{d}])d\bar{a}\,\bar{c}\,D_{\bar{2}\bar{1}}$
$[T]_1^{16}$	$1 \leftarrow 2 \leftarrow 1 \leftarrow 3 \rightarrow 2 \rightarrow 1 \leftarrow 3 \leftarrow 2 \leftarrow 1 \rightarrow 3 \rightarrow 1 \rightarrow 2 \leftarrow 3 \leftarrow 1 \rightarrow 2 \rightarrow 3 \rightarrow 1$ $D_{33'2'\bar{1}\bar{3}2'1'3'} - D_{2'1'3'\bar{2}\bar{1}3'2'2} = acb\bar{a}bdc - bdccb = a[cb, \bar{a}bdc]$
$[T]_2^{16}$	$1 \leftarrow 2 \leftarrow 1 \leftarrow 2 \leftarrow 1 \rightarrow 3 \rightarrow 2 \rightarrow 1 \leftarrow 3 \leftarrow 2 \rightarrow 1 \rightarrow 2 \leftarrow 3 \leftarrow 1 \rightarrow 2 \rightarrow 3 \rightarrow 1$ $D_{33'33'\bar{2}\bar{1}\bar{3}2'} - D_{2'1'3'\bar{2}\bar{1}3'3\bar{1}'} = bdc[\bar{c}\,\bar{d}\,\bar{b}\,a, cac]\bar{a}bD_{\bar{2}}$
$[T]_3^{16}$	$1 \leftarrow 2 \leftarrow 3 \leftarrow 2 \leftarrow 1 \leftarrow 3 \leftarrow 2 \leftarrow 1 \rightarrow 3 \rightarrow 2 \rightarrow 1 \rightarrow 2 \rightarrow 3 \leftarrow 1 \rightarrow 2 \rightarrow 3 \rightarrow 1$ $D_{311'3'2'1'3'\bar{2}} - D_{2'1'3'\bar{2}1'3'31} = (adcbdc - bdcdca)D_{\bar{2}} = a[dca, \bar{a}bdc]D_{\bar{2}}$
$[T]_4^{16}$	$1 \leftarrow 2 \leftarrow 3 \leftarrow 2 \leftarrow 1 \rightarrow 3 \leftarrow 2 \rightarrow 1 \rightarrow 2 \rightarrow 1 \leftarrow 3 \leftarrow 1 \leftarrow 2 \rightarrow 3 \rightarrow 2 \rightarrow 3 \rightarrow 1$ $D_{311'3'\bar{2}1'\bar{3}\bar{3}'} - D_{2'1'11'\bar{3}\bar{2}\bar{2}'3} = b([\bar{b}a, dc]d\bar{a} + dc[\bar{b}ac, \bar{c}d\bar{a}])\bar{c}D_{\bar{2}\bar{1}}$
$[T]_5^{16}$	$1 \leftarrow 2 \leftarrow 3 \leftarrow 2 \leftarrow 3 \leftarrow 2 \leftarrow 1 \rightarrow 3 \rightarrow 2 \rightarrow 1 \leftarrow 3 \rightarrow 2 \rightarrow 3 \leftarrow 1 \rightarrow 2 \rightarrow 3 \rightarrow 1$ $D_{311'11'3'\bar{2}\bar{1}} - D_{2'1'3'\bar{2}1'1\bar{2}'3} = a([d, \bar{a}bdc] + d[dc, \bar{a}b])\bar{b}aD_{\bar{2}\bar{1}}$

(Table continued)

TABLE II. *(Continued)*

TDI combination	Laser link trajectory and residual noise
$[T]^{16}_{6}$	$1 \leftarrow 2 \leftarrow 3 \leftarrow 1 \leftarrow 3 \leftarrow 2 \rightarrow 1 \leftarrow 3 \leftarrow 2 \leftarrow 1 \rightarrow 3 \rightarrow 2 \rightarrow 3 \rightarrow 1 \rightarrow 2 \rightarrow 3 \rightarrow 1$ $D_{3122'1'\bar{3}2'1'} - D_{2'1'3'2'1'12\bar{3}'} = ([abd\bar{a}, bd]c + bda[bd, \bar{a}c])\bar{c}D_{\bar{2}\bar{1}}$
$[T]^{16}_{7}$	$1 \leftarrow 2 \leftarrow 3 \leftarrow 1 \leftarrow 3 \leftarrow 1 \leftarrow 3 \leftarrow 2 \leftarrow 1 \rightarrow 3 \rightarrow 2 \rightarrow 1 \rightarrow 3 \rightarrow 1 \rightarrow 2 \rightarrow 3 \rightarrow 1$ $D_{3122'22'1'3'} - D_{2'1'3'2'2312} = a([b, \bar{a}bdc] + b[bdc, \bar{a}])a$
$[T]^{16}_{8}$	$1 \leftarrow 2 \leftarrow 3 \leftarrow 1 \leftarrow 3 \rightarrow 2 \leftarrow 1 \rightarrow 3 \rightarrow 2 \rightarrow 1 \leftarrow 3 \leftarrow 2 \leftarrow 3 \rightarrow 1 \rightarrow 2 \rightarrow 3 \rightarrow 1$ $D_{3122'\bar{1}3'\bar{2}\bar{1}} - D_{2'1'3'2'\bar{1}\bar{1}'\bar{2}'3} = bd([\bar{d}\,\bar{b}\,a\bar{c}, cbc] + cb[c\bar{d}\,\bar{b}\,a, \bar{c}])D_{\bar{2}\bar{1}}$
$[T]^{16}_{9}$	$1 \leftarrow 2 \leftarrow 1 \leftarrow 2 \rightarrow 3 \rightarrow 2 \rightarrow 1 \leftarrow 3 \leftarrow 2 \rightarrow 1 \leftarrow 3 \leftarrow 1 \rightarrow 2 \leftarrow 3 \rightarrow 1 \rightarrow 3 \rightarrow 1$ $D_{33'3\bar{1}'\bar{1}\bar{3}2'1'} - D_{2'22'\bar{1}3'\bar{2}\bar{2}'3} = a(\bar{a}bb[\bar{b}a\bar{d}, c] + [ca\bar{d}, \bar{a}b\bar{c}]c)dD_{\bar{2}\bar{1}}$
$[T]^{16}_{10}$	$1 \leftarrow 2 \leftarrow 3 \leftarrow 2 \rightarrow 1 \rightarrow 2 \rightarrow 1 \leftarrow 3 \leftarrow 1 \leftarrow 3 \rightarrow 2 \leftarrow 1 \rightarrow 3 \rightarrow 1 \leftarrow 2 \rightarrow 3 \rightarrow 1$ $D_{311'\bar{3}\bar{3}'\bar{3}2'2} - D_{2'1'\bar{3}2'2\bar{3}'1\bar{2}'} = a(\bar{a}b[\bar{c}, d\bar{a}b] + \bar{a}b\bar{c}[c\bar{b}a, d\bar{a}\,\bar{c}]\bar{a}bb)\bar{b}$
$[T]^{16}_{11}$	$1 \leftarrow 2 \leftarrow 1 \leftarrow 3 \leftarrow 2 \rightarrow 1 \rightarrow 3 \rightarrow 2 \rightarrow 1 \leftarrow 3 \leftarrow 1 \leftarrow 2 \rightarrow 3 \rightarrow 1 \rightarrow 2 \leftarrow 3 \rightarrow 1$ $D_{33'2'1'\bar{3}\bar{2}\bar{1}\bar{3}} - D_{2'\bar{1}3'2'1'\bar{3}\bar{2}\bar{2}'} = acbd\bar{a}\,\bar{a} - bcbd\bar{a}\,\bar{b} = a[cbd\bar{a}, \bar{a}b]\bar{b}$
$[T]^{16}_{12}$	$1 \leftarrow 2 \leftarrow 1 \leftarrow 3 \leftarrow 2 \rightarrow 1 \leftarrow 3 \rightarrow 2 \leftarrow 1 \rightarrow 3 \rightarrow 1 \rightarrow 2 \rightarrow 3 \leftarrow 1 \rightarrow 2 \leftarrow 3 \rightarrow 1$ $D_{33'2'1'\bar{3}2'\bar{1}3'} - D_{2'\bar{1}3'\bar{2}1'3'2'2} = a([cbd, \bar{a}]bc + \bar{a}bc([\bar{c}\,\bar{b}\,cb, d] + d[\bar{c}\,\bar{b}, cb])bc)$
$[T]^{16}_{13}$	$1 \leftarrow 2 \leftarrow 1 \leftarrow 3 \rightarrow 2 \rightarrow 1 \rightarrow 3 \leftarrow 2 \rightarrow 1 \leftarrow 3 \leftarrow 1 \leftarrow 2 \leftarrow 3 \rightarrow 1 \rightarrow 2 \rightarrow 3 \rightarrow 1$ $D_{33'2'\bar{1}\bar{3}\bar{2}1'\bar{3}} - D_{2'1'3'2'\bar{1}\bar{3}\bar{2}\bar{2}'} = b([\bar{b}a, dcb\bar{a}] + \bar{b}a[cb\bar{a}, d])\bar{a}$
$[T]^{16}_{14}$	$1 \leftarrow 2 \leftarrow 1 \leftarrow 3 \rightarrow 2 \rightarrow 1 \leftarrow 3 \rightarrow 2 \rightarrow 3 \leftarrow 1 \rightarrow 2 \leftarrow 3 \leftarrow 2 \leftarrow 3 \rightarrow 1 \rightarrow 3 \rightarrow 1$ $D_{33'2'\bar{1}\bar{3}2'\bar{1}\bar{1}'} - D_{2'22'\bar{1}\bar{1}'\bar{1}3'\bar{2}} = a([cb, \bar{a}d]\bar{d}b\bar{d} + \bar{a}([dcb\bar{d}, bc]\bar{c}\,\bar{d} + b[cdc, b\bar{d}\,\bar{c}\,\bar{d}\,\bar{c}]c))D_{\bar{2}}$
$[T]^{16}_{15}$	$1 \leftarrow 2 \leftarrow 1 \leftarrow 2 \leftarrow 1 \rightarrow 3 \leftarrow 2 \rightarrow 1 \leftarrow 3 \rightarrow 2 \rightarrow 1 \rightarrow 2 \rightarrow 3 \leftarrow 1 \rightarrow 2 \leftarrow 3 \rightarrow 1$ $D_{33'33'\bar{2}1'\bar{3}2'} - D_{2'\bar{1}3'\bar{2}1'3'31} = ([a, bcd]\bar{a}ca\bar{b} + a[ca\bar{b}, bcd\bar{a}])bD_{\bar{2}}$
$[T]^{16}_{16}$	$1 \leftarrow 2 \leftarrow 1 \leftarrow 2 \leftarrow 3 \rightarrow 1 \rightarrow 2 \rightarrow 1 \rightarrow 3 \leftarrow 2 \rightarrow 1 \leftarrow 3 \leftarrow 1 \leftarrow 3 \rightarrow 2 \rightarrow 3 \rightarrow 1$ $D_{33'31\bar{2}'\bar{3}'\bar{3}\bar{2}} - D_{2'1'1\bar{2}'\bar{2}\bar{2}'3\bar{1}'} =$ $(((([ac, bcdc]\bar{c}\,\bar{d}\,\bar{c}\,\bar{b}\,ac\bar{d}b + bcdc([a\bar{d}, \bar{c}\,\bar{d}\,ac] + a\bar{d}\,\bar{c}[\bar{b}ac, \bar{d}b]))\bar{b}d - bd[\bar{b}d, \bar{d}caca\bar{d}])\bar{c}\,\bar{b}\,\bar{c}\,\bar{a} + bd\bar{b}[cacb, \bar{b}a\bar{d}\,\bar{c}]\bar{b}\,\bar{c}\,\bar{a})D_{\bar{2}}$
$[T]^{16}_{17}$	$1 \leftarrow 2 \leftarrow 1 \leftarrow 2 \rightarrow 3 \rightarrow 2 \rightarrow 1 \leftarrow 3 \leftarrow 1 \leftarrow 3 \leftarrow 2 \rightarrow 1 \rightarrow 3 \rightarrow 1 \rightarrow 2 \leftarrow 3 \rightarrow 1$ $D_{33'3\bar{1}'\bar{1}\bar{3}2'2} - D_{2'\bar{1}3'2'23\bar{1}'\bar{2}'} = a(\bar{a}b[a\bar{d}, cb] + c[a\bar{d}c, \bar{c}\,\bar{a}\,b]b)\bar{b}$
$[T]^{16}_{18}$	$1 \leftarrow 2 \leftarrow 1 \rightarrow 3 \leftarrow 2 \rightarrow 1 \rightarrow 2 \leftarrow 3 \leftarrow 2 \rightarrow 1 \leftarrow 3 \rightarrow 2 \rightarrow 3 \leftarrow 1 \leftarrow 2 \rightarrow 3 \rightarrow 1$ $D_{33'\bar{2}1'\bar{3}\bar{3}'11'} - D_{2'1'\bar{3}\bar{2}1'1\bar{2}'3} = ac([d\bar{a}, \bar{c}\,\bar{b}]bd\bar{a}b + \bar{c}[\bar{b}d, \bar{a}bd\bar{a}b])\bar{b}aD_{\bar{2}\bar{1}}$

response function are distinct from those derived in the literature. In particular, we choose the second monomial's six-tuple to be (1,1,0,1,1,0). Therefore, we have $Y_1 = 2$, $Y_2 = 2$, $Y_3 = 0$ and $Y_1 + Y_2 + Y_3 \neq 0$. By taking $k_1 = -1, k_2 = 0$ on the first row of Eq. (41), one derive the commutator as $[abd\bar{a}, bd]$ which can be decomposed as

$$
\begin{aligned}
[abd\bar{a}, bd] = {} & (bd - bdabd\bar{a} - 1 + abd\bar{a})(1 - a) \\
& + (bda + 1 - a - abd\bar{a})(1 - b) \\
& + (bdab + b - abd\bar{a}b - ab)(1 - d). \qquad (76)
\end{aligned}
$$

Thus, the coefficients q_i and $q_{i'}$ read

$$
\begin{aligned}
& q_1 = D_{2'1'12} - D_{2'1'123122'1'\bar{3}} - 1 + D_{3122'1'\bar{3}}; \\
& q_2 = D_{2'1'} - D_{3122'1'\bar{3}2'1'} + D_{2'1'123} - D_3; \\
& q_3 = (D_{2'1} - D_{3122'1'\bar{3}2'1'} + D_{2'1'123} - D_3)D_1; \\
& q_{1'} = D_{2'1'12312} + 1 - D_{312} - D_{3122'1'\bar{3}}; \\
& q_{2'} = 0; \\
& q_{3'} = (D_{2'1'12312} + 1 - D_{3122'1'\bar{3}})D_{2'}. \qquad (77)
\end{aligned}
$$

To evaluate its response function and sensitivity curve, we adopt typical parameters in the LISA mission and the general results of the sensitivity functions derived in [26,27]. Specifically, one assumes that armlength $L = 2.5 \times 10^6$ km and the corresponding amplitude spectral densities of the test mass and shot noise: $s_a^{\mathrm{LISA}} = 3 \times 10^{-15}\ \mathrm{m\,s^{-2}}/\sqrt{\mathrm{Hz}}$ and $s_x^{\mathrm{LISA}} = 15 \times 10^{-12}\ \mathrm{m}/\sqrt{\mathrm{Hz}}$ [28]. The resultant sensitivity curve and response function can be evaluated numerically, and they are shown in Figs. 4 and 5. The obtained sensitivity of the solution Eq. (77) is distinct from those of the combinations derived using the geometric TDI approach [7,29]. This is manifestly shown by Fig. 6.

As pointed out in [29], the sensitivity possesses a degree of degeneracy, so algebraically different solutions may still have identical sensitivity. Therefore, the obtained distinct sensitivity curve implies that the combination Eq. (77) is likely independent of those already explored in the literature. In practice, different sensitivities can be exploited to furnish an overall better performance for the detector, which is an interesting subject in its own right.

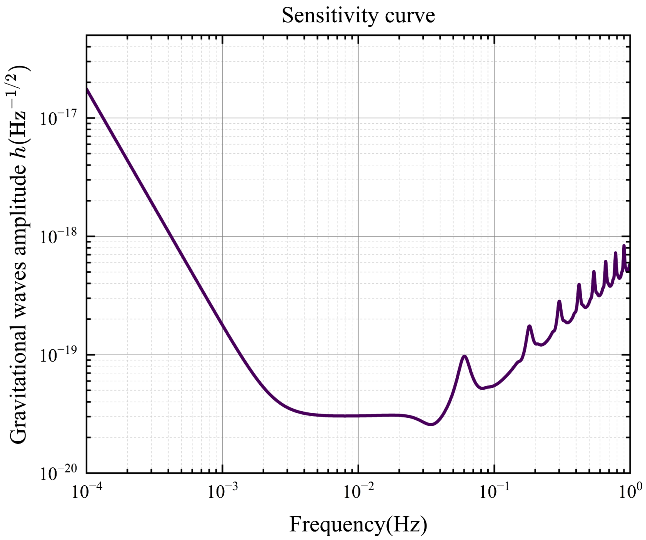

FIG. 4. Sensitivity curve of the combination Eq. (77).

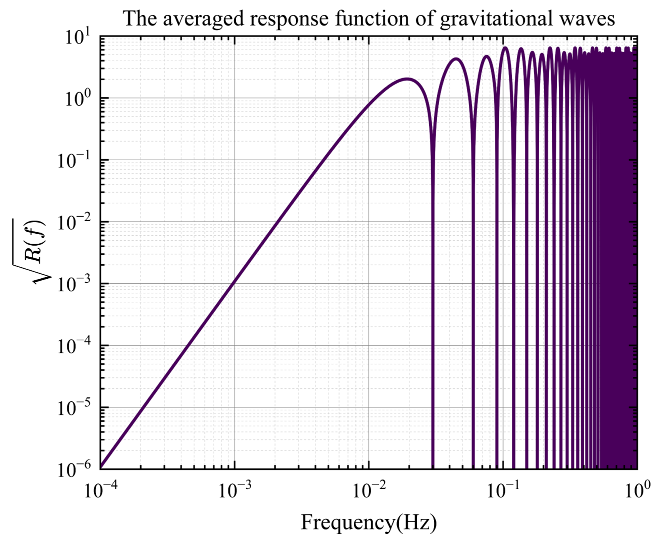

FIG. 5. The averaged gravitational waves response functions of the combination Eq. (77).

VI. FURTHER DISCUSSIONS AND CONCLUDING REMARKS

This work presented an extended version of the combinatorial algebraic method for modified second-generation TDI. When compared against existing combinatorial algorithms, the following generalizations have been made:

(i) The constraint equations are lifted. The TDI combinations derived using the original algorithm that employs some constraints can be considered special solutions in the present context.

(ii) The vanishing condition of the commutator was promoted to a system of equations whose general solution can be expressed as a linear combination of a few bases in the form of four-tuples. The original version of the algorithm where the subscripts of the monomials are related by permutation is a particular case.

(iii) To encounter the TDI coefficients, an extended scheme is elaborated. It involves four independent variables instead of two, as in the original algorithm.

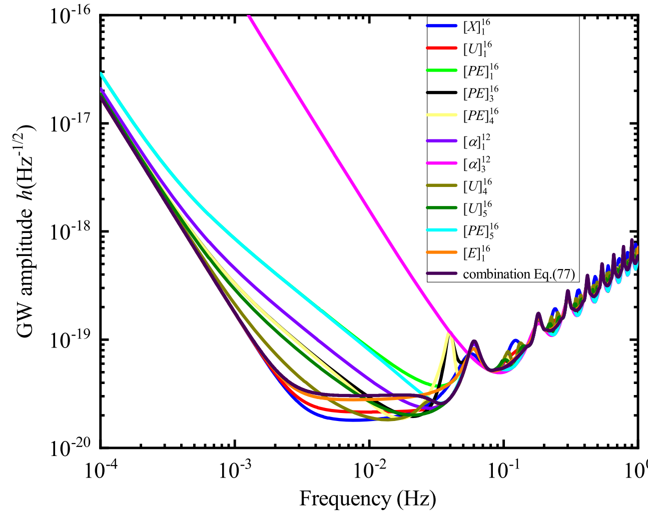

FIG. 6. Sensitivity curve of the combination Eq. (77) compared against those obtained by geometric TDI approach [7,29].

The proposed scheme aims at the second-generation TDI solutions defined by Eq. (10). This differs from the modified second-generation ones defined by Eq. (11), which has been extensively explored in the literature [7,15,19–22]. Nonetheless, by definition, a modified second-generation TDI combination is a second-generation one. Also, it is not difficult to generalize the present scheme to explicitly deal with modified-second generation TDI solutions, which can be achieved by considering six different rates of change in armlengths when deriving Eq. (37).

Compared to the prevailing geometric TDI approach, which utilizes an algorithm to exhaust possible parameter space, the present scheme is an interesting alternative. It employs an intrinsically different strategy which derives the TDI coefficients from a valid form for the laser-noise residual. As an algebraic approach, it also possesses an advantage in terms of computational efficiency. The computational time increases geometrically with the number of links for a method of exhaustion, while the algebraic approach enumerates possible solutions and generates the results almost instantly.

Although all known geometric TDI solutions up to sixteen links have been manifestly recuperated using the present scheme, it is not a proof of exhaustion. We have assumed that the laser-noise residual is in the form of the commutator. Moreover, the monomials that furnish the commutator are exclusively formed by a, b, c, d and their inverses. Apparently, these are only sufficient conditions for a valid TDI solution. In other words, the obtained solutions do not necessarily cover the entire solution space. For instance, as discussed in [21], novel solutions might be encountered as one proceeds to consider higher-order commutators. Another intriguing challenge is, from the perspective of geometric TDI, that the solution obtained by the current approach is apparently not "irreducible." Besides, recent studies also explored how TDI solutions can be classified in terms of their sensitivity functions [22,29], a feature not to be analyzed straightforwardly in the present framework. Nonetheless, a systematic algebraic approach for second-generation TDI is still missing. Also, as we showed in the main text, the proposed algorithm spans much further in the solution space when compared with its predecessors. We plan to address some of the relevant issues in further studies.

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APPENDIX: A DERIVATION OF EQ. (41)

This appendix gives a detailed account of the solution Eqs. (41) of (40) presented in the main text.

We first assume that $Y_1 \neq 0$. By substituting the form of n_b derived from Eqs. (40a) into (40b), we have

$$((Y_3 - Y_1)n_a - (Y_1 + Y_3)n_c + 2Y_3 n_d)(Y_1 + Y_2 + Y_3) = 0. \tag{A1}$$

This case can be further divided into two possibilities. When $Y_1 + Y_2 + Y_3 \neq 0$, one need to solve

$$(Y_3 - Y_1)n_a - (Y_1 + Y_3)n_c + 2Y_3 n_d = 0, \tag{A2}$$

which can be written as

$$(Y_3 - Y_1)(n_a + n_c) + 2Y_3(n_d - n_c) = 0, \tag{A3}$$

and it gives

$$\frac{n_a + n_c}{n_d - n_c} = \frac{-2Y_3}{Y_3 - Y_1}, \tag{A4}$$

for $Y_3 \neq Y_1$.

Since the rhs of Eq. (A4) is a given constant, it is convenient to express the numerator and denominator in terms of their greatest common divisor $\mathrm{GCD}(2Y_3, Y_3 - Y_1)$. We therefore have

$$\begin{aligned} n_a + n_c &= -\frac{2Y_3}{\mathrm{GCD}(2Y_3, Y_3 - Y_1)} k_1, \\ n_d - n_c &= \frac{Y_3 - Y_1}{\mathrm{GCD}(2Y_3, Y_3 - Y_1)} k_1. \end{aligned} \tag{A5}$$

where k_1 is an arbitrary proportional constant. Thus, n_b can be expressed as

$$n_b = n_c + \frac{Y_3 - Y_2}{\mathrm{GCD}(2Y_3, Y_3 - Y_1)} k_1. \tag{A6}$$

In particular, even if $Y_3 - Y_1 = 0$, Eq. (A6) still holds since $\mathrm{GCD}(2Y_3, 0) = |2Y_3|$. Besides, n_b and n_c are integers, implying that the second term on the rhs of Eq. (A6) must also be an integer. It can be adapted to the above equations by replacing $\mathrm{GCD}(2Y_3, Y_3 - Y_1)$ with $\mathrm{GCD}(2Y_3, Y_3 - Y_1, Y_3 - Y_2)$ in Eqs. (A5)–(A6). In other words, by choosing two arbitrary integers k_1 and $k_2 \equiv n_c$, we have

$$\begin{aligned} n_a &= -k_2 - \frac{2Y_3}{\mathrm{GCD}(2Y_3, Y_3 - Y_1, Y_3 - Y_2)} k_1, \\ n_b &= k_2 + \frac{Y_3 - Y_2}{\mathrm{GCD}(2Y_3, Y_3 - Y_1, Y_3 - Y_2)} k_1. \\ n_c &= k_2 \\ n_d &= k_2 + \frac{Y_3 - Y_1}{\mathrm{GCD}(2Y_3, Y_3 - Y_1, Y_3 - Y_2)} k_1. \end{aligned} \tag{A7}$$

Equation (A7) is essentially the first row of Eq. (41) given in the main text when expressed as the four-tuple.

When $Y_1 + Y_2 + Y_3 = 0$, one substitutes the above relation into Eqs. (40) to have

$$\begin{aligned} -3Y_1 n_a - 2Y_1 n_b + Y_1 n_c - 2Y_1 n_d = 0, \\ -3Y_2 n_a - 2Y_2 n_b + Y_2 n_c - 2Y_2 n_d = 0. \end{aligned} \tag{A8}$$

Since $Y_1 \neq 0$, the first line of Eq. (A8) gives

$$-3n_a - 2n_b + n_c - 2n_d = 0, \tag{A9}$$

which implies that the second line of Eq. (A8) will always be satisfied. This gives rise to a solution with three free integers, for which we choose $k_1 \equiv n_a$, $k_2 \equiv n_b$, and $k_3 \equiv n_d$, and

$$\begin{aligned} n_a &= k_1, \\ n_b &= k_2, \\ n_c &= 3k_1 + 2k_2 + 2k_3, \\ n_d &= k_3. \end{aligned} \tag{A10}$$

Equation (A10) is essentially the third row of Eq. (41) when given in the form of a four-tuple.

Second, for $Y_1 = 0$, Eqs. (40) gives

$$(Y_2 + Y_3)(n_a - n_c + 2n_d) = 0, \tag{A11a}$$

$$2Y_2(n_a + n_d) = Y_3(n_a - n_c + 2n_b). \tag{A11b}$$

This case can be further divided into the following three possibilities. When $Y_2 + Y_3 = 0$ and $Y_2 \neq 0$, Eq. (A11) gives

$$n_c = 2n_b + 2n_d + 3n_a,$$

which is nothing but Eq. (A9) which implies the solution Eq. (A10). By joining the two conditions, the solution is given by the third row of Eq. (41) as long as $Y_1 + Y_2 + Y_3 = 0$, and the three quantities do not vanish identically.

When $Y_2 + Y_3 \neq 0$, one substitutes the form of n_c derived from Eq. (A11a) into Eq. (A11b) to have

$$Y_2(n_a + n_b) + (Y_2 + Y_3)(n_d - n_b) = 0. \tag{A12}$$

By employing a similar prescription for Eq. (A3), one finds

$$\begin{aligned} n_a + n_b &= \frac{Y_2 + Y_3}{\mathrm{GCD}(Y_2, Y_2 + Y_3)} k_1, \\ n_d - n_b &= -\frac{Y_2}{\mathrm{GCD}(Y_2, Y_2 + Y_3)} k_1, \\ n_c &= n_a + 2n_d. \end{aligned} \tag{A13}$$

By choosing two independent integers k_1 and $k_2 \equiv n_b$, we have

$$\begin{aligned} n_a &= -k_2 + \frac{Y_2 + Y_3}{\mathrm{GCD}(Y_2, Y_2 + Y_3)} k_1, \\ n_b &= k_2, \\ n_c &= k_2 + \frac{Y_3 - Y_2}{\mathrm{GCD}(Y_2, Y_2 + Y_3)} k_1, \\ n_d &= k_2 - \frac{Y_2}{\mathrm{GCD}(Y_2, Y_2 + Y_3)} k_1. \end{aligned} \tag{A14}$$

This is essentially the second row of Eq. (41) given in the main text.

When $Y_1 = Y_2 = Y_3 = 0$, Eq. (A11) no longer poses any constraint. This implies that its solution is an arbitrary four-tuple, as given by the last row of Eq. (41). This concludes the derivation of Eq. (41) given in the main text.

Nonetheless, we note that the classification in Eq. (41) is not unique. For instance, as long as $Y_2 + Y_3 \neq 0$, it is not difficult to show that the following expression is a solution of Eq. (40)

$$\begin{aligned} n_a &= k_1 \frac{Y_2 + Y_3}{\mathrm{GCD}(Y_1 - Y_2, Y_2 - Y_3, Y_2 + Y_3)}, \\ n_b &= k_2 \frac{Y_2 + Y_3}{\mathrm{GCD}(Y_1 + Y_3, 2Y_3, Y_2 + Y_3)}, \\ n_c &= k_1 \frac{Y_3 - Y_2}{\mathrm{GCD}(Y_1 - Y_2, Y_2 - Y_3, Y_2 + Y_3)} + k_2 \frac{2Y_3}{\mathrm{GCD}(Y_1 + Y_3, 2Y_3, Y_2 + Y_3)}, \\ n_d &= k_1 \frac{Y_1 - Y_2}{\mathrm{GCD}(Y_1 - Y_2, Y_2 - Y_3, Y_2 + Y_3)} + k_2 \frac{Y_1 + Y_3}{\mathrm{GCD}(Y_1 + Y_3, 2Y_3, Y_2 + Y_3)}, \end{aligned} \tag{A15}$$

where k_1 and k_2 are two arbitrary integers. In particular, Eq. (A15) generalizes to the third row of Eq. (41) when $Y_1 + Y_2 + Y_3 = 0$. Subsequently, one may proceed further by assuming $Y_2 + Y_3 = 0$.

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