Few thoughts on θ and the electric dipole moments

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I highlight a few thoughts on the contribution to the dipole moments from the so-called θ parameter. The dipole moments are known can be generated by θ. In fact, the renowned strong $\mathcal{CP}$ problem was formulated as a result of nonobservation of the dipole moments. What is less known is that there is another parameter of the theory, the $\theta_{\rm QED}$ which becomes also a physical and observable parameter of the system when some conditions are met. This claim should be contrasted with conventional (and very naive) viewpoint that the $\theta_{\rm QED}$ is unphysical and unobservable. A specific manifestation of this phenomenon is the so-called Witten effect when the magnetic monopole becomes the dyon with induced electric charge $e'=-e\frac{\theta_{\rm QED}}{2\pi}$. We argued that the similar arguments suggest that the electric magnetic dipole moment μ of any microscopical configuration in the background of $\theta_{\rm QED}$ generates the electric dipole moment $\langle d_{\rm ind}\rangle$ proportional to $\theta_{\rm QED}$, i.e., $\langle d_{\rm ind}\rangle = -\frac{\theta_{\rm QED}\cdot\alpha}{\pi}\mu$. We also argue that many $\mathcal{CP}$ odd correlations such as $\langle \vec{B}_{\rm ext}\cdot\vec{E}\rangle = -\frac{\alpha\theta_{\rm QED}}{\pi}\vec{B}^2_{\rm ext}$ will be generated in the background of an external magnetic field $\vec{B}_{\rm ext}$ as a result of the same physics.

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I. INTRODUCTION AND MOTIVATION

The leitmotiv of the present work is related to the fundamental parameter θ in quantum chromodynamics (QCD), as well as the axion field related to this parameter. The θ parameter was originally introduced in the 1970s. Although the θ term can be represented as a total derivative and does not change the equation of motion, it is known that this parameter is a fundamental physical parameter of the system on the nonperturbative level. It is known that the $\theta\neq 0$ introduces $\mathcal{P}$ and $\mathcal{CP}$ violation in QCD, which is most well captured by the renowned strong $\mathcal{CP}$ problem.

In particular, what is the most important element for the present notes is that the θ parameter generates the neutron (and proton) dipole moment which is known to be very small, $d_n\lesssim 10^{-26}$ e cm, see e.g., review in Physics Today [1]. It can be translated to the upper limit for $\theta\lesssim 10^{-10}$. The strong $\mathcal{CP}$ problem is formulated as follows: why parameter θ is so small in strongly coupled gauge theory? The proton electric dipole moment d_p, similar to the neutron dipole moment d_n will be also generated as a result of non-vanishing θ. In particular, a future measurement of the d_p on the level $d_p\lesssim 10^{-29}$ e cm will be translated to much better upper limit for $\theta\lesssim 10^{-13}$.

The strong $\mathcal{CP}$ problem in QCD problem was resolved by promoting the fundamental parameter θ to a dynamical axion $\theta(x)$ field, see original papers [2–8] and review articles [9–14]. However, the axion has not yet been discovered 45 years after its initial formulation. Still, it remains the best resolution of the strong $\mathcal{CP}$ problem to date, which has also led to numerous proposals for direct dark matter searches.

On the other hand, one may also discuss a similar theta term in QED. It is normally assumed that the $\theta_{\rm QED}$ parameter in the abelian Maxwell Electrodynamics is unphysical and can be always removed from the system. The arguments are based on the observation that the $\theta_{\rm QED}$ term does not change the equation of motion, which is also correct for non-abelian QCD. However, in contrast with QCD when $\pi_3[SU(3)]=\mathbb{Z}$, the topological mapping for the abelian gauge group $\pi_3[U(1)]=0$ is trivial. This justifies the widely accepted view that $\theta_{\rm QED}$ does not modify the equation of motions (which is correct) and does not affect any physical observables and can be safely removed from the theory (which is incorrect as we argue below). We emphasize here that the claim is not that $\theta_{\rm QED}$ vanishes. Instead, the (naive) claim is that the physics cannot depend on $\theta_{\rm QED}$ irrespective to its value.

While these arguments are indeed correct for a trivial vacuum background when the theory is defined on an infinitely large $3+1$ dimensional Minkowski space-time, it has been known for quite sometime that the $\theta_{\rm QED}$ parameter is in fact a physical parameter of the system when the theory

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is formulated on a nonsimply connected, compact manifold with nontrivial $\pi_1[U(1)] = \mathbb{Z}$, when the gauge cannot be uniquely fixed, see the original Refs. [15,16] and review [17]. Such a construction can be achieved, for example, by putting a system into a background of the magnetic field or defining a system on a compact manifold with nontrivial topology. In what follows we treat $\theta_{\rm QED}$ as a new fundamental (unknown) parameter of the theory.

Roughly speaking, the phenomena, in all respects, are very similar to the Aharonov-Bohm and Aharonov Casher effects when the system is highly sensitive to pure gauge (but topologically nontrivial) configurations. In such circumstances the system cannot be fully described by a single ground state.[1] Instead, there are multiple degenerate states which are classified by a topological index. The physics related to pure gauge configurations describing the topological sectors is highly nontrivial. In particular, the gauge cannot be fixed and defined uniquely in such systems. This is precisely a deep reason why $\theta_{\rm QED}$ parameter enters the physical observables in the axion Maxwell electrodynamics in full agreement with very generic arguments [15–17]. Precisely these contributions lead to the explicit $\theta_{\rm QED}$-dependent effects, which cannot be formulated in terms of conventional propagating degrees of freedom (propagating photons with two physical polarizations).

The possible physical effects from $\theta_{\rm QED}$ have also been discussed previously [19,20] in the spirit of the present notes. We refer to our paper [21] with explicit and detail computations of different observable effects (such as induced dipole moment, induced current on a ring, generating the potential difference on the plates, etc) when the system is defined on a nontrivial manifold, or placed in the background of the magnetic field.

It is important to emphasize that some effects can be proportional to $\theta_{\rm QED}$, as opposed to $\dot{\theta}_{\rm QED}$ as commonly assumed or discussed for perturbative computations. Precisely this feature has the important applications when some observables are proportional to the static time-independent $\theta_{\rm QED}$, and, in general, do not vanish even when $\dot{\theta}_{\rm QED} \equiv 0$, see below.

II. AXION θ FIELD AND VARIETY OF TOPOLOGICAL PHENOMENA

Our starting point is the demonstration that the $\theta_{\rm QED}$ indeed does not enter the equations of motion. As a direct consequence of this observation, the corresponding Feynman diagrams at any perturbation order will produce vanishing result for any physical observable at constant $\theta_{\rm QED}$. Indeed,

$$\vec{j}_a = -\dot{\theta}_{\rm QED} \frac{\alpha}{2\pi} \vec{B}, \qquad \alpha \equiv \frac{e^2}{4\pi}, \tag{1}$$

which shows that $\dot{\theta}_{\rm QED}$ and not $\theta_{\rm QED}$ itself enters the equations of motion. In our analysis we ignored spatial derivatives $\partial_i \theta_{\rm QED}$ as they are small for nonrelativistic axions. This anomalous current (1) points along magnetic field in contrast with ordinary E & M, where the current is always orthogonal to $\vec{B}$. Most of the recent proposals [9–14] to detect the dark matter axions are precisely based on this extra current (1) when $\dot{\theta}$ is identified with propagating axion field oscillating with frequency m_a.

We would like to make a few comments on the unusual features of this current. First of all, the generation of the very same nondissipating current (1) in the presence of θ has been very active area of research in recent years. However, it is with drastically different scale of order $\Lambda_{\rm QCD}$ instead of m_a. The main driving force for this activity stems from the ongoing experimental results at RHIC (relativistic heavy ion collider) and the LHC (Large Hadron Collider), which can be interpreted as the observation of such anomalous current (1).

The basic idea for such an interpretation can explained as follows. It has been suggested by [22,23] that the so-called $\theta_{\rm ind}$-domain can be formed in heavy ion collisions as a result of some nonequilibrium dynamics. This induced $\theta_{\rm ind}$ plays the same role as fundamental θ and leads to a number of $\mathcal{P}$ and $\mathcal{CP}$ odd effects, such as chiral magnetic effect, chiral vortical effect, and charge separation effect, to name just a few. This field of research initiated in [24] became a hot topic in recent years as a result of many interesting theoretical and experimental advances, see recent review papers [25,26] on the subject.

In particular, the charge separation effect mentioned above can be viewed as a result of generating of the induced electric field

$$\langle \vec{E} \rangle_{\rm ind} = -\frac{\alpha \theta_{\rm QED}}{\pi} \vec{B}_{\rm ext} \tag{2}$$

in the background of the external magnetic field $\vec{B}_{\rm ext}$ and $\theta_{\rm QED} \neq 0$. This induced electric field $\langle \vec{E} \rangle_{\rm ind}$ separates the electric charges, which represents the charge separation effect. Then formula (2) essentially implies that the electric field locally emerges in every location where magnetic field is present in the background of the $\theta_{\rm QED} \neq 0$.

The effect of separation of charges can be interpreted as a generation of the electric dipole moment in such unusual background. Indeed, for a table-top type experiments it has been argued in [21] that in the presence of the $\theta_{\rm QED}$ the

[1]We refer to [18] with physical explanation (in contrast with very mathematical papers mentioned above) of why the gauge cannot be uniquely fixed in such circumstances. In paper [18] the so-called "modular operator" has been introduced into the theory. The $\exp(i\theta)$ parameter in QCD is the eigenvalue of the large gauge transformation operator, while $\exp(i\theta_{\rm QED})$ is the eigenvalue of the modular operator from [18]. This analogy explicitly shows why $\theta_{\rm QED}$ becomes a physically observable parameter in some circumstances.

electric and magnetic dipole moments of a topologically nontrivial configuration (such as a ring or torus) are intimately related:

$$\langle d_{\rm ind}\rangle = -\frac{\theta_{\rm QED}\cdot\alpha}{\pi}\langle m_{\rm ind}\rangle, \qquad \alpha\equiv\frac{e^2}{4\pi} \tag{3}$$

which obviously resembles the Witten's effect [27] when the magnetic monopole becomes the dion with electric charge $e' = -(e\theta_{\rm QED}/2\pi)$.

To support this interpretation we represent the magnetic dipole moment $\langle m_{\rm ind}\rangle$ as a superposition of two magnetic charges g and $-g$ at distance L_3 apart, where L_3 can be viewed as the size of the compact manifold in construction [21] along the third direction.[2] As the magnetic charge g is quantized, $g=\frac{2\pi}{e}$, formula (3) can be rewritten as

$$\langle d_{\rm ind}\rangle = -\frac{\theta_{\rm QED}e^2}{4\pi^2}\frac{2\pi L_3}{e} = -\left(\frac{e\theta_{\rm QED}}{2\pi}\right)L_3 = e'L_3 \tag{4}$$

This configuration becomes an electric dipole moment $\langle d_{\rm ind}\rangle$ with the electric charges $e' = -(e\theta_{\rm QED}/2\pi)$ which precisely coincides with the Witten's expression for $e' = -(e\theta_{\rm QED}/2\pi)$ in terms of the $\theta_{\rm QED}$ according to [27]. This construction is justified as long as magnetic monopole size is much smaller than the size of the entire configuration L_3 such that the topological sectors from monopole and anti-monopole do not overlap and cannot untwist themselves. The orientation of the axis L_3 also plays a role as it defines the L_1L_2 plane with nontrivial mapping determined by $\pi_1[U(1)]=\mathbb{Z}$, see below. If our arguments on justification of this formula are correct it can be applied to all fundamental particles including electrons, neutrons, and protons because the typical scale $L_3\sim m_e^{-1}\sim 10^{-11}$ cm, while magnetic monopole itself can be assumed to be much smaller in size. In this case the expression (3) derived in terms of the path integral in [21] assumes the form

$$\langle d_{\rm ind}\rangle = -\frac{\theta_{\rm QED}\cdot\alpha}{\pi}\mu, \tag{5}$$

where μ is the magnetic moment of any configuration, including the elementary particles: μ_e, μ_p, μ_n. As emphasized in [21,28] the corresponding expression can be represented in terms of the boundary terms, which normally emerge for all topological effects.

The observed upper limit for $d_e < 10^{-29}$ e cm implies that $\theta_{\rm QED} < 10^{-16}$. We do not have a good explanation of why this parameter is so small. This question is not addressed in the present work. It is very possible that a different axion field must be introduced into the theory which drives $\theta_{\rm QED}$ to zero, similar to conventional axion resolution of the strong $\mathcal{CP}$ problem [2–8].

The equation similar to (5), relating the electric and magnetic dipole moments of the elementary particles was also derived in [29,30] where it has been argued that for time-dependent axion background the electric dipole moment of the electron d_e will be generated,[3] and it must be proportional to the magnetic moment of the electron μ_e and the axion field $\theta(t)$. The absolute value for the axion field $\theta_0\approx 3.7\times 10^{-19}$ was fixed by assuming the axions saturate the dark matter density today. While the relation (5) and the one derived in [29,30] look identically the same (in the static limit $m_a\to 0$ and proper normalization) the starting points are dramatically different: we begin with canonically defined fundamental unknown constant $\theta_{\rm QED}\neq 0$ while computations of [29,30] are based on assumption of time dependent axion fluctuating field saturating the DM density today, which obviously implies a different normalization for θ. Still, both expressions identically coincide in the static $m_a\to 0$ limit.

The identical expressions with precisely the same coefficients (for time dependent [29,30] and time independent (5) formulas) in static limit $m_a\to 0$ relating the electric dipole and magnetic dipole moments strongly suggest that the time dependent expression [29,30] can be smoothly extrapolated to (5) with constant $\theta_{\rm QED}$. This limiting procedure can be viewed as a slow adiabatic process when $\dot\theta\propto m_a\to 0$ and the θ becomes the time-independent parameter, $\theta\to\theta_{\rm QED}$ when the same normalization is implemented.[4]

We want to present one more argument suggesting that the constant $\theta_{\rm QED}$ may produce physical effects including the generating of the electric dipole moment. Indeed the S_θ term in QED in the background of the uniform static magnetic field along z direction can be rewritten as follows

$$S_\theta \propto \theta_{\rm QED}e^2\int {\rm d}^4x\vec{E}\cdot\vec{B} = 2\pi\kappa\theta_{\rm QED}\cdot\left[e\int dzdtE_z\right].$$
$$\text{where } 2\pi\kappa\equiv\left[e\int d^2x_\perp B_z\right] \tag{6}$$

The expression on the right hand side is still a total divergence, and does not change the equation of motion. In fact, the expression in the brackets is identically the same

[2] This construction should be thought as a pure mathematical one. The absence of the real magnetic monopoles in Nature cannot prevent us from such fictitious theoretical construction.

[3] We also refer to paper [31] with criticism of this result and [32] responding to this criticism.

[4] A different approach on computation of the time dependent dipole moment due to the fluctuating θ parameter was developed recently in [33]. The corresponding expression given in [33] approaches a finite nonvanishing constant value if one takes the consecutive limits $t\to\infty$ and after that the static limit $m_a\to 0$ by representing $e/(2m)=\mu$ in terms of the magnetic moment of a fermion. In this form it strongly resembles the expression derived in [29,30].

as the θ term in 2d Schwinger model, where it is known to be a physical parameter of the system as a result of nontrivial mapping $\pi_1[U(1)] = \mathbb{Z}$, see e.g., [34] for a short overview of the θ term in 2d Schwinger model in the given context.[5]

The expression (6) shows once again that $\theta_{\rm QED}$ parameter in 4d Maxwell theory becomes the physical parameter of the system in the background of the magnetic field.[6] In such circumstances the electric field will be induced along the magnetic field in the region of space where the magnetic field is present according to (2). This relation explains why the electric dipole moment of any configuration becomes related to the magnetic dipole moment of the same configuration as Eq. (5) states.

The topological arguments for special case (6) when the external magnetic field is present in the system suggest that the corresponding configurations cannot "unwind" as the uniform static magnetic field B_z enforces the system to become effectively two-dimensional, when the $\theta_{\rm QED}$ parameter is obviously a physical parameter, similar to analogous analysis in the well-known 2d Schwinger model, see footnote 5.

The practical implication of this claim is that there are some $\theta_{\rm QED}$-dependent contributions to the dipole moments of the particles. While the $\theta_{\rm QED}$ does not produce any physically measurable effects for QED with trivial topology, or in vacuum, we expect that in many cases as discussed in [21] and in present work the physics becomes sensitive to the $\theta_{\rm QED}$ which is normally "undetectable" in a typical scattering experiment based on perturbative analysis of QED. We want to list below several $\mathcal{CP}$ odd correlations which will be generated in the presence of $\theta_{\rm QED}$, and which could be experimentally studied by a variety of instruments.

The generation of the induced electric field (2) unambiguously implies that the following $\mathcal{CP}$ odd correlation will be generated

$$\langle \vec{B}_{\rm ext} \cdot \vec{E} \rangle = -\frac{\alpha\theta_{\rm QED}}{\pi} \vec{B}^2_{\rm ext}. \tag{7}$$

Another $\mathcal{CP}$ odd correlation which can be also studied is as follows:

$$\left\langle \sum_i \vec{\mu}_i \cdot \vec{E} \right\rangle = -\frac{\alpha\theta_{\rm QED}}{\pi} \sum_i \vec{B}_{\rm ext} \cdot \vec{\mu}_i, \tag{8}$$

where one should average over entire ensemble of particles with magnetic moments $\vec{\mu}_i$, which are present in the region of a nonvanishing magnetic field $\vec{B}_{\rm ext}$. The induced electric field (2) will coherently accelerate the charged particles along $\vec{B}_{\rm ext}$ direction such that particles will assume on average nonvanishing momentum $\vec{p}_i$ along $\vec{B}_{\rm ext}$. As a result of this coherent behavior the following $\mathcal{CP}$ odd correlation for entire ensemble of particles is expected to occur

$$\left\langle \sum_i \vec{\mu}_i \cdot \vec{p}_i \right\rangle \propto \frac{\alpha\theta_{\rm QED}}{\pi} \sum_i \vec{B}_{\rm ext} \cdot \vec{\mu}_i. \tag{9}$$

One should add that the dual picture when the external magnetic field $\vec{B}_{\rm ext}$ is replaced by external electric field $\vec{E}_{\rm ext}$ also holds. For example, instead of (2) the magnetic field will be induced in the presence of the strong external electric field $\vec{E}_{\rm ext}$, as, e.g., in the proposal [36] to measure the proton EDM when the $\vec{E}_{\rm ext}$ is directed along the radial component,

$$\langle \vec{B} \rangle_{\rm ind} = \frac{\alpha\theta_{\rm QED}}{\pi} \vec{E}_{\rm ext}, \tag{10}$$

such that the correlation similar to (7) will be also generated

$$\langle \vec{B} \cdot \vec{E}_{\rm ext} \rangle = \frac{\alpha\theta_{\rm QED}}{\pi} \vec{E}^2_{\rm ext}. \tag{11}$$

III. CONCLUSION AND FUTURE DIRECTIONS

The topic of the present notes on the dipole moments of the particles and antiparticles in the presence of the $\theta_{\rm QED}$ is largely motivated by the recent experimental advances in the field, see [36,37]. There are many other $\mathcal{CP}$ odd phenomena which accompany the generation of the dipole moments. All the relations discussed in the present notes, including (5) or (7) are topological in nature and related to impossibility to uniquely describe the gauge fields over entire system, as overviewed in the Introduction.

Essentially the main claim is that the $\theta_{\rm QED}$ should be treated as a new fundamental parameter of the theory when the system is formulated on a topologically nontrivial manifold, and in particular, in the background of a magnetic field which enforces a nontrivial topology, as argued in this work.

I believe that the very nontrivial relations such as (5) or (7) which apparently emerge in the system at nonvanishing θ and $\theta_{\rm QED}$ is just the tip of the iceberg of much deeper physics rooted to the topological features of the gauge theories.

In particular, the θ dependent portion of the vacuum energy could be the source of the dark energy today (at $\theta = 0$) in the de Sitter expanding space as argued in [38,39]. Furthermore, these highly nontrivial topological phenomena in strongly coupled gauge theories can be

[5]In this exactly solvable 2D Schwinger model one can explicitly see why the gauge cannot be uniquely fixed, and, as the consequence of this ambiguity, the θ becomes observable parameter of the system. The same 2D Schwinger model also teaches us how this physics can be formulated in terms of the so-called Kogut-Susskind ghost [35] which is the direct analog of the Veneziano ghost in 4D QCD.

[6]The parameter κ which classifies our states is arbitrary real number. It measures the magnetic physical flux, which not necessary assumes the integer values.

tested in the QED tabletop experiments where the very same gauge configurations which lead to the relation similar to (5) or (7) may generate an additional Casimir forces, as well as many other effects as discussed in [28,34,40]. What is even more important is that many of these effects in axion electrodynamics can be in principle measured, see [41–45] with specific suggestions and proposals. I finish on this optimistic note.

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