Fractional topological charge in $SU(N)$ gauge theories without dynamical quarks

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In $SU(N)$ gauge theories without dynamical quarks, we discuss how configurations with fractional $\mathbb{Z}_N$ magnetic charge also have fractional topological charge, $\sim 1/N$, and dominate topologically nontrivial fluctuations in the confining vacuum. They are not solutions of the classical equations of motion, but arise as quantum solutions of the effective Lagrangian, whose size is essentially fixed, on the order of the confinement scale. We give both a general mathematical analysis and illustrative solutions. We discuss strong evidence for this from numerical simulations on the lattice, and suggest definitive tests. We also speculate how these objects change with the introduction of dynamical quarks, and their effects especially at low temperature and nonzero density.

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I. INTRODUCTION

It is well-known that topologically nontrivial configurations play an essential role in quantum chromodynamics (QCD). For $J^P = 0^-$ mesons, such configurations must be present in order to split the isosinglet η' meson from the octet of pions, kaons, and the η meson [1–3]. They also affect the spectrum of mesons with higher spin [4,5], and contribute to the proton and photon structure functions in polarized deep-inelastic scattering [6–14].

In weak coupling for a $SU(N)$ gauge theory, the dominant configurations are instantons, which are self-dual solutions to the classical equations of motion. By asymptotic freedom instanton effects can be reliably computed at high temperature, T, or quark chemical potential, $\mu_{\rm qk}$ [15–22]. The action for a single instanton with unit topological charge is $= 8\pi^2/g^2$. Since by asymptotic freedom the running coupling constant, $g^2(T) \sim 8\pi^2/(c\log(T))$, [23] the topological susceptibility falls off sharply at high temperature,

$$\chi_{\rm top}(T) \sim T^4 \exp(-8\pi^2/g^2(T)) \sim 1/T^{c-4}, \quad T \to \infty. \tag{1}$$

Remarkably, numerical simulations in lattice QCD find that this power law holds down to temperatures as low as $\approx$300 MeV [24–35]. This is valid when $\mu_{\rm qk} = 0$; at the end of the paper we discuss what might occur for cold, dense quarks, where $T \ll \mu_{\rm qk} \neq 0$.

For $T < 300$ MeV, in QCD the topological susceptibility is not that of a dilute instanton gas. To understand the what generates the topological susceptibility at low temperature and in vacuum, it is useful to consider a $SU(N)$ gauge theory without dynamical quarks. In this case, a global $\mathbb{Z}_N$ symmetry is spontaneously broken above the temperature for deconfinement, $T_{\rm deconf}$ [36]. Numerical simulations of the pure gauge theory find that the dependence on N is rather weak. Forming a dimensionless ratio between the topological susceptibility and the square of the string tension, at zero temperature $\chi_{\rm top}(0)/\sigma^2$, only varies by $\approx$10% between $N = 3$ [37–42] and higher N [43–62]. These results suggest that as $N \to \infty$, the topological susceptibility does not vary with temperature in the confined phase. Numerical simulations find that the deconfining phase transition is of first order for three or more colors, with $T_{\rm deconf} \approx 270$ MeV for $N = 3$ [63–67]. For $N \geq 3$, $\chi_{\rm top}(T)$ jumps at $T_{\rm deconf}$, and then falls off rapidly with increasing T, dominated by instantons above some temperature close to $T_{\rm deconf}$.

It is difficult to see how the topological susceptibility could be due to instantons in the confined phase at large N [68]. Holding $g^2N \equiv \lambda$ fixed as $N \to \infty$, the action of a single instanton in the partition function is exponentially suppressed, $\sim \exp(-(8\pi^2/\lambda)N)$. This quandary was recognized originally by Witten [1] and Veneziano [2], who argued nevertheless that in vacuum the topological susceptibility is not exponentially suppressed at large N [72].

The most natural possibility is that there are objects with fractional topological charge $\sim 1/N$, whose contribution directly survives at infinite N, $\sim \exp(-8\pi^2/\lambda)$. On a torus,

’t Hooft constructed explicit solutions with fractional topological charge $\sim 1/N$, with twists in $Z(N)$ electric and magnetic charge [73–78]. Since they depend upon $Z(N)$ twisted boundary conditions, however, they are limited to finite volume.

From numerical simulations on the lattice, Gonzalez-Arroyo and Martinez [79], and then with Montero [80], used cooling techniques to isolate configurations with nontrivial $Z(N)$ electric and magnetic charge. They argue that in infinite volume, that objects with fractional topological charge dominate, and produce a topological susceptibility and string tension of the correct magnitude. See also, Refs. [81–85].

At nonzero temperature in the deconfined phase, analytically Kraan, van Baal, Lee, and Lu (KvBLL) [86–99] showed that at nonzero temperature instantons can be viewed as made of N constituents, each with topological charge $1/N$. The constituents of KvBLL instantons have nontrival holonomy at spatial infinity, though, and thus cannot be pulled arbitrarily far apart.

A useful limit is to study gauge theories on a femtoslab, where one spatial dimension, L, is very small, with $L\Lambda_{\mathrm{QCD}} \ll 1$, where Λ_{QCD} is the renormalization mass scale of QCD. Over large distances the theory reduces to one in $2+1$ dimensions [100–103]. On a femtoslab, semiclassical techniques demonstrate that monopole instantons with topological charge $\sim 1/N$ are ubiquitous [104–116]. Presumably this survives when the size of the slab increases to distances where $L\Lambda_{\mathrm{QCD}} \sim 1$.

In this paper we consider configurations with fractional topological charge $\sim 1/N$ in both the $\mathbb{CP}^{N-1}$ model in $1+1$ dimensions [71,93–95,114,117–127], and for $SU(N)$ gauge theories, without dynamical quarks, in $3+1$ dimensions.

The $\mathbb{CP}^{N-1}$ model is useful. Instantons with integral topological charge are solutions of the classical equations of motion [128]. For both models, the classical action is invariant under a scale symmetry, which implies that instantons have a scale size, ρ, which ranges from zero to infinity. In Sec. II, we construct solutions which stationary points of a quantum effective action, whose size is manifestly nonperturbative, on the order of the confinement scale. They only exist as multivalued solutions.

In Sec. III we give a general mathematical analysis, extending that of Refs. [73–78]. The crucial element is that the configuration space of a pure $SU(N)$ gauge theory has a global $Z(N)$ symmetry, which is absent when dynamical quarks are present. We demonstrate generally how objects with integral topological charge are composed of those with fractional $Z(N)$ magnetic charge. As in the $\mathbb{CP}^{N-1}$ model, the relevant solutions are necessarily multivalued; similarly, we also suggest that in gauge theories that their size is fixed, on the order of the confinement scale.

In the deconfined phase we outline the construction of a solution with topological charge $1/N$ in the deconfined phase in Sec. III E. It is distinct from KvBLL instantons, which have nontrivial holonomy at spatial infinity, and integral magnetic charge. Instead, our solution, a type of $Z(N)$ dyon, has trivial holonomy at spatial infinity, but being multivalued, has fractional $Z(N)$ magnetic charge.

In Sec. IV we discuss results from numerical simulations on the lattice. Notably, with significant effort the N dependence of higher moments of fluctuations in the topological charge can be measured [44,45,49,52,54,58–60]. For example, the kurtosis coefficient b_2 is the ratio of fluctuations between the fourth and second moments in the topological charge. We discuss how results by Bonanno *et al.* [60], on the N dependence of b_2, strongly suggest that there is a dense liquid of fractionally charged objects. While intriguing, it is an indirect measurement. We then review how objects with fractional topological charge can be measured directly on the lattice, following Edwards *et al.* [129].

We conclude with a discussion of how quarks and $\mathbb{Z}_N$ dyons might interact in QCD. In this we depend crucially upon recent results from the lattice by Biddle, Kamleh, and Leinweber [130–133], and speculate how $\mathbb{Z}_N$ dyons might generate topologically nontrivial fluctuations in cold, dense quark matter.

Some of our conclusions are familiar. To measure a system whose total topological charge is fractional, $\mathbb{Z}_N$ twisted boundary conditions must be used [73–85]. Even so, following Gonzalez-Arroyo and Martinez [79–85], and as on a femtoslab [100–116], the vacuum is a condensate of objects with fractional topological charge.

In other ways we differ from previous analysis. The multivalued nature of our configurations is rather unlike known solutions, although we suggest tests to distinguish them from, e.g., KvBLL instantons. While we argue that our configurations dominate the topological susceptibility, our methods are not adequate to demonstrate that they produce confinement, as in Refs. [79–85]. Lastly, for our configurations, which are on the order of the confinement scale, the really crucial question is how dense the condensate is, and whether they can distinguished from other fluctuations. In this analyzing how the confingurations evaporate as the temperature is raised, especially near the deconfining transition, will be essential.

II. $\mathbb{CP}^{N-1}$ MODEL

Consider a nonlinear sigma model in two spacetime dimensions, with the target space the complex projective space $\mathbb{CP}^{N-1}$. The target space is formed by N complex variables z^i, the so-called homogeneous coordinates, identifying $z^i \sim w z^i$, $w \in \mathbb{C} - \{0\}$. The magnitude of w can be fixed by setting $\sum_{i=1}^{N} \bar{z}^i z^i = \bar{z}\cdot z = 1$, and the phase of w removed by gauging the overall $U(1)$ symmetry. The Lagrangian density is

$$\mathcal{L} = \frac{1}{g^2}\sum_{i=1}^{N} |D_\mu z^i(x)|^2; \qquad D_\mu = \partial_\mu - iA_\mu(x), \quad (2)$$

where the z^is satisfy $\bar{z}\cdot z = 1$ at each point in two spacetime dimensions, x_μ. The gauge field A_μ ensures that the z's are also invariant under local $U(1)$ transformations, $z^i(x) \to \mathrm{e}^{i\alpha(x)} z^i(x)$ [134]. The absence of a kinetic term for the gauge field $A_\mu(x)$ ensures that no new degrees of freedom are introduced via the gauging. Classically A_μ can be eliminated by its equation of motion,

$$A_\mu = -\frac{i}{2}\left(\frac{\bar{z}^i\partial_\mu z^i - \partial_\mu \bar{z}^i z^i}{\bar{z}\cdot z}\right), \tag{3}$$

so that $\mathcal{L}$ can be reexpressed entirely in terms of the z^i and $\bar{z}^i$. The only coupling constant in Eq. (2) is g^2, which is dimensionless.

The Lagrangian in Eq. (2) is obviously invariant under global $SU(N)$ transformations, $z^i \to U^i{}_j z^j$, where $U \in SU(N)$. Elements of the center of $SU(N)$, which is $\mathbb{Z}_N$, are special. These are the $U_k = \mathrm{e}^{2\pi i k/N}\mathbf{1}$, where $k = 0, 1\ldots(N-1)$, underwhich the z^i's transform as $z^i \to e^{2\pi i k/N} z^i$. Since the z^i are homogeneous coordinates, though, this $\mathbb{Z}_N$ rotation can be eliminated by a global $U(1)$ rotation. This reduces the full global symmetry to $SU(N)/\mathbb{Z}_N$ [71,114,118].

The topological winding number is

$$Q = \frac{1}{2\pi}\int d^2x\, \epsilon^{\mu\nu}\partial_\mu A_\nu. \tag{4}$$

For fields where z^i approaches a constant at infinity, Q is an integer. All classical configurations with $Q \neq 0$ are known [118]. Keeping g^2N fixed as $N \to \infty$, the value of the classical action is uniformly $\sim N$. The fluctuations about arbitrary instanton configurations have also been computed. While they simplify for $N = 2$, where it reduces to an $O(3)$ model [117,120,121], for $N > 2$ the integration over the collective coordinates of the instantons is not tractable. Even so, at large N they appear to be exponentially suppressed.

As discussed in the seminal papers [71,118], the large N analysis in the quantum theory can be carried out by introducing a Lagrange multiplier field $\lambda(x)$ to impose the constraint $\bar{z}\cdot z - 1 = 0$ and then integrating out the z^i fields. This leads to an effective action for A_μ and λ,

$$\mathcal{S}_{\mathrm{eff}} = N\,\mathrm{tr}\,\log\left(-D_\mu^2 + i\lambda\right) - i\int d^2x \frac{\lambda(x)}{g^2}. \tag{5}$$

The corresponding equations of motion are

$$N\,\mathrm{tr}\, D_\mu^{\mathrm{cl}} \frac{1}{-(D_\mu^{\mathrm{cl}})^2 + m^2(x)} = 0, \tag{6}$$

and

$$N\,\mathrm{tr}\frac{1}{-(D_\mu^{\mathrm{cl}})^2 + m^2(x)} - \frac{i}{g^2} = 0, \tag{7}$$

for arbitrary solutions $A_\mu(x) = A_\mu^{\mathrm{cl}}(x)$ and $i\lambda(x) = m^2(x)$. In vacuum $A_\mu^{\mathrm{cl}} = 0$ and $m^2(x)$ is constant, with the dynamically generated mass m related to the coupling constant g^2 through dimensional transmutation [71,118].

The quantum dynamics of the model, defined by Eq. (5), is rather different from that expected from the classical analysis of Eq. (2). Classically, the constraint $\bar{z}\cdot z = 1$ necessarily breaks the $SU(N)$ global symmetry, and the z fields are massless. In contrast, the quantum vacuum is invariant under the $SU(N)/\mathbb{Z}_N$ symmetry, in accord the Mermin-Wagner-Coleman theorem. The expectation values of the z^i vanish at infinity, and are massive fields.

As shown by Witten, Eq. (16) of Ref. [71], an effective theory for the quantum theory can be written in a derivative expansion,

$$\mathcal{S}_{\mathrm{eff}} = \int |(\partial_\mu - iA_\mu)Z^i_{\mathrm{qu}}|^2 - m^2 \bar{Z}^i_{\mathrm{qu}} Z^i_{\mathrm{qu}} - \frac{N}{48\pi m^2} F_{\mu\nu}F^{\mu\nu} + \cdots. \tag{8}$$

Here the Z^i_{qu} are effective fields to describe the low energy behavior of Eq. (5), and are not the original z^i fields of Eq. (2); the Z^i_{qu} are massive fields, which vanish at infinity. As mentioned above, the mass squared m^2 is dynamically generated, fixed by the solution of Eqs. (6) and (7).

There are many other terms which contribute to the derivative expansion in Eq. (8). These include terms with higher (covariant) derivatives of Z^i_{qu}; higher derivatives of A_μ, which by gauge invariance must enter as powers of $F^2_{\mu\nu}$; and lastly, derivatives of the constraint field $\lambda(x)$. None of these higher-order terms qualitatively change our discussion of the stationary points of Eq. (8). We stress that in this effective theory, the global symmetry of $SU(N)/\mathbb{Z}_N$ is unbroken, so by confinement the only allowed states are $\mathbb{Z}_N$ invariant [71].

We now turn to consider topologically nontrivial field configurations of the quantum effective action in Eq. (8). A general analysis of how to proceed in general field theories is outlined in the Appendix. Adopting polar coordinates (r, φ) in two dimensions, at large r the solution for the gauge field must satisfy

$$A_\mu dx^\mu \sim Q d\varphi; \qquad \text{i.e.,}\ A_\varphi \sim \frac{Q}{r} \quad \text{as}\ r \to \infty, \tag{9}$$

so that $\int F \sim Q \neq 0$. This should be accompanied by a suitable ansatz for $m^2(x)$ is a function of r, but we do not elaborate on this since it is not important for the main thread of our arguments. Determining the solution of the nonlocal equations of motion in Eqs. (6) and (7) is not elementary. But there is one aspect of any such solution which is worth of note, and which in fact is a recurrent point throughout our analysis. While the classical action is invariant under scale transformations, at large N the quantum effective action is not. Thus while Eq. (9) fixes the behavior at infinity, the nature of the full solution varies over a distance $\sim 1/m$.

We next turn to the possibility of configurations with fractional topological charge. On a femtoslab classical instantons were constructed by Unsal [114]; their size is necessarily on the order of the width of the slab. In contrast, we consider quantum instantons in vacuum. As a first step, consider the spherically symmetric configuration

$$F_{12} = \begin{cases} 2/(Na^2) & r < a \\ 0 & r > a. \end{cases} \tag{10}$$

This corresponds to $Q = \int (F/2\pi) = 1/N$, and the gauge potential

$$A_\mu dx^\mu = -\frac{1}{N\pi a^2}\int d^2x' \frac{\epsilon_{\mu\nu}(x-x')^\nu}{|x-x'|^2}\rho(x')dx^\mu, \tag{11}$$

where $\rho(x')$ is equal to 1 in a small disk of radius a, and zero elsewhere. This configuration is a slightly thickened vortex, with Eq. (11) consistent with the asymptotic behavior of Eq. (9), except that now the topological charge is fractional, with $Q = 1/N$. The contribution of this configuration to the action (8) is

$$\frac{N}{48\pi m^2}\int F^2 = \frac{1}{6Nm^2a^2}. \tag{12}$$

The action for the higher terms will be similarly suppressed, since they must involve powers of $F_{\mu\nu}$.

The ansatz of Eq. (11) may be written for ρ with support around the origin, as

$$A_\mu dx^\mu = f(r)d\varphi = \frac{1}{N}\begin{cases} (r^2/a^2)d\varphi & r \le a \\ d\varphi & r > a \end{cases}. \tag{13}$$

Turning to the $Z^i_{\rm qu}$-dependent part of the action, the only point of subtlety is about the phase of $Z^i_{\rm qu}$. With the background of Eqs. (10) and (11), the parallel transport of $Z^i_{\rm qu}$ in a full circle around the origin (or the location of the vortex) gives $Z^i_{\rm qu} \to e^{2\pi i/N}Z^i_{\rm qu}$. The phase may also be viewed as the Aharonov-Bohm phase acquired by $Z^i_{\rm qu}$ in a circuit around the vortex. While the $Z^i_{\rm qu}$ are not single valued, this phase can be removed by an $SU(N)$ transformation in its center $\mathbb{Z}_N$.

We can now supplement the ansatz Eqs. (11) or (13) with a suitable ansatz for $Z^i_{\rm qu}$, such as

$$Z^1_{\rm qu} = e^{i\varphi/N}h(r), \qquad Z^i_{\rm qu} = 0, \quad i = 2, 3, \ldots, N, \tag{14}$$

or any $SU(N)/\mathbb{Z}_N$ transformation of this. We have incorporated the aperiodicity in φ mentioned above, namely, $Z^i_{\rm qu}(r, 2\pi) = e^{2\pi i/N}Z^i_{\rm qu}(r, 0)$. This multivaluedness is where we differ from previous analysis by Berg and Lüscher [120] and Fateev *et al.* [121].

Taking the matter part of the action as in Eq. (8), we find

$$S_{\rm eff} = 2\pi\int dr\, r\left[\left(\frac{\partial h}{\partial r}\right)^2 + \frac{h^2}{r^2}\left(f - \frac{1}{N}\right)^2 + m^2h^2\right] + \cdots. \tag{15}$$

The behavior of $h(r)$ for small and large values of r can be inferred from the equation of motion for h, namely,

$$-\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial h}{\partial r}\right) + \left(f - \frac{1}{N}\right)^2\frac{h}{r^2} + m^2h + \cdots = 0. \tag{16}$$

By examining the small r and large r limits of this equation, we can see that

$$h(r) \sim \begin{cases} r^{\frac{1}{N}} & r \to 0 \\ e^{-mr} & r \to \infty. \end{cases} \tag{17}$$

Notice that h vanishes exponentially as $r \to \infty$. This is a significant point. While the gauge part of the configuration (13) is like an Abrikosov-Nielsen-Olesen vortex, the asymptotic behavior of $Z^i_{\rm qu}$ is very different. We may also note that the vanishing of $Z^i_{\rm qu}$ at spatial infinity is consistent with the fact that any configuration of finite action should reproduce vacuum behavior at spatial infinity.

Introducing a scale factor r_0, a simple ansatz consistent with Eq. (17) is

$$h(r) = C\frac{u^{\frac{1}{N}}}{1 + u^{\frac{1}{N}}}e^{-\mu u}, \qquad u = \frac{r}{r_0}, \qquad \mu = mr_0. \tag{18}$$

It is easy to verify that $S_{\rm eff}$ is finite with this ansatz and that the term in Eq. (15) involving both f and h depends on a. Along with the gauge-field contribution in Eq. (12), we get a nonlinear expression involving a and C. Treating these as variational parameters, we can obtain values which minimize the action, at least within the class of ansätze in Eqs. (13), (14), and (18).

A few comments are in order at this point. Notice that, even if $C = 0$, we do have a vortexlike configuration in Eq. (13). Although extremization with respect to a with just this term leads to $a \to \infty$, there are terms with higher powers of F in the action, indicated by ellipsis in Eq. (8). Including them and extremizing will lead to a finite value for a, which can only be set by the single dimensionful parameter in the model, the mass m. The inclusion of higher $Z^i_{\rm qu}$-dependent terms produces terms which are of order C^4 and higher. Thus, we expect that extremization including such terms gives finite values to both a and C.

As noted, this is equivalent to solving the nonlocal equations of motion in Eqs. (5) and (6).

To frame this more generally, at large N, we can again consider expanding Eq. (5) in powers of A_φ which is of order $1/N$, based on our ansatz. As for the solution with integral topological charge, and as in the example above, as a solution of the quantum action the size is $\sim 1/m$. The term linear in A_φ vanishes by the equation of motion for the gauge potential. Taking $m^2(x) = m^2$, then, the term in the action ~ 1 automatically vanishes. The expansion of the effective action to quadratic order in A_φ, i.e., as in Eq. (8), shows that the nonzero contribution of the A-part of the action will be of order $1/N$.

This demonstrates that there are configurations with fractional topological charge, $\sim 1/N$. We have not computed the exact configuration, for reasons we now discuss. At nonzero θ, the energy of the vacuum is an even function in θ [1,2,44,45,54],

$$E(\theta) - E(0) = \frac{\chi}{2}\theta^2(1 + b_2\theta^2 + \cdots); \tag{19}$$

χ is the topological susceptibility, $\chi = \langle Q^2\rangle/V$, where V is the volume of space-time. The second coefficient, b_2, is the kurtosis of the topological charge,

$$b_2 = -\frac{1}{12}\frac{\langle Q^4\rangle - 3\langle Q^2\rangle^2}{\langle Q^2\rangle}. \tag{20}$$

For both χ and b_2, all expectation values are computed at $\theta = 0$.

The topological susceptibility is a dimensional quantity, and so by dimensional transmutation $\chi \sim m^2$. The N dependence of the coefficients can be understood by assuming that fluctuations in the topological charge are fractional, $\Delta Q \sim 1/N$. Since there are N ways of inserting a charge $1/N$ in the theory, $\chi \sim N(1/N)^2 \sim 1/N$. Similarly, $b_2 \sim (1/N)^4/(1/N)^2 \sim 1/N^2$, etc. We did not compute the exact configurations with fractional topological charge because that can be computed from the free energy in a constant background field for $F_{\mu\nu}$ [45,54]. Thus the θ dependence is certainly described by a dense liquid of fractionally charged instantons.

III. TOWARDS FRACTIONAL INSTANTONS IN 4D

We now turn to non-Abelian gauge theories in four dimensions. One of the key steps in understanding configurations of fractional topological charge is the identification of what is meant by the gauge group. Although this question has been analyzed before, it is useful to collect some of the basic ideas here. We will first consider the boundary values for gauge transformations based on the Gauss law (or the nature of the test functions to be used in implementing the Gauss law) and how these are related to charge quantization conditions. This will clarify the nature of the configuration space and will naturally lead to the possibility of fractional topological charges.

A. Gauss law in the E representation

We consider the gauge theory in the $A_0 = 0$ gauge. We must then impose the Gauss law on the wave functions. Quantization conditions on the electric charge will be important for us, so it is more appropriate to consider wave functions in the representation which are eigenstates of the electric-field operators E^a. In other words, the wave functions are functionals of the electric field. The Gauss law operator is given by

$$G^a(x) = \nabla_i E_i^a + f^{abc} A_i^b E_i^c, \tag{21}$$

where f^{abc} are the structure constants of the Lie algebra of G. The gauge potential and the electric field obey the usual commutation rule $[A_i^a(x), E_j^b(y)] = i\delta^{ab}\delta_{ij}\delta(x-y)$, so that, in the E-representation, $A_i^a = i(\delta/\delta E_i^a)$. The physical wave functions Ψ are selected by the condition that the Gauss law operator must annihilate them. This condition can be written as

$$\int_M \theta^a(x) G^a(x)\,\Psi = \int_M \theta^a(x)\left[\nabla_i E_i^a - i f^{abc} E_i^b \frac{\delta}{\delta E_i^c}\right]\Psi = 0. \tag{22}$$

(The integral is over the spatial manifold M). This law should be required only for test functions $\theta^a(x)$ obeying certain conditions; the nature of these conditions will be clear from the following discussion. Treating $\theta^a(x)$ as an infinitesimal group parameter, (22) may be written as

$$\delta\Psi \equiv \Psi(U^{-1}EU) - \Psi(E) = -\left[i\int_M \theta^a(x)\nabla_i E_i^a\right]\Psi, \tag{23}$$

where $E_i = T^a E_i^a$, $U = \exp(iT^a\theta^a) \approx 1 + iT^a\theta^a$, T^a being hermitian matrices which form a basis for the Lie algebra of G, with $[T^a, T^b] = if^{abc}T^c$. For the fundamental representation, we write $T^a = t^a$ and normalize them by $\mathrm{Tr}(t^a t^b) = \frac{1}{2}\delta^{ab}$. The quantity $U^{-1}E_i U$ is the gauge transform of E_i and hence $\delta\Psi$ measures the change of Ψ under a gauge transformation with parameter $\theta^a(x)$. Obviously, if Ψ is a solution to (23), then so is $\Psi f(E)$ where $f(E)$ is a gauge-invariant function of E_i. The general solution to (23) may therefore be written as $\Psi = \rho\Phi(E)$, where $\Phi(E)$ is an arbitrary gauge-invariant function and ρ is a particular solution to

$$\delta\rho + \left[i\int_x \theta^a(x)\nabla_i E_i^a\right]\rho = 0. \qquad (24)$$

A finite transformation, and the corresponding variation of ρ, can be obtained by composition of infinitesimal transformations. Assume that, for an electric field E_i, we have started from the identity and built up a finite transformation U. At this point, the electric field is given by $\mathcal{E}_i = U^{-1}E_iU$. A further infinitesimal transformation would be given by $iT^a\theta^a = U^{-1}\delta U$. Thus (24), written for an arbitrary point on the space of U's, becomes

$$\begin{aligned}\delta\rho + 2\int_M \mathrm{Tr}(\nabla_i\mathcal{E}_iU^{-1}\delta U)\rho = 0,\\ \delta(\log\rho) = -2\int_M \mathrm{Tr}(\nabla_i\mathcal{E}_iU^{-1}\delta U) \equiv \Omega.\end{aligned} \qquad (25)$$

One can integrate this equation along a curve in the space of U's from the identity to U to obtain the change of ρ under a finite transformation. With δ interpreted as a derivative on the space of U's, $U^{-1}\delta U$ is a covariant vector (or one-form) and the result of the integration is generally path dependent. For the result to be independent of the path of integration, the curl of $\mathrm{Tr}(\nabla\cdot\mathcal{E}U^{-1}\delta U)$, viewed as a covariant vector or as a one-form on the space of the U's, must vanish. Thus the integrability condition for (25), or the path independence for the change in ρ, becomes

$$\delta\Omega = \delta\left[-2\int \mathrm{Tr}(\nabla_i\mathcal{E}_iU^{-1}\delta U)\right] = 0. \qquad (26)$$

Here we take δ to signify the exterior derivative, so that δ acting on a one-form (or covariant vector) gives the curl. We now write $\Omega = \Omega_1 + \Omega_2$ with

$$\begin{aligned}\Omega_1 &= 2\int_M \mathrm{Tr}(\mathcal{E}_i\nabla_i(U^{-1}\delta U))\\ &= 2\int_M \mathrm{Tr}[E_i(\nabla_i(\delta UU^{-1}) - [\nabla_iUU^{-1}, \delta UU^{-1}])],\\ \Omega_2 &= -2\oint_{\partial M} \mathrm{Tr}(\mathcal{E}_iU^{-1}\delta U)dS^i\\ &= -2\oint_{\partial M} \mathrm{Tr}(E_i\delta U\,U^{-1})dS^i.\end{aligned} \qquad (27)$$

It is easily checked, using $\delta(\delta UU^{-1}) = (\delta UU^{-1})^2$, without the need of any integration by parts on M, that $\delta\Omega_1 = 0$. For the second term, we find

$$\delta\Omega_2 = -2\oint_{\partial M} \mathrm{Tr}(E_i\delta UU^{-1}\delta UU^{-1})dS^i. \qquad (28)$$

This is in general not zero. Indeed if U is constant on ∂M, $\delta\Omega_2 = -2\mathrm{Tr}[Q(\delta UU^{-1})^2]$, $Q = \oint E_i dS^i$. In this case, $\delta\Omega_2$ has the form of the coadjoint orbit two-form on G/H, where $H \subset G$ is the subgroup which commutes with the charge Q. This form, well-known as the basis for the Borel-Weil-Bott theory on group representations, is a nondegenerate two-form on G/H. In order to have $\delta\Omega = 0$, we must therefore implement the Gauss law only for those U's which obey the restriction

$$\oint_{\partial M} \mathrm{Tr}[E_i\delta UU^{-1}]dS^i = 0 \qquad (29)$$

This is basically the cocycle condition which allows us to build up finite transformations using sequences of infinitesimal transformations. If E_i on ∂M can be arbitrary, this condition (29) would require fixing U to some value, say, U_∞ on ∂M. [If U_∞ is held fixed, $\delta U_\infty = 0$, so that the requirement (29) is trivially satisfied.] This clarifies the nature of the test functions θ^a in (24) in imposing the Gauss law; The test functions must be so chosen that they lead to U_∞ on ∂M.

The key question for us is then; What are the allowed values of U_∞? This will be determined by the charge quantization conditions. But before we take up this issue, a comment on the asymptotic behavior of U is in order. Although we argued using constant U on ∂M, generically, we cannot impose the Gauss law for U's which are not constant on ∂M as well, since $\delta\Omega_2$ will not vanish for such cases. In fact, U's which are not constant on ∂M correspond to degrees of freedom which are physical and generate the "edge modes" of a gauge theory. If we consider the boundary to be at spatial infinity, such edge modes are irrelevant. This will be the case for our analysis in this paper.

Returning to constant values of U on ∂M, and the identification of the possible values of U_∞, we start with the question; how does Ψ change under transformations which go to a constant $U \neq U_\infty$? It is easily seen that the action of a general infinitesimal transformation

$$\delta A_i^a = -\partial_i\theta^a - f^{abc}A_i^b\theta^c, \qquad \delta E_i^a = -f^{abc}E_i^b\theta^c, \qquad (30)$$

is given by

$$\delta\Psi = \left[i\int_M D_i\theta^a(x)E_i^a\right]\Psi = \exp[iQ^a\theta^a(r=\infty)]\Psi, \qquad (31)$$

where Q^a is the electric charge $Q^a = \oint E_i^a dS_i$. Thus transformations which go to a constant $\neq U_\infty$ act as a Noether symmetry, under which the charged states undergo a phase transformation. If the only charges in the theory correspond to the adjoint representation of G and its products, i.e., if the states are invariant under $\mathbb{Z}_N \in SU(N)$, then the wave functions are invariant for those U's which go to an element of the center $\mathbb{Z}_N$ at spatial infinity. We have seen that we can implement the Gauss law only for transformations which go to a fixed element U_∞ at spatial infinity. Now we see that the allowed choices for U_∞ correspond to an element of the center $\mathbb{Z}_N$.

To recapitulate briefly, we have seen that the true gauge transformations of the theory, in the sense of corresponding to a redundancy of description, are of the form $U(\vec{x})$ with:

(a) $U \to$ a constant U_∞ at spatial infinity; and
(b) $U_\infty \in \mathbb{Z}_N$ for a theory with charges which are $\mathbb{Z}_N$-invariant.

B. Charge quantization and U_∞: An alternate argument

There is another way to arrive at the conclusion of the previous subsection, namely, by a direct analysis of the charge quantization conditions. Notice that, for U's obeying (29), we can write Ω as

$$\begin{aligned}\Omega &= 2\int_M \mathrm{Tr}[E_i U \nabla_i (U^{-1}\delta U) U^{-1}] \\ &= 2\int_M \mathrm{Tr}[E_i \delta(\nabla_i U U^{-1})] \\ &= \delta\left(2\int_M \mathrm{Tr}[E_i \nabla_i U U^{-1}]\right). \end{aligned} \tag{32}$$

Using this and integrating (25) from the identity to U, we obtain

$$\rho(U^{-1}EU) = \rho(E)\exp\left(2\int_M \mathrm{Tr}(E_i\nabla_i U U^{-1})\right). \tag{33}$$

This equation will be important for us; it will have a key role in subsequent analysis. So another comment and another derivation will be appropriate before proceeding. One concern about (33) might be that we have used integration from the identity to U. In three spatial dimensions, since $\Pi_3(G) = \mathbb{Z}$, there are U's which are not connected to the identity. Even though the derivation given above does not quite make it clear, the result (33) holds even for U's which are not in the connected component. This can be seen by the following alternate derivation borrowed from [135]:

$$\begin{aligned}\Psi(E) &= \int d\mu(A)\exp\left[2\int_M \mathrm{Tr}(E_i A_i)\right]\Psi(A) \\ &= \int d\mu(A)\exp\left[2\int_M \mathrm{Tr}(E_i A_i)\right]\Psi(U^{-1}AU + U^{-1}\nabla U) \\ &= \exp\left(-2\int_M \mathrm{Tr}(E_i \nabla_i U U^{-1})\right) \\ &\quad \times \int d\mu(A)\exp\left[2\int_M \mathrm{Tr}(U^{-1}E_i U A_i)\right]\Psi(A) \\ &= \exp\left(-2\int_M \mathrm{Tr}(E_i\nabla_i U U^{-1})\right)\Psi(U^{-1}EU), \end{aligned} \tag{34}$$

where we have first used the gauge invariance of the wave functions in the A-representation [i.e. $\Psi(A) = \Psi(U^{-1}AU + U^{-1}\nabla U)$] and then changed the variable of integration from A to $U^{-1}AU + U^{-1}\nabla U$. With $\Psi = \rho\Phi(E)$, (34) gives (33). (This derivation is simpler but the earlier analysis does reveal some interesting aspects of imposing the Gauss law.)

Equation (33) contains certain charge quantization requirements which can be used to see why the boundary values of U can be an element of the center, rather than strictly being the identity. We can show that $\rho(E)$ of (33) will vanish unless certain conditions are satisfied by E_i. For this, it is adequate to examine some special configurations. The basic strategy is to choose an electric field configuration and a U which commutes with the chosen configuration for E_i. Equation (33) then gives an identity of the form $\rho = \rho e^{i\lambda}$ where the phase λ is given by the integral $2\int_M \mathrm{Tr}(E_i\nabla_i U U^{-1})$. This would imply that ρ must vanish unless the phase is an integral multiple of 2π; this is the constraint for the chosen type of field configuration. For simplicity, we shall use $G = SU(2)$ for the example below; generalization to other groups is straightforward.

For our example, we choose polar coordinates (r, θ, φ) and take

$$E_\theta = E_\varphi = 0, \quad E_r = \frac{\sigma_3}{2}\frac{q}{4\pi r^2}, \quad U = \exp(i\sigma_3 f(r)). \tag{35}$$

This field corresponds to a point charge at $r = 0$. To avoid the singularity, we shall remove the point $r = 0$ from M. Thus the boundary ∂M consists of a small sphere around $r = 0$ and the sphere at spatial infinity. Even though U is not constant in space, we have chosen it to commute with the given E_i. Evaluating the phase factor in (33), we obtain

$$\rho(U^{-1}EU) = \rho(E)\,\exp(2i(\Delta f)q), \tag{36}$$

where $\Delta f = f(\infty) - f(0)$. As for the values of $f(0), f(\infty)$, they should be integral multiples of π to be consistent with the trivial action of U on states at the boundaries. If we require U to go to the identity (and not just an element of the center) at the boundary, $\Delta f = 2\pi n,\ n \in \mathbb{Z}$. Equation (36) then tells us that we can have nonzero ρ for $q = \frac{1}{2}n$. The Gauss law for, say fermion sources, may be written as

$$\nabla\cdot E^a + f^{abc}A^b\cdot E^c = \bar{\psi}T^a\psi. \tag{37}$$

For the fundamental representation, this gives, for a point source with T^3 charge, $E^3 = \frac{1}{2}(1/4\pi r^2)$. This is consistent with the quantization of q. On the other hand, if we allow U to go to -1, then we only need $\Delta f = \pi n$. Correspondingly, (36) tells us that q should be quantized as $q = n$. Equation (37) also tells us that this is consistent with sources transforming under $\mathbb{Z}_2$-invariant representations.

The result of the arguments presented here is that wave functions are invariant under gauge transformations which go to an element of the center in theories where the charges are in $\mathbb{Z}_N$-invariant representations. Such transformations therefore characterize the redundancy of the variables (A_i, E_i) in the theory.

The configuration we have used for obtaining charge quantization has a divergent kinetic energy $T = \frac{1}{2}\int E^2$. It is possible to find nonsingular configurations which lead to the same result; it is just that the argument will be a little more elaborate.

C. Nature of the configuration space

The E-representation of the wave functions was useful in elucidating the nature of the allowed boundary values for U. However, for the analysis and formulation of ansätze for the configurations with fractional topological charge, the A representation is more appropriate, so this is the representation we will use for the rest of this paper.

We can now formalize the situation with the gauge transformations as follows. Staying within the $A_0 = 0$ gauge, let

$$\begin{aligned}\mathcal{A} &\equiv \{\text{Set of all gauge potentials}\, A_i\}\\ &\equiv \{\text{Set of all Lie-algebra-valued vector fields}\\ &\qquad \text{on space}\ \mathbb{R}^3\}.\end{aligned}$$

Further, let

$$\begin{aligned}\mathcal{G} &\equiv \{\text{Set of all}\, g(\vec{x})\colon \mathbb{R}^3 \to SU(N), \text{such that}\\ &\qquad g(\vec{x}) \to \text{constant} \in SU(N)\ \text{as}\ |\vec{x}| \to \infty\}\\ \mathcal{G}_\omega &\equiv \{\text{Set of all}\, g(\vec{x})\colon \mathbb{R}^3 \to SU(N),\ \text{such that}\\ &\qquad g(\vec{x}) \to \omega \in \mathbb{Z}_N\ \text{as}\ |\vec{x}| \to \infty\}.\end{aligned}$$

Evidently, $\mathcal{G}/\mathcal{G}_1 = SU(N)$, the set of rigid transformations or the set of constant boundary values for elements g in $\mathcal{G}$. Our discussion of the Gauss law shows that the gauge group, namely, the set of transformations which leave the wave functions invariant, is given by $\mathcal{G}_1$ in a theory without $\mathbb{Z}_N$-invariance. However, in a theory with $\mathbb{Z}_N$ invariance, $\mathcal{G}_\omega$ leaves Ψ invariant for any ω, so that the gauge group is $\mathcal{G}_* = \cup_{\omega\in\mathbb{Z}_N}\mathcal{G}_\omega$. Since the difference between $\mathcal{G}_1$ and $\mathcal{G}_\omega$ is in the boundary value, we may also consider any element of $\mathcal{G}_\omega$ to be of the form $g(\vec{x})\omega$, where $g(\vec{x})$ goes to the identity at spatial infinity.

The physical configuration space, for theories with charges in the fundamental representation, i.e., without $\mathbb{Z}_N$-invariance, is given by $\mathcal{A}/\mathcal{G}_1$. It is easy to see that this space is multiply connected. Consider a sequence of configurations $A_i(\vec{x}, \tau)$ with $0 \le \tau \le 1$ given by

$$\begin{aligned}A_i(\vec{x}, \tau) &= A_i(\vec{x})(1-\tau) + \tau A_i^g(\vec{x}),\\ A_i^g(\vec{x}) &= g^{-1}A_i(\vec{x})g + g^{-1}\partial_i g,\end{aligned} \tag{38}$$

where $g(\vec{x}) \in \mathcal{G}_1$. Thus $g(\vec{x}) \to 1$ at spatial infinity. The starting point and ending point of this sequence of gauge fields are gauge equivalent, so that (38) gives a closed curve in $\mathcal{A}/\mathcal{G}_1$. If this curve is contractible, then we will be able to transform the entire sequence into gauge-equivalent configurations, writing

$$A_i(\vec{x}, \tau) = g^{-1}(\vec{x}, \tau)A_i(\vec{x})g(\vec{x}, \tau) + g^{-1}(\vec{x}, \tau)\partial_i g(\vec{x}, \tau). \tag{39}$$

The transformations $g(\vec{x}, \tau)$ give a homotopic deformation of the identity (at $\tau = 0$) to $g(\vec{x})$ at $\tau = 1$. The homotopy classes of transformations $g \in \mathcal{G}_1$ are characterized by the winding number

$$Q[g] = \frac{1}{24\pi^2}\int \mathrm{Tr}(g^{-1}dg)^3. \tag{40}$$

Thus, if g is chosen to have nonzero winding number, then we do not have the possibility (39), leading to the conclusion that there are noncontractible paths in $\mathcal{A}/\mathcal{G}_1$. In other words, if $g(\vec{x})$ has nonzero winding number, the configuration (38) traces out a noncontractible path in $\mathcal{A}/\mathcal{G}_1$ as τ changes from 0 to 1. The usual instanton is an example of such a path, which, although it is not captured by the simple parametrization given in (38), is deformable to (38). In general, the noncontractible paths are topologically nontrivial configurations with nonzero instanton number, but not necessarily self-dual (or antiself-dual). In fact, evaluating the instanton number on the configurations (38), we find

$$\begin{aligned}\nu[A] &\equiv -\frac{1}{8\pi^2}\int_{M\times[0,1]} \mathrm{Tr}(F\,F)\\ &= \frac{1}{24\pi^2}\int_{M,\tau=1} \mathrm{Tr}(g^{-1}dg)^3,\end{aligned} \tag{41}$$

where we used the fact that g goes to the identity at spatial infinity.

D. Fractional values of ν

It is now easy to see how one may get fractional values of ν. We consider a path in the space of gauge potentials $\mathcal{A}$ of the form (39), say with $g = U(\vec{x}, \tau)$, where $U(\vec{x}, 1)$ is such that it goes to $\omega = \exp(2\pi i/N)$ as $|\vec{x}| \to \infty$. In other words, $U(\vec{x}, 1) \in \mathcal{G}_\omega$. Therefore the path $A^U = U^{-1}AU + U^{-1}\nabla U$ is closed in the $\mathbb{Z}_N$-invariant theory. The instanton number of this configuration can be evaluated explicitly, but before doing that, a comment is in order. The configuration $U^{-1}AU + U^{-1}\nabla U$ looks similar to (39), but there is an important difference. In (39), $g(\vec{x}, \tau)$ gives a homotopy between the identity and $g(\vec{x})$, so that $A_i(\vec{x}, \tau)$ is gauge equivalent to $A_i(\vec{x})$ for any value of τ. Further, the value of $g(\vec{x},\tau)$ as $|\vec{x}| \to \infty$ is identity. To get a noncontractible path, one needs to consider A_i which depend on τ as in (38) (or in the usual self-dual instanton configurations). In the present case, the boundary value of U changes from the identity to ω, so that at $\tau \neq 0$, 1, U is not an element of $\mathcal{G}_1$ or $\mathcal{G}_\omega$. This is why the configurations $U^{-1}AU + U^{-1}\nabla U$ can still give a nonzero ν.

Turning to details, it is useful to have an explicit construction of such a $U(\vec{x},\tau)$. Let t^a, $a = 1, 2, \ldots, (N^2 - 1)$, denote a basis of Hermitian $N \times N$ matrices for the Lie algebra of $SU(N)$, normalized so that $\mathrm{Tr}(t^a t^b) = \frac{1}{2}\delta^{ab}$. We can take t^{N^2-1} to be diagonal and given by

$$(t^{N^2-1})_{ij} = \sqrt{\frac{N}{2(N-1)}}\begin{cases} \frac{1}{N}\delta_{ij} & i,j = 1,2,\ldots,(N-1) \\ \frac{1}{N} - 1 & i = j = N. \end{cases} \tag{42}$$

This is the $SU(N)$ version of the usual hypercharge matrix. It is easy to see that

$$g = \exp(i2\pi\tau\sqrt{2(N-1)/N}\, t_{N^2-1}) \tag{43}$$

is a path from $g = 1$ to $g = \omega$ in $SU(N)$ as τ varies from zero to 1. Thus it is a closed path in the pure $SU(N)$ gauge theory. Keeping in mind that instantons are essentially in an $SU(2)$ subgroup of $SU(N)$, we define the $N \times N$ matrix

$$Y_{ij} = \begin{cases} \frac{1}{N}\delta_{ij} & i,j = 1,2,\ldots,(N-2) \\ \frac{1}{2}(\sigma\cdot\hat{x})_{ij} + (\frac{1}{N} - \frac{1}{2})_{ij} & i,j = N-1, N. \end{cases} \tag{44}$$

We can then define

$$U(\vec{x},\tau) = \exp(iY\Theta(r,\tau)) \tag{45}$$

with $\Theta(r,0) = 0$, $\Theta(0,\tau) = 0$ and $\Theta(\infty,\tau) = 2\pi\tau$. [One example of such a function is $\Theta(r,\tau) = 2\pi\tau r/(r + r_0)$. There are obviously infinitely many Θ's consistent with the required boundary behavior.] This gives a spherically symmetric ansatz for an element of $\mathcal{G}_\omega$. It is easy to verify that $U(\infty,\tau)$ traces out a path from the identity to ω in $SU(N)$. Also, since $U(\infty,\tau) \to 1, \omega$ at $\tau = 0$, 1, it qualifies as a gauge transformation at the two ends in the $SU(N)/\mathbb{Z}_N$ theory.

Returning to the configurations $A_i^U = U^{-1}A_iU + U^{-1}\partial_i U$ in the space of potentials, we see that this corresponds to a closed path in $\mathcal{A}/\mathcal{G}_\omega$. Since U depends on $\vec{x},\tau$, but A_i depends only on $\vec{x}$,

$$\begin{aligned}\mathcal{F} &= dA^U + A^U A^U = U^{-1}FU + d\tau\frac{\partial}{\partial\tau}A^U \\ &= U^1(F - Da)U, \end{aligned} \tag{46}$$

where F involves only the spatial components of the field strength tensor and $a = d\tau\dot{U}U^{-1}$. (For this calculation, τ can also be viewed as the time coordinate, so that Da is essentially the electric field.) From (46),

$$\begin{aligned}\nu &= -\frac{1}{8\pi^2}\int \mathrm{Tr}[(F - Da)(F - Da)] = \frac{1}{4\pi^2}\int \mathrm{Tr}(DaF) \\ &= \frac{1}{4\pi^2}\oint \mathrm{Tr}(aF). \end{aligned} \tag{47}$$

The indicated boundary integration is over spatial infinity and over all τ. This shows that we will need a nonzero magnetic flux to obtain a nonzero value for ν. Therefore, we consider monopolelike configurations with the asymptotic behavior

$$\begin{aligned}F &= \frac{1}{2}F_{ij}dx^i \wedge dx^j \\ &\to -\frac{i}{2}(\sigma\cdot\hat{x})\frac{M}{2}\epsilon_{ijk}\frac{\hat{x}^k}{r^2}dx^i \wedge dx^j, \end{aligned} \tag{48}$$

where σ_i are in the 2×2 block of $i, j = (N - 1), N$ viewed as an $N \times N$ matrix. M is (electric charge e times) the monopole charge. We then find

$$\nu = M. \tag{49}$$

M must be quantized according to the Dirac quantization condition. This condition, for a general gauge group is the Goddard-Nuyts-Olive (GNO) quantization condition [136] and amounts to the following. If the electric charges correspond to representations of G, then the magnetic charges M take values in the dual group $\tilde{G}$. For our case, we note that the GNO dual of $SU(N)$ is $SU(N)/\mathbb{Z}_N$. Thus if the electric charges are $\mathbb{Z}_N$ invariant, taking values corresponding to $SU(N)/\mathbb{Z}_N$ representations, then the fundamental charges of $SU(N)$ are allowed values for M. They are thus quantized in units of $1/N$.

Thus we see that we can indeed obtain fractional values of ν. The problem however, is that for the nonsingular 't Hooft-Polyakov ('t H-P) monopoles, the quantization condition is not quite the Dirac (or GNO) condition. In fact, for the case of $SU(2)$, M is an integer for 't H-P monopoles, whereas the GNO condition would suggest that it is possible to get $M = \frac{1}{2}$. It is, however, possible to construct nonsingular configurations of separated GNO monopoles which have a total flux consistent with the 't Hooft-Polyakov condition. These will look like some split versions of the 't H-P monopole.

We can construct an ansatz for the split monopole for the case of $SU(2)$ as follows. Let A_D be the Dirac form of the monopole given by

$$A_D = \frac{(\hat{x}_1 d\hat{x}_2 - \hat{x}_2 d\hat{x}_1)}{(1 + \hat{x}_3)}. \tag{50}$$

The 't Hooft-Polyakov form of the monopole is then given by

$$\begin{aligned} A &= (1-K(r))\left[g^{-1}i\left(\frac{\sigma_3}{2}\right)A_D g + g^{-1}dg\right] \\ &= i\left(\frac{\sigma^a}{2}\right)(1-K)\epsilon_{abc}\frac{x^b}{r^2}dx^c, \end{aligned} \tag{51}$$

where g is the matrix

$$g = \frac{1}{\sqrt{1+z\bar{z}}}\begin{bmatrix} 1 & z \\ -\bar{z} & 1 \end{bmatrix} \tag{52}$$

and $z = \tan(\theta/2)e^{-i\varphi}$ and

$$\hat{x}_1 = \frac{z+\bar{z}}{1+z\bar{z}}, \qquad \hat{x}_2 = \frac{i(z-\bar{z})}{1+z\bar{z}}, \qquad \hat{x}_3 = \frac{1-z\bar{z}}{1+z\bar{z}}. \tag{53}$$

We may also note that $g^{-1}\sigma_3 g = \sigma\cdot\hat{x}$. The function $K(r)$ vanishes exponentially outside of the core of the monopole, and $1-K(r)\sim r^2$ for small r. The advantage of writing it as in (51) is that, for large r, we can trivially calculate F as

$$F = ig^{-1}\frac{\sigma_3}{2}g\,dA_D = \frac{i}{2}\sigma\cdot\hat{x}\,\sin\theta\,d\theta d\varphi. \tag{54}$$

Thus, $F^a = -\hat{x}^a \sin\theta\, d\theta d\varphi$, with $\int F^a\hat{x}^a = -4\pi$. We can now modify this ansatz with some of the flux piped away from the monopole by a vortex. We consider an Abelian vortex given by

$$A_v = \frac{1}{2}f(\rho, x_3)\frac{x_1 dx_2 - x_2 dx_1}{\rho^2}, \tag{55}$$

where $\rho^2 = x_1^2 + x_2^2$. This is a vortex along the x_3-axis. We also have $f(\rho, x_3)\to 1$ as ρ becomes large, essentially outside the core of the vortex. The factor of $\frac{1}{2}$ tells us that the flux carried by this vortex is $2\pi/2$; it is a $\mathbb{Z}_N$ vortex, for $N=2$. We will consider a vortex of finite length L by taking as an ansatz

$$f(\rho, x_3) = \frac{1}{2}\tanh\lambda\rho\,[\tanh\tilde{\lambda}x_3 - \tanh\tilde{\lambda}(x_3 - L)]. \tag{56}$$

This function vanishes exponentially for $x_3 \ll 0$ and for $x_3 \gg L$. The core of the vortex has an extent in ρ of the order of $1/\lambda$. Our modified ansatz is now given by

$$A = (1-K(r))[g^{-1}i\sigma_3(A_D - A_v)g + g^{-1}dg]. \tag{57}$$

Consider a large sphere of radius R much larger than the core of the monopole and the core of the vortex. If $R \ll L$, then the sphere intersects the vortex. The flux may be computed by taking $K\to 0$, so that

$$F = \frac{i}{2}\sigma\cdot\hat{x}(dA_D - 2dA_v). \tag{58}$$

The flux is then $-(4\pi - 2\pi) = -2\pi$. This is what we expect for a GNO monopole, and is equivalent to $M=\frac{1}{2}$. If we consider a sphere of radius much larger than L, then the contribution from A_v is zero, since f vanishes and we get -4π for the total flux. In this sense, we can view the configuration (57) as a split monopole.

The relevance of the split monopole can be understood from the following question; In a calculation or simulation of the vacuum-to-vacuum transition amplitude, can we see configurations with fractional values of ν? For this it is useful to write ν in terms of the Chern-Simons integral

$$S_{CS}(M) = -\frac{1}{8\pi^2}\int_M \mathrm{Tr}\left(A\,dA + \frac{2}{3}A^3\right). \tag{59}$$

The topological charge ν, which is the integral of the exterior derivative of the Chern-Simons over spacetime, can then be written as

$$\begin{aligned} \nu &= S_{CS}(M, \tau=1) - S_{CS}(M, \tau=0) \\ &\quad + \frac{1}{8\pi^2}\oint_{\partial M}\mathrm{Tr}(A_i E_j)dx^0\wedge dx^i\wedge dx^j. \end{aligned} \tag{60}$$

As the representative of the vacuum at $\tau=0$, we may take $A_i = 0$. The final configuration is also the vacuum, so it must be a gauge transform of $A_i=0$, say, $A_i = g^{-1}dg$. Further, if we consider spatial boundary conditions (periodic, Dirichlet, etc.) which lead to vanishing of the integral with the electric flux on ∂M, we find

$$\begin{aligned} \nu &= S_{CS}(M, \tau=1) - S_{CS}(M, \tau=0) \\ &= \frac{1}{24\pi^2}\int_M \mathrm{Tr}(g^{-1}dg)^3 = Q[g]. \end{aligned} \tag{61}$$

Since $Q[g]$ is an integer, even for g's such that $g\to\omega$ on ∂M, we get integral values of ν in the vacuum-to-vacuum amplitude. However, we can have configurations like $A_i^U = U^{-1}A_i U + U^{-1}\partial_i U$ where A_i is a split monopole configuration as in (57). We get separated configurations, each of which in isolation may be considered as having a fractional value of ν, but the total value of ν is integral.

E. Simple solution

We will now illustrate the analysis given above in a related but slightly different way and also comment on the situation with finite nonzero temperature T. It is convenient to frame this discussion in terms of a nonzero A_0, by replacing the field configuration $(A_0 = 0, U^{-1}A_i U + U^{-1}\partial_i U)$ by its gauge equivalent version $(A_0 = \dot{U}U^{-1}, A_i)$. For A_0, at $r=\infty$ our choice is then

$$A_0 = \frac{2\pi T}{N}\mathbf{k}. \tag{62}$$

Here $\mathbf{k}$ is a diagonal $SU(N)$ matrix related to $\mathbb{Z}_N$ transformations, so their elements are integers. There are two choices,

$$\mathbf{k}_1 = \begin{pmatrix} \mathbf{1}_{N-1} & 0 \\ 0 & -(N-1) \end{pmatrix}, \tag{63}$$

$$\mathbf{k}_2 = \begin{pmatrix} \mathbf{1}_{N-2} & 0 & 0 \\ 0 & -(N-1) & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{64}$$

These are obviously related to the matrix $(t^{N^2-1})_{ij}$ in Eq. (42). For $\mathbf{k}$ equal to either $\mathbf{k}_i$, the Wilson line in the imaginary time direction, t, is

$$\Omega = \exp\left(i\int_0^{1/T} A_0\, dt\right) = \exp\left(\frac{2\pi i}{N}\mathbf{k}\right), \tag{65}$$

has nontrivial holonomy, as these values represent $\mathbb{Z}_N$ degenerate vacum.

For the spatial components, construct a split ‘t H-P monopole, as in the previous section. Divide a sphere into an upper and a lower hemisphere, with gauge potentials on each, $A^{\pm}$, and take

$$A^{\pm}_{\phi} = \frac{1}{2Nr}\mathbf{m}\frac{(\pm 1 - \cos\theta)}{\sin\theta}. \tag{66}$$

To see this is a $\mathbb{Z}_N$ monopole, compute the Wilson line for a special closed path, $\vec{s}$. Since the vector potential is specified by two patches, we compute the Wilson line with A^+, going around by 2π in ϕ; then, take the Wilson line with A^-, running in the opposite direction,

$$\exp\left(i\oint \vec{A}^+\cdot d\vec{s}\right)\left(\exp\left(i\oint \vec{A}^-\cdot d\vec{s}\right)\right)^\dagger = \exp\left(\frac{2\pi i}{N}\mathbf{m}\right). \tag{67}$$

This is manifestly gauge invariant, and $=1$ if the configuration is trivial, $A^+ = A^-$. For the $\mathbb{Z}_N$ monopole, instead one obtains a nontrivial element of $\mathbb{Z}_N$. For this to be true, $\mathbf{m}$ must be one of the two matrices, $\mathbf{c}_1$ or $\mathbf{c}_2$.

The above are the boundary conditions at spatial infinity, $r\to\infty$. At the origin, $r=0$, we require all A_μ's to vanish, at least like $\sim r^2$, so that $F_{\mu\nu}\sim r$ as $r\to 0$.

As argued in Sec. II, in general we expect that this exists only as a quantum instanton, on the order of the confinement scale. At nonzero temperature, however, $1/T$ provides an alternate length scale. While the solution is approximately self-dual over distances $\sim 1/T$, because of the presence of the Debye screening mass, it is not self-dual over larger distances. This generates corrections $\sim\sqrt{g^2}$ to the action.

It is straightforward to compute the topological charge. For large r,

$$A_0(r) = \frac{2\pi T}{N}\mathbf{k} - \frac{1}{2Nr}\mathbf{m} + \ldots. \tag{68}$$

For a static configuration,

$$Q = \frac{1}{4\pi^2}\int d^4x\partial_i\,\mathrm{tr}\,(A_0 B_i) = \frac{1}{N^2}\mathbf{m}\cdot\mathbf{k}. \tag{69}$$

This was first derived by ’t Hooft [73].

There are only two cases to consider. Either the $\mathbb{Z}_N$ charges are the same, or they are different. If they are the same, $\mathbf{m} = \mathbf{k_1}$,

$$Q = \frac{N-1}{N}. \tag{70}$$

If they charges are different, such as $\mathbf{m} = \mathbf{k}_1$ and $\mathbf{k} = \mathbf{k}_2$, then

$$Q = -\frac{1}{N}. \tag{71}$$

We conclude this section by discussing the relationship between the configuration above and that of Kraan, van Baal, Lee, and Lu (KvBLL) [86–99] Like ours, their solution carries magnetic charge and has nontrivial holonomy. Our ansatz, however, carries $\mathbb{Z}_N$ magnetic charge, and so must be represented by a multivalued function, Eq. (66), while that of KvBLL has integral magnetic charge. For our solution, the boundary condition for holonomy at spatial infinity ensures that it is a vacuum which is degenerate with the vacuum. Thus when one-loop corrections are included, the action for our solution will remain finite. In contrast, for the solution of KvBLL, the holonomy is at a maximum of the holonomous potential. When one-loop corrections are included, then, the action for a constituent with charge $1/N$ diverges as the spatial volume. The action for an instanton with integral charge remains finite, which is why on the quantum level, these constituents cannot be pulled apart. In the next section, we discuss how to distinguish between our configurations and those of KvBLL.

IV. $\mathbb{Z}_N$ DYONS ON THE LATTICE

In the previous section we argued that configurations with fractional magnetic charge can generate fractional topological charge. From this the N dependence of $E(\theta)$ in Eq. (19) follows immediately, and accords with general expectation [1,2]. Assuming that fluctuations in the topological charge are $\Delta Q\sim 1/N$, since there are N^2 ways of inserting a fractional charge in an $SU(N)$ gauge theory, the topological susceptibility $\chi\sim N^2(1/N)^2\sim 1$. As in the

$\mathbb{CP}^{N-1}$ model, $b_2 = \tilde{b}_2/N^2$, where $\tilde{b}_2$ is a number of order one.

The precise value of $\tilde{b}_2$ is rather interesting. By numerical simulations on the lattice, Bonanno *et al.* [60] computed b_2 for $N = 4$ and 6, and compared to known results for $N = 3$. They exclude a constant value of b_2, as expected for a dilute gas of instantons. Instead, their results strongly favor $b_2 = \tilde{b}_2/N^2$. Comparing to Eq. (20), $\tilde{b}_2 = -1/12 \approx -0.08$ for a dilute gas of fractionally charged objects. Instead, Ref. [60] find a value which is more than twice as large, $\tilde{b}_2 \approx -0.19$. This indicates that fractional instantons do not form a dilute gas, but a dense liquid. That the vacuum of a $SU(N)$ gauge theory is complicated, with a dense liquid of $\mathbb{Z}_N$ dyons, is to be expected.

While the value of $\tilde{b}_2$ is most suggestive, it does not comprise definitive evidence for fractional topological charge. We then discuss a way of measuring fractional topological charge directly. In the continuum, an instanton in a $SU(N)$ gauge theory with topological charge one, coupled to single massless Dirac quark in the fundamental representation, has two zero modes, one for each chirality. In the adjoint representation, however, there are $2N$ zero modes. Thus, a single $\mathbb{Z}_N$ dyon has two zero modes for a quark in the adjoint representation.

On the lattice, in the pure gauge theory one can use an external quark propagator to look for isolated zero modes. To ensure these are not lattice artifacts, it is imperative to use a Dirac propagator with exact chiral symmetry, such as the overlap operator [137–143].

Using an external quark propagator in the adjoint representation, then, one can look for isolated $\mathbb{Z}_N$ dyons. This was first done by Edwards *et al.* [129], who found evidence for fractional topological charge. Their lattices were coarse, however. With present techniques and much finer lattices it should be possible to establish the existence of fractional topological charge close to the continuum limit [144]. With the overlap operator, one would look for configurations with (almost) zero modes; from the eigenvector, one could estimate the position and size of the object.

We stress that unless there are boundary conditions which are twisted with respect to $\mathbb{Z}_N$ [73,74], then the net topological charge will always be integral. Even so, it should be possible from the eigenvectors to see if N $\mathbb{Z}_N$ dyons are tightly bound into instantons, or if $\mathbb{Z}_N$ dyons and antidyons form a dense liquid of objects with fractional topological charge.

In the confining phase of a gauge theory surely the worldlines of the $\mathbb{Z}_N$ dyons are tangled, both with themselves and those of other dyons and anti-dyons. This is especially true if the size of the dyons is on the order of the confinement scale. This may help explain why lattice studies by Horvath *et al.* do not find evidence for a simple instanton, concentrated about a single point in spacetime, but for an extended structure [124,125,145–149].

The change in the behavior of $\mathbb{Z}_N$ dyons is especially interesting near the deconfining transition temperature T_d. $\mathbb{Z}_N$ dyons carry $\mathbb{Z}_N$ magnetic charge. This is allowed in the confined phase, where $\mathbb{Z}_N$ magnetic charge is unconfined. In the deconfined phase, though, $\mathbb{Z}_N$ magnetic charge propagating in the temporal direction is confined. Thus $\mathbb{Z}_N$ dyons are only relevant at best in a narrow temperature region above T_d. As the temperature increases, so will the magnetic string tension, binding the $\mathbb{Z}_N$ dyons with increasing strength into instantons with integral topological charge. This window of temperature at $T \geq T_d$ where $\mathbb{Z}_N$ dyons are relevant could vanish as $N \to \infty$, which appears to be suggested by numerical simulations on the lattice [51].

We note that at temperatures just above T_d, it should be possible to distinguish between our configurations, and those of KvBLL, by measuring the value of the Polyakov loop at the location of the near zero mode of the adjoint quark propagator.

Especially interesting to study would be the behavior of $\mathbb{Z}_2$ dyons in a $SU(2)$ gauge theory, where the deconfining transition is of second order.

V. $\mathbb{Z}_N$ DYONS AND QUARKS

We have concentrated exclusively on a gauge theory without dynamical quarks. In this section we discuss what might occur with their introduction.

For a $\mathbb{Z}_N$ magnetic monopole (or dyon), a Wilson loop in the fundamental representation picks up a phase of $\exp(2\pi i/N)$ as it encircles the worldline of the monopole. The same is true for dynamical quarks in the fundamental representation, and so it is not obvious how the monopole density changes as quarks are introduced.

There are recent results about the density of $\mathbb{Z}_N$ monopoles in the presence of dynamical quarks. While the definition of $\mathbb{Z}_N$ monopoles, and so their density, is gauge dependent, changes in the density should be meaningful and gauge invariant. Biddle, Kamleh, and Leinweber [130–133] have studied the change in the density of $\mathbb{Z}_N$ monopoles as quarks are introduced, and find that the monopole density strongly increases as the mass of the quarks *de*creases.

While the sign of the effect is unexpected, we can use this to suggest how anomalous interactions might change as a function of the temperature, T, and quark chemical potential, $\mu_{\rm qk}$.

At $\mu_{\rm qk} = 0$ and $T \neq 0$, the lattice finds that the topological susceptiblity is consistent with a dilute gas of instantons, as in Eq. (1), for $T > 300$ MeV [24–34]. Notice that this temperature is close to the deconfining temperature for the pure glue theory, of $T_d \approx 270$ MeV [150].

The lattice finds that the crossover temperature for chiral symmetry is at $T_\chi \approx 156 \pm 2$ MeV [152–154]. Thus for $T_\chi \leq T \leq 300$ MeV, massless quarks interact with what is surely a dense liquid of $\mathbb{Z}_N$ dyons. At a temperature

$T < T_\chi$, the topological susceptibility changes slowly with temperature, as massive quarks, or equivalently hadrons, interact with this dense liquid of $\mathbb{Z}_N$ dyons. Thus, in QCD there are demonstrably three regimes for the topological suceptibility.

Similarly, at low temperature and nonzero quark chemical potential, μ, it is natural to suggest that there are again three regimes for the topological susceptibility. While this regime is not accessible to classical computers because of the sign problem, eventually it will be measured using quantum computers. Nevertheless, we can at least speculate.

Because the number of degrees of freedom for quarks and gluons at nonzero temperature is so much greater than that at $T = 0$ and $\mu_{\rm qk} \neq 0$, estimates with a dilute instanton gas indicate that at zero temperature, instantons do not dominate until very high chemical potential, at least $\mu_{\rm qk} \sim 2$ GeV [18]. We note that this bound uses the incomplete result for the instanton density at zero temperature and $\mu_{\rm qk} \neq 0$; a better bound would follow from the full instanton density [21].

Consider then the opposite limit, moving up in the quark chemical potential. The chemical potential has no effect upon the free energy until it exceeds one-third the mass of the nucleon, minus the binding energy of nuclear matter, at something like $\mu_{\rm qk} \sim 300$ MeV. Assuming that this regime is like that for $\mu_{\rm qk} = 0$ and $T < T_\chi$, the topological susceptibility presumably varies little as μ increases. This should hold until chiral symmetry is restored at $\mu_{\rm qk} = \mu_\chi$, and the quarks are (essentially) massless. Then the topological susceptiblity should vary significantly, as the Fermi sea of massless quarks interacts strongly with a dense liquid of $\mathbb{Z}_N$ dyons. This includes both a chirally symmetric hadronic phase, a chirally symmetric quarkyonic phase [155,156], and perhaps even into the perturbative regime, for $\mu_{\rm qk} > 1$ GeV [157–159]. In the latter, for $1 \le \mu_{\rm qk} \le 2$ GeV, color superconductivity is dominant near the Fermi surface, but the effects of the axial anomaly can still affect the possible pairing mechanisms [160].

In summary, while the interactions between massless, dynamical quarks and a dense liquid of $\mathbb{Z}_N$ dyons dominate for $\mu_{\rm qk} = 0$ and the intermediate temperature region of $T_\chi \approx 156 \le T \le 300$ MeV, the analogous regime for $T = 0$ could be *much* broader, from $\mu_\chi \le \mu_{\rm qk} \le 2$ GeV.

This suggestion is obviously conjecture, and so we do not bother with considering how the entire phase diagram in T and $\mu_{\rm qk}$ might fill out. It does indicate that this phase diagram is exceedingly rich.

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APPENDIX: CRITICAL POINTS OF THE QUANTUM ACTION

In this appendix we discuss how topologically nontrivial configurations arise not as solutions to the classical equations of motion, but as critical points of the effective quantum action. Before discussing details of such configurations, it is useful to comment briefly on the role of such critical points. Denoting the fields generically by the symbol φ, with Minkowski signature the effective action $\Gamma(\Phi)$ is given by the functional integral

$$e^{i\Gamma(\Phi)} = \int [d\varphi] \exp\left(iS(\varphi + \Phi) - i \int \varphi \frac{\delta\Gamma}{\delta\Phi} \right) \quad \text{(A1)}$$

Consider then a solution Φ of $\frac{\delta\Gamma}{\delta\Phi} = 0$ with boundary values $\phi \to \Phi_1$ as $t \to -\infty$, Φ_2 as $t \to +\infty$. From Eq. (A1), $\Gamma(\Phi)$ evaluated on this solution is

$$e^{i\Gamma(\Phi)} = \int [d\varphi] \exp\left(iS(\varphi + \Phi)\right). \quad \text{(A2)}$$

Independently, we can see that the transition amplitude from a configuration Φ_1 at $t \to -\infty$ to Φ_2 at $t \to +\infty$ is

$$\begin{aligned} \langle \Phi_2 | \Phi_1 \rangle &= \langle \Phi_2 | e^{-iH(t-\tau)} | \Phi_1 \rangle \Big|_{t\to\infty, \tau\to-\infty} \\ &= \int [d\varphi] e^{iS(\varphi)} \Big|_{\varphi(t=-\infty)=\Phi_1, \varphi(t=\infty)=\Phi_2} \\ &= \int [d\varphi] e^{iS(\varphi+\Phi)}, \end{aligned} \quad \text{(A3)}$$

where we shift $\varphi \to \varphi + \Phi$ in the last line and integrate over φ's which vanish as $t \to \pm\infty$. The boundary values Φ_1, Φ_2 for $\varphi + \Phi$ are carried by Φ. Comparing Eqs. (A2) and (A3), we see that the solution of $\frac{\delta\Gamma}{\delta\Phi} = 0$ gives the transition amplitude for $\Phi_1 \to \Phi_2$. (This is essentially the result that the S-matrix is given by Γ evaluated on its critical points; here we are using φ-diagonal states, rather than specifying the incoming and outgoing states by spins and momenta of the particles.) By the same reasoning, by obtaining solutions with specific boundary behavior at $t \to \pm\infty$, or more generally, specific asymptotic behavior in spacetime, we get information about transition amplitudes. The analysis in text on the critical points of the effective action should be viewed with this interpretation in mind. We will be using Euclidean signature in this section, as is appropriate for tunneling transitions.

A related point, perhaps worth emphasizing, is that the solution Φ should not be interpreted as the vacuum expectation value of the quantum field. (Notice that, to get the vacuum-to-vacuum amplitude, one has to integrate Eq. (A3) over Φ_1, Φ_2 after taking the product with the vacuum wave functions $\Psi_0(\Phi_1)$ and $\Psi_0^*(\Phi_2)$. The configurations we discuss will have boundary behaviors which correspond to the vacuum in the $SU(N)/\mathbb{Z}_N$ theory, but are distinct before modding out by $\mathbb{Z}_N$.

[1] E. Witten, Current algebra theorems for the U(1) "Goldstone boson", Nucl. Phys. **B156**, 269 (1979).
[2] G. Veneziano, U(1) without instantons, Nucl. Phys. **B159**, 213 (1979).
[3] G. 't Hooft, How instantons solve the U(1) problem, Phys. Rep. **142**, 357 (1986).
[4] F. Giacosa, A. Koenigstein, and R. D. Pisarski, How the axial anomaly controls flavor mixing among mesons, Phys. Rev. D **97**, 091901 (2018).
[5] F. Giacosa, S. Jafarzade, and R. Pisarski, Anomalous interactions for heterochiral mesons with $J^{PC} = 1^{+-}$ and 2^{-+}, arXiv:2309.00086.
[6] G. Veneziano, Is there a QCD spin crisis?, Mod. Phys. Lett. A **04**, 1605 (1989).
[7] G. M. Shore and G. Veneziano, The U(1) Goldberger-Treiman relation and the two components of the proton "spin", Phys. Lett. B **244**, 75 (1990).
[8] G. M. Shore and G. Veneziano, The U(1) Goldberger-Treiman relation and the proton "spin": A renormalization group analysis, Nucl. Phys. **B381**, 23 (1992).
[9] G. M. Shore and G. Veneziano, Testing target independence of the 'proton spin' effect in semiinclusive deep inelastic scattering, Nucl. Phys. **B516**, 333 (1998).
[10] S. Narison, G. M. Shore, and G. Veneziano, Topological charge screening and the 'proton spin' beyond the chiral limit, Nucl. Phys. **B546**, 235 (1999).
[11] S. D. Bass, The spin structure of the proton, Rev. Mod. Phys. **77**, 1257 (2005).
[12] G. M. Shore, The U(1)(A) anomaly and QCD phenomenology, Lect. Notes Phys. **737**, 235 (2008).
[13] A. Tarasov and R. Venugopalan, Role of the chiral anomaly in polarized deeply inelastic scattering: Finding the triangle graph inside the box diagram in Bjorken and Regge asymptotics, Phys. Rev. D **102**, 114022 (2020).
[14] A. Tarasov and R. Venugopalan, Role of the chiral anomaly in polarized deeply inelastic scattering. II. Topological screening and transitions from emergent axionlike dynamics, Phys. Rev. D **105**, 014020 (2022).
[15] D. J. Gross, R. D. Pisarski, and L. G. Yaffe, QCD and instantons at finite temperature, Rev. Mod. Phys. **53**, 43 (1981).
[16] C. P. Korthals Altes and A. Sastre, Thermal instanton determinant in compact form, Phys. Rev. D **90**, 125002 (2014).
[17] C. P. Korthals Altes and A. Sastre, Caloron correction to the effective potential in thermal gluodynamics (2015).
[18] R. D. Pisarski and F. Rennecke, Multi-instanton contributions to anomalous quark interactions, Phys. Rev. D **101**, 114019 (2020), and references therein.
[19] F. Rennecke, Higher topological charge and the QCD vacuum, Phys. Rev. Res. **2**, 033359 (2020).
[20] A. Boccaletti and D. Nogradi, The semi-classical approximation at high temperature revisited, J. High Energy Phys. 03 (2020) 045.
[21] D. Nogradi, J. Papavasilliou, and R. D. Pisarski (to be published).
[22] The computation at $T \neq 0$ is complete to one loop order, while that at $\mu_{qk} \neq 0$ is lacking, although doable [21].
[23] For an $SU(N)$ gauge theory coupled to N_f flavors of massless quarks, $c = (11N - 4N_f C_f)/3$, $C_f = (N^2 - 1)/(2N)$. The factor of T^4 in Eq. (1) arises from the integral over the instanton scale size, ρ.
[24] S. Borsanyi, M. Dierigl, Z. Fodor, S. D. Katz, S. W. Mages, D. Nogradi, J. Redondo, A. Ringwald, and K. K. Szabo, Axion cosmology, lattice QCD and the dilute instanton gas, Phys. Lett. B **752**, 175 (2016).
[25] G. Grilli di Cortona, E. Hardy, J. Pardo Vega, and G. Villadoro, The QCD axion, precisely, J. High Energy Phys. 01 (2016) 034.
[26] C. Bonati, M. D'Elia, M. Mariti, G. Martinelli, M. Mesiti, F. Negro, F. Sanfilippo, and G. Villadoro, Axion phenomenology and θ-dependence from $N_f = 2 + 1$ lattice QCD, J. High Energy Phys. 03 (2016) 155.
[27] S. Borsanyi *et al.*, Calculation of the axion mass based on high-temperature lattice quantum chromodynamics, Nature (London) **539**, 69 (2016).
[28] J. Frison, R. Kitano, H. Matsufuru, S. Mori, and N. Yamada, Topological susceptibility at high temperature on the lattice, J. High Energy Phys. 09 (2016) 021.
[29] P. Petreczky, H.-P. Schadler, and S. Sharma, The topological susceptibility in finite temperature QCD and axion cosmology, Phys. Lett. B **762**, 498 (2016).
[30] Y. Taniguchi, K. Kanaya, H. Suzuki, and T. Umeda, Topological susceptibility in finite temperature (2 + 1)-flavor QCD using gradient flow, Phys. Rev. D **95**, 054502 (2017).
[31] M. P. Lombardo and A. Trunin, Topology and axions in QCD, Int. J. Mod. Phys. A **35**, 2030010 (2020).
[32] P. T. Jahn, P. M. Junnarkar, G. D. Moore, and D. Robaina, Multicanonical reweighting for the QCD topological susceptibility, Phys. Rev. D **104**, 014502 (2021).
[33] S. Borsanyi and D. Sexty, Topological susceptibility of pure gauge theory using density of states, Phys. Lett. B **815**, 136148 (2021).

[34] Y.-C. Chen, T.-W. Chiu, and T.-H. Hsieh, Topological susceptibility in finite temperature QCD with physical $(u/d, s, c)$ domain-wall quarks, Phys. Rev. D **106**, 074501 (2022).

[35] The overall magnitude of the topological susceptibility is about an order of magnitude greater than the instanton result at one loop order, but surely it is necessary to include the corrections to the instanton density at two loop order for $T \neq 0$. Multi-instanton configurations can also contribute at low T [19].

[36] D. Gaiotto, A. Kapustin, N. Seiberg, and B. Willett, Generalized global symmetries, J. High Energy Phys. 02 (2015) 172.

[37] B. Alles, M. D'Elia, and A. Di Giacomo, Topological susceptibility at zero and finite T in SU(3) Yang-Mills theory, Nucl. Phys. **B494**, 281 (1997); Nucl. Phys. **B679**, 397(E) (2004).

[38] S. Durr, Z. Fodor, C. Hoelbling, and T. Kurth, Precision study of the SU(3) topological susceptibility in the continuum, J. High Energy Phys. 04 (2007) 055.

[39] M. Luscher and F. Palombi, Universality of the topological susceptibility in the SU(3) gauge theory, J. High Energy Phys. 09 (2010) 110.

[40] G.-Y. Xiong, J.-B. Zhang, Y. Chen, C. Liu, Y.-B. Liu, and J.-P. Ma, Topological susceptibility near T_c in SU(3) gauge theory, Phys. Lett. B **752**, 34 (2016).

[41] P. T. Jahn, G. D. Moore, and D. Robaina, $\chi_{\mathrm{top}}(T \gg T_{\mathrm{c}})$ in pure-glue QCD through reweighting, Phys. Rev. D **98**, 054512 (2018).

[42] L. Giusti and M. Lüscher, Topological susceptibility at $T > T_{\mathrm{c}}$ from master-field simulations of the SU(3) gauge theory, Eur. Phys. J. C **79**, 207 (2019).

[43] B. Lucini and M. Teper, SU(N) gauge theories in four-dimensions: Exploring the approach to N = infinity, J. High Energy Phys. 06 (**2001**) 050.

[44] L. Del Debbio, G. M. Manca, H. Panagopoulos, A. Skouroupathis, and E. Vicari, Theta-dependence of the spectrum of SU(N) gauge theories, J. High Energy Phys. 06 (2006) 005.

[45] E. Vicari and H. Panagopoulos, Theta dependence of SU(N) gauge theories in the presence of a topological term, Phys. Rep. **470**, 93 (2009).

[46] M. Garcia Perez, A. Gonzalez-Arroyo, and A. Sastre, Adjoint fermion zero-modes for SU(N) calorons, J. High Energy Phys. 06 (2009) 065.

[47] Z. Fodor, K. Holland, J. Kuti, D. Nogradi, and C. Schroeder, Topology and higher dimensional representations, J. High Energy Phys. 08 (2009) 084.

[48] Z. Fodor, K. Holland, J. Kuti, D. Nogradi, and C. Schroeder, Chiral properties of SU(3) sextet fermions, J. High Energy Phys. 11 (**2009**) 103.

[49] H. Panagopoulos and E. Vicari, The 4D SU(3) gauge theory with an imaginary θ term, J. High Energy Phys. 11 (2011) 119.

[50] B. Lucini and M. Panero, SU(N) gauge theories at large N, Phys. Rep. **526**, 93 (2013).

[51] C. Bonati, M. D'Elia, H. Panagopoulos, and E. Vicari, Change of θ Dependence in 4D SU(N) Gauge Theories Across the Deconfinement Transition, Phys. Rev. Lett. **110**, 252003 (2013).

[52] C. Bonati, M. D'Elia, and A. Scapellato, θ dependence in $SU(3)$ Yang-Mills theory from analytic continuation, Phys. Rev. D **93**, 025028 (2016).

[53] M. Cè, M. García Vera, L. Giusti, and S. Schaefer, The topological susceptibility in the large-N limit of SU(N) Yang–Mills theory, Phys. Lett. B **762**, 232 (2016).

[54] C. Bonati, M. D'Elia, P. Rossi, and E. Vicari, θ dependence of 4D $SU(N)$ gauge theories in the large-N limit, Phys. Rev. D **94**, 085017 (2016).

[55] R. Kitano, T. Suyama, and N. Yamada, $\theta = \pi$ in $SU(N)/\mathbb{Z}_N$ gauge theories, J. High Energy Phys. 09 (2017) 137.

[56] E. Itou, Fractional instanton of the SU(3) gauge theory in weak coupling regime, J. High Energy Phys. 05 (2019) 093.

[57] C. Bonanno, C. Bonati, and M. D'Elia, Topological properties of CP^{N-1} models in the large-N limit, J. High Energy Phys. 01 (2019) 003.

[58] C. Bonati, M. Cardinali, and M. D'Elia, θ dependence in trace deformed $SU(3)$ Yang-Mills theory: A lattice study, Phys. Rev. D **98**, 054508 (2018).

[59] C. Bonati, M. Cardinali, M. D'Elia, and F. Mazziotti, θ-dependence and center symmetry in Yang-Mills theories, Phys. Rev. D **101**, 034508 (2020).

[60] C. Bonanno, C. Bonati, and M. D'Elia, Large-N $SU(N)$ Yang-Mills theories with milder topological freezing, J. High Energy Phys. 03 (**2021**) 111.

[61] R. Kitano, R. Matsudo, N. Yamada, and M. Yamazaki, Peeking into the θ vacuum, Phys. Lett. B **822**, 136657 (2021).

[62] E. Bennett, D. K. Hong, J.-W. Lee, C. J. D. Lin, B. Lucini, M. Piai, and D. Vadacchino, Color dependence of the topological susceptibility in Yang-Mills theories, Phys. Lett. B **835**, 137504 (2022).

[63] G. Boyd, J. Engels, F. Karsch, E. Laermann, C. Legeland, M. Lutgemeier, and B. Petersson, Thermodynamics of SU(3) lattice gauge theory, Nucl. Phys. **B469**, 419 (1996).

[64] S. Borsanyi, G. Endrodi, Z. Fodor, S. D. Katz, and K. K. Szabo, Precision SU(3) lattice thermodynamics for a large temperature range, J. High Energy Phys. 07 (2012) 056.

[65] M. Shirogane, S. Ejiri, R. Iwami, K. Kanaya, and M. Kitazawa, Latent heat at the first order phase transition point of SU(3) gauge theory, Phys. Rev. D **94**, 014506 (2016).

[66] M. Caselle, A. Nada, and M. Panero, QCD thermodynamics from lattice calculations with nonequilibrium methods: The SU(3) equation of state, Phys. Rev. D **98**, 054513 (2018).

[67] S. Borsanyi, Z. Fodor, D. A. Godzieba, R. Kara, P. Parotto, and D. Sexty, Precision study of the continuum SU(3) Yang-Mills theory: How to use parallel tempering to improve on supercritical slowing down for first order phase transitions, Phys. Rev. D **105**, 074513 (2022).

[68] There could be a dense liquid or crystal of instantons [69,70]. However, as noted by Witten [71], for the topological susceptibility not to be exponentially suppressed in N, it is necessary to show that the action of such a liquid or crystal vanishes at $\sim N$, and so instead is ~ 1. Consider how this could occur semi-classically. The effective action for a classical instanton of size ρ is $\sim N$, and is a power series in $\lambda = g^2 N$:

$$\mathcal{S}_{\mathrm{eff}}(\rho\Lambda) = N \frac{8\pi^2}{\lambda}(1 + \lambda \log(\rho\Lambda) + \cdots),$$

up to terms $\sim N^0$, where Λ is the renormalization mass scale. At large N, one minimizes the effective action with respect to ρ, with the dominant configuration an instanton of a single scale size, ρ_∞. This is not sufficient, though: It is necessary that the effective action at this scale size *vanishes*, with $\mathcal{S}_{\rm eff}(\rho_\infty \Lambda) = 0$, up to corrections ~ 1. It is not obvious how this might come about.

[69] G. W. Carter and E. V. Shuryak, Do instantons and strings cluster when the number of colors is large?, Phys. Lett. B **524**, 297 (2002).

[70] Y. Liu, E. Shuryak, and I. Zahed, Dense instanton-dyon liquid model: Diagrammatics, Phys. Rev. D **98**, 014023 (2018).

[71] E. Witten, Instantons, the quark model, and the 1/N expansion, Nucl. Phys. **B149**, 285 (1979).

[72] E. Witten, Theta Dependence in the Large N Limit of Four-Dimensional Gauge Theories, Phys. Rev. Lett. **81**, 2862 (1998).

[73] G. 't Hooft, Confinement and topology in non-Abelian gauge theories, Acta Phys. Austriaca Suppl. **22**, 531 (1980).

[74] G. 't Hooft, Some twisted self-dual solutions for the Yang-Mills equations on a hypertorus, Commun. Math. Phys. **81**, 267 (1981).

[75] P. van Baal, Some results for SU(N) gauge fields on the hypertorus, Commun. Math. Phys. **85**, 529 (1982).

[76] S. Sedlacek, A direct method for minimizing the Yang-Mills functional over four manifolds, Commun. Math. Phys. **86**, 515 (1982).

[77] C. Nash, Gauge potentials and bundles over the four torus, Commun. Math. Phys. **88**, 319 (1983).

[78] T. P. Killingback, The Gribov ambiguity in gauge theories on the 4 torus, Phys. Lett. B **138**, 87 (1984).

[79] A. Gonzalez-Arroyo and P. Martinez, Investigating Yang-Mills theory and confinement as a function of the spatial volume, Nucl. Phys. **B459**, 337 (1996).

[80] A. Gonzalez-Arroyo, P. Martinez, and A. Montero, Gauge invariant structures and confinement, Phys. Lett. B **359**, 159 (1995).

[81] A. Gonzalez-Arroyo and A. Montero, Do classical configurations produce confinement?, Phys. Lett. B **387**, 823 (1996).

[82] M. Garcia Perez, A. Gonzalez-Arroyo, and C. Pena, Perturbative construction of self-dual configurations on the torus, J. High Energy Phys. 09 (2000) 033.

[83] A. González-Arroyo, Constructing SU(N) fractional instantons, J. High Energy Phys. 02 (2020) 137.

[84] J. Dasilva Golán and M. García Pérez, SU(N) fractional instantons and the Fibonacci sequence, J. High Energy Phys. 12 (2022) 109.

[85] A. Gonzalez-Arroyo, On the fractional instanton liquid picture of the Yang-Mills vacuum and confinement, arXiv:2302.12356.

[86] K.-M. Lee and P. Yi, Monopoles and instantons on partially compactified D-branes, Phys. Rev. D **56**, 3711 (1997).

[87] K.-M. Lee, Instantons and magnetic monopoles on $R^3 \times S^1$ with arbitrary simple gauge groups, Phys. Lett. B **426**, 323 (1998).

[88] K.-M. Lee and C. Lu, SU(2) calorons and magnetic monopoles, Phys. Rev. D **58**, 025011 (1998).

[89] T. C. Kraan and P. van Baal, Periodic instantons with nontrivial holonomy, Nucl. Phys. **B533**, 627 (1998).

[90] T. C. Kraan and P. van Baal, Monopole constituents inside SU(n) calorons, Phys. Lett. B **435**, 389 (1998).

[91] M. Garcia Perez, A. Gonzalez-Arroyo, A. Montero, and P. van Baal, Calorons on the lattice: A new perspective, J. High Energy Phys. 06 (**1999**) 001.

[92] D. Diakonov, Instantons at work, Prog. Part. Nucl. Phys. **51**, 173 (2003).

[93] M. Eto, Y. Isozumi, M. Nitta, K. Ohashi, and N. Sakai, Instantons in the Higgs phase, Phys. Rev. D **72**, 025011 (2005).

[94] M. Eto, Y. Isozumi, M. Nitta, K. Ohashi, and N. Sakai, Solitons in the Higgs phase: The moduli matrix approach, J. Phys. A **39**, R315 (2006).

[95] M. Eto, T. Fujimori, Y. Isozumi, M. Nitta, K. Ohashi, K. Ohta, and N. Sakai, Non-Abelian vortices on cylinder: Duality between vortices and walls, Phys. Rev. D **73**, 085008 (2006).

[96] F. Bruckmann, S. Dinter, E.-M. Ilgenfritz, M. Muller-Preussker, and M. Wagner, Cautionary remarks on the moduli space metric for multi-dyon simulations, Phys. Rev. D **79**, 116007 (2009).

[97] D. Diakonov, Topology and confinement, Nucl. Phys. B, Proc. Suppl. **195**, 5 (2009).

[98] E. Poppitz and M. Unsal, Index theorem for topological excitations on $R^3 \times S^1$ and Chern-Simons theory, J. High Energy Phys. 03 (2009) 027.

[99] M. M. Anber and E. Poppitz, Nonperturbative effects in the standard model with gauged 1-form symmetry, J. High Energy Phys. 12 (2021) 055.

[100] A. M. Polyakov, Quark confinement and topology of gauge groups, Nucl. Phys. **B120**, 429 (1977).

[101] G. V. Dunne, I. I. Kogan, A. Kovner, and B. Tekin, Deconfining phase transition in $(2+1)$-dimensions: The Georgi-Glashow model, J. High Energy Phys. 01 (2001) 032.

[102] I. I. Kogan and A. Kovner, Monopoles, vortices and strings: Confinement and deconfinement in $(2+1)$-dimensions at weak coupling, in *At the Frontier of Particle Physics* (World Scientific, Singapore, 2022).

[103] Y. V. Kovchegov and D. T. Son, Critical temperature of the deconfining phase transition in $(2+1)$-d Georgi-Glashow model, J. High Energy Phys. 01 (2003) 050.

[104] M. Unsal and L. G. Yaffe, (In)validity of large N orientifold equivalence, Phys. Rev. D **74**, 105019 (2006).

[105] M. Unsal, Abelian Duality, Confinement, and Chiral Symmetry Breaking in SU(2) QCD-Like Theory, Phys. Rev. Lett. **100**, 032005 (2008).

[106] M. Unsal, Magnetic bion condensation: A new mechanism of confinement and mass gap in four dimensions, Phys. Rev. D **80**, 065001 (2009).

[107] M. Unsal and L. G. Yaffe, Center-stabilized Yang-Mills theory: Confinement and large N volume independence, Phys. Rev. D **78**, 065035 (2008).

[108] M. Shifman and M. Unsal, QCD-like theories on $R^3 \times S^1$: A smooth journey from small to large r(S(1)) with double-trace deformations, Phys. Rev. D **78**, 065004 (2008).

[109] D. Simic and M. Unsal, Deconfinement in Yang-Mills theory through toroidal compactification with deformation, Phys. Rev. D **85**, 105027 (2012).

[110] M. M. Anber, E. Poppitz, and M. Unsal, 2d affine XY-spin model/4d gauge theory duality and deconfinement, J. High Energy Phys. 04 (2012) 040.

[111] E. Poppitz, T. Schäfer, and M. Unsal, Continuity, deconfinement, and (super) Yang-Mills theory, J. High Energy Phys. 10 (2012) 115.

[112] M. M. Anber, S. Collier, E. Poppitz, S. Strimas-Mackey, and B. Teeple, Deconfinement in $\mathcal{N}=1$ super Yang-Mills theory on $\mathbb{R}^3 \times \mathbb{S}^1$ via dual-Coulomb gas and "affine" XY-model, J. High Energy Phys. 11 (2013) 142.

[113] K. Aitken, A. Cherman, E. Poppitz, and L. G. Yaffe, QCD on a small circle, Phys. Rev. D **96**, 096022 (2017).

[114] M. Ünsal, Strongly coupled QFT dynamics via TQFT coupling, J. High Energy Phys. 11 (2021) 134.

[115] E. Poppitz, Notes on confinement on $R^3 \times S^1$: From Yang-Mills, Super-Yang-Mills, and QCD(adj) to QCD(F), Symmetry **14**, 180 (2022).

[116] Y. Tanizaki and M. Ünsal, Center vortex and confinement in Yang-Mills theory and QCD with anomaly-preserving compactifications, Prog. Theor. Exp. Phys. **2022**, 04A108 (2022).

[117] D. J. Gross, Meron configurations in the two-dimensional O(3) sigma model, Nucl. Phys. **B132**, 439 (1978).

[118] A. D'Adda, M. Luscher, and P. Di Vecchia, A 1n expandable series of nonlinear sigma models with instantons, Nucl. Phys. **B146**, 63 (1978).

[119] P. Di Vecchia, An effective Lagrangian with no U(1) problem in CP$^{(n-1)}$ models and QCD, Phys. Lett. **85B**, 357 (1979).

[120] B. Berg and M. Luscher, Computation of quantum fluctuations around multi-instanton fields from exact Green's functions: The CP$^{(n-1)}$ case, Commun. Math. Phys. **69**, 57 (1979).

[121] V. A. Fateev, I. V. Frolov, and A. S. Schwarz, Quantum fluctuations of instantons in the nonlinear σ model, Nucl. Phys. **B154**, 1 (1979).

[122] K. D. Rothe and J. A. Swieca, Fractional winding numbers and the $u(1)$ problem, Nucl. Phys. **B168**, 454 (1980).

[123] A. R. Zhitnitsky, Fractional topological charge, torons and breaking of discrete chiral symmetry in the supersymmetric O(3) σ model, Sov. Phys. JETP **67**, 1095 (1988).

[124] S. Ahmad, J. T. Lenaghan, and H. B. Thacker, Coherent topological charge structure in CP^{N-1} models and QCD, Phys. Rev. D **72**, 114511 (2005).

[125] Y. Lian and H. B. Thacker, Small instantons in CP^1 and CP^2 sigma models, Phys. Rev. D **75**, 065031 (2007).

[126] F. Bruckmann, Instanton Constituents in the O(3) Model at Finite Temperature, Phys. Rev. Lett. **100**, 051602 (2008).

[127] W. Brendel, F. Bruckmann, L. Janssen, A. Wipf, and C. Wozar, Instanton constituents and fermionic zero modes in twisted CP^n models, Phys. Lett. B **676**, 116 (2009).

[128] M. F. Atiyah, N. J. Hitchin, V. G. Drinfeld, and Y. I. Manin, Construction of Instantons, Phys. Lett. A **65**, 185 (1978).

[129] R. G. Edwards, U. M. Heller, and R. Narayanan, Evidence for fractional topological charge in SU(2) pure Yang-Mills theory, Phys. Lett. B **438**, 96 (1998).

[130] J. C. Biddle, W. Kamleh, and D. B. Leinweber, Impact of dynamical fermions on the center vortex gluon propagator, Phys. Rev. D **106**, 014506 (2022).

[131] J. C. Biddle, W. Kamleh, and D. B. Leinweber, Static quark potential from center vortices in the presence of dynamical fermions, Phys. Rev. D **106**, 054505 (2022).

[132] D. Leinweber, J. Biddle, and W. Kamleh, Centre vortex structure of QCD-vacuum fields and confinement, SciPost Phys. Proc. **6**, 004 (2022).

[133] J. C. Biddle, W. Kamleh, and D. B. Leinweber, Centre vortex structure in the presence of dynamical fermions, Phys. Rev. D **107**, 094507 (2023).

[134] Notice that the constraint $\bar{z} \cdot z = 1$ defines a sphere S^{2N-1}, while the gauge symmetry removes a phase, i.e., S^1, which leaves S^{2N-1}/S^1. This is another way to define $\mathbb{CP}^{N-1}$.

[135] R. Jackiw, Gauge theories in the momentum/curvature representation, arXiv:hep-th/9604040.

[136] P. Goddard, J. Nuyts, and D. I. Olive, Gauge theories and magnetic charge, Nucl. Phys. **B125**, 1 (1977).

[137] P. H. Ginsparg and K. G. Wilson, A remnant of chiral symmetry on the lattice, Phys. Rev. D **25**, 2649 (1982).

[138] D. B. Kaplan, A method for simulating chiral fermions on the lattice, Phys. Lett. B **288**, 342 (1992).

[139] R. Narayanan and H. Neuberger, Infinitely many regulator fields for chiral fermions, Phys. Lett. B **302**, 62 (1993).

[140] R. Narayanan and H. Neuberger, Chiral determinant as an overlap of two vacua, Nucl. Phys. **B412**, 574 (1994).

[141] R. Narayanan and H. Neuberger, Chiral Fermions on the Lattice, Phys. Rev. Lett. **71**, 3251 (1993).

[142] R. Narayanan and H. Neuberger, A construction of lattice chiral gauge theories, Nucl. Phys. **B443**, 305 (1995).

[143] M. Luscher, Exact chiral symmetry on the lattice and the Ginsparg-Wilson relation, Phys. Lett. B **428**, 342 (1998).

[144] Fodor *et al.* performed simulations in a $SU(3)$ gauge theory using an external quark propagator in the sextet representation [47]. As they discuss, the sextet representation is sensitive to the presence of objects with topological charge $1/5$, and for which they see no evidence. However, this does not exclude the appearance of objects with charge $1/3$.

[145] K. D. Rothe and J. A. Swieca, On the local structure of topological charge fluctuations in QCD, Phys. Rev. D **67**, 011501 (2003).

[146] I. Horvath *et al.*, Inherently global nature of topological charge fluctuations in QCD, Phys. Lett. B **612**, 21 (2005).

[147] H. B. Thacker, Tachyonic crystals and the laminar instability of the perturbative vacuum in asymptotically free gauge theories, Phys. Rev. D **81**, 125006 (2010).

[148] H. B. Thacker, C. Xiong, and A. Kamat, Chiral quark dynamics and topological charge: The role of the Ramond-Ramond U(1) gauge field in holographic QCD, Phys. Rev. D **84**, 105011 (2011).

[149] A. Alexandru and I. Horváth, Unusual Features of QCD Low-Energy Modes in the Infrared Phase, Phys. Rev. Lett. **127**, 052303 (2021).

[150] We also comment that by looking at the spectra of higher spin mesons, the region between $300 \leq T \leq 600$ MeV is still far from a perturbative quark-gluon plasma [151].

[151] L. Y. Glozman, O. Philipsen, and R. D. Pisarski, Chiral spin symmetry and the QCD phase diagram, Eur. Phys. J. A **58**, 247 (2022).

[152] A. Bazavov *et al.* (HotQCD Collaboration), Chiral crossover in QCD at zero and non-zero chemical potentials, Phys. Lett. B **795**, 15 (2019).

[153] S. Borsanyi, Z. Fodor, J. N. Guenther, R. Kara, S. D. Katz, P. Parotto, A. Pasztor, C. Ratti, and K. K. Szabo, QCD Crossover at Finite Chemical Potential from Lattice Simulations, Phys. Rev. Lett. **125**, 052001 (2020).

[154] J. N. Guenther, An overview of the QCD phase diagram at finite T and μ, Proc. Sci. LATTICE2021 (**2022**) 013.

[155] L. McLerran and R. D. Pisarski, Phases of cold, dense quarks at large N_c, Nucl. Phys. **A796**, 83 (2007).

[156] M. Lajer, R. M. Konik, R. D. Pisarski, and A. M. Tsvelik, When cold, dense quarks in $1+1$ and $3+1$ dimensions are not a Fermi liquid, Phys. Rev. D **105**, 054035 (2022).

[157] T. Gorda, A. Kurkela, P. Romatschke, M. Säppi, and A. Vuorinen, Next-to-Next-to-Next-to-Leading Order Pressure of Cold Quark Matter: Leading Logarithm, Phys. Rev. Lett. **121**, 202701 (2018).

[158] T. Gorda, A. Kurkela, R. Paatelainen, S. Säppi, and A. Vuorinen, Soft Interactions in Cold Quark Matter, Phys. Rev. Lett. **127**, 162003 (2021).

[159] T. Gorda, A. Kurkela, R. Paatelainen, S. Säppi, and A. Vuorinen, Cold quark matter at N^3LO: Soft contributions, Phys. Rev. D **104**, 074015 (2021).

[160] R. D. Pisarski, Critical line for H superfluidity in strange quark matter?, Phys. Rev. C **62**, 035202 (2000).