

Multipole moments of a charged rotating disc of dust in general relativity

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The gravitational and electromagnetic multipole moments of the charged rotating disc of dust, which is an axisymmetric, stationary solution of the Einstein-Maxwell equations in terms of a post-Newtonian expansion, are calculated and discussed. It turns out that the individual mass, angular momentum, electric and magnetic moments are ordered in the sense that higher moments have a lower absolute value. There is an interesting conjecture stating that the absolute values of all higher multipole moments of a uniformly rotating perfect fluid body are always greater than those of the corresponding Kerr spacetime, which we generalize to include charged bodies. We find that for the charged rotating disc of dust, the conjecture holds (within the limits of accuracy of the post-Newtonian expansion).

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I. INTRODUCTION

The exterior spacetime of an astrophysical body can be characterized by its multipole moments. In 1970, Geroch introduced multipole moments in a coordinate independent way for static, asymptotically flat, vacuum spacetimes [1], and Hansen later generalized them to stationary (asymptotically flat, vacuum) spacetimes [2]. In case of axisymmetry, Hansen further showed that the multipole moment tensors can be represented by scalars P_n [2]. Relying on an asymptotic expansion of the Ernst potential on the symmetry axis, Fodor, Hoenselaers and Perjés presented, in 1989, an explicit algorithm to compute those scalar multipole moments [3].

A generalization to stationary, electro-vacuum spacetimes with gravitational and electromagnetic multipole moments, P_n and Q_n , respectively, was published by Simon [4], and a corresponding calculation scheme, in the axisymmetric case, was given by Hoenselaers and Perjés [5]. As it turned out, [5] contained two nontrivial mistakes, however. The first one was corrected by Sotiriou and Apostolatos [6] and the second one recently, in 2021, by Fodor *et al.* [7].

In the present paper we calculate and discuss the gravitational and electromagnetic multipole moments of the charged rotating disc of dust (with constant angular velocity and constant specific charge) using the latest calculation procedure by Fodor *et al.* [7]. The charged rotating disc of dust is an axisymmetric, stationary and physically reasonable solution of the Einstein-Maxwell equations expressed in terms of a post-Newtonian expansion up to tenth order [8–10]. An exact solution to this problem is not available so far; in particular it is not

contained in the solution classes discussed in [11]. A study of the multipole moments of the exact solution of the uncharged disc of dust [12–14] was published by Kleinwächter *et al.* [15].

We additionally formulate a generalized version of the multipole conjecture by Filter and Kleinwächter [16] and test it using the multipole moments of the charged rotating disc of dust. The conjecture states that the absolute values of all higher multipole moments of an axisymmetric, stationary, physically well-defined body are always greater than those of the corresponding multipole moments of the Kerr-Newman spacetime with the same mass, angular momentum and charge. We show that the generalized multipole conjecture holds for the charged rotating disc of dust.

II. CHARGED ROTATING DISC OF DUST

In general relativity it is rather easy to construct purely mathematical solutions without any physical meaning, however, the charged rotating disc of dust is also physically relevant. We model the disc by an infinitesimally thin equilibrium configuration of dust with constant specific charge $\epsilon \in [-1, 1]$ (electric charge density over baryonic mass density) that is rigidly rotating around the axis of symmetry with constant angular velocity Ω . Dust is a perfect fluid with vanishing pressure. The disc is axisymmetric, stationary, and it obeys reflection symmetry [8,9,17].

Using axisymmetry and stationarity, the corresponding metric can be expressed globally in terms of Weyl-Lewis-Papapetrou coordinates¹:

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¹We use units in which $c = G = 4\pi\epsilon_0 = 1$.

$$ds^2 = f^{-1}[h(d\rho^2 + d\zeta^2) + \rho^2 d\varphi^2] - f(dt + a d\varphi)^2. \quad (1)$$

The electromagnetic four-potential A_a takes the form:

$$A_a = (0, 0, A_\varphi, A_t). \quad (2)$$

In axistationary spacetimes, the coupled Einstein-Maxwell equations in electro-vacuum can be reduced to the Ernst equations [18]:

$$(\Re\mathcal{E} + |\Phi|^2)\Delta\mathcal{E} = (\nabla\mathcal{E} + 2\bar{\Phi}\nabla\Phi) \cdot \nabla\mathcal{E}, \quad (3)$$

$$(\Re\mathcal{E} + |\Phi|^2)\Delta\Phi = (\nabla\mathcal{E} + 2\bar{\Phi}\nabla\Phi) \cdot \nabla\Phi, \quad (4)$$

where

$$\mathcal{E} = (f - |\Phi|^2) + ib \quad \text{and} \quad \Phi = \alpha + i\beta \quad (5)$$

are the Ernst potentials. Here, $\alpha := \Re\Phi = -A_t$, and the potentials β and b are defined by

$$\beta_{,\rho} = \frac{f}{\rho}(A_{\varphi,\zeta} - aA_{t,\zeta}), \quad \beta_{,\zeta} = -\frac{f}{\rho}(A_{\varphi,\rho} - aA_{t,\rho}), \quad (6)$$

and

$$b_{,\rho} = -\frac{f^2}{\rho}a_{,\zeta} - 2(\beta A_{t,\rho} - A_t\beta_{,\rho}),$$

$$b_{,\zeta} = \frac{f^2}{\rho}a_{,\rho} - 2(\beta A_{t,\zeta} - A_t\beta_{,\zeta}). \quad (7)$$

With the help of reflection symmetry, one can formulate a well-defined boundary value problem to the Ernst equations for the charged rotating disc of dust, which was solved in terms of a post-Newtonian expansion up to eighth order by Palenta and Meinel [8] and up to tenth by Breithaupt *et al.* [9]:

$$f = 1 + \sum_{k=1}^{10} f_{2k} g^{2k}, \quad b = \sum_{k=1}^{10} b_{2k+1} g^{2k+1}, \quad (8)$$

$$\alpha = \sum_{k=1}^{10} \alpha_{2k} g^{2k}, \quad \beta = \sum_{k=1}^{10} \beta_{2k+1} g^{2k+1}. \quad (9)$$

The coefficient functions f_{2k} , b_{2k+1} , α_{2k} , and β_{2k+1} are functions of the elliptic coordinates $\eta \in [-1, 1]$ and $\nu \in [0, \infty]$, defined via

$$\rho = \rho_0 \sqrt{(1 - \eta^2)(1 + \nu^2)}, \quad \zeta = \rho_0 \eta \nu. \quad (10)$$

Here, $g := \sqrt{\gamma}$ and γ is the relativity parameter, originally introduced by Bardeen and Wagoner [19], defined by:

$$\gamma := 1 - \sqrt{f_c}, \quad \text{with} \quad f_c := f(\rho = 0, \zeta = 0). \quad (11)$$

Note that the metric function f can also be written as $f = e^{2U}$, where U can be interpreted as a generalized Newtonian potential. Equivalently, the relativity parameter can be expressed in terms of the redshift Z_c of a photon traveling from the center to infinity:

$$\gamma = \frac{Z_c}{1 + Z_c}. \quad (12)$$

The parameter space of the disc solution is therefore given by $g \in [0, 1]$, $\epsilon \in [0, 1]$ and the coordinate radius ρ_0 . For $g \ll 1$, one obtains a Newtonian solution, and $g \rightarrow 1$ corresponds to the ultra-relativistic limit in which we assume black hole formation. Without loss of generality, we restrict to positive charges and ρ_0 serves as a scaling parameter. All multipole moments defined in the subsequent section are functions of g , ϵ , and ρ_0 only.

III. MULTIPOLE MOMENTS

In order to derive multipole moments, we introduce new potentials

$$\Xi := \frac{1 - \mathcal{E}}{1 + \mathcal{E}} \quad \text{and} \quad \Lambda := \frac{2\Phi}{1 + \mathcal{E}}. \quad (13)$$

Those can be expressed as an asymptotic expansion on the upper symmetry axis:

$$\Xi_+ = \frac{1}{\zeta} \sum_{n=0}^{\infty} \frac{m_n}{\zeta^n}, \quad \Lambda_+ = \frac{1}{\zeta} \sum_{n=0}^{\infty} \frac{q_n}{\zeta^n}. \quad (14)$$

According to Fodor *et al.* [7], the gravitational and electromagnetic multipole moments, P_n and Q_n , respectively, of an axisymmetric and stationary spacetime can be obtained from the coefficients m_n and q_n by the following procedure:

$$P_0 = m_0, \quad (15)$$

$$P_1 = m_1, \quad (16)$$

$$P_2 = m_2, \quad (17)$$

$$P_3 = m_3 + \frac{1}{5} \bar{q}_0 S_{10}, \quad (18)$$

$$P_4 = m_4 - \frac{1}{7} \bar{m}_0 M_{20} + \frac{3}{35} \bar{q}_1 S_{10} + \frac{1}{7} \bar{q}_0 (3S_{20} - 2H_{20}), \quad (19)$$

$$\begin{aligned}
 P_5 = m_5 - \frac{1}{21}\bar{m}_1 M_{20} - \frac{1}{3}\bar{m}_0 M_{30} + \frac{1}{21}\bar{q}_2 S_{10} \\
 + \frac{1}{21}\bar{q}_1(4S_{20} - 3H_{20}) + \frac{1}{21}\bar{q}_0(\bar{q}_0 q_0 S_{10} \\
 - \bar{m}_0 m_0 S_{10} + 14S_{30} + 13S_{21} - 7H_{30})
 \end{aligned} \quad (20)$$

and

$$Q_0 = q_0, \quad (21)$$

$$Q_1 = q_1, \quad (22)$$

$$Q_2 = q_2, \quad (23)$$

$$Q_3 = q_3 - \frac{1}{5}\bar{m}_0 H_{10}, \quad (24)$$

$$Q_4 = q_4 + \frac{1}{7}\bar{q}_0 Q_{20} - \frac{3}{35}\bar{m}_1 H_{10} - \frac{1}{7}\bar{m}_0(3H_{20} - 2S_{20}), \quad (25)$$

$$\begin{aligned}
 Q_5 = q_5 + \frac{1}{21}\bar{q}_1 Q_{20} + \frac{1}{3}\bar{q}_0 Q_{30} - \frac{1}{21}\bar{m}_2 H_{10} \\
 - \frac{1}{21}\bar{m}_1(4H_{20} - 3S_{20}) + \frac{1}{21}\bar{m}_0(\bar{m}_0 m_0 H_{10} \\
 - \bar{q}_0 q_0 H_{10} - 14H_{30} - 13H_{21} + 7S_{30}),
 \end{aligned} \quad (26)$$

where

$$M_{ij} = m_i m_j - m_{i-1} m_{j+1}, \quad Q_{ij} = q_i q_j - q_{i-1} q_{j+1}, \quad (27)$$

$$S_{ij} = m_i q_j - m_{i-1} q_{j+1}, \quad H_{ij} = q_i m_j - q_{i-1} m_{j+1}. \quad (28)$$

Higher multipole moments become increasingly complicated. The gravitational and electromagnetic multipole moments are closely related to each other. In fact, by interchanging $m_n \leftrightarrow q_n$ and $\bar{m}_n \leftrightarrow -\bar{q}_n$ (correspondingly $M_{ij} \leftrightarrow Q_{ij}$ and $S_{ij} \leftrightarrow H_{ij}$), P_n transforms into Q_n .

As mentioned in Sec. II, the disc of dust additionally obeys reflection symmetry. Expressed on the upper symmetry axis, reflection symmetry is equivalent to [20–24]

$$\mathcal{E}_+(\zeta)\bar{\mathcal{E}}_+(-\zeta) = 1, \quad \Phi_+(\zeta) = -\bar{\Phi}_+(-\zeta)\mathcal{E}_+(\zeta), \quad (29)$$

or in terms of the new potentials to

$$\Xi_+(\zeta) = -\bar{\Xi}_+(-\zeta), \quad \Lambda_+(\zeta) = -\bar{\Lambda}_+(-\zeta). \quad (30)$$

This means that P_n and Q_n are real for even n and imaginary for odd n . The real and imaginary parts of P_n are called mass and angular momentum moments and those of Q_n are referred to as electric and magnetic moments. We denote:

$$P_n = M_n + iJ_n, \quad (31)$$

$$Q_n = E_n + iB_n, \quad (32)$$

where due to reflection symmetry,

$$\text{for even } n: P_n = M_n, \quad Q_n = E_n, \quad (33)$$

$$\text{for odd } n: P_n = iJ_n, \quad Q_n = iB_n. \quad (34)$$

Note that on the upper symmetry axis $\eta = 1$, $\zeta = \rho_0 \nu$, and thus

$$\Xi_+ = \frac{1}{\nu} \sum_{n=0}^{\infty} \frac{m_n^*}{\nu^n}, \quad \Lambda_+ = \frac{1}{\nu} \sum_{n=0}^{\infty} \frac{q_n^*}{\nu^n}, \quad (35)$$

with dimensionless $m_n^* := \frac{m_n}{\rho_0^{n+1}}$ and $q_n^* := \frac{q_n}{\rho_0^{n+1}}$. In order to obtain coordinate independent expressions for the multipole moments, we normalize m_n and q_n by the disc's proper radius $R_0 := \int_0^{\rho_0} \sqrt{g_{\rho\rho}} d\rho$, where $g_{\rho\rho} = f^{-1}h$, see also [25]:

$$m_n^\circ := \frac{m_n}{R_0^{n+1}}, \quad q_n^\circ := \frac{q_n}{R_0^{n+1}}. \quad (36)$$

The first multipole moments are the gravitational mass, $P_0 = M_0 = M$, the angular momentum, $P_1/i = J_1 = J$, the electric charge, $Q_0 = E_0 = Q$, and the magnetic dipole moment, $Q_1/i = B_1 = D$.

Inserting the potentials f , b , α , and β , Eqs. (8) and (9), into Ξ and Λ , Eq. (13), and applying the above mentioned procedure reveals the multipole moments of the charged rotating disc of dust in terms of a post-Newtonian expansions up to tenth order. The first multipole moments, normalized by the disc's proper radius R_0 , up to third order ($k = 3$), read²:

$$\begin{aligned}
 M^\circ = \frac{4g^2}{3\pi} - \frac{4(\epsilon^2 - 1)g^4}{45\pi} \\
 + \frac{1}{30240\pi^3} ((2790\epsilon^4 - 23699\epsilon^2 + 20464)\pi^2 \\
 - 35840\epsilon^4 + 247296\epsilon^2 - 211456)g^6 + \mathcal{O}(g^8),
 \end{aligned} \quad (37)$$

$$\begin{aligned}
 J^\circ = \sqrt{1 - \epsilon^2} \left[\frac{8g^3}{15\pi} - \frac{2(34\epsilon^2 - 7)g^5}{315\pi} \right. \\
 + \frac{1}{453600\pi^3} ((59894\epsilon^4 - 40131\epsilon^2 - 126312)\pi^2 \\
 \left. - 322560\epsilon^4 + 207872\epsilon^2 + 1189888)g^7 + \mathcal{O}(g^9) \right],
 \end{aligned} \quad (38)$$

²Note that the notation with the circle superscript introduced here, i.e., $P_n^\circ := \frac{P_n}{R_0^{n+1}}$ and $Q_n^\circ := \frac{Q_n}{R_0^{n+1}}$, has a different meaning than that in [9].

$$Q^\circ = \epsilon \left[\frac{4g^2}{3\pi} - \frac{16(\epsilon^2 - 1)g^4}{45\pi} + \frac{1}{30240\pi^3} ((5862\epsilon^4 - 32147\epsilon^2 + 25840)\pi^2 - 35840\epsilon^4 + 247296\epsilon^2 - 211456)g^6 + \mathcal{O}(g^8) \right], \quad (39)$$

$$D^\circ = \epsilon \sqrt{1 - \epsilon^2} \left[\frac{4g^3}{15\pi} - \frac{(34\epsilon^2 - 115)g^5}{315\pi} + \frac{1}{907200\pi^3} ((58614\epsilon^4 - 177091\epsilon^2 - 200232)\pi^2 - 322560\epsilon^4 - 867328\epsilon^2 + 3340288)g^7 + \mathcal{O}(g^9) \right]. \quad (40)$$

Below, we also list the normalized gravitational and electromagnetic quadrupole and octupole moments (up to $k = 3$) of the charged rotating disc of dust:

$$P_2^\circ = M_2^\circ = -\frac{4g^2}{15\pi} + \frac{4(5\epsilon^2 + 8)g^4}{105\pi} + \frac{1}{453600\pi^3} ((-46258\epsilon^4 + 23337\epsilon^2 - 71376)\pi^2 + 107520\epsilon^4 + 68096\epsilon^2 - 175616)g^6 + \mathcal{O}(g^8), \quad (41)$$

$$\frac{P_3^\circ}{i} = J_3^\circ = \sqrt{1 - \epsilon^2} \left[-\frac{8g^3}{35\pi} + \frac{2(68\epsilon^2 + 171)g^5}{945\pi} - \frac{1}{4989600\pi^3} ((449242\epsilon^4 + 1015547\epsilon^2 - 256536)\pi^2 - 1520640\epsilon^4 - 4764672\epsilon^2 + 10543104)g^7 + \mathcal{O}(g^9) \right], \quad (42)$$

$$Q_2^\circ = E_2^\circ = \epsilon \left[-\frac{4g^2}{15\pi} + \frac{4(4\epsilon^2 + 9)g^4}{105\pi} - \frac{1}{453600\pi^3} ((36018\epsilon^4 - 26537\epsilon^2 + 84816)\pi^2 - 107520\epsilon^4 - 68096\epsilon^2 + 175616)g^6 + \mathcal{O}(g^8) \right], \quad (43)$$

$$\frac{Q_3^\circ}{i} = B_3^\circ = \epsilon \sqrt{1 - \epsilon^2} \left[-\frac{4g^3}{35\pi} + \frac{(62\epsilon^2 + 51)g^5}{945\pi} - \frac{1}{9979200\pi^3} ((394522\epsilon^4 + 54107\epsilon^2 - 2568216)\pi^2 - 1520640\epsilon^4 - 9022464\epsilon^2 + 19058688)g^7 + \mathcal{O}(g^9) \right]. \quad (44)$$

The mass, angular momentum, electric and magnetic moments are alternating positive and negative, to be more precise,

$$\text{for } l = 0, 2, 4, \dots: \quad \begin{array}{ll} M_{2l} \geq 0, & J_{2l+1} \geq 0, \\ E_{2l} \geq 0, & B_{2l+1} \geq 0, \end{array} \quad (45)$$

$$\text{for } l = 1, 3, 5, \dots: \quad \begin{array}{ll} M_{2l} \leq 0, & J_{2l+1} \leq 0, \\ E_{2l} \leq 0, & B_{2l+1} \leq 0. \end{array} \quad (46)$$

It should be emphasized that the global prefactors, whereby

$$J_n \sim \sqrt{1 - \epsilon^2}, \quad E_n \sim \epsilon, \quad B_n \sim \epsilon \sqrt{1 - \epsilon^2}, \quad (47)$$

ensure that for vanishing charge, i.e., $\epsilon = 0$, all electromagnetic multipole moments Q_n become zero, and for vanishing rotation, i.e., $\epsilon = 1$, all angular momentum and magnetic moments, J_n and B_n , respectively.³

Note that disc configurations with $\epsilon = 0$ rotate with maximal angular velocity and ones with $\epsilon = 1$ have no rotation at all. This becomes obvious in the Newtonian limit where each dust particle in the disc is in an equilibrium of gravitational, electric and centrifugal force.

The influence of the global prefactors is also reflected in Figs. 1–4. There, the dependence of the multipole moments normalized by the proper radius, M_n° , J_n° , E_n° , and B_n° , on the specific charge ϵ is depicted for $n = 0, \dots, 7$. The relativity parameter is set to $g = 0.6$.

Figures 5 and 6 show the dependence of the normalized gravitational multipole moments on the relativity parameter g , where $\epsilon = 0.3$. As can be seen there, in the limit $g \rightarrow 0$, all gravitational multipole moments P_n° vanish, and with increasing g the first ones (to be more precise, $n \leq 1$ for all ϵ) grow monotonically. The electromagnetic multipole moments Q_n show a completely analogous behavior.

³Additionally, mass and electric moments contain only even powers of g and angular momentum and magnetic moments only odd, due to a symmetric and antisymmetric transformation behavior, respectively, under a change of sense of rotation. For further details, see [8] or [25].

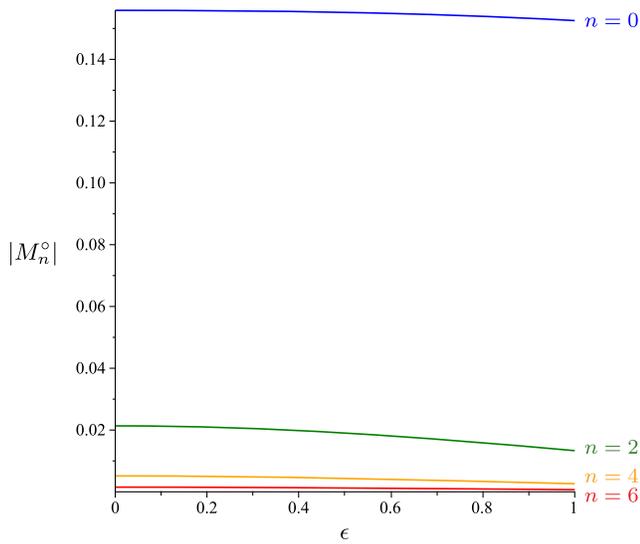


FIG. 1. Normalized mass moments M_n^o for $n = \{0, 2, 4, 6\}$ as functions of the specific charge ϵ plotted for $g = 0.6$.

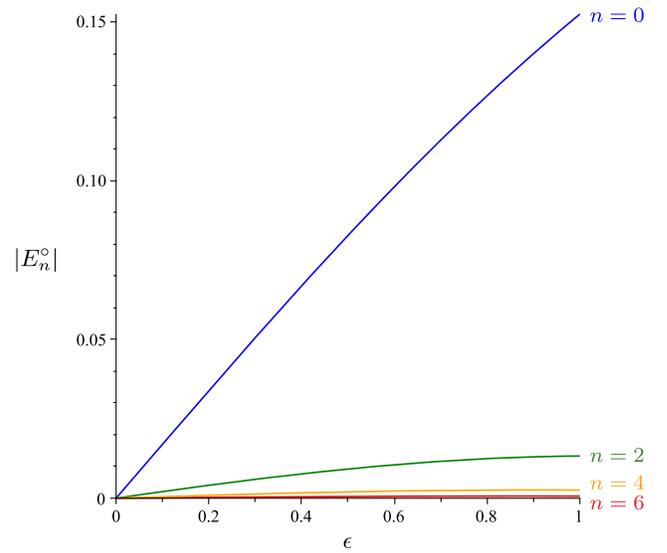


FIG. 3. Normalized electric moments E_n^o for $n = \{0, 2, 4, 6\}$ as functions of the specific charge ϵ plotted for $g = 0.6$.

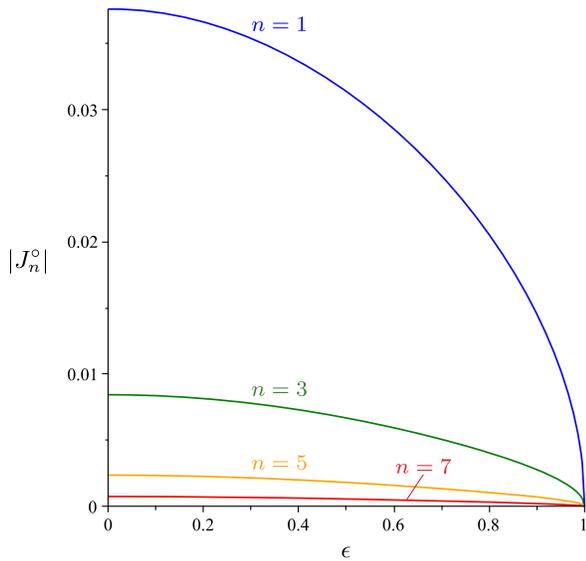


FIG. 2. Normalized angular momentum moments J_n^o for $n = \{1, 3, 5, 7\}$ as functions of the specific charge ϵ plotted for $g = 0.6$.

Remarkably, as can be seen in Figs. 1–6, all multipole moments are perfectly ordered:

$$|M_n| \geq |M_{n+2}|, \quad (48)$$

$$|J_n| \geq |J_{n+2}|, \quad (49)$$

$$|B_n| \geq |B_{n+2}|, \quad (50)$$

$$|E_n| \geq |E_{n+2}|. \quad (51)$$

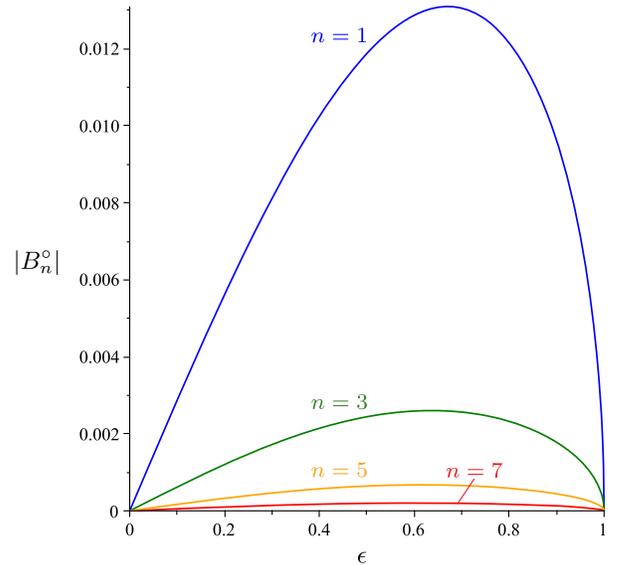


FIG. 4. Normalized magnetic moments B_n^o for $n = \{1, 3, 5, 7\}$ as functions of the specific charge ϵ plotted for $g = 0.6$.

Notice that this ordering as well as the alternating sign can also be observed for the (exact) gravitational multipole moments of the uncharged disc of dust [15].

Interesting are also the ratios $\frac{P_n^o}{P_{n-2}^o}$ and $\frac{Q_n^o}{Q_{n-2}^o}$ of the individual moments. According to Figs. 7 and 8, the absolute values of the ratios are greater, the higher the gravitational multipole moments are. The corresponding plots of the ratios of the electromagnetic multipole moments look almost identical to those of the gravitational moments and in fact agree at $g = 0$.⁴ In addition, there is a

⁴This is not a coincidence, as in the Newtonian limit $\Lambda = \epsilon \Xi$ holds.

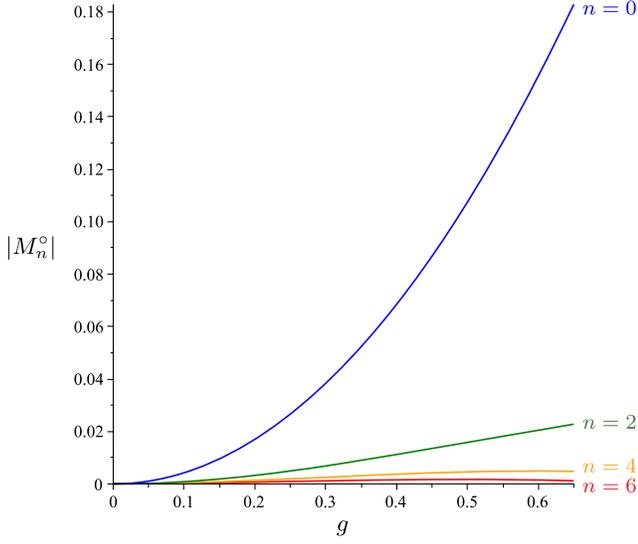


FIG. 5. Normalized mass moments M_n° for $n = \{0, 2, 4, 6\}$ as functions of the relativity parameter g plotted for $\epsilon = 0.3$.

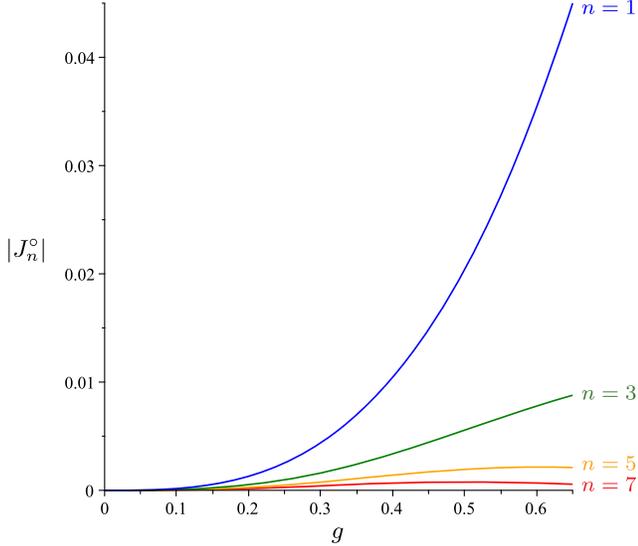


FIG. 6. Normalized angular momentum moments J_n° for $n = \{1, 3, 5, 7\}$ as functions of the relativity parameter g plotted for $\epsilon = 0.3$.

pairwise agreement at $g = 0$: $\frac{P_3^\circ}{P_1^\circ} = \frac{P_4^\circ}{P_2^\circ} = -\frac{3}{7}$ ($= \frac{Q_3^\circ}{Q_1^\circ} = \frac{Q_4^\circ}{Q_2^\circ}$) and $\frac{P_5^\circ}{P_3^\circ} = \frac{P_6^\circ}{P_4^\circ} = -\frac{5}{9}$ ($= \frac{Q_5^\circ}{Q_3^\circ} = \frac{Q_6^\circ}{Q_4^\circ}$).

The decision to restrict our discussion to the first eight multipole moments P_n and Q_n and the relativity parameter to $g \leq 0.65$ is based on the convergence behavior of the multipole moments. As an example, to estimate the convergence of the electric moments, we calculate the resulting change by adding the K th order to the post-Newtonian expansion at order $K - 1$ relative to the solution at K th order:

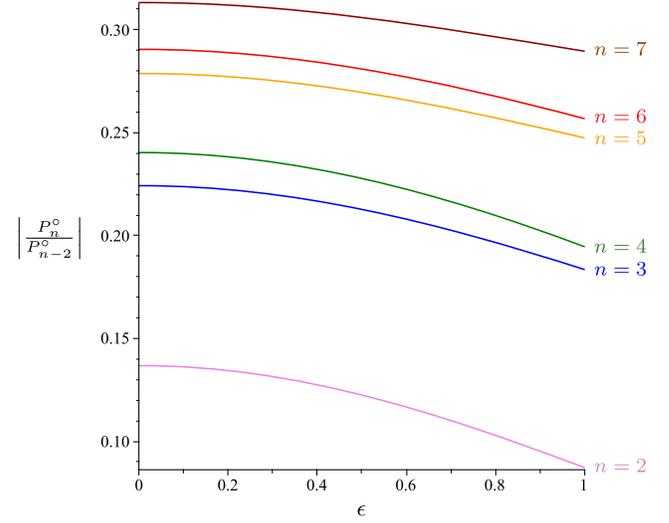


FIG. 7. Ratios $\frac{P_n^\circ}{P_{n-2}^\circ}$ for $n = \{2, 3, \dots, 7\}$ as functions of the specific charge ϵ plotted for $g = 0.6$.

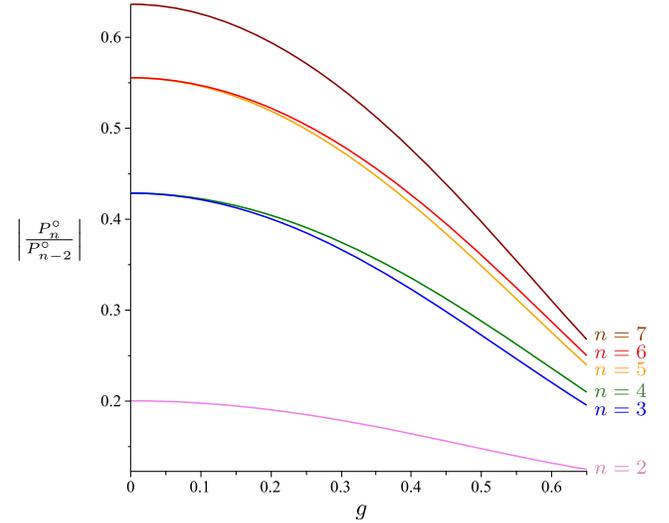


FIG. 8. Ratios $\frac{P_n^\circ}{P_{n-2}^\circ}$ for $n = \{2, 3, \dots, 7\}$ as functions of the relativity parameter g plotted for $\epsilon = 0.3$.

$$\delta E_{n|K}^\circ := \frac{|E_{n|K}^\circ - E_{n|K-1}^\circ|}{|E_{n|K}^\circ|}, \quad (52)$$

$$\text{where } E_{n|K}^\circ := \sum_{k=1}^K E_{n2k}^\circ g^{2k}. \quad (53)$$

According to Fig. 9, the relative change by adding the last order is for all E_n° ($n = 0, \dots, 6$) less than or equal to 2.2×10^{-4} for $g = 0.65$. (In this case, $\epsilon = 0$ shows the worst convergence behavior.) Moreover, as can already be guessed from the calculation scheme, the higher the multipole moments, the less good the convergence.

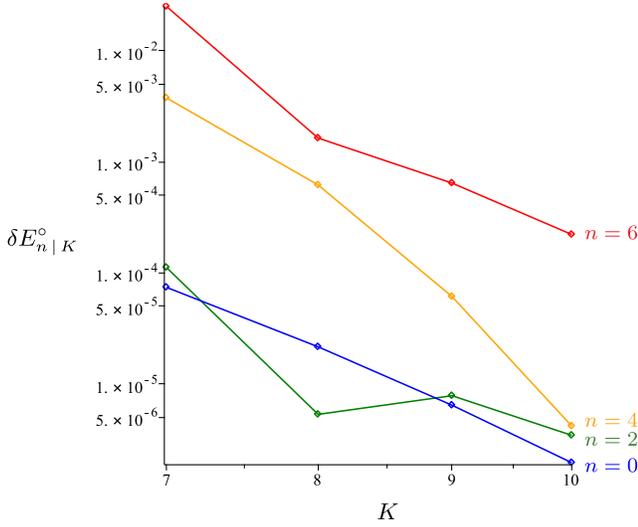


FIG. 9. Convergence estimate of the normalized multipole moments E_0^o , E_2^o , E_4^o , and E_6^o at order $k = K$ plotted for $g = 0.65$ and $\epsilon = 0$.

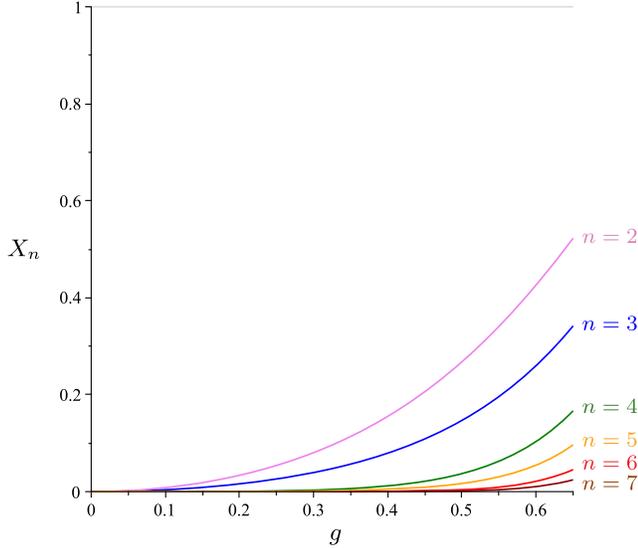


FIG. 10. X_n for $n = \{2, 3, \dots, 7\}$ as functions of the relativity parameter g plotted for $\epsilon = 0$.

For the mass, angular momentum and magnetic moments, we get an even better, but otherwise analogous convergence behavior.⁵

A direct comparison of the derived multipole moments (in terms of the post-Newtonian expansion) in the limit $\epsilon = 0$ with the exact solutions of the uncharged rotating disc of dust [15] shows an excellent agreement. In Table I, we compare the sixth and seventh gravitational multipole moments for different values of the relativity parameter.

⁵For $\epsilon = 0$ and $g = 0.65$, one gets: $\delta M_{6|10}^o = 7.5 \times 10^{-5}$, $\delta J_{7|10}^o = 1.9 \times 10^{-4}$, $\delta B_{7|10}^o = 9.7 \times 10^{-5}$.

TABLE I. Normalized multipole moments M_6^o, J_7^o : charged disc in the limit $\epsilon = 0$ (post-Newtonian expansion up to tenth order) versus uncharged disc (exact solution).

		Charged disc $\epsilon=0$	Uncharged disc
$g = 0.55$	$ M_6^o $	1.6286550×10^{-3}	1.6286551×10^{-3}
	$ J_7^o $	8.0663923×10^{-4}	8.0663952×10^{-4}
$g = 0.6$	$ M_6^o $	1.4875997×10^{-3}	1.4876000×10^{-3}
	$ J_7^o $	7.3636251×10^{-4}	7.3636450×10^{-4}
$g = 0.65$	$ M_6^o $	1.3141025×10^{-3}	1.3141033×10^{-3}
	$ J_7^o $	6.3590588×10^{-4}	6.3591807×10^{-4}

In particular, the relative error of the multipole moments of the charged rotating disc of dust in the limit $\epsilon = 0$ with respect to the exact solutions at $g = 0.65$ is 6.1×10^{-7} for M_6^o and 1.9×10^{-5} for J_7^o .

IV. MULTIPOLE CONJECTURE

Filter and Kleinwächter [16] formulated an interesting conjecture about the multipole moments of (uncharged) rigidly rotating perfect fluid bodies: The absolute values of all higher multipole moments P_n ($n \geq 2$) of an axisymmetric, rigidly rotating, perfect fluid body, surrounded by vacuum, are always greater than those of the corresponding moments of the Kerr spacetime with the same mass and angular momentum.

An obvious question now is whether this conjecture can be extended to more general, particularly charged, bodies.

The multipole moments of the Kerr-Newman spacetime, with mass M^{KN} , angular momentum J^{KN} and charge Q^{KN} , are given by [6]:

$$P_n^{\text{KN}} = m_n^{\text{KN}} = M^{\text{KN}} \left(i \frac{J^{\text{KN}}}{M^{\text{KN}}} \right)^n, \quad (54)$$

$$Q_n^{\text{KN}} = q_n^{\text{KN}} = Q^{\text{KN}} \left(i \frac{J^{\text{KN}}}{M^{\text{KN}}} \right)^n. \quad (55)$$

With these we state:

Generalized multipole conjecture. *For the gravitational and electromagnetic multipole moments, P_n and Q_n , of an isolated, axisymmetric, stationary, physically well-defined body of ordinary matter, with mass M , angular momentum J , and charge Q , holds for all $n \geq 2$:*

$$|P_n| \geq \left| \frac{J^n}{M^{n-1}} \right|, \quad |Q_n| \geq \left| \frac{QJ^n}{M^n} \right|. \quad (56)$$

Furthermore, in case of $J \neq 0$, equalities apply if and only if the body reaches a black hole limit.

The goal of this section is to test the generalized multipole conjecture using the charged rotating disc of dust as a concrete and physically meaningful candidate that

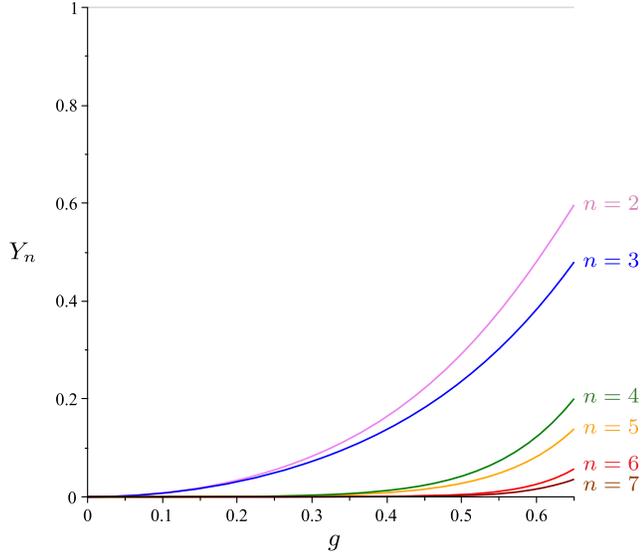


FIG. 11. Y_n for $n = \{2, 3, \dots, 7\}$ as functions of the relativity parameter g plotted for $\epsilon = 0$.

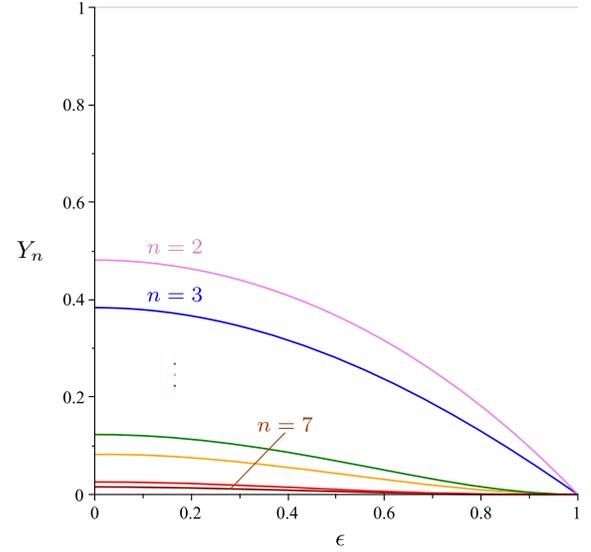


FIG. 13. Y_n for $n = \{2, 3, \dots, 7\}$ as functions of the specific charge ϵ plotted for $g = 0.6$.

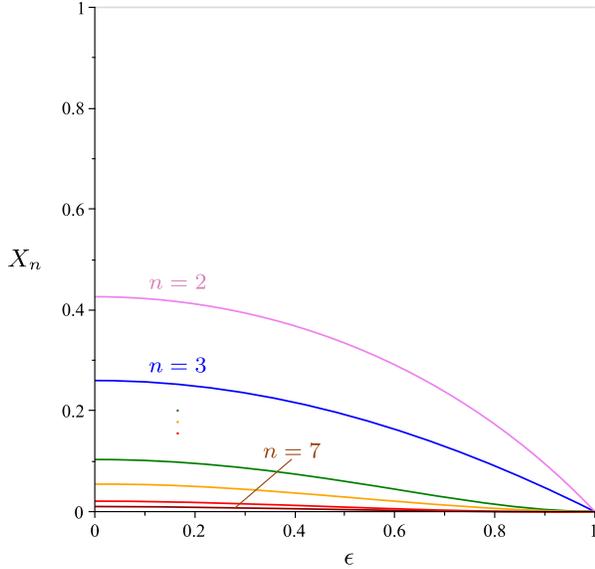


FIG. 12. X_n for $n = \{2, 3, \dots, 7\}$ as functions of the specific charge ϵ plotted for $g = 0.6$.

satisfies the requirements of the conjecture.⁶ To this end, we plot the quantities X_n and Y_n that have to be less than or equal to 1 if the conjecture is valid:

⁶Note that the Kerr-Newman spacetime furnished with the mass M , angular momentum J , and charge Q of the charged rotating disc of dust does not describe a black hole but a hyperextreme solution ($Q^2 + \frac{J^2}{M^2} > M^2$).

$$X_n := \left| \frac{J^n}{M^{n-1} P_n} \right| \leq 1, \quad Y_n := \left| \frac{Q J^n}{M^n Q_n} \right| \leq 1. \quad (57)$$

Notice that due to the global prefactors of P_n and Q_n the quantities X_n and Y_n stay regular in the limits $\epsilon \rightarrow 1$ and $\epsilon \rightarrow 0$. In fact, X_n and Y_n vanish for $\epsilon \rightarrow 1$ and take finite values in the limit $\epsilon \rightarrow 0$ (see also Figs. 12 and 13).

In the limit $\epsilon \rightarrow 1$, the angular momentum J vanishes and therefore also all Kerr-Newman multipole moments, P_n^{KN} and Q_n^{KN} , with $n \geq 1$. Since the multipole moments of the disc, P_n and Q_n , become zero only for odd, but not for even n , the disc spacetime clearly does not attain the Reissner-Nordström solution in this limit.

As can be seen in Figs. 10–13, indeed all X_n and Y_n are smaller than 1 in the region $g \in [0, 0.65]$. Furthermore, there is a similar ordering as for the individual moments:

$$X_n \geq X_{n+1}, \quad Y_n \geq Y_{n+1}. \quad (58)$$

Therefore, most critical are X_2 and Y_2 , and for increasing n , the conjecture is even better fulfilled.

On the other hand, the first X_n and Y_n are also the quantities with the best convergence behavior. As n becomes larger, the convergence of X_n and Y_n (as it contains P_n and Q_n , respectively) becomes less and less good. The relative change [as defined in Eq. (52)] by adding the last order is $\delta X_{7|10} = 1.9 \times 10^{-2}$ and $\delta Y_{7|10} = 9.7 \times 10^{-3}$, where $g = 0.65$ and $\epsilon = 0$.⁷

⁷Specifically for X_7 and Y_7 , the convergence gets slightly worse for higher ϵ : $\delta X_{7|10} = 4.1 \times 10^{-2}$ and $\delta Y_{7|10} = 3.4 \times 10^{-2}$, for $\epsilon = 1$ and $g = 0.65$. However, as X_n and Y_n for high n are not decisive for the verification of the conjecture, this is not problematic.

The good convergence of X_2 and Y_2 allows verification of the conjecture for slightly higher values of the relativity parameter until $g \approx 0.8$ and $g \approx 0.75$, respectively.⁸ The conjecture remains true there. For the limit $g \rightarrow 1$, Breithaupt *et al.* [9] provided strong evidence that the multipole moments of the charged rotating disc of dust converge to those of the (extreme) Kerr-Newman spacetime.

Additionally, one can observe from Figs. 12 and 13 that all values of X_n and Y_n decrease equally with increasing ϵ . This fact is very convenient for our purposes, since for the most critical case, $\epsilon = 0$, the exact multipole moments [15] of the uncharged disc of dust [12–14] are available. The relative error of $X_2|_{\epsilon=0}$ with respect to the corresponding exact value of the uncharged disc of dust is 1.3×10^{-2} for $g = 0.8$. For the gravitational multipole moments of the uncharged disc of dust, Filter and Kleinwächter showed that the conjecture is fulfilled [16].

In summary, the conjecture holds at least up to $g \approx 0.8$ for the gravitational and up to $g \approx 0.75$ for the electromagnetic multipole moments. In the limit $g \rightarrow 1$, the quantities X_n and Y_n converge to 1 according to Breithaupt *et al.*, and for the most critical case, $\epsilon = 0$, by means of the exact solution of the uncharged disc of dust, it was proven that $X_n \leq 1$. All this taken together, we conclude that (within the scope of accuracy of the post-Newtonian expansion) the generalized multipole conjecture stated above is fulfilled for the multipole moments of the charged rotating disc of dust.

V. CONCLUSIONS AND OUTLOOK

The individual mass, angular momentum, electric and magnetic moments of the charged rotating disc of dust, in

⁸For $\epsilon = 0$: $\delta X_{2|10} = 1.9 \times 10^{-2}$ at $g = 0.8$ and $\delta Y_{2|10} = 2.8 \times 10^{-2}$ at $g = 0.75$.

agreement with the uncharged disc [15], follow an ordering in which the higher they are, the less they contribute to the gravitational and electromagnetic field. This effect is in addition to the suppression of higher orders in the asymptotic expansions. Therefore, primarily the first multipole moments are relevant for the far field behavior of the gravitational and electromagnetic field of the charged rotating disc of dust.

Of course, it would be desirable to have a proof of the generalized multipole conjecture. This would first require a precise notion of physical well-definedness.⁹ In case of nonvanishing angular momentum, the conjecture could serve as a powerful tool to distinguish ordinary physical bodies from black holes by measuring their multipole moments in the far field. In fact, if the necessary condition $Q^2 + \frac{J^2}{M^2} \leq M^2$ for a black hole is satisfied, the measurement of the multipole moments for $n = 2$ would already make the difference.

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⁹Perhaps one should consider a stronger condition, like uniformly rotating, charged perfect fluid bodies with convective electric currents only, see [26].

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