Dilaton chiral perturbation theory at next-to-leading order

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(Received 26 July 2023; accepted 15 September 2023; published 11 October 2023)

We apply dilaton chiral perturbation theory (dChPT) at next-to-leading order to lattice data from the LatKMI Collaboration for the eight-flavor SU(3) gauge theory. In previous work, we found that leading-order dChPT does not account for these data, but that a model extension of leading-order dChPT with a varying mass anomalous dimension describes these data well. Here we calculate the next-to-leading order corrections for the pion mass and decay constant. We focus on these quantities, as data for the dilaton mass are of poorer quality. The application of next-to-leading order dChPT is difficult because of the large number of new low-energy constants, and the results of our fits turn out to be inconclusive. They suggest—yet cannot firmly establish—that the LatKMI mass range might be outside the scope of dChPT.

DOI: 10.1103/PhysRevD.108.074506

I. INTRODUCTION

Gauge theories with a large number of light fermion degrees of freedom have attracted a lot of attention in recent years. Notable examples include the SU(3) theory with eight fundamental Dirac fermions [1–4], or with two sextet Dirac fermions [5–9]. While it has not been firmly established whether chiral symmetry is broken in the massless limit or that, alternatively, there is an infrared fixed point for either of these fermion contents, both theories share a number of interesting features. First, for the accessible fermion masses, the spectrum contains light pions. Second, in the same fermion mass range, the spectrum of both theories also contains a flavor-singlet scalar meson which is roughly degenerate with the pions, and is thus much lighter than all other excitations. Last, the spectrum shows signs of approximate hyperscaling, with a roughly constant or slowly varying ratio of hadron masses to the pion decay constant. The latter two features are qualitatively different from QCD, and suggest that both theories are either inside the conformal window, or alternatively, below the sill but relatively close to it.

In a series of papers [10–14] we proposed an effective field theory (EFT) framework which extends ordinary chiral perturbation theory (ChPT) to account systematically for both the pions and the light scalar state. The new EFT, called dilaton chiral perturbation theory, or dChPT for short, applies to confining theories close to the sill of the conformal window. Intuitively, the coupling of these theories is "walking," instead of running, and therefore such theories exhibit approximate scale symmetry. dChPT is based on the assumption that the light singlet scalar is a pseudo Nambu-Goldstone boson arising from the spontaneous breaking of the approximate scale symmetry, much like pions are pseudo Nambu-Goldstone bosons of a spontaneously broken approximate chiral symmetry when the fermion mass is nonzero. Two small parameters control the systematic low-energy expansion: one is the fermion mass, as in ordinary ChPT; the other is the distance to the sill of the conformal window in theory space.

We have found that dChPT has a *large-mass* regime in which the theory predicts approximate hyperscaling. This might explain the unique features of the spectrum of the theories described above. In marked distinction from ordinary ChPT, in the large-mass regime the fermion mass need not be small compared with the confinement and chiral symmetry breaking scale of the massless theory. Yet dChPT still admits a systematic expansion, now thanks only to the smallness of the theory-space parameter controlling the distance to the conformal sill [13].

In Ref. [15] we applied dChPT to data of the LSD Collaboration [4] for the $N_f = 8$, SU(3) gauge theory (assuming that this theory is below the sill of the conformal window). We found that at leading order (LO), dChPT provides a good description of these data, with a mass anomalous dimension $\simeq 0.93$. We also tried to fit data of the LatKMI Collaboration [1] for the same theory. This dataset spans a wider range of larger fermion masses. We found that the full LatKMI mass range cannot be described by LO dChPT, but that a model extension of LO dChPT with a variable mass anomalous dimension accounts well for these data [16].

This model description of the LatKMI data may be regarded as an *ad hoc* partial resummation of higher orders in dChPT. An obvious question is whether dChPT can describe the LatKMI data *systematically*. In order to address this question, the next task is to go to next-to-leading order (NLO) in dChPT. This is the main goal of this paper. We focus on the NLO expressions for the pion mass and decay constant, because existing data for the dilaton mass are much less precise while data for the dilaton decay constant are not available at all.

This paper is organized as follows. In Sec. II we review dChPT at leading order, limiting ourselves to the elements we will need in this paper. The vacuum expectation value (VEV) of the dilaton field is a function of the fermion mass. In comparison with ordinary ChPT at LO, this entails a stronger dependence of hadronic quantities on the fermion mass, notably including the pion (and dilaton) decay constants.

In Sec. III we calculate the NLO corrections for the pion mass M_π and decay constant F_π. We introduce an external gauge field a_μ which serves as a source for the nonsinglet axial current, and calculate the effective action at NLO as a function of a_μ, following the strategy of Ref. [17]. This allows us to extract the axial-vector two-point function, from which both F_π and M_π may be obtained. In addition, we consider the dilaton effective potential to NLO, in order to determine the dilaton VEV at this order, which in turn leads to additional corrections to F_π and M_π. Some technical details are relegated to Appendix A.

In Sec. IV we employ the NLO expressions to fit the LatKMI data. This turns out to be difficult, and rather inconclusive, because of the large number of low-energy constants (LECs) that appear in dChPT at NLO. We further discuss our findings in the concluding section, Sec. V, and comment on what will be needed to make progress beyond the current state of the art.

dChPT at LO was also applied to the sextet model in Refs. [8,9], and to the split mass ten-flavor theory in Ref. [18].

Extensions at NLO of ordinary ChPT which include a singlet scalar were also considered in Refs. [19,20]. The approach followed in these works differs from our approach, and therefore a direct comparison is not useful. In particular, Refs. [19,20] do not appear to establish a systematic power counting. Also, the key role of the dilaton VEV in determining the dependence of physical quantities on the fermion mass (already at tree level) is not considered. In addition, the primary application of Ref. [19] is to a possible alternative EFT for QCD, in which the $f_0(500)$ is assumed to be a dilaton. To consider this proposal, we address in Appendix B the question of whether dChPT might be valid for two-flavor QCD. Our estimate for the dChPT decay width of a dilaton into two pions turns out to be smaller than the actual QCD decay width by about a factor of 25. This casts serious doubt on the interpretation of the $f_0(500)$ as a dilaton.

A framework which is somewhat closer to our work was discussed in Refs. [21–23]. However, as we pointed out in Ref. [16], the power counting introduced in Ref. [22] is incorrect. For a recent application of this approach, see Ref. [24].

II. DILATON ChPT AT LEADING ORDER

In this section we review dChPT at LO. In Sec. II A we present and discuss the tree-level Lagrangian, and define our power counting. In Sec. II B we consider the dilaton potential, and show how it leads to a dilaton VEV that depends on the fermion mass m. In Sec. II C we give the LO results for the fermion-mass dependence of M_π and F_π, the pion mass and decay constant, and M_τ and F_τ, the dilaton mass and decay constant. We use these results to review the existence of the large-mass regime. In Sec. II D we rederive the LO results for M_π and F_π from the axial-current two-point function, using this to set the stage for Sec. III.

A. Leading-order Lagrangian

We consider an $\mathrm{SU}(N_c)$ gauge theory with N_f fermions in the fundamental representation. At LO, the Lagrangian for dChPT is given by[1]

$$\begin{aligned}\mathcal{L}_{\mathrm{LO}} &= \frac{1}{4} f_\pi^2 e^{2\tau} \mathrm{tr}(\partial_\mu \Sigma^\dagger \partial_\mu \Sigma) + \frac{1}{2} f_\tau^2 e^{2\tau} \partial_\mu \tau \partial_\mu \tau \\ &\quad - \frac{1}{2} f_\pi^2 B_\pi m e^{(3-\gamma^*)\tau} \mathrm{tr}(\Sigma + \Sigma^\dagger) \\ &\quad + f_\tau^2 c_1 B_\tau e^{4\tau} \left(-\frac{1}{4} + \tau \right). \end{aligned} \tag{2.1}$$

Here $\Sigma \in \mathrm{SU}(N_f)$ is the usual nonlinear pion field,

$$\Sigma = e^{2i\pi/f_\pi}, \qquad \pi = \pi^a T^a, \tag{2.2}$$

with π^a representing the $N_f^2 - 1$ pions,[2] while τ is the (dimensionless) dilaton field. There are altogether five LO LECs. Four of them include f_π and B_π, the familiar LO parameters of ordinary ChPT, and f_τ and $c_1 B_\tau$, which play a parallel role for the dilaton field. In writing Eq. (2.1) we assume that the dilaton field τ has been shifted such that, at tree level, its expectation value vanishes in the chiral limit, $m \to 0$ (see Sec. II B).

The effective theory is defined in the Veneziano limit [25], in which $N_f, N_c \to \infty$ holding $n_f = N_f/N_c$ constant. We assume the following power counting:

$$p^2/\Lambda^2 \sim m/\Lambda \sim (n_f^* - n_f) \sim 1/N_c, \tag{2.3}$$

[1]Throughout this paper we use the Euclidean metric.
[2]The $SU(N_f)$ generators are normalized by $\mathrm{tr}(T^a T^b) = \frac{1}{2}\delta^{ab}$.

where p is a typical pion or dilaton momentum (e.g., in a scattering process). As in ordinary ChPT, Λ is a parameter with dimension of mass measuring the scale of chiral symmetry breaking in the massless limit. n_f^* is the value of n_f at the sill of the conformal window in the Veneziano limit. Since infrared conformality is recovered above the sill, the small parameter $n_f^* - n_f$ controls deviations from conformality below the sill, where the coupling "walks," but the theory ultimately confines. Defining the theory in the Veneziano limit allows us to treat $n_f^* - n_f$ as a continuous parameter. This technicality will not play an important role in the present paper, as we will only consider dChPT for a fixed choice of N_f and N_c. In the LO Lagrangian, this small parameter occurs in the form of

$$c_1 B_\tau \propto n_f^* - n_f. \tag{2.4}$$

The fifth LO parameter, γ^*, is interpreted as the mass anomalous dimension at the infrared fixed point at the nearby sill of the conformal window. For a detailed discussion of the assumptions underlying the construction of this EFT, the proof that Eq. (2.3) defines a systematic power counting, and the resulting form of the LO Lagrangian (2.1) we refer to Refs. [10–13].

With m the renormalized fermion mass defined at some renormalization scale μ in the underlying gauge theory, only the combination $B_\pi m$ is renormalization-group invariant. Physical quantities can thus depend only on this combination. In practice, when we will fit lattice data in Sec. IV, it will be convenient to choose a renormalization scale $\mu = O(1/a)$, where a is the lattice spacing. Consistent with this choice, we will identify m with the (staggered) bare fermion mass.

B. Dilaton potential and vacuum expectation value

Let us consider the classical potential for the dilaton field τ. Assuming $m > 0$, the term proportional to m in Eq. (2.1) is minimized by setting $\Sigma = 1$, and the dilaton potential takes the form

$$\mathcal{V}_{\rm LO} = f_\tau^2 c_1 B_\tau \left(e^{4\tau} \left(-\frac{1}{4} + \tau \right) - \frac{m}{c_1 \mathcal{M}} e^{(3-\gamma^*)\tau} \right). \tag{2.5}$$

For $\mathcal{V}_{\rm LO}$ to be bound from below, we need $c_1 B_\tau > 0$ and $\gamma^* > -1$. Here

$$\mathcal{M} = \frac{f_\tau^2 B_\tau}{N_f f_\pi^2 B_\pi}. \tag{2.6}$$

$c_1 \mathcal{M}$ defines a quantity of order Λ, and is order 1 in the Veneziano limit, since $f_\pi \sim \sqrt{N_c}$ and $f_\tau \sim N_c$ [10]. Minimizing $\mathcal{V}_{\rm LO}$ as a function of $v = \langle \tau \rangle$ leads to the saddle-point equation

$$4 v e^{(1+\gamma^*)v} = \frac{(3-\gamma^*)m}{c_1 \mathcal{M}}. \tag{2.7}$$

This equation determines the classical solution $v_0 = v_0(m)$ as a monotonically increasing function of m with $v_0(0) = 0$. The solution $v_0(m)$ can be expressed in terms of the Lambert W function [15].

The right-hand side of Eq. (2.7) is of order 1 in the power counting defined in Eq. (2.3). The numerical value of the right-hand side can be small, in which case the approximate solution is

$$v_0(m) \approx \frac{(3-\gamma^*)m}{4 c_1 \mathcal{M}}, \tag{2.8}$$

which defines the *small-mass* regime. The right-hand side of Eq. (2.7) can also be large, leading to a different approximate solution

$$v_0(m) \approx \frac{1}{1+\gamma^*} \log\left(\frac{(3-\gamma^*)m}{4 c_1 \mathcal{M}} \right), \tag{2.9}$$

which defines the *large-mass* regime. We discuss the physical relevance of these two regimes in the following subsection.

C. Masses and decay constants at LO

The tree-level masses and decay constants are given by

$$M_\pi^2 = 2 B_\pi m e^{(1-\gamma^*)v_0} = \frac{8 c_1 \mathcal{M} B_\pi v}{3-\gamma^*} e^{2v_0}, \tag{2.10a}$$

$$M_\tau^2 = 4 c_1 B_\tau e^{2v_0} (1 + (1+\gamma^*) v_0), \tag{2.10b}$$

$$F_\pi = f_\pi e^{v_0}, \tag{2.10c}$$

$$F_\tau = f_\tau e^{v_0}. \tag{2.10d}$$

The predictions for M_π and M_τ are obtained using the saddle-point equation, while the expressions for F_π and F_τ are read off from the LO Lagrangian. Unlike in ordinary ChPT, the tree-level decay constants depend on the fermion mass via the dependence of the classical solution for the dilaton VEV, v_0, on m.

In the small-mass regime, where $v_0 \sim m$, the dilaton decouples in the sense that $M_\pi / M_\tau \ll 1$, and we recover the usual predictions of standard ChPT at LO. In contrast, in the large-mass regime, in which v_0 is approximated by Eq. (2.9), we find

$$M_\pi \sim M_\tau \sim F_\pi \sim F_\tau \sim \left(\frac{m}{c_1 \mathcal{M}} \right)^{1/(1+\gamma^*)}, \tag{2.11}$$

i.e., the theory exhibits approximate hyperscaling [13]. Intuitively, the breaking of scale invariance is dominated by the fermion mass m, which, in turn, is large compared with the confinement (and chiral symmetry breaking) scale of the theory, $\Lambda \sim c_1\mathcal{M}$. If we consider the typical expansion parameter of ChPT, then, using Eq. (2.9),

$$\frac{M_\pi^2}{(4\pi F_\pi)^2} = \frac{c_1\mathcal{M}B_\pi v_0}{2\pi^2(3-\gamma^*)f_\pi^2} \approx \frac{c_1\mathcal{M}B_\pi}{2\pi^2(1+\gamma^*)(3-\gamma^*)f_\pi^2}\log\left(\frac{(3-\gamma^*)m}{4c_1\mathcal{M}}\right). \quad (2.12)$$

The expansion parameter is small provided that

$$\frac{c_1\mathcal{M}B_\pi}{8\pi^2 f_\pi^2}\log\left(\frac{m}{2c_1\mathcal{M}}\right) \quad (2.13)$$

is small (for definiteness we assumed $\gamma^* \approx 1$). Hence, as long as $c_1 \propto n_f^* - n_f$ is small, the fermion mass can be large compared with the dynamical scale of the massless theory, characterized by $c_1\mathcal{M}$. The systematic expansion on which dChPT is based is then an expansion in terms of the only small parameter $n_f^* - n_f$. In particular, this ensures that M_π and M_τ are still parametrically small compared with F_π and F_τ in the large-mass regime [13].

D. The axial-current two-point function

It will be useful to review how to obtain the LO pion mass and decay constant from the nonsinglet axial-current two-point function, as we will employ this method at NLO, following Ref. [17]. The axial-current two-point function can be obtained by coupling the theory to a Hermitian external source a_μ for the axial current, and taking the second derivative of the effective action with respect to a_μ. At LO, the effective action is obtained by solving the equations of motion in the presence of a_μ, and substituting the solution back into the tree-level action.

We couple $\mathcal{L}_{\rm LO}$ to a_μ through the introduction of the covariant derivative

$$D_\mu\Sigma = \partial_\mu\Sigma - i\{a_\mu, \Sigma\},$$
$$D_\mu\Sigma^\dagger = (D_\mu\Sigma)^\dagger = \partial_\mu\Sigma^\dagger + i\{a_\mu, \Sigma^\dagger\}, \quad (2.14)$$

where

$$a_\mu = a_\mu^a T^a. \quad (2.15)$$

The natural power counting is obtained by taking $a_\mu \sim \partial_\mu$. The equations of motion for the classical fields $\Sigma = U$ and $\tau = v$ in the presence of the external field a_μ are then

$$0 = e^{2v}(D_\mu D_\mu U U^\dagger - U D_\mu D_\mu U^\dagger) + 2e^{2v}\partial_\mu v(D_\mu U U^\dagger - U D_\mu U^\dagger) - 2B_\pi m e^{yv}(U - U^\dagger) + \frac{2B_\pi m}{N_f}e^{yv}\mathrm{tr}(U - U^\dagger), \quad (2.16a)$$

$$0 = 4c_1 f_\tau^2 B_\tau e^{4v} v - f_\tau^2 e^{2v}(\partial_\mu v)^2 - f_\tau^2 e^{2v}\partial^2 v + \frac{1}{2}f_\pi^2 e^{2v}\mathrm{tr}(D_\mu U D_\mu U^\dagger) - \frac{1}{2}f_\pi^2 B_\pi y m e^{yv}\mathrm{tr}(U + U^\dagger), \quad (2.16b)$$

where

$$y = 3 - \gamma^*. \quad (2.17)$$

Under "intrinsic" parity,

$$U \to U^\dagger, \qquad v \to v, \qquad a_\mu \to -a_\mu, \quad (2.18)$$

the LO Lagrangian and Eq. (2.16b) are even while Eq. (2.16a) is odd. If we wish to obtain the effective action to quadratic order in a_μ, it is thus sufficient to expand Eq. (2.16a) to linear order in a_μ and π [using Eq. (2.2) with $\Sigma = U$]. Intrinsic parity implies also that $\partial_\mu v$ is of order a_μ^2; hence the second term in Eq. (2.16a) can be dropped. The last term does not contribute either, and the equation simplifies to

$$e^{2v}(\partial^2\pi - \partial_\mu a_\mu) - 2B_\pi m e^{(3-\gamma^*)v}\pi = 0. \quad (2.19)$$

We can rewrite Eq. (2.16b) as an equation for $\delta v = v - v_0$, with v_0 the solution of Eq. (2.7). This will lead to a solution for δv which is at least quadratic in a_μ. However, since δv solves the equation of motion, its contribution to the action will be of order $(\delta v)^2$, i.e., order a_μ^4, and we can thus ignore δv. Therefore we can set v equal to v_0 [verifying that Eq. (2.16b) recovers Eq. (2.7)], and, using Eq. (2.10a), Eq. (2.19) thus leads to

$$\pi^a = \frac{\partial_\mu a_\mu^a}{\partial^2 - M_\pi^2} \Rightarrow \pi^a(p) = \frac{-ip_\mu a_\mu^a(p)}{p^2 + M_\pi^2}, \quad (2.20)$$

in momentum space. Substituting $v = v_0$ and Eq. (2.20) back into the leading-order action $S_{\rm LO}$ we obtain

$$S_{\rm LO} = \frac{1}{2}F_\pi^2\int\frac{d^4p}{(2\pi)^4}a_\mu^a(-p)\left(\delta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2 + M_\pi^2}\right)a_\nu^a(p), \quad (2.21)$$

where we used Eqs. (2.10a) and (2.10c). Equation (2.21) yields the expected form for the pion contribution to the axial-current two-point function. This means that we have recovered the expressions for M_π and F_π from the quadratic term in a_μ in the effective action at LO. In the next section, we will extend this approach to obtain F_π and M_π at NLO.

III. DILATON ChPT AT NEXT-TO-LEADING ORDER

There are three parts to the NLO calculation of M_π and F_π. First, as in ordinary ChPT, we need to calculate the one-loop diagrams obtained from the LO Lagrangian. We will do so in Sec. III A by calculating the one-loop corrections to Eq. (2.21) following the method of Ref. [17]. The UV divergences encountered in this calculation are renormalized by the NLO Lagrangian, which constitutes the second part. We will identify the relevant NLO operators and obtain their contributions to M_π and F_π in Sec. III B. These parts of the calculation are similar to ordinary ChPT. The last step, which has no counterpart in ordinary ChPT, is to calculate the NLO corrections to the dilaton VEV, v. This also influences M_π and F_π, as we have already seen at tree level; cf. Eq. (2.10). We will obtain the corrections to v from the effective potential for a constant dilaton field at NLO in Sec. III C, where we also discuss the necessary counterterms. In Sec. III D we introduced the renormalized LECs, while in Sec. III E we assemble all the NLO contributions to M_π and F_π, as well as to the dilaton's VEV.

As usual, the LO Lagrangian is $\mathcal{O}(p^2)$ in the power counting (2.3), and the NLO Lagrangian will be $\mathcal{O}(p^4)$.

A. Axial-current two-point function

We begin with expanding the fields Σ and τ as

$$\begin{aligned} \Sigma &= u e^{i\xi} u, \\ \xi &= 2\pi^a T^a / f_\pi, \\ \tau &= v + \tilde{\tau} = v + \hat{\tau}/f_\tau. \end{aligned} \tag{3.1}$$

Here $u^2 = U$, where U is the solution of the equations of motion (2.16) in the presence of the axial gauge field a_μ, while v is the corresponding dilaton VEV. As explained in Sec. II D, however, we can take $v = v_0$, namely, constant and equal to its LO value, if we are interested in the effective action to NLO and to order a_μ^2. Expanding the action to second order in the fluctuation fields ξ and $\tilde{\tau}$, we obtain

$$\begin{aligned} S_2 = \frac{1}{2}\int d^4x \Big(&-\frac{1}{4} f_\pi^2 e^{2v_0} \mathrm{tr}[D_\mu U D_\mu (u^\dagger \xi^2 u^\dagger) + D_\mu (u \xi^2 u) D_\mu U^\dagger - 2 D_\mu (u\xi u) D_\mu (u^\dagger \xi u^\dagger)] \\ &- i f_\pi^2 e^{2v_0} \mathrm{tr}[D_\mu U D_\mu (u^\dagger \xi u^\dagger) - D_\mu (u\xi u) D_\mu U^\dagger]\tilde{\tau} + f_\pi^2 e^{2v_0} [D_\mu U D_\mu U^\dagger]\tilde{\tau}^2 + f_\tau^2 e^{2v_0} (\partial_\mu \tilde{\tau})^2 \\ &+ \frac{1}{2} f_\pi^2 B_\pi m e^{y v_0} \mathrm{tr}[(U + U^\dagger)\xi^2] - i f_\pi^2 B_\pi m y e^{y v_0} \mathrm{tr}[(U - U^\dagger)\xi]\tilde{\tau} \\ &- \frac{1}{2} f_\pi^2 B_\pi m y^2 e^{y v_0} \mathrm{tr}[U + U^\dagger]\tilde{\tau}^2 + 4 f_\tau^2 B_\tau c_1 e^{4v_0} [1 + 4v_0]\tilde{\tau}^2 \Big) \\ = S_\pi + S_\tau + S_{\mathrm{mix}}. \end{aligned} \tag{3.2}$$

Following Ref. [17], and using Eq. (3.1), the "pionic" part S_π can be written as

$$S_\pi = \frac{1}{2} e^{2v_0} \int d^4x \pi^a D_\pi^{ab} \pi^b, \tag{3.3}$$

with

$$D_\pi^{ab}\pi^b = -d_\mu d_\mu \pi^a + 4\hat{\sigma}_\pi^{ab}\pi^b, \tag{3.4a}$$

$$d_\mu \pi^a = \partial_\mu \pi^a + \hat{\Gamma}_\mu^{ab}\pi^b, \tag{3.4b}$$

$$\hat{\Gamma}_\mu^{ab} = -2\mathrm{tr}[[T^a, T^b]\Gamma_\mu], \tag{3.4c}$$

$$\hat{\sigma}_\pi^{ab} = -\frac{1}{2}\mathrm{tr}[[T^a, \Delta_\mu][T^b, \Delta_\mu]] + \frac{1}{4} e^{(y-2)v_0} \mathrm{tr}[\{T^a, T^b\}\sigma], \tag{3.4d}$$

in which

$$\Gamma_\mu = \frac{1}{2}(u^\dagger \partial_\mu u - \partial_\mu u u^\dagger - i u^\dagger a_\mu u + i u a_\mu u^\dagger), \tag{3.5a}$$

$$\Delta_\mu = \frac{1}{2}(u^\dagger \partial_\mu u + \partial_\mu u u^\dagger - i u^\dagger a_\mu u - i u a_\mu u^\dagger), \tag{3.5b}$$

$$\sigma = B_\pi m (U + U^\dagger). \tag{3.5c}$$

The mixed part containing the terms bilinear in π^a and $\hat{\tau}$ can be written as

$$S_{\mathrm{mix}} = \frac{1}{2}\frac{f_\pi}{f_\tau} e^{2v_0} \int d^4x \hat{\tau} D_{\mathrm{mix}}^a \pi^a, \tag{3.6}$$

with

$$D_{\mathrm{mix}}^a \pi^a = (-2i\mathrm{tr}[\Delta_\mu T^a] d_\mu + \sigma_{\mathrm{mix}}^a)\pi^a, \tag{3.7a}$$

$$\sigma_{\mathrm{mix}}^a = -\frac{i}{2} B_\pi m y e^{(y-2)v_0} \mathrm{tr}[T^a (U - U^\dagger)]. \tag{3.7b}$$

Finally, the terms quadratic in $\hat{\tau}$ can be written as

$$S_\tau = \frac{1}{2} e^{2v_0} \int d^4x \hat{\tau} D_\tau \hat{\tau}, \tag{3.8}$$

with

$$\begin{aligned} D_\tau \hat{\tau} &= -\partial^2 \hat{\tau} - \sigma_\tau \hat{\tau}, \\ \sigma_\tau &= -4B_\tau c_1 e^{2v_0}(1+4v_0) + \frac{1}{2} y^2 e^{(y-2)v_0} \frac{f_\pi^2}{f_\tau^2} \mathrm{tr}(\sigma) \\ &\quad - \frac{f_\pi^2}{f_\tau^2} \mathrm{tr}(D_\mu U D_\mu U^\dagger). \end{aligned} \tag{3.9}$$

Defining for convenience $\hat{\tau}$ as the zeroth component of π^a, the one-loop effective action $S^{(1)} = S^{(1)}(a_\mu)$ is now defined by a Gaussian integral

$$e^{-S^{(1)}} = \int [d\pi] \exp\left(-\frac{1}{2} e^{2v_0} \int d^4x \pi^T \mathcal{D} \pi\right), \tag{3.10}$$

in which

$$\mathcal{D} = \begin{pmatrix} D_\tau & D_{\mathrm{mix}}^T \\ D_{\mathrm{mix}} & D_\pi \end{pmatrix}, \tag{3.11}$$

and thus

$$S^{(1)} = \frac{1}{2} \mathrm{Tr} \log \mathcal{D}, \tag{3.12}$$

up to an irrelevant constant.

What we need is $S_2^{(1)}$, the part of $S^{(1)}$ which is quadratic in a_μ. To obtain it, we write

$$\mathcal{D} = \begin{pmatrix} -\partial^2 + M_\tau^2 & 0 \\ 0 & (-\partial^2 + M_\pi^2)\mathbf{1} \end{pmatrix} + \begin{pmatrix} \delta_\tau & \delta_{\mathrm{mix}}^T \\ \delta_{\mathrm{mix}} & \delta_\pi \end{pmatrix}, \tag{3.13}$$

where $\delta = 0$ for $a_\mu = 0$, and $\mathbf{1}$ is the $(N_f^2 - 1) \times (N_f^2 - 1)$ unit matrix. δ_π and δ_τ both start at quadratic order in a_μ, while δ_{mix} starts at linear order in a_μ. By expanding the logarithm in Eq. (3.12) to quadratic order in a_μ, we find

$$\begin{aligned} S_2^{(1)} &= \frac{1}{2} D_{0\tau}^{-1}(0) \int d^4x \delta_\tau(x) + \frac{1}{2} D_{0\pi}^{-1}(0) \int d^4x \mathrm{tr}(\delta_\pi(x)) \\ &\quad - \frac{1}{2} \int d^4x \int d^4y \delta_{\mathrm{mix}}^a(x) D_{0\tau}^{-1}(x-y) \delta_{\mathrm{mix}}^a(y) D_{0\pi}^{-1}(y-x), \end{aligned} \tag{3.14}$$

where $D_{0\tau}^{-1}(x-y)$ and $D_{0\pi}^{-1}(x-y)$ are the tree-level dilaton and pion propagators, respectively. Defining

$$\nabla_\mu \pi^a = \partial_\mu \pi^a - a_\mu^a, \tag{3.15}$$

we have to leading order in $\pi^a \sim a_\mu^a$ [cf. Eq. (2.20)]

$$\begin{aligned} \mathrm{tr}\delta_\pi &= -N_f \nabla_\mu \pi^a \nabla_\mu \pi^a - \frac{N_f^2 - 1}{N_f} M_\pi^2 \pi^a \pi^a, \\ \delta_{\mathrm{mix}}^a &= -\frac{F_\pi}{F_\tau}\left(\nabla_\mu \pi^a \partial_\mu - \frac{1}{4} y M_\pi^2 \pi^a\right), \\ \delta_\tau &= \frac{F_\pi^2}{F_\tau^2}\left(2\nabla_\mu \pi^a \nabla_\mu \pi^a + \frac{1}{2} y^2 M_\pi^2 \pi^a \pi^a\right). \end{aligned} \tag{3.16}$$

The first two terms in Eq. (3.14) give tadpole contributions

$$\begin{aligned} S_\pi(a_\mu) &= -\frac{1}{2} K(M_\pi^2) \int \frac{d^4q}{(2\pi)^4} a_\mu^a(-q) \left[N_f\left(\delta_{\mu\nu} - \frac{q_\mu q_\nu}{q^2 + M_\pi^2}\right)\right. \\ &\quad \left. - \frac{1}{N_f} \frac{M_\pi^2 q_\mu q_\nu}{(q^2 + M_\pi^2)^2}\right] a_\nu^a(q), \end{aligned} \tag{3.17}$$

and

$$\begin{aligned} S_\tau(a_\mu) &= \frac{F_\pi^2}{F_\tau^2} K(M_\tau^2) \int \frac{d^4q}{(2\pi)^4} a_\mu^a(-q) \left[\delta_{\mu\nu} - \frac{q_\mu q_\nu}{q^2 + M_\pi^2}\right. \\ &\quad \left. + \left(\frac{y^2}{4} - 1\right) \frac{M_\pi^2 q_\mu q_\nu}{(q^2 + M_\pi^2)^2}\right] a_\nu^a(q), \end{aligned} \tag{3.18}$$

in which

$$K(M^2) = \int \frac{d^dp}{(2\pi)^d} \frac{1}{p^2 + M^2} = \frac{M^2}{16\pi^2}\left(-\lambda + \log\left(\frac{M^2}{\mu^2}\right)\right), \tag{3.19a}$$

$$\lambda = \frac{2}{\epsilon} - \gamma + \log(4\pi) + 1, \tag{3.19b}$$

where $d = 4 - \epsilon$ and γ is the Euler constant. The mixed contribution in Eq. (3.14) has two insertions of δ_{mix}, which is linear in a_μ. There is no contribution to $S_2^{(1)}$ linear in a_μ, while the quadratic contribution is given by

$$S_{\rm mix}(a_\mu) = -\frac{F_\pi^2}{F_\tau^2}\int \frac{d^4q}{(2\pi)^4} a_\mu^a(-q) \times \left[\left(\frac{q_\mu q_\rho}{q^2+M_\pi^2}-\delta_{\mu\rho}\right) I_{\rho\sigma}(q,M_\pi^2,M_\tau^2)\left(\frac{q_\sigma q_\nu}{q^2+M_\pi^2}-\delta_{\sigma\nu}\right) + \frac{1}{4}\frac{q_\mu y M_\pi^2}{q^2+M_\pi^2} I_\rho(q,M_\pi^2,M_\tau^2)\left(\frac{q_\rho q_\nu}{q^2+M_\pi^2}-\delta_{\rho\nu}\right) - \frac{1}{4}\left(\frac{q_\mu q_\rho}{q^2+M_\pi^2}-\delta_{\mu\rho}\right) I_\rho(q,M_\tau^2,M_\pi^2)\frac{q_\nu y M_\pi^2}{q^2+M_\pi^2} + \frac{1}{16}\frac{q_\mu y M_\pi^2}{q^2+M_\pi^2}\frac{q_\nu y M_\pi^2}{q^2+M_\pi^2} I(q^2,M_\pi^2,M_\tau^2)\right] a_\nu^a(q), \quad (3.20)$$

in which

$$I_{\rho\sigma}(q,M_\pi^2,M_\tau^2) = \int \frac{d^4p}{(2\pi)^4}\frac{q_\rho(p+q)_\sigma}{(p+q)^2+M_\tau^2}\frac{1}{p^2+M_\pi^2},$$
$$I_\rho(q,M_\pi^2,M_\tau^2) = \int \frac{d^4p}{(2\pi)^4}\frac{q_\rho}{(p+q)^2+M_\tau^2}\frac{1}{p^2+M_\pi^2},$$
$$I(q^2,M_\pi^2,M_\tau^2) = \int \frac{d^4p}{(2\pi)^4}\frac{1}{(p+q)^2+M_\tau^2}\frac{1}{p^2+M_\pi^2}. \quad (3.21)$$

All integrals can be evaluated using dimensional regularization. We find that

$$I_{\rho\sigma}(q,M_\pi^2,M_\tau^2) = B(q^2,M_\pi^2,M_\tau^2)\delta_{\rho\sigma} + C(q^2,M_\pi^2,M_\tau^2)q_\rho q_\sigma,$$
$$I_\rho(q,M_\pi^2,M_\tau^2) = A(q^2,M_\pi^2,M_\tau^2)q_\rho, \quad (3.22)$$

with

$$I(q^2,M_\pi^2,M_\tau^2) = \frac{1}{16\pi^2}\int_0^1 dx\left(\lambda - \log\left(\frac{D}{\mu^2}\right)-1\right),$$
$$A(q^2,M_\pi^2,M_\tau^2) = -\frac{1}{16\pi^2}\int_0^1 dx\,x\left(\lambda - \log\left(\frac{D}{\mu^2}\right)-1\right),$$
$$B(q^2,M_\pi^2,M_\tau^2) = -\frac{1}{2}\frac{1}{16\pi^2}\int_0^1 dx\,D\left(\lambda - \log\left(\frac{D}{\mu^2}\right)\right),$$
$$C(q^2,M_\pi^2,M_\tau^2) = -\frac{1}{16\pi^2}\int_0^1 dx\,x(1-x)\left(\lambda - \log\left(\frac{D}{\mu^2}\right)-1\right), \quad (3.23)$$

and

$$D \equiv D(q^2,M_\pi^2,M_\tau^2) = x(1-x)q^2 + (1-x)M_\pi^2 + xM_\tau^2. \quad (3.24)$$

Collecting these results, we obtain

$$S_2^{(1)}(a_\mu) = S_\pi(a_\mu) + S_\tau(a_\mu) + S_{\rm mix}(a_\mu) = \int \frac{d^4q}{(2\pi)^4} a_\mu^a(-q)a_\nu^a(q)\left[\left(\frac{F_\pi^2}{F_\tau^2}(K(M_\tau^2) - B(q^2,M_\pi^2,M_\tau^2)) - \frac{1}{2}N_f K(M_\pi^2)\right)\left(\delta_{\mu\nu} - \frac{q_\mu q_\nu}{q^2+M_\pi^2}\right) + \left(\frac{F_\pi^2}{F_\tau^2}\left(\frac{y^2}{4}-1\right)K(M_\tau^2) + \frac{1}{2N_f}K(M_\pi^2) - \frac{F_\pi^2}{F_\tau^2}\left(\frac{y^2M_\pi^2}{16}I(q^2,M_\pi^2,M_\tau^2) - B(q^2,M_\pi^2,M_\tau^2) + M_\pi^2 C(q^2,M_\pi^2,M_\tau^2) + \frac{yM_\pi^2}{4}(A(q^2,M_\tau^2,M_\pi^2) - A(q^2,M_\pi^2,M_\tau^2))\right)\right)\frac{M_\pi^2 q_\mu q_\nu}{(q^2+M_\pi^2)^2}\right]. \quad (3.25)$$

Returning to $S_{\rm LO}$, Eq. (2.21), if we vary $F_\pi \to F_\pi + \delta F_\pi$ and $M_\pi^2 \to M_\pi^2 + \delta M_\pi^2$ we find for the variation

$$\delta S_{\rm LO} = \int \frac{d^4p}{(2\pi)^4} a_\mu^a(-p)\left[\delta F_\pi F_\pi\left(\delta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2+M_\pi^2}\right) + \frac{1}{2}F_\pi^2\frac{\delta M_\pi^2 p_\mu p_\nu}{(p^2+M_\pi^2)^2}\right] a_\nu^a(p). \quad (3.26)$$

Comparing with Eqs. (3.25) and (3.26) we finally obtain

$$\left.\frac{\delta F_\pi}{F_\pi}\right|_{\text{1-loop}} = -\frac{1}{2}\frac{N_f}{F_\pi^2}K(M_\pi) + \frac{1}{F_\tau^2}(K(M_\tau^2) - B(-M_\pi^2,M_\pi^2,M_\tau^2)), \quad (3.27)$$

and

$$\left.\frac{\delta M_\pi^2}{M_\pi^2}\right|_{\text{1-loop}} = \frac{1}{N_f F_\pi^2} K(M_\pi^2) + \frac{2}{F_\tau^2}\left(\frac{y^2}{4} - 1\right) K(M_\tau^2) - \frac{2}{F_\tau^2}\left(\frac{y^2 M_\pi^2}{16} I(-M_\pi^2, M_\pi^2, M_\tau^2) + \frac{y M_\pi^2}{4}\left(A(-M_\pi^2, M_\tau^2, M_\pi^2) - A(-M_\pi^2, M_\pi^2, M_\tau^2)\right) - B(-M_\pi^2, M_\pi^2, M_\tau^2) + M_\pi^2 C(-M_\pi^2, M_\pi^2, M_\tau^2)\right). \tag{3.28}$$

B. $\mathcal{O}(p^4)$ Lagrangian

The $\mathcal{O}(p^4)$ Lagrangian for dChPT was constructed in Ref. [10]. Here we will list only those operators that contribute to the axial-current two-point function.

The first class of operators corresponds to the standard $\mathcal{O}(p^4)$ Lagrangian [17]. Their coupling to the dilaton field is detailed in Ref. [10]. Those that are relevant here are

$$Q_4^\pi = 2B_\pi m L_4 e^{(y-2)v_0} \mathrm{tr}(D_\mu U^\dagger D_\mu U)\mathrm{tr}(U + U^\dagger), \tag{3.29a}$$

$$Q_5^\pi = 2B_\pi m L_5 e^{(y-2)v_0} \mathrm{tr}(D_\mu U^\dagger D_\mu U(U + U^\dagger)), \tag{3.29b}$$

$$Q_6^\pi = -(2B_\pi m)^2 L_6 e^{2(y-2)v_0} (\mathrm{tr}(U + U^\dagger))^2, \tag{3.29c}$$

$$Q_8^\pi = -(2B_\pi m)^2 L_8 e^{2(y-2)v_0} \mathrm{tr}(UU + U^\dagger U^\dagger), \tag{3.29d}$$

where we already substituted the classical VEV v_0 for τ. To quadratic order in a_μ, these lead to the contribution

$$S_L(a_\mu) = 4M_\pi^2 \int \frac{d^4q}{(2\pi)^4} a_\mu^a(-q)\left(L_{45}\left(\delta_{\mu\nu} - \frac{q_\mu q_\nu}{q^2 + M_\pi^2}\right) + (2L_{68} - L_{45})\frac{M_\pi^2 q_\mu q_\nu}{(q^2 + M_\pi^2)^2}\right) a_\nu(q), \tag{3.30}$$

in which

$$L_{45} = N_f L_4 + L_5, \qquad L_{68} = N_f L_6 + L_8. \tag{3.31}$$

Next, we consider the "mixed" operators of Ref. [10]. The relevant ones are

$$Q_1^{\text{mix}} = \frac{1}{4} f_\pi^2 c_{01}^\pi e^{2v_0} \mathrm{tr}(D_\mu U^\dagger D_\mu U), \tag{3.32a}$$

$$Q_2^{\text{mix}} = \frac{1}{4} f_\pi^2 c_{11}^\pi v_0 e^{2v_0} \mathrm{tr}(D_\mu U^\dagger D_\mu U), \tag{3.32b}$$

$$Q_3^{\text{mix}} = -\frac{1}{2} f_\pi^2 B_\pi m c_{01}^M e^{yv_0} \mathrm{tr}(U + U^\dagger), \tag{3.32c}$$

$$Q_4^{\text{mix}} = -\frac{1}{2} f_\pi^2 B_\pi m c_{11}^M v_0 e^{yv_0} \mathrm{tr}(U + U^\dagger). \tag{3.32d}$$

Again, we already substituted v_0 for τ. The first index of c_{nk}, namely n, refers to the power of τ multiplying the exponential containing τ; the second index, k, is the power of $n_f^* - n_f$ contained in c_{nk}. Hence the new LECs c_{01}^π, c_{11}^π, c_{01}^M and c_{11}^M all contain one power of $n_f^* - n_f$ (like c_1). These operators appear at $\mathcal{O}(p^4)$ because, in addition to one power of $n_f^* - n_f$, they contain also one power of the fermion mass m or two derivatives.[3] To quadratic order in a_μ, these operators lead to the contribution

$$S_{\text{mix}}(a_\mu) = \frac{1}{2} F_\pi^2 \int \frac{d^4q}{(2\pi)^4} a_\mu^a(-q) a_\nu(q) \times \left((c_{01}^\pi + v_0 c_{11}^\pi)\left(\delta_{\mu\nu} - \frac{q_\mu q_\nu}{q^2 + M_\pi^2}\right) + (c_{01}^M - c_{01}^\pi + (c_{11}^M - c_{11}^\pi)v_0)\frac{M_\pi^2 q_\mu q_\nu}{(q^2 + M_\pi^2)^2}\right). \tag{3.33}$$

Equations (3.30) and (3.33) together give rise to the corrections

$$\left.\frac{\delta F_\pi}{F_\pi}\right|_{\mathcal{O}(p^4)} = 4M_\pi^2 L_{45} + \frac{1}{2}(c_{01}^\pi + v_0 c_{11}^\pi), \tag{3.34}$$

and

$$\left.\frac{\delta M_\pi^2}{M_\pi^2}\right|_{\mathcal{O}(p^4)} = 8\frac{M_\pi^2}{F_\pi^2}(2L_{68} - L_{45}) + c_{01}^M - c_{01}^\pi + (c_{11}^M - c_{11}^\pi)v_0. \tag{3.35}$$

We have listed all the $\mathcal{O}(p^4)$ operators which contribute to the NLO part of the effective action at quadratic order in a_μ. Other $\mathcal{O}(p^4)$ operators contribute to the NLO correction of the dilaton VEV v, and ultimately to M_π^2 and F_π at NLO, as we discuss in the next subsection.

C. Dilaton effective potential

In this subsection we calculate $\mathcal{V}_{\text{NLO}}$, the effective potential for the constant mode v of the dilaton field τ, at NLO. The saddle-point equation that follows from this effective potential, to be discussed in Sec. III E below, yields the NLO correction v_1 to the dilaton VEV. This

[3] Apart from their dependence on N_f and N_c, the operators in Eqs. (3.32a) and (3.32c) are identical to the corresponding tree-level operators. We return to this point in Sec. IV A below.

correction, in turn, contributes to M_π^2 and F_π at NLO, as follows from Eq. (2.10).

The NLO effective potential consists of three parts,

$$\mathcal{V}_{\rm NLO} = \mathcal{V}_{\rm LO} + \mathcal{V}^{(1)} + \mathcal{V}_{\mathcal{O}(p^4)}, \tag{3.36}$$

where the tree-level potential $\mathcal{V}_{\rm LO}$ is given in Eq. (2.5). In order to derive the one-loop contribution, $\mathcal{V}^{(1)}$, we expand the LO action to quadratic order in $\hat{\tau}$ and π, using $\tau = v + \hat{\tau}/f_\tau$ and Eq. (2.2), obtaining

$$S_2 = \frac{1}{2}\int d^4x (e^{2v}\partial_\mu\pi^a\partial_\mu\pi^a + 2B_\pi m e^{yv}\pi^a\pi^a + e^{2v}(\partial_\mu\hat{\tau})^2 + \mathcal{V}''_{\rm LO}(v)\hat{\tau}^2), \tag{3.37}$$

where $\mathcal{V}''_{\rm LO}$ is the second derivative of $\mathcal{V}_{\rm LO}$ with respect to v.[4] Integrating over π and $\hat{\tau}$ yields the one-loop effective potential

$$\mathcal{V}^{(1)} = -\frac{1}{64\pi^2}\left((e^{-2v}\mathcal{V}''(v))^2\left(\lambda + \frac{1}{2} - \log\left(\frac{e^{-2v}\mathcal{V}''(v)}{\mu^2}\right)\right) + (N_f^2-1)(2B_\pi m e^{(y-2)v})^2 \times \left(\lambda + \frac{1}{2} - \log\left(\frac{2B_\pi m e^{(y-2)v}}{\mu^2}\right)\right)\right). \tag{3.38}$$

The first term on the right-hand side was already obtained in Ref. [10].

Next, we consider the contribution of $\mathcal{O}(p^4)$ operators to the dilaton effective potential. There are three pure-dilaton operators,

$$Q_1^\tau = c_{02} f_\tau^2 B_\tau e^{4\tau},$$
$$Q_2^\tau = c_{12}\tau f_\tau^2 B_\tau e^{4\tau},$$
$$Q_3^\tau = c_{22}(\tau^2/2) f_\tau^2 B_\tau e^{4\tau}, \tag{3.39}$$

where the coefficients c_{n2} are of order $(n_f^* - n_f)^2$. Additional contributions come from Eqs. (3.32c) and (3.32d), as well as from Eqs. (3.29c) and (3.29d). Finally, there is a contribution from the $\mathcal{O}(p^4)$ operator

$$Q_{H_2}^\pi = -(2B_\pi m)^2 H_2 N_f e^{2(y-2)v}. \tag{3.40}$$

In ordinary ChPT the corresponding operator derives from a contact term; but in dChPT this operator becomes dependent on the dilaton field, and thus contributes to the dilaton effective potential. The contribution of all these operators yields

$$\mathcal{V}_{\mathcal{O}(p^4)} = -8B_\pi^2 \hat{L} N_f m^2 e^{2(y-2)v} + f_\tau^2 B_\tau e^{4v}\left(c_{02} + c_{12}v + \frac{1}{2}c_{22}v^2\right) - N_f f_\pi^2 B_\pi m e^{yv}(c_{01}^M + v c_{11}^M), \tag{3.41}$$

where

$$\hat{L} = L_8 + 2N_f L_6 + \frac{1}{2}H_2. \tag{3.42}$$

D. Renormalization

We define the following renormalized $\mathcal{O}(p^4)$ LECs:

$$c_{02} = c_{02}^r + \frac{c_1^2 B_\tau}{4\pi^2 f_\tau^2}\lambda, \tag{3.43a}$$

$$c_{12} = c_{12}^r + \frac{2c_1^2 B_\tau}{\pi^2 f_\tau^2}\lambda, \tag{3.43b}$$

$$c_{22} = c_{22}^r + \frac{8c_1^2 B_\tau}{\pi^2 f_\tau^2}\lambda, \tag{3.43c}$$

$$c_{01}^\pi = c_{01}^{\pi,r} + \frac{3c_1 B_\tau}{8\pi^2 f_\tau^2}\lambda, \tag{3.43d}$$

$$c_{11}^\pi = c_{11}^{\pi,r} + \frac{3c_1 B_\tau}{2\pi^2 f_\tau^2}\lambda, \tag{3.43e}$$

$$c_{01}^M = c_{01}^{M,r} + \frac{c_1 B_\tau y^2}{8\pi^2 f_\tau^2}\lambda, \tag{3.43f}$$

$$c_{11}^M = c_{11}^{M,r} + \frac{c_1 B_\tau y^2}{2\pi^2 f_\tau^2}\lambda, \tag{3.43g}$$

$$L_{45} = L_{45}^r - \frac{1}{128\pi^2}\left(N_f + \frac{f_\pi^2}{3f_\tau^2} + \frac{3N_f y^2 f_\pi^4}{4f_\tau^4}\right)\lambda, \tag{3.43h}$$

$$L_{68} = L_{68}^r - \frac{1}{256\pi^2}\left(\frac{N_f^2-1}{N_f} - \frac{(3y^2-8)f_\pi^2}{24f_\tau^2} + \frac{N_f y^4 f_\pi^4}{4f_\tau^4}\right)\lambda, \tag{3.43i}$$

$$\hat{L} = \hat{L}^r - \frac{1}{128\pi^2}\left(\frac{N_f^2-1}{N_f} + \frac{N_f y^4 f_\pi^4}{4f_\tau^4}\right)\lambda, \tag{3.43j}$$

where λ is defined in Eq. (3.19b). This amounts to using the so-called "$\overline{\rm MS}+1$" scheme of Ref. [17]. Employing these expressions removes all divergences from Eqs. (3.27), (3.28) and (3.38), and replaces bare LECs by renormalized LECs in Eqs. (3.34), (3.35) and (3.41).

[4] In the calculation of $\mathcal{V}_{\rm NLO}$ we set a_μ to zero. Equation (3.37) may be obtained by setting $a_\mu = 0$ and (thus) $u = 1$ in Eq. (3.2), while keeping v arbitrary.

E. M_π^2 and F_π at $\mathcal{O}(p^4)$

The saddle-point equation at NLO is obtained by minimizing the effective potential (3.36) with respect to v. In the derivatives of $\mathcal{V}^{(1)}$ and $\mathcal{V}_{\mathcal{O}(p^4)}$ we can set $v = v_0$ immediately. In the tree-level term we substitute $v = v_0 + v_1$ and expand to linear order in v_1, using that v_0 solves the tree-level equation (2.7). We obtain an equation for v_1, the NLO correction for the dilaton VEV, viz.,

$$\left.\frac{\partial^2 \mathcal{V}_{\mathrm{LO}}}{\partial v^2}\right|_{v=v_0} v_1 + \left.\frac{\partial \mathcal{V}^{(1)}}{\partial v}\right|_{v=v_0} + \left.\frac{\partial \mathcal{V}_{\mathcal{O}(p^4)}}{\partial v}\right|_{v=v_0} = 0. \quad (3.44)$$

The solution is

$$\begin{aligned} v_1 = -\frac{1}{F_\tau^2 M_\tau^2}\Bigg[&\frac{M_\tau^2}{64\pi^2 F_\tau^2} N_f y(y-4)^2 F_\pi^2 M_\pi^2 + \frac{M_\tau^2}{16\pi^2 F_\tau^2}\left(3F_\tau^2 M_\tau^2 - \frac{1}{4} N_f y(y-4)^2 F_\pi^2 M_\pi^2\right)\log\left(\frac{M_\tau^2}{\mu^2}\right) \\ &+ B_\tau F_\tau^2 e^{2v_0}(4c_{02}^r + c_{12}^r(1+4v_0) + c_{22}^r v_0(2v_0+1)) - \frac{1}{2} N M_\pi^2 F_\pi^2 (c_{01}^{M,r} y + c_{11}^{M,r}(y v_0 + 1)) \\ &+ \frac{(N_f^2-1)(y-2)M_\pi^4}{32\pi^2}\log\left(\frac{M_\pi^2}{\mu^2}\right) - 4N_f(y-2)\hat{L}^r M_\pi^4\Bigg]. \end{aligned} \quad (3.45)$$

At NLO, v_1 contributes to F_π and M_π^2 through the dependence of the LO expressions on the dilaton VEV. Technically, we need to restore $v_0 \to v$ in Eq. (2.10), substitute $v = v_0 + v_1$, and then expand to linear order in v_1.

The final expressions for F_π and M_π^2 to NLO are given by

$$\begin{aligned} \frac{F_\pi^{\mathrm{NLO}}}{F_\pi} = 1 + v_1 - \frac{N_f}{32\pi^2}\frac{M_\pi^2}{F_\pi^2}\log\left(\frac{M_\pi^2}{\mu^2}\right) + \frac{M_\tau^2}{16\pi^2 F_\tau^2}\log\left(\frac{M_\tau^2}{\mu^2}\right) + \frac{4M_\pi^2}{F_\pi^2}L_{45}^r + \frac{1}{2}(c_{01}^{\pi,r} + c_{11}^{\pi,r} v_0) \\ - \frac{1}{32\pi^2 F_\tau^2}(M_\pi^2(J_0(M_\pi,M_\tau) - 2J_1(M_\pi,M_\tau) + J_2(M_\pi,M_\tau)) + M_\tau^2 J_1(M_\pi,M_\tau)), \end{aligned} \quad (3.46)$$

and

$$\begin{aligned} \left(\frac{M_\pi^{\mathrm{NLO}}}{M_\pi}\right)^2 = 1 + (1-\gamma^*)v_1 + \frac{M_\pi^2}{16\pi^2 N_f F_\pi^2}\log\left(\frac{M_\pi^2}{\mu^2}\right) + \frac{(y^2-4)M_\tau^2}{32\pi^2 F_\tau^2}\log\left(\frac{M_\tau^2}{\mu^2}\right) + c_{01}^{M,r} - c_{01}^{\pi,r} + v_0(c_{11}^{M,r} - c_{11}^{\pi,r}) \\ - \frac{8M_\pi^2}{F_\pi^2}(L_{45}^r - 2L_{68}^r) + \frac{1}{16\pi^2 F_\tau^2}\Bigg(\frac{1}{24}(3y^2-8)M_\pi^2 + \frac{1}{8}(y^2-4y+8)M_\pi^2 J_0(M_\pi,M_\tau) \\ + (M_\tau^2 + (y-4)M_\pi^2)J_1(M_\pi,M_\tau) + 3M_\pi^2 J_2(M_\pi,M_\tau)\Bigg), \end{aligned} \quad (3.47)$$

where

$$J_n(M_1,M_2) = \int_0^1 dx x^n \log\left(\frac{M_1^2(1-x)^2 + M_2^2 x}{\mu^2}\right). \quad (3.48)$$

These results follow from combining Eqs. (2.10), (3.27), (3.28), (3.34), (3.35), (3.43), and (3.45). Explicit expressions for the J_n integrals for $n = 0$, 1, 2 are given in Appendix A.

IV. FITS OF NLO dChPT TO LatKMI DATA

In previous work we applied LO dChPT to lattice data for the pion mass M_π and decay constant F_π in the eight-flavor SU(3) gauge theory. In Ref. [15] we fitted data obtained by the LSD Collaboration [4], finding that LO dChPT describes these data well, with a mass anomalous dimension $\gamma^* = 0.93(2)$. In Ref. [16] we attempted to fit data from the LatKMI Collaboration [1]. The fermion masses considered by LatKMI span a wider range than those considered by LSD. In addition, when measured in units of the gradient flow scale t_0 [26], the LatKMI mass range lies well above that of LSD.[5] We found that LO dChPT cannot describe the LatKMI data over the full fermion mass range. But we also showed that extending LO dChPT with a model for a varying mass anomalous dimension can describe these data reasonably well.

Both LSD and LatKMI measured also the dilaton mass M_τ (in the case of LatKMI, for a subset of their fermion masses). But the quality of these data is significantly poorer, and thus the dilaton mass data have little effect on the fit results. No measurements of the dilaton decay constant F_τ exist to date.

[5]As t_0 itself is strongly mass dependent, this statement amounts to the use of a mass-dependent scale setting.

In view of the inability to describe the LatKMI data using LO dChPT, the question arises whether NLO dChPT will do better. This question will be addressed in this section. In Sec. IV A we recast the NLO results of Sec. III in a form which is more convenient for the fits. We also discuss redundancies among the parameters that arise when one considers a theory with fixed N_f and N_c. Then, in Sec. IV B we will present and discuss the results of our fits.

A. Fit form and parameters

We introduce the following combinations of the LO parameter [15]:

$$d_1 = \frac{(3-\gamma^*)f_\pi^2}{8B_\pi c_1 \mathcal{M}}, \qquad d_2 = \frac{f_\pi^2}{2B_\pi}, \qquad d_3 = \frac{4c_1 B_\tau}{f_\pi^2}, \tag{4.1}$$

with $\mathcal{M}$ defined in Eq. (2.6). In terms of these parameters, the LO results (2.10) can be re-expressed as

$$F_\pi = f_\pi e^{v_0}, \tag{4.2a}$$

$$M_\pi^2 = \frac{f_\pi^2}{d_1} v_0 e^{2v_0}, \tag{4.2b}$$

$$M_\tau^2 = f_\pi^2 d_3 (1 + (1+\gamma^*)v_0) e^{2v_0}, \tag{4.2c}$$

$$\theta \equiv \frac{F_\pi^2}{F_\tau^2} = \frac{f_\pi^2}{f_\tau^2} = \frac{2d_1 d_3}{(3-\gamma^*)N_f}, \tag{4.2d}$$

where in the last equation we defined the decay-constant ratio θ in terms of the parameters (4.1), substituting Eq. (2.10d) for F_τ. The LO VEV v_0 solves Eq. (2.7) and can be written as a function of the fermion mass m in terms of the Lambert function W_0 as

$$v_0(m) = \frac{1}{1+\gamma^*} W_0\left(\frac{(1+\gamma^*)d_1}{d_2} m\right). \tag{4.3}$$

Using Eqs. (4.2) and (4.3) we recast the NLO results of Eqs. (3.46), (3.47) and (3.45) as

$$\begin{aligned}\frac{F_\pi^{\rm NLO}}{F_\pi} = {}& 1 + v_1 - \frac{N_f v_0}{32\pi^2 d_1}\log\left(\frac{M_\pi^2}{\mu^2}\right) + \frac{\theta d_3(1+(4-y)v_0)}{16\pi^2}\log\left(\frac{M_\tau^2}{\mu^2}\right) + \frac{4v_0}{d_1}L_{45}^r + \frac{1}{2}(c_{01}^{\pi,r} + c_{11}^{\pi,r} v_0) \\ & - \frac{\theta}{32\pi^2}\left(\frac{v_0}{d_1}(J_0(M_\pi,M_\tau) - 2J_1(M_\pi,M_\tau) + J_2(M_\pi,M_\tau)) + d_3(1+(4-y)v_0)J_1(M_\pi,M_\tau)\right),\end{aligned} \tag{4.4}$$

$$\begin{aligned}\left(\frac{M_\pi^{\rm NLO}}{M_\pi}\right)^2 = {}& 1 + (y-2)v_1 + \frac{v_0}{16\pi^2 N_f d_1}\log\left(\frac{M_\pi^2}{\mu^2}\right) + \frac{(y^2-4)\theta d_3(1+(4-y)v_0)}{32\pi^2}\log\left(\frac{M_\tau^2}{\mu^2}\right) \\ & + c_{01}^{M,r} - c_{01}^{\pi,r} + v_0(c_{11}^{M,r} - c_{11}^{\pi,r}) - \frac{8v_0}{d_1}(L_{45}^r - 2L_{68}^r) \\ & + \frac{\theta}{16\pi^2}\left(\frac{1}{24}(3y^2-8)\frac{v_0}{d_1} + \frac{1}{8}(y^2-4y+8)\frac{v_0}{d_1}J_0(M_\pi,M_\tau)\right. \\ & \left. + \left(d_3(1+(4-y)v_0) + (y-4)\frac{v_0}{d_1}\right)J_1(M_\pi,M_\tau) + 3\frac{v_0}{d_1}J_2(M_\pi,M_\tau)\right),\end{aligned} \tag{4.5}$$

and

$$\begin{aligned}v_1 = {}& \frac{N_f y(y-4)^2\theta^2 v_0}{64\pi^2 d_1}\left(\log\left(\frac{M_\tau^2}{\mu^2}\right) - 1\right) - \frac{3\theta d_3(1+(4-y)v_0)}{16\pi^2}\log\left(\frac{M_\tau^2}{\mu^2}\right) \\ & - \frac{1}{d_3(1+(4-y)v_0)f_\pi^2}(4\bar{c}_{02}^r + \bar{c}_{12}^r(1+4v_0) + \bar{c}_{22}^r v_0(1+2v_0)) + \frac{1}{2}\frac{N_f\theta v_0}{d_1 d_3(1+(4-y)v_0)}(c_{01}^{M,r} y + c_{11}^{M,r}(1+yv_0)) \\ & - \frac{(N_f^2-1)(y-2)\theta v_0^2}{32\pi^2 d_1^2 d_3(1+(4-y)v_0)}\log\left(\frac{M_\pi^2}{\mu^2}\right) + \frac{4N_f(y-2)\theta v_0^2}{d_1^2 d_3(1+(4-y)v_0)}\hat{L}^r,\end{aligned} \tag{4.6}$$

where

$$\bar{c}_{i2}^r = B_\tau c_{i2}^r, \quad i = 0, 1, 2. \tag{4.7}$$

Inside the logarithms, Eqs. (4.2b) and (4.2c) should be used for M_π^2 and M_τ^2.

We next consider possible redundancies among the parameters appearing in the NLO predictions. These redundancies occur because we have data only from the eight-flavor SU(3) theory, and thus we do not get to vary N_f or N_c. We start with Eq. (3.32c), which is identical to the tree-level pion mass term [cf. Eq. (2.1)], except that it

TABLE I. Reproduction of LO fits of the LatKMI data [16]. Each fit is done in a "window" of five successive fermion masses.

Fit Range	A 0.012–0.04	B 0.015–0.05	C 0.02–0.06	D 0.03–0.07	E 0.04–0.08	F 0.05–0.1
$\chi^2/\mathrm{d.o.f.}$	11.7/10	12.2/9	7.2/9	4.8/8	4.7/7	4.0/6
p value	0.30	0.20	0.62	0.77	0.69	0.68
γ_*	0.608(8)	0.589(10)	0.543(10)	0.534(12)	0.527(8)	0.498(13)
$10^2 af_\pi$	0.50(7)	0.67(6)	0.89(8)	1.0(2)	1.07(13)	1.12(14)
$10aB_\pi$	4.7(2)	4.99(14)	5.09(14)	5.3(4)	5.3(2)	5.1(2)
$10af_\tau$	0.23(4)	0.31(4)	0.41(4)	0.44(11)	0.44(7)	0.47(11)
$10^4 c_1 a^2 B_\tau$	0.16(6)	0.33(9)	0.70(17)	1.1(6)	1.3(5)	1.5(7)
$10d_1$	1.72(10)	1.52(6)	1.34(5)	1.27(13)	1.24(6)	1.21(7)
$-\log(ad_2)$	10.5(3)	10.0(2)	9.45(15)	9.2(4)	9.1(2)	9.0(2)
d_3	2.6(3)	3.0(4)	3.5(5)	4.2(7)	4.7(10)	4.7(1.9)

contains an extra factor of c_{01}^M. In principle, the LEC c_{01}^M is $\mathcal{O}(p^2)$ in the power counting (2.3), having in general an $\mathcal{O}(n_f - n_f^*)$ piece plus an $\mathcal{O}(1/N_c)$ piece. But since our data come from a single theory, the fit cannot resolve Eq. (3.32c) from the tree-level pion mass term. Hence we must not include c_{01}^M in the fit. Similarly, the operator in Eq. (3.32a) is identical to the tree-level pion kinetic term; hence c_{01}^π must not be included in a single-theory fit either.

Finally, after the τ shift that fixed the form of the dilaton potential in Eq. (2.1) [10], this potential is a linear combination of the operators $f_\tau^2 B_\tau e^{4\tau}$ and $f_\tau^2 B_\tau \tau e^{4\tau}$ [compare Eq. (3.39)]. Therefore, only one additional linear combination of these operators should be kept in the fit. In the actual fits discussed below, $\bar{c}_{02}^r$ was kept.

In summary, to eliminate the single-theory redundancies we set $c_{01}^{\pi,r}$, $c_{01}^{M,r}$ and $\bar{c}_{12}^r$ to zero in Eqs. (4.4)–(4.6). This leaves us with seven independent NLO parameters, in addition to the five parameters that appear in LO dChPT. The total number of parameters to be considered at NLO is thus twelve.

B. Fit results

As we will see, a complete, meaningful NLO fit to the LatKMI data [1] turns out to be impossible. We thus begin by describing our strategy.

In order to keep the mass dependence fully explicit in our application of dChPT, we assume a mass-independent scheme for setting the scale. The ensembles of Ref. [1] share a common bare coupling, and thus, by definition, a common lattice spacing a as well. The LatKMI calculations were done for 10 different bare fermion masses,

$$am = \{0.012, 0.015, 0.02, 0.03, 0.04, 0.05, 0.06, 0.07, 0.08, 0.1\}. \tag{4.8}$$

In Ref. [16] we fitted data for M_π^2/F_π^2 and aF_π to LO dChPT in "sliding windows" of five successive fermion masses, ranging from $\{0.012, 0.015, 0.02, 0.03, 0.04\}$ to $\{0.05, 0.06, 0.07, 0.08, 0.1\}$. Trying to add more fermion masses to a given LO fit led to a rapid deterioration of the quality of these fits, and an LO fit to all ten masses yielded an unacceptably low p value (of about 10^{-11}).

We reproduce the LO fits in Table I. In the new LO fits we also included data for M_τ^2/F_π^2. The dilaton mass M_τ was computed for only a subset of the fermion masses, $am \in \{0.012, 0.015, 0.02, 0.03, 0.04, 0.06\}$. Moreover, the errors of M_τ are significantly larger than those of F_π and M_π. Thus, the inclusion of M_τ data in the LO fits results in negligible changes in the previous fit predictions.[6] But it constrains the LO parameter d_3, which does not occur in the LO expressions for aM_π and aF_π (more below).

Moving on to NLO, the primary quantities that we fit are aF_π and aM_π.[7] The LatKMI dataset thus provides 20 data points. While we have 12 parameters in the NLO fit, it turns out that a fit with all of them is unable to determine even the LO parameters. In fact, the fit's predictions for most of the LO parameters contain huge errors on a logarithmic scale. Our first conclusion is thus that significantly better data will be needed to carry out a complete NLO dChPT fit.

Facing this situation, we narrowed the scope of our fits. First, while d_3 is a LO fit parameter, it appears in the expressions for aF_π and aM_π only at NLO. In order to better constrain d_3, we included also data for M_τ^2/F_π^2 in the fit, which we fitted to the corresponding LO expression. We did not include NLO corrections for M_τ^2/F_π^2, both because of the low quality of M_τ data, and because this would introduce even more NLO parameters than already present in expressions (4.4), (4.5) and (4.6).

In addition, we attempted to find good fits to the data at a fixed renormalization scale $a\mu = 1$ using only a few of the

[6] The p values of the new LO fits are substantially higher, because the χ^2 increases by only a little, while the number of degrees of freedom is larger.

[7] More precisely, the fitted quantities are aF_π and $(aM_\pi)^2$.

TABLE II. Fits to aF_π, M_π^2/F_π^2 (NLO) and M_τ^2/F_π^2 (LO, see text). Fits D, E and F are the same as fits A, B and C, respectively, but with the data at $am = 0.02$ omitted from the fit. The last column shows the range of each parameter in the LO fits of Table I.

Fit Masses	A 0.012–0.07	B 0.012–0.08	C 0.012–0.1	D A no 0.02	E B no 0.02	F C no 0.02	LO range
χ^2/d.o.f.	13.0/15	20.5/17	37.6/19	8.0/12	9.6/14	20.2/16	
p value	0.61	0.25	0.007	0.79	0.79	0.21	
γ^*	0.658(7)	0.654(10)	0.650(13)	0.659(10)	0.659(12)	0.656(13)	0.5–0.6
$10^2 af_\pi$	3.8(2)	4.33(18)	4.70(17)	4.2(3)	4.54(18)	4.86 (17)	0.5–1.1
aB_π	1.17(5)	1.138(44)	1.13(4)	1.18(5)	1.17(4)	1.17(3)	0.47–0.53
$10af_\tau$	1.06(15)	0.98(14)	0.99(13)	1.03(15)	1.01(14)	1.03(13)	0.23–0.47
$10^3 c_1 a^2 B_\tau$	1.6(3)	1.9(3)	2.2(4)	2.1(4)	2.4(4)	2.7(5)	0.016–0.15
d_1	0.29(9)	0.44(13)	0.52(15)	0.33(11)	0.41(11)	0.45(11)	0.12–0.17
$-\log(ad_2)$	7.37(12)	7.10(8)	6.94(8)	7.20(15)	7.03(8)	6.90(7)	9–10.5
d_3	4.4(6)	4.2(5)	4.1(5)	4.7(6)	4.7(6)	4.6(6)	2.6–4.7
$\tilde{c}_{22}^r$	0.023(5)	0.026(5)	0.030(6)	0.030(7)	0.033(7)	0.038(8)	
$\hat{L}^r$	2.0(5)	2.9(8)	3.4(9)	2.3(7)	2.8(7)	3.1(8)	

NLO parameters, setting the remaining NLO parameters to zero.[8] After considerable experimentation, we found that keeping only the NLO parameters $\tilde{c}_{22}^r$ and $\hat{L}^r$, we are able to obtain good fits to the LatKMI data over essentially the entire mass range. We note that these two NLO parameters appear in the expression for v_1, Eq. (4.6), while aF_π and aM_π depend on these parameters only indirectly, through the dependence of the NLO corrections on v_1 [Eqs. (4.4) and (4.5)].

The NLO fits with the parameters $\tilde{c}_{22}^r$ and $\hat{L}^r$ are shown in Table II, and we will next discuss them in detail. Fit C includes data from all ten masses in Eq. (4.8), while in fit B we omitted the highest mass, and in fit A the highest two. The p values of fits A and B are good; fit C is marginal, but still drastically better than the LO fit for the entire mass range. Inspecting differences between data and fit predictions, what stands out in fit C is a 3σ discrepancy for aF_π at $am = 0.02$. The same data point is also relatively poorly fitted in fits A and B as well (albeit with a smaller discrepancy). This suggests a possible issue with the data at $am = 0.02$. We thus repeated fits A, B and C omitting the data at $am = 0.02$, obtaining fits D, E, and F of Table II. The new fits have a higher p value than their companion fits in which the $am = 0.02$ data are kept. Notably, fit F, which includes data from all masses except $am = 0.02$ is now also a good fit. We thus find that the expressions predicted by NLO dChPT can describe the data of Ref. [1].

While the NLO fits of Table II are technically good, nonetheless this is not the behavior expected from a systematic order-by-order expansion, for several reasons. First, as already discussed above, we could not fit all the NLO parameters simultaneously. Instead, we were driven to include only a small subset of the NLO parameters in the fit in an essentially *ad hoc* way. The origin of this problem is clearly that the data are not good enough.

A different problem surfaces when we compare the LO "sliding window" fits of Table I with the values for the LO parameters predicted by the NLO fits of Table II. This problem has two facets. First, if LO dChPT is to provide a reasonable first approximation of the data, we would expect the variation of the LO parameters across the collection of window fits to be modest. A caveat is that LatKMI's full mass range is large: the ratio of the largest to the smallest mass is about 10. An examination of the actual results reveals that γ^* and B_π vary by about 20% or less across the window fits, which is certainly a small variation. Next, f_π and f_τ vary by about a factor of 2, which, considering the wide range of LatKMI masses, might not be entirely unreasonable.

The last LO parameter, $c_1 B_\tau$, varies by a factor of 10 in the window fits.[9] This is a huge variation, which, already by itself, indicates that there is no way that the NLO contribution can be a small correction in comparison with LO. The reason is, simply, that the predictions of the different LO window fits for $c_1 B_\tau$ are already in gross disagreement with each other. Hence, any given NLO result for this parameter cannot be in agreement with *all* the fits of Table I simultaneously. To make things worse, while the values of $c_1 B_\tau$ in the fits of Table II are quite stable, they are yet larger than the largest value obtained in Table I by at least another order of magnitude. The values of af_π in Table II are also larger by about a factor of 4 or more in

[8]The values of all NLO parameters at a different renormalization scale can be obtained using Eqs. (3.45)–(3.47).

[9]The values of $c_1 B_\tau$ were not reported in the fits of Ref. [16], in which M_τ data were not included.

TABLE III. Comparison of values of v_0 and v_1 from an NLO fit (Table II), with the value of v_0 from a LO fit (Table I).

	NLO fit A			LO fit A
am	v_0	v_1	$v_0 + v_1$	v_0
0.012	1.01(11)	−0.847(19)	0.16(11)	2.21(14)
0.04	1.50(12)	−0.7631(15)	0.74(12)	2.81(15)
0.07	1.75(13)	−0.697(15)	1.05(13)	⋯

comparison with the largest value obtained in the LO window fits.

Another acute problem has to do with the predictions of the NLO fits for the dilaton vacuum expectation value, $v(m)$. This is illustrated in Table III. We compare values obtained in NLO fit A (Table II), with those from the LO fit A (Table I), at selected mass values: $am = 0.012$, which is the smallest mass used by LatKMI, and also in both fits; $am = 0.04$, the largest mass in the LO fit; and $am = 0.07$, the largest mass in the NLO fit. The results for other masses fall in between the values shown in the table.

Examining the first two rows of Table III, one can see that the results for v_0 disagree by roughly a factor of 2 between the LO and NLO fits. Another problem is that v_1 is large and negative,[10] leading to a large cancellation in the sum $v_0 + v_1$. The discrepancy between that sum and the value of v_0 in the LO fit is even larger. In fact, v_1 is so large, that the question arises if it should be resummed, given that the difference between $1 + v_1$ and $\exp(v_1)$ is substantial. We note that $|v_1/v_0|$ is largest for the smallest mass. However, this by itself is not necessarily a problem, because we enforced $v_0 \to 0$ for $m \to 0$ via the τ shift, but we do not readjust the τ shift at NLO.[11]

Finally, the NLO corrections for M_π and F_π are dominated by the contribution of the v_1 term, and are also by themselves too big to be comfortable with.

In summary, we conclude that dChPT, as a systematic order-by-order expansion, with the bulk of the physics captured at LO, cannot account for the LatKMI mass range.

If we were dealing with ordinary ChPT, the natural conclusion would have been that the data come from a mass range which is too high, at least in part. However, in dChPT, large masses *per se* do not necessarily lead to a failure of the expansion. As we showed in Ref. [13], dChPT has a *large-mass* regime in which the fermion masses are not small in comparison with the chiral symmetry breaking scale of the massless theory; and yet, dChPT still provides a systematic expansion, now thanks (only) to the smallness of the other expansion parameter, $n_f - n_f^*$.

The reason why dChPT fails to account systematically for the LatKMI data is thus probably that the LatKMI mass range is too far from the influence of the fixed point at the sill of the conformal window. In more concrete terms, it suggests a too-large beta function and/or a mass anomalous dimension that varies too fast over the LatKMI mass range. By contrast, for the LSD data, which also come from the $N_f = 8$ theory but at a lower mass range, we were able to obtain good LO fits for the entire mass range, indicating that dChPT is applicable in that range [15].

In spite of all the issues discussed above, the goodness of the NLO fits we presented in this section means that the corresponding NLO expressions provide a good *model* of the full LatKMI dataset. For completeness, we briefly mention here the alternative model we proposed in Ref. [16], which centers around a variable mass anomalous dimension that goes beyond dChPT.

The model of Ref. [16] replaces $\gamma^*\tau$ in Eq. (2.1) by a function

$$F(\tau) = \gamma_0\tau - \frac{1}{2}b\tau^2 + \frac{1}{3}c\tau^3, \tag{4.9}$$

where γ_0, b and c are phenomenological parameters. The corresponding mass anomalous dimension is

$$\gamma_m = \frac{\partial F}{\partial\tau} = \gamma_0 - b\tau + c\tau^2, \tag{4.10}$$

which eventually becomes a function of m, when $v(m)$ is substituted for the τ field. Expanding

$$e^{3\tau - F(\tau)} = e^{(3-\gamma_0)\tau}\left(1 + \frac{1}{2}b\tau^2 + \mathcal{O}(\tau^3)\right), \tag{4.11}$$

we see that formally γ_0 can be identified with γ^*, and that b can be interpreted as a next-to-next-to-leading order (NNLO) LEC, etc. This model therefore amounts to a specific resummation of dChPT. However, while the model provides good fits to the LatKMI data (see Table 2 of Ref. [16]), the values of γ_0 obtained in such fits are very different from any of the values for γ^* in both Tables I and II.

We conclude this section by recalling that Ref. [1] considered only one lattice spacing, and thus we are not in the position to discuss the continuum limit, or say much about lattice spacing effects. As discussed in more detail in Refs. [1,16], the pion taste splittings are significant, suggesting that lattice spacing effects are not small. While in Refs. [15,16] we considered the extension of dChPT to staggered dChPT at LO, it is not possible to do so with the available LatKMI data at NLO. In our present analysis, we assumed that lattice spacing effects for the staggered Goldstone pion, as well as finite-volume effects, are small enough to apply continuum, infinite-volume dChPT.

[10] Interestingly, the bulk of the contribution to v_1 comes from the two NLO LECs, $\tilde{c}_{22}^r$ and $\hat{L}^r$.

[11] In principle, we might consider shifting τ again to achieve $v_0 + v_1 = 0$ for $m = 0$. However, for the fits of Table II, we find, using Eq. (4.6), that $v_1(m = 0)$ is very small compared with the values shown in Table III. Such a shift would thus have very little impact in practice.

V. CONCLUSION

We have extended dChPT to next-to-leading order, and calculated the NLO corrections to the pion mass M_π and decay constant F_π. These quantities have been computed on the lattice with relatively high precision in the eight-flavor SU(3) theory by the LatKMI [1] and LSD [4] Collaborations. While both collaborations also reported results for the dilaton mass M_τ, these results have much larger errors in comparison with the pionic quantities.

Our main goal in this paper was to investigate to what extent NLO dChPT can account for the LatKMI data. While we found [15] that LO dChPT provides a very good description of the LSD data,[12] the same is not true for the LatKMI data [16]. The LatKMI simulations were performed at much larger fermion masses (in physical units) than the LSD ones, and cover a wider range of fermion masses. Both factors ostensibly play a role in the failure of LO dChPT to describe the LatKMI data. It is thus natural to ask whether the situation might improve when dChPT is extended to NLO.

The results of our investigation are somewhat inconclusive. The LatKMI data are not precise enough to carry out a full-fledged NLO fit of the pion mass and decay constant. Instead, we found that “truncated” NLO fits, in which several of the NLO LECs are arbitrarily set to zero (at a given renormalization scale) can describe the M_π and F_π data over the full LatKMI mass range. However, a more detailed analysis of the LO fits, in which we also considered the (limited) data for M_τ, as well as further scrutiny of the nominally successful but truncated NLO fits, suggest that the LatKMI data might be outside the scope of dChPT. The likely reason is that these data live at a scale at which the renormalized coupling runs too fast, and the same goes for the mass anomalous dimension (more below). This conclusion would be in line with our previous work [16], where we found that a model based on LO dChPT with a varying mass anomalous dimension successfully describes the LatKMI data over the entire mass range. That said, given the quality of the LatKMI data on the one hand, and the difficulty of carrying out dChPT fits beyond LO on the other hand, we cannot rule out that the LatKMI mass range could still lie within the domain of validity of dChPT, and that more precise data from the same mass range could be described by NLO dChPT, with potentially small NNLO corrections.

We end with a few remarks. First, we are using the distance to the conformal sill, $|n_f - n_f^*|$, as the small parameter controlling the hard breaking of scale invariance. But as we explained in detail in our first paper [10] (see also appendix B of Ref. [16]), this parameter is actually a proxy for the magnitude of the trace anomaly, hence, of the beta function. Thus, regardless of the behavior of the $N_f = 8$ theory in the deep infrared, it is possible that at the scale probed by the LatKMI data the beta function is just too large for dChPT to work.

Second, we did not consider the dilaton mass at NLO in dChPT, even though both LatKMI and LSD reported results for M_τ. The reasons are that M_τ data are much less precise than M_π and F_π data, and, additionally, that even more NLO LECs would be required if we include the NLO expression for M_τ.

Clearly, more precise data, preferably at smaller fermion masses, will be needed in order to continue investigating whether dChPT is the correct EFT for the light meson sector of the eight-flavor SU(3) theory. We are looking forward to analyzing the new refined data recently obtained by the LSD Collaboration [27].

Finally, both the LatKMI and LSD Collaborations reported results at only one value of the bare coupling, i.e., at only a single lattice spacing. It is thus very difficult to investigate the effects of scaling violations. Lattice results for the staggered taste splittings, obtained by both collaborations, suggest that scaling violations are not small. While in Refs. [15,16] we were able to extend the LO fits to explore the inclusion of taste splittings, this is not feasible at NLO. Results at a different lattice spacing would be very helpful, but may not be easy to obtain in the face of potentially slow running of the coupling, if indeed the eight-flavor theory is close to the conformal sill.

ACKNOWLEDGMENTS

The work of A. F. and M. G. is supported by the U.S. Department of Energy, Office of Science, Office of High Energy Physics, under Award No. DE-SC0013682. Y. S. is supported by the Israel Science Foundation under Grant No. 1429/21.

APPENDIX A: INTEGRALS

The integrals defined in Eq. (3.48) for $n = 0, 1, 2$ are given by

$$J_0(M_1, M_2) = -2 - \frac{1}{2}\frac{M_2^2}{M_1^2}\log\frac{M_1^2}{M_2^2} + \log\frac{M_1^2}{\mu^2} + f(M_1, M_2), \tag{A1a}$$

$$J_1(M_1,M_2) = -\frac{3}{2} + \frac{M_2^2}{2M_1^2} - \frac{M_2^2}{4M_1^4}(4M_1^2 - M_2^2)\log\frac{M_1^2}{M_2^2} + \frac{1}{2}\log\frac{M_1^2}{\mu^2} + \frac{2M_1^2 - M_2^2}{2M_1^2} f(M_1,M_2), \tag{A1b}$$

$$J_2(M_1,M_2) = -\frac{11}{9} + \frac{3M_2^2}{2M_1^2} - \frac{M_2^4}{3M_1^4} + \frac{1}{3}\log\frac{M_1^2}{\mu^2} - \frac{M_2^2}{6M_1^6}(3M_1^2 - M_2^2)^2\log\frac{M_1^2}{M_2^2} + \frac{(3M_1^2 - M_2^2)(M_1^2 - M_2^2)}{3M_1^4} f(M_1,M_2). \tag{A1c}$$

[12]Other approaches provide a good description of these data as well. In particular, Refs. [21–23] considered a model generalization of LO dChPT which involves a new continuous parameter Δ, and this approach was successfully applied to the LSD data; see Refs. [8,16].

The function f is defined by

$$f(M_1, M_2) = \frac{M_2\sqrt{4M_1^2 - M_2^2}}{M_1^2}\left(\arctan\left[\frac{M_2}{\sqrt{4M_1^2 - M_2^2}}\right] - \arctan\left[\frac{M_2^2 - 2M_1^2}{M_2\sqrt{4M_1^2 - M_2^2}}\right]\right) \tag{A2}$$

for $M_1 \geq M_2/2$. The same expression can be used by analytic continuation for $M_1 < M_2/2$, which yields

$$f(M_1, M_2) = \frac{1}{2}\frac{M_2\sqrt{M_2^2 - 4M_1^2}}{M_1^2}\log\left(\frac{M_2 - \sqrt{M_2^2 - 4M_1^2}}{M_2 + \sqrt{M_2^2 - 4M_1^2}}\right). \tag{A3}$$

For $q^2 = -M_\pi^2$, the integrals of Eq. (3.23) can be expressed in terms of the J functions (A1) as

$$\begin{aligned}
I(-M_\pi^2, M_\pi^2, M_\tau^2) &= \frac{1}{16\pi^2}(\lambda - 1 - J_0(M_\pi, M_\tau)),\\
A(-M_\pi^2, M_\pi^2, M_\tau^2) &= -\frac{1}{32\pi^2}(\lambda - 1 - 2J_1(M_\pi, M_\tau)),\\
A(-M_\pi^2, M_\tau^2, M_\pi^2) &= -\frac{1}{32\pi^2}(\lambda - 1 + 2(J_1(M_\pi, M_\tau) - J_0(M_\pi, M_\tau))),\\
B(-M_\pi^2, M_\pi^2, M_\tau^2) &= -\frac{1}{192\pi^2}((2M_\pi^2 + 3M_\tau^2)\lambda - 6M_\tau^2 J_1(M_\pi, M_\tau) - 6M_\pi^2(J_0(M_\pi, M_\tau) - 2J_1(M_\pi, M_\tau) + J_2(M_\pi, M_\tau))),\\
C(-M_\pi^2, M_\pi^2, M_\tau^2) &= -\frac{1}{96\pi^2}(\lambda - 1 - 6(J_1(M_\pi, M_\tau) - J_2(M_\pi, M_\tau))).
\end{aligned} \tag{A4}$$

APPENDIX B: $\tau \to \pi\pi$ DECAY

An interesting question is whether two-flavor QCD might be close enough to the conformal window to be within the domain of dChPT. Identifying the dilaton with the $f_0(500)$ resonance, one would then expect LO dChPT to give a reasonably accurate prediction of the $f_0(500)$ decay width into two pions.

The $\tau \to \pi\pi$ decay rate is fixed in terms of the LO quantities: the pion and dilaton masses and decay constants, and the mass anomalous dimension γ^*. The $\tau\pi\pi$ vertex can be read off from the LO Lagrangian (2.1). In terms of the rescaled fields

$$\tau_r = \tau/F_\tau, \qquad \pi_r = e^{v_0}\pi = (F_\pi/f_\pi)\pi, \tag{B1}$$

this 3-point vertex is

$$\frac{1}{F_\tau}\tau_r\left(\partial_\mu\pi_r^a\partial_\mu\pi_r^a + \frac{1}{2}(3-\gamma_*)M_\pi^2\pi_r^a\pi_r^a\right). \tag{B2}$$

The amplitude for $\tau \to \pi^a\pi^b$ decay is

$$\mathcal{M}^{ab} = -\frac{1}{F_\tau}\delta^{ab}(M_\tau^2 + (1-\gamma_*)M_\pi^2), \tag{B3}$$

and the total decay width is

$$\Gamma_{\tau\to\pi\pi} = \frac{1}{32\pi}\frac{N_f^2 - 1}{M_\tau F_\tau^2}\sqrt{1 - \frac{4M_\pi^2}{M_\tau^2}}(M_\tau^2 + (1-\gamma_*)M_\pi^2)^2. \tag{B4}$$

Let us try to apply this result to $N_f = 2$ QCD, using $M_\pi = 135$ MeV, $F_\pi = 92.2$ MeV and $M_\tau = 441$ MeV [28]. Assuming that γ_* is in the range 0.5–1.0, this yields

$$\Gamma_{\tau\to\pi\pi} = 249(11)\left(\frac{F_\pi}{F_\tau}\right)^2 \text{MeV}, \tag{B5}$$

where the "error" in the prefactor accounts for the assumed range of γ^*. F_τ has not been computed directly on the lattice. But the ratio $F_\pi^2/F_\tau^2 = f_\pi^2/f_\tau^2$ is independent of the fermion mass. Also, while the decay constants exhibit scaling with N_c, they are not expected to depend on N_f in a significant way. Our successful LO dChPT fits [15] to the $N_f = 8$ data of Ref. [4] suggest that $f_\pi^2/f_\tau^2 \approx 0.09$. Using this estimate, we obtain

$$\Gamma_{\tau\to\pi\pi} \approx 22 \text{ MeV}. \tag{B6}$$

This result is to be compared to the predicted width in QCD, which is 544(22) MeV [28]. Our estimate for the width predicted by LO dChPT is thus roughly 25 times smaller than the actual width! We conclude that two-flavor QCD must be too far from the conformal sill to be described by dChPT.

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