Linearized second law of black hole thermodynamics for gravitational theory with higher-order interactions of matter fields

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In the gravitational theory containing the second-order interaction of gravity with the scalar and electromagnetic fields at most, the black hole entropy obeying the linearized second law of black hole thermodynamics has been obtained. To generally study the linearized second law of black hole thermodynamics, we extend the gravitational theory to include higher-order interactions of gravity with the scalar and electromagnetic fields. We derive the general expression of black hole entropy satisfying the linearized second law, which is expressed as the Wald entropy with correction terms. It is worth noting that the correction terms consist of both the minimal and nonminimal coupling interactions between gravity and the scalar field, and the contribution of the electromagnetic field is not involved. Since the black hole entropy satisfying the linearized second law is determined only by the nonminimum coupling interactions in gravity according to the previous perspective, this result upends our understanding of the linearized second law in general gravitational theory.

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I. INTRODUCTION

Black holes as particular spacetime structures have been predicted by general relativity. For black holes in classical general relativity, the singularity lies at the center of black holes and is surrounded by a specific null hypersurface. The null hypersurface is called the event horizon and is regarded as the boundary of black holes. The event horizon plays a critical role in the investigation of black hole physics because many essential properties of black holes, especially black hole thermodynamics, are reflected by the event horizon in an equilibrium state or a state of dynamic evolution. The area law of black holes was suggested first by Hawking [1], which states that the area of the event horizon of black holes never decreases along the direction of time evolution. Based on the area law of black holes, Bekenstein [2] proposed that the area of the event horizon of black holes could be identified with the entropy of the classical adiabatic thermodynamic system directly because the evolution tendency of the area of the event horizon is similar to the evolution of the entropy constrained by the second law of thermodynamics. Utilizing the quantum field theory in curved spacetime, Hawking [3] first proved that the entropy and the temperature of black holes are defined, respectively, by the area of the event horizon and the surface gravity of black holes. The entropy and the area of the event horizon of black holes satisfy a simple proportional relationship, i.e., $S_{\mathrm{BH}} = A/4$, where A is the area of the event horizon of black holes. The black hole entropy satisfying the proportional relationship is called Bekenstein-Hawking entropy. According to the definitions of the temperature and the entropy of black holes, the four laws of black hole thermodynamics are established [4–6]. Since the four laws of black hole thermodynamics are identical to the four laws of thermodynamics satisfied by the classical thermodynamic system, black holes can be regarded as thermodynamic systems rather than pure spacetime structures. In the four laws of black hole thermodynamics, the two profound laws for black holes are the first and second laws of black hole thermodynamics, respectively. If black holes are regarded as thermodynamic systems, these two laws of black hole thermodynamics should be seen first as robust features of black holes. Although black holes in classical general relativity automatically satisfy the two laws of black hole thermodynamics, the two laws are not necessarily guaranteed to hold for black holes in an arbitrary diffeomorphism invariant gravitational theory. An issue of whether the first and second laws of black hole thermodynamics are still the best features of black holes in general gravitational theory is raised naturally. Starting with this issue, Wald and Iyer [7,8] proposed the Noether charge method to investigate black hole thermodynamics in general diffeomorphism invariant gravitational theory. The result shows that the first law of black hole thermodynamics is generally suitable for black holes in an arbitrary gravitational theory and that the entropy always obeying the first law of black hole thermodynamics is called the Wald entropy rather than Bekenstein-Hawking entropy. Although it has been

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demonstrated that the Wald entropy is generally suitable for the first law of black hole thermodynamics in general diffeomorphism invariant gravitational theory, whether it always satisfies the second law of black hole thermodynamics has not been investigated enough. Therefore, we will mainly discuss the matching relationship between Wald entropy and the second law of thermodynamics.

So far, a suitable scheme to quantize the gravitational theory has not been established faultlessly. If the self-interactions of gravity or the interactions between gravity and matter fields are included in the gravitational theory, the two categories of interactions are not studied rigorously under the complete quantum regime. One of the most fruitful methods is to construct the low-energy efficient gravitational theory corresponding to quantum gravity to investigate the two categories of interactions in the frame of quantum gravity approximately. This method introduces higher curvature terms and minimal or nonminimal coupling interaction terms between gravity and matter fields essentially, which correspond to two categories of interactions and are called the quantum correction terms, into the effective Lagrangian of the quantum gravity theory at a low-energy scale. In other words, when the low-energy efficient scheme is adopted to deal with the issue of the quantization of gravity, some quantum correction terms, which are the higher curvature terms and the minimal or nonminimal coupling interaction terms between gravity and matter fields, should be added to the Lagrangian of the gravitational theory [9–11].

The expression of the Wald entropy is closely dependent on the Lagrangian of the gravitational theory according to the definition. When the quantum correction terms that correspond to the two categories of interactions are included in the Lagrangian, the nonminimal coupling interaction terms in these quantum correction terms will sufficiently influence the expression of the Wald entropy. We can infer that a substantial change in the specific expression of the Wald entropy will inevitably affect the matching relationship between the Wald entropy and the second law of black hole thermodynamics. Therefore, to investigate whether the Wald entropy generally satisfies the second law of black hole thermodynamics in any diffeomorphism invariant gravity, one should first consider the effect of each quantum correction term describing nonminimal coupled interactions in the Lagrangian of the gravitational theory on the matching relationship between the Wald entropy and the second law before finding a general research method. For the gravitational theory containing the first category of interactions, only higher curvature terms in quantum correction terms are involved in the Lagrangian. Considering a perturbation process caused by matter fields in spacetime, the second law of black hole thermodynamics under the linear-order approximation, called the linearized second law in the following, in the Gauss-Bonnet gravity and the Lovelock gravity are investigated [12–14]. The results show that the entropy of black holes that obeys the linearized second law in two gravitational theories is the Jacobson-Myers entropy rather than the Wald entropy. Subsequently, a general research technique that can investigate the linearized second law in the higher curvature gravity is proposed by Wall [15]. The expression of the black hole entropy that always satisfies the linearized second law in the gravitational theory with partial higher curvature terms is further obtained, which can be written as the Wald entropy with correction terms. It indicates that the Wald entropy does not always obey the linearized second law for an arbitrary higher curvature gravity. When the gravitational theory contains the second category of interactions, the minimal and nonminimal coupling terms between gravity and matter fields in quantum correction terms appear in the Lagrangian. The linearized second law has not been investigated enough in this theory of gravity, and the general expression of the black hole entropy that satisfies the linearized second law has not been obtained yet. In previous works, we have mainly focused on the linearized second law in gravity with the second category of interactions. For the Horndeski gravity and the general quadric corrected Einstein-Maxwell gravity, we have shown that the Wald entropy obeys the linearized second law of thermodynamics during the matter field perturbation [16,17]. However, when investigating the linearized second law in the general second-order scalar-tensor gravity, we found that the evolution of Wald entropy during the perturbation process no longer satisfies the requirements of the linearized second law. The expression of black hole entropy obeying the linearized second law can also be written as the Wald entropy with relevant correction terms [18]. According to this result, we can infer that the Wald entropy also does not always obey the linearized second law for an arbitrary diffeomorphism invariant gravitational theory with the second category of interactions. Although the linearized second law in the Horndeski gravity, the general quadric corrected Einstein-Maxwell gravity, and the general second-order scalar-tensor gravity have been investigated in previous works, these three gravitational theories involve the second-order interactions of gravity with matter fields at most. To adequately study the linearized second law in general diffeomorphism invariant gravity, one should further consider the higher-order interactions between gravity and matter fields in the gravitational theory.

The scalar field in gravitational theory has been a topic of great interest in recent years because the scalar field dynamics can help us understand some detailed features of the Universe. From an empirical motivation mainly related to astronomical observations, the scalar field can be regarded as a powerful tool to explain many phenomena at the Galactic and cosmological scales. It means that the gravitational theories incorporating scalar fields may help us understand these phenomena, such as the origin of the early Universe and its late-time accelerated expansion, as well as the presence of dark matter and dark energy [19]. Meanwhile, the properties of these phenomena also confirm that the scalar field is a suitable candidate to solve these unknown phenomena in the Universe. Therefore, many gravitational theories containing the scalar field, such as Brans-Dicke theory [20], inflation theory, and several other cosmological models [21–24], are gradually established. Additionally, from the perspective of astronomical observations again, many celestial bodies in our Universe commonly take electric charge, and outside spacetime fills with the electromagnetic field rather than the vacuum. It implies that the electromagnetic field should be considered in the theory of gravity. Therefore, according to the above facts and gravitational theories used to study the linearized second law in our previous works, we will further consider a more general gravity, which contains partial higher curvature terms, the scalar field with its derivatives, and the electromagnetic field. It means that the gravitational theory includes higher-order interactions of gravity with the scalar and electromagnetic fields rather than only limiting to second-order interactions of gravity with the scalar and electromagnetic fields. In the following, this gravitational theory is abbreviated as the gravitational theory with higher-order interactions for simplification. From the gravitational theory, we will investigate the linearized second law of black holes and derive the general expression of the black hole entropy always obeying the linearized second law during the matter fields perturbation process.

The organization of the paper is as follows. In Sec. II, the gravitational theory with higher-order interactions is introduced, and the definition of the Wald entropy of black holes is given. In Sec. III, a perturbation process is further considered, which comes from the additional matter fields outside black holes in spacetime, to investigate the linearized second law. From the Wald entropy of black holes in the gravitational theory with higher-order interactions, based on the assumptions that the matter fields should satisfy the null energy condition and that a regular bifurcation surface exists in the background spacetime, we will derive the expression of black hole entropy commonly obeying the linearized second law in the gravitational theory during the perturbation process. The paper ends with discussions and conclusions in Sec. IV.

II. GRAVITATIONAL THEORY WITH HIGHER-ORDER INTERACTIONS AND THE WALD ENTROPY

We will consider a diffeomorphism invariant gravitational theory with higher-order interactions to investigate the linearized second law of black hole thermodynamics and derive the expression of the black hole entropy commonly satisfying the second law in the gravitational theory. The Lagrangian of the diffeomorphism invariant gravitational theory can be expressed formally as

$$\mathcal{L} = \mathcal{L}(g_{ab}, R_{abcd}, F_{ab}, \phi, \nabla_a\phi, \nabla_a\nabla_b\phi). \tag{1}$$

To investigate the linearized second law in this gravitational theory with higher-order interactions, the additional matter fields, which are minimal coupling with gravity, in spacetime should be introduced. Moreover, a quasistationary process is further involved, which states that the matter fields existing outside black holes pass through the event horizon and fall into the interior of black holes. The spacetime configuration of black holes can be perturbed through the matter fields during the process. This point of view indicates that the matter fields and the spacetime of black holes can be regarded as a complete dynamical system, and the perturbation process is an evolutionary process of the dynamical system. Therefore, when the additional matter fields appear in spacetime, the Lagrangian of the gravitational theory with higher-order interactions can be expanded as

$$\mathcal{L} = \mathcal{L}(g_{ab}, R_{abcd}, F_{ab}, \phi, \nabla_a\phi, \nabla_a\nabla_b\phi) + \mathcal{L}_{\rm mt}, \tag{2}$$

where $\mathcal{L}_{\rm mt}$ represents the Lagrangian of the additional matter fields in spacetime. After calculating the variation of the expanded Lagrangian with respect to the metric g_{ab}, the equation of motion of the gravitational part can be formally expressed as

$$H_{ab} = 8\pi T_{ab}. \tag{3}$$

The left-hand side of Eq. (3) can be further written as a linear combination of four components,

$$H_{ab} = H^1_{ab} + H^2_{ab} + H^3_{ab} + H^4_{ab}. \tag{4}$$

The first component corresponds to the derivative of the Lagrangian to the Riemann curvature R_{abcd}. The second component comes from the derivative of the Lagrangian to the first-order covariant derivative of the scalar field $\nabla_a\phi$. The third component is derived from the derivation of the Lagrangian to the second-order covariant derivative of the scalar field $\nabla_a\nabla_b\phi$. The fourth component is the derivative of the Lagrangian to the electromagnetic field F_{ab}. The specific expressions of four terms on the right-hand side of Eq. (4) are expressed, respectively, as

$$\begin{aligned}
H^1_{ab} &= (E_R)_a^{\ cde} R_{bcde} + 2\nabla^c\nabla^d(E_R)_{acbd},\\
H^2_{ab} &= \frac{1}{2}(E_1)_a\nabla_b\phi,\\
H^3_{ab} &= -\nabla^c(E_2)_{cb}\nabla_a\phi + \frac{1}{2}\nabla^c(E_2)_{ab}\nabla_c\phi\\
&\quad + \frac{1}{2}(E_2)_{ab}\nabla_c\nabla^c\phi,\\
H^4_{ab} &= (E_F)_a^{\ c}F_{bc},
\end{aligned} \tag{5}$$

where

$$(E_R)^{abcd} = \frac{\partial\mathcal{L}}{\partial R_{abcd}}, \qquad (E_1)^a = \frac{\partial\mathcal{L}}{\partial\nabla_a\phi},$$
$$(E_2)^{ab} = \frac{\partial\mathcal{L}}{\partial\nabla_a\nabla_b\phi}, \qquad (E_F)^{ab} = \frac{\partial\mathcal{L}}{\partial F_{ab}}. \tag{6}$$

In addition, T_{ab} on the right-hand side of Eq. (3) is the stress-energy tensor of the theory of gravity, which only contains the stress-energy tensor of the minimal coupling additional matter fields $T^{\rm mt}_{ab}$. From the physical perspective, we assume that the minimal coupling matter fields should satisfy the null energy condition. According to the assumption and the fact that the total stress-energy tensor of the theory of gravity only contains the stress-energy tensor of the additional matter fields, for any null vector field n^a in spacetime, the null energy condition can be expressed as

$$T^{\rm mt}_{ab}n^a n^b = T_{ab}n^a n^b \geq 0. \tag{7}$$

In the $(n+2)$-dimensional diffeomorphism invariant gravitational theory with higher-order interactions described by the Lagrangian in Eq. (2), the Wald entropy of black holes in stationary background spacetime of the gravitational system can be defined as

$$S_{\rm W} = \frac{1}{4}\int_s d^n y\sqrt{\gamma}\rho_{\rm W}, \tag{8}$$

where γ is the determinant of the induced metric on any slice of the event horizon, y is introduced to label the transverse coordinates on the cross section of the event horizon, and $\rho_{\rm W}$ is the entropy density of the Wald entropy. The entropy density is further given as

$$\rho_{\rm W} = -8\pi\frac{\partial\mathcal{L}}{\partial R_{abcd}}\epsilon_{ab}\epsilon_{cd}, \tag{9}$$

in which $\mathcal{L}$ is the Lagrangian of the gravitational theory, R_{abcd} is the tensor of the Riemann curvature, and ϵ_{ab} is the binormal on any cross section of the event horizon.

The quantum correction terms in the Lagrangian of the low-energy efficient gravitational theory corresponding to quantum gravity can be divided into two types. These two types of quantum correction terms are the higher curvature terms and minimal or nonminimal coupling interaction terms between gravity and matter fields and correspond to the self-interactions and the interactions between gravity and matter fields in quantum gravity. According to the definition of the Wald entropy, one can see that only the higher curvature terms and nonminimal coupling interaction terms in the quantum correction terms can sufficiently influence the expression of the Wald entropy, and the minimal interaction terms do not affect the Wald entropy because it does not include the Riemann curvature terms. It means that some new higher curvature or nonminimal coupling terms introduced into the Lagrangian of the gravitational theory will affect the expression of the Wald entropy. Although the Wald entropy commonly obeys the first law of black hole thermodynamics, whether the expression of the Wald entropy that contains the influence of the new higher curvature or nonminimal coupling terms in the Lagrangian of gravity still satisfies the second law of black hole thermodynamics should be further examined. So far, the general expression of the black hole entropy always obeying the linearized second law in the gravitational theory with partial higher curvature terms has been given, which can be written as the Wald entropy with correction terms. The result shows that the Wald entropy should be corrected to satisfy the linearized second law when only the self-interaction of gravity is involved in the gravitational theory. However, the general expression of black hole entropy that obeys the linearized second law in an arbitrary diffeomorphism invariant gravity with only the interactions between gravity and matter fields has not been given until now. Since the scalar field in spacetime plays an essential role in the research of cosmology and quantum gravity, while celestial bodies in our Universe are always taking the electric charge based on the perspective of astronomical observations, we would like to investigate the linearized second law in an arbitrary diffeomorphism invariant gravitational theory with higher-order interactions and derive the general expression of the black hole entropy always obeying the linearized second law in the gravitational theory.

III. LINEARIZED SECOND LAW OF BLACK HOLE THERMODYNAMICS FOR THE GRAVITATIONAL THEORY WITH HIGHER-ORDER INTERACTIONS

As mentioned above, a physical quasistationary accretion process of black holes is introduced to investigate the linearized second law of black hole thermodynamics. The accretion process describes the dynamical process where the additional matter fields that are minimal coupling to gravity fall into black holes and perturb the spacetime geometry of black holes. To obtain the expression of the black hole entropy commonly satisfying the linearized second law in diffeomorphism invariant gravitational theory with higher-order interactions, we should further assume that black holes will finally settle down to a stationary state after the matter field perturbation process. This assumption is called the stability assumption for simplicity.

For the $(n+2)$-dimensional diffeomorphism invariant gravitational theory described by the Lagrangian in Eq. (2), the event horizon of black holes that is $(n+1)$-dimensional null hypersurface in spacetime is denoted as $\mathcal{H}$. A parameter "u" is introduced as an affine parameter to parametrize the event horizon. Furthermore, a null vector field

$k^a = (\partial/\partial u)^a$ can be chosen to generate the event horizon and satisfies the geodesic equation $k^b \nabla_b k^a = 0$. Any specific value of the parameter u corresponds to an n-dimensional cross section on the event horizon. We can establish coordinates with two null vectors, i.e., $\{k^a, l^a, y^a\}$, on the cross section of the event horizon. In the coordinates, the null vector l^a is another null vector different from k^a, and another parameter "v" is chosen to represent the null vector l^a as $l^a = (\partial/\partial v)^a$. These null vectors in the coordinates satisfy the following two relationships:

$$k^a k_a = l^a l_a = 0, \qquad k^a l_a = -1. \tag{10}$$

According to the two null vector fields, the binormal on any cross section is defined as $\epsilon_{ab} = 2k_{[a} l_{b]}$, and the induced metric on any slice of the event horizon is defined as

$$\gamma_{ab} = g_{ab} + 2k_{(a} l_{b)}. \tag{11}$$

From the induced metric and the null vector fields k^a, the extrinsic curvature of the event horizon can be defined by

$$B_{ab} = \gamma_a{}^c \gamma_b{}^d \nabla_c k_d. \tag{12}$$

Using the definition of the induced metric and the expression of the extrinsic curvature in Eqs. (11) and (12), the evolution of the induced metric along the direction of the future event horizon can be given as

$$\gamma_a{}^c \gamma_b{}^d \mathcal{L}_k \gamma_{cd} = 2\left(\sigma_{ab} + \frac{\theta}{n}\gamma_{ab}\right) = 2B_{ab}, \tag{13}$$

where σ_{ab} and θ represent the shear and the expansion of the event horizon during the evolutionary process, respectively. Moreover, the evolution of the extrinsic curvature is obtained as

$$\gamma_a{}^c \gamma_b{}^d \mathcal{L}_k B_{cd} = B_{ac} B_b{}^c - \gamma_a{}^c \gamma_b{}^d R_{ecfd} k^e k^f. \tag{14}$$

Utilizing the evolution property of the extrinsic curvature, the Raychaudhuri equation can be further given as

$$\frac{d\theta}{d\lambda} = -\frac{\theta^2}{n-2} - \sigma_{ab}\sigma^{ab} - R_{uu}, \tag{15}$$

where the quantity R_{uu} is the abbreviation of $k^a k^b R_{ab}$. In the following, we will use some latin letters at the beginning of the alphabet, i.e., $a, b, c, \ldots$, to represent the abstract index in any tensor and will use some latin letters that start from the letter i, i.e., $i, j, k, \ldots$, to represent the spatial index in any tensor. Meanwhile, a convention will be further introduced to simplify the expressions of equations, which can be stated as follows.

(1) An index in any tensor can be replaced by the parameter u or v directly when the index contracts with the null vector k^a or l^a.
(2) An index in any tensor can be replaced by one of the spatial indices, i.e., $i, j, k, \ldots$, when the index contracts with the induced metric on the cross section of the event horizon.

For any tensor $X_{a_1 b_1 \cdots a_2 b_2 \cdots a_3 b_3 \cdots}$, the first category of indices in the tensor, $a_1, b_1 \ldots$, contracts with the null vector field k^a; the second category of indices in the tensor, $a_2, b_2, \ldots$, contracts with the vector field l^a; and the third category of indices in the tensor, $a_3, b_3, \ldots$, contracts with the induced metric on the cross section of the event horizon. According to the above convention, the tensor contracting all indices with two null vector fields, k^a and l^a, and the induced metrics can be simplified as

$$k^{a_1} k^{b_1} \cdots l^{a_2} l^{b_2} \cdots \gamma^{\;a_3}_{c_3(i)} \gamma^{\;b_3}_{d_3(j)} \cdots X_{a_1 b_1 \cdots a_2 b_2 \cdots a_3 b_3 \cdots} = X_{uu\cdots vv\cdots ij\cdots}. \tag{16}$$

To explicitly depict the perturbation process caused by the additional matter fields outside black holes, a sufficient small parameter ϵ is introduced to represent the order of approximation of the perturbation. We can assume that the three quantities, which are the extrinsic curvature, the expansion, and the shear of the event horizon, contribute only under the first-order approximation of the matter field perturbation. Based on the small parameter ϵ, the relationship of the three quantities under the first-order approximation can be written as $B_{ab} \sim \theta \sim \sigma_{ab} \sim \mathcal{O}(\epsilon)$. Since we hope to find out the expression of the black hole entropy commonly obeying the linearized second law, the symbol "$\simeq$" will be used to represent the identity under the first-order approximation of the perturbation process. According to the above conventions, the extrinsic curvature of the event horizon and the evolution of the extrinsic curvature along the future event horizon, which have been given in Eqs. (12) and (14), can be rewritten as

$$B_{ij} \simeq D_i k_j, \qquad \mathcal{L}_k B_{ij} \simeq -R_{uiuj}, \tag{17}$$

under the linear-order approximation, where the derivative operator D_a is the pure spatial derivative operator. For any tensor $X_{a_1 a_2 \cdots}$, the spatial derivative operator can be defined as

$$D_a X_{a_1 a_2 \cdots} = \gamma_a{}^b \gamma_{a_1}{}^{b_1} \gamma_{a_2}{}^{b_2} \cdots \nabla_b X_{b_1 b_2 \cdots}. \tag{18}$$

The linearized version of the Raychaudhuri equation can be further written as

$$\frac{d\theta}{d\lambda} \simeq -R_{uu}. \tag{19}$$

According to the induced metric and the null vector fields l^a, a new quantity C_{ij} can be defined as $C_{ij} = D_i l_j$. Following the calculation method of the evolution of the extrinsic curvature along the future direction of the event horizon, the evolution of C_{ij} along the same orientation on the background spacetime can be given as

$$\mathcal{L}_k C_{ij} = -R_{iujv}. \tag{20}$$

Additionally, from the definition of the Wald entropy and the above conventions, the density of the Wald entropy ρ_{W} in coordinates with two null vectors k^a and l^a can be written as

$$\rho_{\mathrm{W}} = -2(E_R)_{uvuv}. \tag{21}$$

The null energy condition of the total stress-energy tensor of the gravitational theory with higher-order interactions in the same coordinates can be rewritten as

$$T_{ab}k^a k^b = T_{uu} \geq 0. \tag{22}$$

Next, we will investigate the linearized second law of black holes in the diffeomorphism invariant gravitational theory with higher-order interactions. Starting with the definition of the Wald entropy, we ultimately expect to derive the expression of the black hole entropy always satisfying the linearized second law in the gravitational theory. If the linearized second law holds, in other words, the value of the black hole entropy monotonously increases under the first-order approximation of the matter fields perturbation process, the expression of the entropy should satisfy the following relationship under the linear-order approximation [13], i.e.:

$$\mathcal{L}_k^2 S \simeq -\frac{1}{4}\int_s \tilde{\epsilon} H_{uu} = -2\pi \int_s \tilde{\epsilon} T_{uu} \leq 0, \tag{23}$$

where S on the left-hand side of Eq. (23) represents the black hole entropy. The equation of motion of the gravitational part and the null energy condition of the stress-energy tensor in Eqs. (3) and (22) have been used in the second and the third steps. The stability assumption requires that the rate of change of the black hole entropy gradually decreases to zero after the perturbation process. It implies that the variation tendency of the rate of change of the entropy can be expressed equivalently, as the second-order Lie derivative of the entropy is always negative during the perturbation process, i.e., $\mathcal{L}_k^2 S \leq 0$. Combining the negative second-order Lie derivative of the entropy with the stability assumption, one can infer that the first-order Lie derivative of the entropy should always be positive during the perturbation process, $\mathcal{L}_k S \geq 0$. It indicates that the value of the entropy is monotonously increasing with the perturbation process. Therefore, if the black hole entropy obeys the relationship in Eq. (23), the entropy will always satisfy the linearized second law of black hole thermodynamics under the perturbation process. Therefore, according to the relationship, to obtain the general expression of the black hole entropy satisfying the linearized second law in the gravitational theory with higher-order interactions, we should calculate the specific expression of H_{uu} under the linear-order approximation first.

So far, two assumptions have been suggested. The first is that the total stress-energy tensor should obey the null energy condition, and the second is the stability assumption. However, the third assumption should be introduced before calculating the expression of H_{uu} under the linear-order approximation, which states that a regular bifurcation surface exists in the background spacetime. The regularity property means that all physical quantities are smooth and finite on the whole Killing horizon, even on the bifurcation surface. According to the coordinates with two null vectors k^a and l^a, i.e., $\{k^a, l^a, y^a\}$, an arbitrary vector z_i^a can be introduced, which represents one of the two null vectors in the coordinates and can be expressed as $z_i^a \in \{k^a, l^a\}$. Considering a quantity written as a contraction of all indices in any tensor $X_{a_1\cdots a_k}$ with the vectors $\{z_i^{a_i}, i = 1\cdots k\}$, i.e., $X_{a_1\cdots a_k} z_1^{a_1}\cdots z_k^{a_k}$, the third assumption indicates that the quantity will vanish on the background spacetime if the number of k^a including in the vector $z_i^{a_i}$ is larger than the number of l^a including in the vector $z_i^{a_i}$. Hence, in this case, the quantity on the background spacetime can be written as

$$X_{a_1\cdots a_k} z_1^{a_1}\cdots z_k^{a_k} = 0. \tag{24}$$

On the other hand, the quantity is still finite and smooth over the whole Killing horizon with the bifurcation surface on the background spacetime if the number of the null vector k^a in $z_i^{a_i}$ is less than or equal to the null vector l^a in $z_i^{a_i}$ [18,25]. In other words, the quantity $X_{a_1\cdots a_k} z_1^{a_1}\cdots z_k^{a_k}$ is a quantity on the background spacetime when the number of k^a in its expression is less than or equal to the number of l^a; the quantity is a quantity under the first-order approximation when the number of k^a in its expression is larger than the number of l^a. This result can be regarded as a criterion to judge whether any quantity in H_{uu} is contributed under the zeroth-order approximation or only under the first-order approximation. For simplicity, the quantity under the zeroth-order approximation (or on the background spacetime) is called the background quantity, and the quantity under the linear-order approximation is called the first-order quantity directly. Moreover, in the following calculations, we will use the symbols $(\ )_n$ or $[\ ]_n$, $(n = 0,\ 1)$, to label every quantity in H_{uu} during the calculation process, where $n = 0$ and $n = 1$ represent the background quantity and the first-order quantity, respectively.

After contracting two null vectors k^a and k^b with the first component on the left-hand side of the equation of motion, the first identity in Eq. (5) can be expressed as

$$H^1_{uu} = k^a k^b (E_R)_{acde} R_b{}^{cde} + 2k^a k^b \nabla^c \nabla^d (E_R)_{acbd}. \tag{25}$$

Expanding the repeated indices by using the definition of the induced metric on the cross section of the event horizon, the specific expression of the first term of Eq. (25) under the first-order approximation is given as

$$\begin{aligned} k^a k^b (E_R)_{acde} R_b{}^{cde} &= [(E_R)_{uijk}]_1 (R_u{}^{ijk})_1 - 2[(E_R)_{uivj}]_0 (R^{vivj})_1 \\ &\quad - 2[(E_R)_{uiuj}]_1 (R^{viuj})_0 + 2[(E_R)_{uivu}]_1 (R_{uvu}{}^i)_1 \\ &\simeq -2(E_R)_{uivj} R^{vivj} - 2(E_R)_{iuju} R^{viuj}. \end{aligned} \tag{26}$$

From the Lagrangian of the gravitational theory, the quantity $(E_R)_{iuju}$ under the first-order approximation can be further calculated as

$$\begin{aligned} (E_R)_{iuju} &\simeq 4 \frac{\partial^2 \mathcal{L}}{\partial R^{vkvl} \partial R^{uiuj}} R^{kvlv} + \frac{\partial^2 \mathcal{L}}{\partial(\nabla_u \nabla_u \phi) \partial R^{uiuj}} \nabla_u \nabla_u \phi \\ &\simeq \mathcal{L}_k \mathcal{P}_{ij}, \end{aligned} \tag{27}$$

where

$$\mathcal{P}_{ij} = -4 \frac{\partial^2 \mathcal{L}}{\partial R^{vkvl} \partial R^{uiuj}} B^{kl} + \frac{\partial^2 \mathcal{L}}{\partial(\nabla_u \nabla_u \phi) \partial R^{uiuj}} \mathcal{L}_k \phi. \tag{28}$$

Using the identities in Eqs. (27) and (20), the result of Eq. (26) under the linear-order approximation can be further simplified as

$$k^a k^b (E_R)_{acde} R_b{}^{cde} \simeq 2\mathcal{L}_k (\mathcal{P}_{ij} \mathcal{L}_k C^{ij}) - 2(E_R)_{uivj} R^{vivj}. \tag{29}$$

Utilizing Leibniz's law, the second term of Eq. (25) can be expanded as

$$\begin{aligned} &2k^a k^b \nabla^c \nabla^d (E_R)_{acbd} \\ &\quad = 2k^a \nabla^c (k^b \nabla^d (E_R)_{acbd}) - 2(k^a \nabla^c k^b) \nabla^d (E_R)_{acbd}. \end{aligned} \tag{30}$$

Since we discuss the linearized second law of black hole thermodynamics in the more general diffeomorphism invariant gravitational theory with higher-order interactions, the specific expression of the quantity $(E_R)_{abcd}$ cannot be written directly. It means that we cannot further derive the expression of the first term on the right-hand side of Eq. (30) obviously under the first-order approximation of the matter fields perturbation. Therefore, a significant identity should be involved first to effectively obtain the expression of this term under the linear-order approximation. From any two-form tensor X^{ab}, we can demonstrate that the identity can be written as

$$\int_s \tilde{\epsilon} k_b \nabla_a X^{ab} = \frac{1}{2} \mathcal{L}_k \int \epsilon_{aba_1 \cdots a_n} X^{ab}. \tag{31}$$

Using this identity twice and Leibniz's law, while according to the density of the Wald entropy in Eq. (21), the integral form of the second term in Eq. (25) on the cross section of the event horizon can be further given as

$$\begin{aligned} 2\int_s \tilde{\epsilon} k^a k^b \nabla^c \nabla^d (E_R)_{acbd} &= -\mathcal{L}_k^2 \int_s \tilde{\epsilon} \rho_W - 2\int_s \tilde{\epsilon} (k^a \nabla^c k^b) \nabla^d (E_R)_{acbd} \\ &\quad + 2\mathcal{L}_k \int_s \tilde{\epsilon} [(k^b \nabla^d l^a) k^c (E_R)_{cabd}] \\ &\quad + 2\mathcal{L}_k \int_s \tilde{\epsilon} [(k^b \nabla^d k^c) l^a (E_R)_{cabd}]. \end{aligned} \tag{32}$$

Before calculating the above expression under the linear-order approximation, three practical identities are introduced to simplify the calculation process, and these identities have been demonstrated in our previous research work [17]. The three identities in the background spacetime can be expressed as

$$\nabla_u l^a = 0, \qquad \nabla_i k_a = 0, \qquad \nabla_a k_i = 0. \tag{33}$$

Meanwhile, the three identities also indicate that each quantity on the left-hand side of each identity is a first-order quantity.

For the integrand of the second integral in Eq. (32), using the three identities in Eq. (33), we can expand the repeated indices and calculate the specific expression under the first-order approximation as

$$\begin{aligned} (k^a \nabla^c k^b) \nabla^d (E_R)_{acbd} &= -[\nabla_u (E_R)_{uivd}]_1 (\nabla^i k^d)_1 - [\nabla_v (E_R)_{uiud}]_1 (\nabla^i k^d)_1 \\ &\quad - (l_a \nabla^i k^a)_1 [\nabla^j (E_R)_{uiju}]_1 + (B^{ij})_1 [\nabla^k (E_R)_{uikj}]_1 \\ &\simeq 0. \end{aligned} \tag{34}$$

The integrand in the third integral of Eq. (32) can be further calculated as

$$\begin{aligned} (k^b \nabla^d l^a) k^c (E_R)_{cabd} &= -[(E_R)_{uiuv}]_1 (\nabla_u l^i)_1 + [(E_R)_{uiuj}]_1 (C^{ij})_0 \\ &\quad + [(E_R)_{uvui}]_1 (l^a \nabla^i k_a)_1 - \frac{1}{2} \rho_W k^a \nabla_u l_a \\ &\simeq (E_R)_{uiuj} C^{ij}, \end{aligned} \tag{35}$$

under the linear-order approximation of the matter fields perturbation. The last term in the first step of Eq. (35) is equal to zero directly according to the geodesic equation $k^b\nabla_b k^a = 0$. Since the integrand in the fourth integral of Eq. (32) under the first-order approximation is

$$\begin{aligned}
&(k^b\nabla^d k^c)l^a(E_R)_{cabd}\\
&\quad = -[(E_R)_{viuj}]_0(B^{ij})_1 - [(E_R)_{uvui}]_1(l^c\nabla^i k_c)_1\\
&\quad \simeq -(E_R)_{viuj}B^{ij}, \qquad (36)
\end{aligned}$$

the fourth integral in the result of Eq. (32) can be finally written as

$$2\mathcal{L}_k\int_s \tilde{\epsilon}[(k^b\nabla^d k^c)\,l^a\,(E_R)_{cabd}] \simeq 2\int_s \tilde{\epsilon}R^{ivjv}\,(E_R)_{viuj}, \qquad (37)$$

where the second identity in Eq. (17) is used in the last step.

In conclusion, utilizing the results in Eqs. (29), (34), (35), and (37), the result of the integral of H^1_{uu} on the cross section under the linear-order approximation can be finally obtained as

$$\begin{aligned}
\int_s \tilde{\epsilon}H^1_{uu} &\simeq -\mathcal{L}_k^2\int_s \tilde{\epsilon}\rho_W + 2\mathcal{L}_k\int_s \tilde{\epsilon}(E_R)_{uiuj}C^{ij} + 2\mathcal{L}_k\int_s \tilde{\epsilon}(\mathcal{P}_{ij}\mathcal{L}_k C^{ij}) + 2\int_s \tilde{\epsilon}R^{ivjv}(E_R)_{viuj} - 2\int_s \tilde{\epsilon}(E_R)_{uivj}R^{vivj}\\
&= -\mathcal{L}_k^2\int_s \tilde{\epsilon}\rho_W + 2\mathcal{L}_k\int_s \tilde{\epsilon}(\mathcal{L}_k\mathcal{P}_{ij})C^{ij} + 2\mathcal{L}_k\int_s \tilde{\epsilon}(\mathcal{P}_{ij}\mathcal{L}_k C^{ij})\\
&= -\mathcal{L}_k^2\int_s \tilde{\epsilon}(\rho_W - 2\mathcal{P}_{ij}C^{ij}), \qquad (38)
\end{aligned}$$

where we have used Eq. (27) in the second step.

For the second identity in Eq. (5), after contracting two null vectors k^a and k^b with the expression of H^2_{ab}, the expression of H^2_{uu} under the linear-order approximation can be directly calculated as

$$H^2_{uu} = \frac{1}{2}k^a k^b (E_1)_a \nabla_b\phi = \frac{1}{2}[(E_1)_u]_1(\mathcal{L}_k\phi)_1 \simeq 0. \qquad (39)$$

Since H^2_{uu} is vanishing under the first-order approximation of the perturbation, its integral on the cross section of the event horizon does not contribute to the final result.

Contracting two null vectors k^a and k^b with the third identity in Eq. (5), the expression of H^3_{uu} can be given as

$$\begin{aligned}
H^3_{uu} &= -k^a k^b \nabla^c (E_2)_{cb}\nabla_a\phi + \frac{1}{2}k^a k^b \nabla^c (E_2)_{ab}\nabla_c\phi\\
&\quad + \frac{1}{2}k^a k^b (E_2)_{ab}\nabla_c\nabla^c\phi. \qquad (40)
\end{aligned}$$

After expanding the repeated indices using the definition of the induced metric on the cross section as well, the expression of H^3_{uu} under the first-order approximation can be further expressed as

$$\begin{aligned}
H^3_{uu} &= -[(E_2)_{uu}]_1(\nabla_v\nabla_u\phi)_0 + (\mathcal{L}_k\phi)_1[\nabla_u(E_2)_{uv}]_1 + \frac{1}{2}(D^i\phi)_0[\nabla_i(E_2)_{uu}]_1 + \frac{1}{2}(\mathcal{L}_k\phi)_1[\nabla_v(E_2)_{uu}]_1 - (\mathcal{L}_k\phi)_1[\nabla^i(E_2)_{ui}]_1\\
&\quad - \frac{1}{2}[\nabla_u(E_2)_{uu}]_1(\nabla_v\phi)_0 + \frac{1}{2}[(E_2)_{uu}]_1(D^i D_i\phi)_0\\
&\simeq \frac{1}{2}(D^i\phi)\nabla_i(E_2)_{uu} + \frac{1}{2}(E_2)_{uu}(D^i D_i\phi) - (E_2)_{uu}(\nabla_v\nabla_u\phi) - \frac{1}{2}(\nabla_v\phi)\nabla_u(E_2)_{uu}. \qquad (41)
\end{aligned}$$

The first term of the result in Eq. (41) is further calculated as

$$\begin{aligned}
\frac{1}{2}(D^i\phi)\nabla_i(E_2)_{uu} &= \frac{1}{2}(D^i\phi)_0 D_i[(E_2)_{uu}]_1 - \frac{1}{2}(D_i\phi)_0[(E_2)_{uj}]_1(B^{ij})_1 + \frac{1}{2}(D^i\phi)_0[(E_2)_{uu}]_1(l^d\nabla_i k_d)_1\\
&\quad - \frac{1}{2}(D_i\phi)_0[(E_2)_{ju}]_1(B^{ij})_1 + \frac{1}{2}(D^i\phi)_0[(E_2)_{uu}]_1(l^d\nabla_i k_d)_1\\
&\simeq \frac{1}{2}(D^i\phi)D_i[(E_2)_{uu}] \qquad (42)
\end{aligned}$$

under the first-order approximation. Combining the result of Eq. (42) with the second term of the result in Eq. (41), we have

$$\frac{1}{2}(D^i\phi)D_i[(E_2)_{uu}]+\frac{1}{2}(E_2)_{uu}(D^iD_i\phi)=\frac{1}{2}D_i[(D^i\phi)(E_2)_{uu}]. \tag{43}$$

When the topology of the event horizon of black holes in the general gravitational theory with higher-order interactions is assumed to be compact, the spatial total derivative term in the integrand, namely, the spatial boundary term of the integral, does not contribute to the final result. Therefore, the result of Eq. (43) can be neglected directly. According to the Lagrangian of the gravitational theory, the expression of $(E_2)_{uu}$ under the first-order approximation can be further given as

$$(E_2)_{uu}\simeq\frac{\partial^2\mathcal{L}}{\partial(\nabla_u\nabla_u\phi)\partial(\nabla_v\nabla_v\phi)}\nabla_u\nabla_u\phi + 4\frac{\partial^2\mathcal{L}}{\partial R^{vivj}\partial(\nabla_v\nabla_v\phi)}R^{ivjv}\simeq\mathcal{L}_k\mathcal{N}, \tag{44}$$

where

$$\mathcal{N}=\frac{\partial^2\mathcal{L}}{\partial(\nabla_u\nabla_u\phi)\partial(\nabla_v\nabla_v\phi)}\mathcal{L}_k\phi-4\frac{\partial^2\mathcal{L}}{\partial R^{vivj}\partial(\nabla_v\nabla_v\phi)}B^{ij}. \tag{45}$$

Using the identity in Eq. (44), the third and fourth terms in the result of Eq. (41) under the linear-order approximation can be further simplified as

$$-[(E_2)_{uu}](\nabla_v\nabla_u\phi)-\frac{1}{2}[\nabla_u(E_2)_{uu}](\nabla_v\phi) \simeq-\mathcal{L}_k^2\left[\frac{1}{2}\mathcal{N}(\nabla_v\phi)\right]. \tag{46}$$

Therefore, the expression of the integral of H^3_{uu} on the cross section under the first-order approximation is given as

$$\int_s\tilde{\epsilon}H^3_{uu}\simeq-\mathcal{L}_k^2\int_s\tilde{\epsilon}\left[\frac{1}{2}\mathcal{N}(\nabla_v\phi)\right]. \tag{47}$$

Finally, after contracting two null vectors k^a and k^b with the fourth identity in Eq. (5) and expanding the repeated index by the induced metric, the expression of H^4_{kk} under the first-order approximation of the perturbation process is directly given as

$$H^4_{uu}=k^ak^b(E_F)_{ac}F_b{}^c=[(E_F)_{ui}]_1(F_u{}^i)_1\simeq0. \tag{48}$$

The result shows that the electromagnetic field part in the gravitational theory with higher-order interactions does not contribute to the expression of the black hole entropy obeying the linearized second law.

Therefore, combining the results in Eqs. (38), (39), (47), and (48), and supplementing the coefficient 1/4, while utilizing the equation of motion of the gravitational part in Eq. (3), the definition of the Wald entropy in Eq. (8), and the null energy condition in Eq. (21), we have

$$\mathcal{L}_k^2(S_{\rm W}+S_{\rm ct})\simeq-\frac{1}{4}\int_s\tilde{\epsilon}H_{uu}=-2\pi\int_s\tilde{\epsilon}T_{uu}\leq0, \tag{49}$$

where

$$S_{\rm ct}=\frac{1}{4}\int_s\tilde{\epsilon}\left[-2\mathcal{P}_{ij}C^{ij}+\frac{1}{2}\mathcal{N}(\nabla_v\phi)\right]. \tag{50}$$

The result shows that the black hole entropy, which always obeys the linearized second law, in the diffeomorphism invariant gravitational theory with higher-order interactions during the matter fields perturbation process can be written as the Wald entropy with two correction terms eventually. The two correction terms include the contributions from the self-interactions of gravity and the minimal and nonminimal coupling interactions between gravity and the scalar field. The contribution of the minimal coupling interactions is only contained in the expression of $\mathcal{N}$ in the second term of Eq. (50). The electromagnetic part in the gravitational theory does not contribute to the expression of the black hole entropy obeying the linearized second law. According to the result in previous literature, the expression of the black hole entropy that always satisfies the linearized second law in gravitational theory with partial higher curvature terms can be expressed as the form of the Wald entropy with relevant correction terms. These correction terms only come from the contribution of the nonminimal coupling self-interactions of gravity. In addition, for the gravitational theory with matter fields used to study the linearized second law in our previous research works, the black hole entropy always obeying the linearized second law can be expressed as the Wald entropy or the Wald entropy with correction terms as well. These correction terms also only come from the nonminimal coupling interaction terms between gravity and matter fields, while the minimal coupling interactions do not influence the specific expression of the black hole entropy. However, when considering the diffeomorphism invariant gravity with higher-order interactions, the minimal and nonminimal coupling interactions between gravity and the scalar field contribute to the correction terms of the Wald entropy at the same time. This result overturns our previous understanding of the expression of the black hole entropy satisfying the linearized second law of thermodynamics in general diffeomorphism invariant gravitational theory.

IV. DISCUSSION AND CONCLUSIONS

For an arbitrary diffeomorphism invariant gravitational theory, the first law of black hole thermodynamics is constructed generally in a thermodynamic equilibrium state, and the black hole entropy in the first law is the Wald entropy. It means that the Wald entropy is the black hole entropy when the gravitational system is in the thermodynamic equilibrium state. However, when a process of thermodynamic evolution is considered, the black holes will be in a thermodynamic evolution state rather than staying in an equilibrium state. In this situation, the actual black hole entropy should be further required to satisfy the second law of black hole thermodynamics. Since the Wald entropy only commonly obeys the first law of black hole thermodynamics, one can reasonably infer that the black hole entropy satisfying both the first and second laws of black hole thermodynamics could be expressed as the Wald entropy with some correction terms. If this inference is correct, it indicates sufficiently that the Wald entropy cannot satisfy the second law of black hole thermodynamics in the dynamic evolution process. In other words, the Wald entropy of black holes cannot fully describe the actual black hole entropy in the gravitational theory, and its expression has a degree of arbitrariness. The degree of arbitrariness of the Wald entropy can be decreased when the Wald entropy with correction terms satisfies the second law of black hole thermodynamics, and the Wald entropy with correction terms will be closer to the actual expression of black hole entropy in the gravitational theory. This inference has been examined according to results in previous works. In the general gravitational theory with partial higher curvature terms, the black hole entropy, which always satisfies both the first and linearized second laws of black hole thermodynamics, can be commonly written as the Wald entropy with correction terms. However, the expression of the black hole entropy, which generally obeys two thermodynamics laws, in the gravitational theory with interactions between gravity and matter fields has not been obtained until now. Therefore, in our research, we investigate the linearized second law in the gravitational theory with higher-order interactions and derive the general expression of black hole entropy that satisfies both the first and linearized second laws of black hole thermodynamics.

The result shows that the black hole entropy that meets the first and linearized second laws can also be expressed as the Wald entropy with two correction terms. The first correction term in Eq. (50) arises from two categories of interaction in the gravitational theory. These interactions are the self-interaction of gravity and the nonminimal coupling interaction between gravity and the scalar field. The second correction term in Eq. (50) also comes from the contributions of two categories of interaction in the gravitational theory. The first category of interaction is the higher-order correction term of the self-interaction of the scalar field in gravitational context, i.e., the minimal coupling interaction between gravity and the scalar field. The second category of interaction is the nonminimal coupling interaction between gravity and the scalar field. From the expression of black hole entropy, one can find that the correction terms in the expression of black hole entropy only obviously appear when black holes in the theory of gravity are in a thermodynamic evolution process. It indicates that the correction terms only describe how the black hole entropy evolves with the dynamical process, while it also ensures that the black hole entropy always meets the requirements of the linearized second law during the evolution process. The black hole will stay in a new equilibrium state at the end of the thermodynamic evolution. At this moment, the correction terms will disappear in the expression of black hole entropy because they only involve the thermodynamic evolution process of black holes, and the expression of black hole entropy will degenerate the Wald entropy automatically. It means that the Wald entropy can only describe the black hole entropy in the equilibrium state, and it cannot describe the black hole entropy when the black hole is in a thermodynamic evolution process. It also shows that the Wald entropy has a degree of arbitrariness and cannot adequately describe the entropy of black holes in the gravitational theory with higher-order interactions. When considering the linearized second law of black hole thermodynamics, the entropy, i.e., the Wald entropy with correction terms, can describe the black hole entropy in both the equilibrium and dynamic evolution state and satisfies the first and linearized second laws of black hole thermodynamics. Therefore, the correction terms decrease the degree of arbitrariness of the Wald entropy, and the Wald entropy with correction terms is approaching the complete expression of black hole entropy in the gravitational theory with higher-order interactions.

The first law of black hole thermodynamics in an arbitrary diffeomorphism invariant gravitational theory has been constructed generally by Wald and Iyer [7,8]. Unlike the first law of black hole thermodynamics in classical general relativity, the entropy in the expression of the first law for the general gravitational theory is called the Wald entropy rather than Bekenstein-Hawking entropy. It indicates that the Wald entropy is commonly suitable for the first law of black hole thermodynamics in general diffeomorphism invariant gravitational theory. However, to deeply understand black hole thermodynamics and further reveal the thermodynamic properties of black holes, whether the Wald entropy is commonly suitable for the second law of black hole thermodynamics should be further investigated. When two categories of interactions, which mainly contain the self-interactions of gravity and the interactions between gravity and matter fields, in the quantum gravitational theory, some relevant quantum correction terms describing the two categories of interactions present to the Lagrangian of the low-energy efficient gravitational theory corresponding to the quantum gravity. Among these quantum correction terms, those describing nonminimal coupling interactions will substantially influence the expression of the Wald entropy according to the definition. Therefore, to generally investigate whether the Wald entropy satisfies the second law of black hole thermodynamics in an arbitrary gravitational theory, we should examine whether the Wald entropy in the general gravitational theory that contains the two categories of interactions, respectively, commonly obeys the requirement of the second law. For the gravitational theory with partial higher curvature terms, the black hole entropy that always satisfies the linearized second law is obtained, which can be written as the Wald entropy with correction terms. It indicates that the Wald entropy does not commonly obey the linearized second law in general higher curvature gravity. Although the linearized second law in the higher curvature gravity has been investigated adequately, the general expression of the black hole entropy satisfying the linearized second law in the gravitational theory with matter fields has not been obtained yet. According to our previous research, the linearized second law of black holes in Horndeski gravity, the general quadratic corrected Einstein-Maxwell gravity, and the general second-order scalar-tensor gravity has been investigated. The results show that the Wald entropy no longer commonly satisfies the linearized second law in these gravitational theories, and the black hole entropy obeying the linearized second law can also be expressed as the Wald entropy with some correction terms. However, in previous research, the theory of gravity that involves the second-order interactions of gravity with scalar and electromagnetic fields at most is heavily considered to investigate the linearized second law of black hole thermodynamics. To generally study the linearized second law, we should generalize the gravitational theory containing the second-order interactions of gravity with the scalar and electromagnetic fields at most to the more general gravitational theory that includes higher-order interactions of gravity with the scalar and electromagnetic fields. Furthermore, we investigate the linearized second law in the gravitational theory and obtain the general expression of the black hole entropy satisfying the linearized second law.

Considering the gravitational theory with higher-order interactions, starting with the definition of the Wald entropy again, we would like to derive the general expression of the black hole entropy that always satisfies the linearized second law in the gravitational theory. A quasistationary accreting process should be introduced first under the linear-order approximation, which states that the additional matter fields that are minimal coupling with gravity outside black holes pass through the event horizon and fall into black holes. It means that the matter fields can perturb the spacetime configuration of black holes during the accreting process. Furthermore, two assumptions are introduced to investigate the linearized second law. The first is that the additional matter fields obey the null energy condition, and the second is the stability assumption. Additionally, to ensure that all physical quantities are smooth and finite on the whole Killing horizon, the third assumption that a regular bifurcation surface exists in the background spacetime is also introduced. From the Raychaudhuri equation, according to the regularity of the bifurcation surface and the null energy condition, we derive the general expression of the black hole entropy commonly obeying the linearized second law in the gravitational theory with higher-order interactions. This entropy can also be written as the Wald entropy with two correction terms. In the expression of black hole entropy, the contribution of the minimal coupling interaction between gravity and the scalar field is also included in one of the two correction terms. This result implies that, when an arbitrary matter field is involved in the gravitational theory, we should consider both the minimal and nonminimal coupling interactions between gravity and the matter field to derive the expression of black hole entropy that obeys the first and linearized second laws of black hole thermodynamics. Therefore, the result overturns our previous cognition of the black hole entropy and gives us a new understanding of the black hole entropy that satisfies the first and linearized second laws in general diffeomorphism invariant gravitational theory.

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