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We study gravitational radiation for a positive value of the cosmological constant  $\Lambda$ . We rely on two battle-tested procedures: (i) We start from the same null coordinate system used by Bondi and Sachs for  $\Lambda = 0$ , but, introduce boundary conditions adapted to allow radiation when  $\Lambda > 0$ ; and (ii) We determine the asymptotic symmetries by studying, *à la* Regge-Teitelboim, the surface integrals generated in the action by these boundary conditions. A crucial difference with the  $\Lambda = 0$  case is that the wave field does not vanish at large distances, but is of the same order as de Sitter space. This novel property causes no difficulty; on the contrary, it makes quantities finite at every step, without any regularization. A direct consequence is that the asymptotic symmetry algebra consists only of time translations and space rotations. Thus, it is not only finite dimensional, but smaller than de Sitter algebra. We exhibit formulas for the energy and angular momentum and their fluxes. In the limit of  $\Lambda$  tending to zero, these formulas go over continuously into those of Bondi, but the symmetry jumps to that of Bondi, Metzner, and Sachs. The expressions are applied to exact solutions, with and without radiation present, and also to the linearized theory.

DOI: [10.1103/PhysRevD.108.064039](https://doi.org/10.1103/PhysRevD.108.064039)**I. INTRODUCTION**

We study gravitational radiation for a positive value of the cosmological constant  $\Lambda$  as generated by compact sources such as stars and black holes. We are guided by, and closely follow the steps of Sachs' classical analysis of the concepts, introduced by Bondi for  $\Lambda = 0$  [1–3]. That analysis gave unambiguous expressions for energy and energy flux, and also established the existence of an infinite-dimensional symmetry algebra, now called the Bondi-Metzner-Sachs (BMS) algebra.

Bondi and Sachs relied only on Einstein's equations with boundary conditions to study a large class of spacetimes, now known as "asymptotically flat" solutions. They did not, and did not need to, invoke the action principle. They required though, that the mass should decrease when radiation is emitted. This demand was essential to arrive

at unambiguous expressions for the mass and its flux. We have not been able to implement the mass diminution requirement for  $\Lambda > 0$ , although we do recover this in the linearized case. In order to arrive at the formulas for the mass and its flux, we have appealed instead to the action principle, including in it appropriate surface integrals *à la* Regge-Teitelboim.

The symmetry algebra consists only of time translations and space rotations, even when gravitational radiation is present. It is not only finite dimensional but smaller than the de Sitter algebra. This result is crucially linked to the  $\mathbb{R} \times \mathbb{S}^2$  topology of the future boundary, and as shown in [4], the symmetry group for asymptotically de Sitter spacetimes depends crucially on the topology. The existence of this smaller symmetry algebra can be attributed to the fact that in the presence of a positive  $\Lambda$  the wave field does not vanish at large distances, in sharp contrast with the asymptotically flat case. As a result, a generic gravitational wave will induce a strong deformation on the geometry of the future boundary. This deformation of the asymptotic region precludes the

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presence of the full de Sitter group or infinite-dimensional extensions thereof. In the limit  $\Lambda \rightarrow 0$ , the mass and flux formulas coincide with those of Bondi and Sachs. In contradistinction, the symmetry algebra has an enormous jump; it becomes the BMS symmetry.

For waves of small amplitude, we recover the energy flux of the linear theory on a de Sitter background. Interestingly, even waves with small amplitudes reach infinity without decay. This is not the case for asymptotically flat spacetimes, in which linear waves decay at least as  $1/r$  at large distances. For special solutions such as Kerr-de Sitter and Robinson-Trautman with  $\Lambda > 0$ , we recover the accepted expressions for the mass and angular momentum. All of our conclusions follow from general relativity, by bringing into it boundary conditions which are the natural extension for  $\Lambda > 0$  of those employed by Sachs for  $\Lambda = 0$ .

Interestingly, all equations in this paper hold also for  $\Lambda < 0$ . In that case, the boundary conditions do not correspond to the typical reflecting ones (see e.g. [5–7]).<sup>1</sup> We find this an appealing issue for exploration, but we will not address it here.

The boundary conditions in this paper are new and have not been explored elsewhere. These new boundary conditions have a finite-dimensional symmetry algebra, lead to finite charges, and accommodate many interesting spacetimes describing radiation in de Sitter-like spacetimes (such as Robinson-Trautman with  $\Lambda > 0$  and linearized waves on a de Sitter background). Some of these properties have been obtained from different boundary conditions and/or different methods as well. We compare our results to other approaches in Sec. VI.

The setup of this paper is as follows. We introduce the Bondi-Sachs coordinates and their falloff rates in Sec. II. Given these coordinates, we study the asymptotic symmetry algebra in Sec. III. In Sec. IV, we compute the energy and angular momentum at future infinity and their fluxes. We apply this formalism to various exact solutions of general relativity with  $\Lambda > 0$  in Sec. V, in which we also study linearized gravitational waves on a de Sitter background. Finally, we compare our approach to various alternative approaches in Sec. VI. The key findings of this paper are summarized in Table I.

## II. BONDI REVISITED FOR $\Lambda > 0$

Although the geometry is very different for  $\Lambda > 0$  and  $\Lambda = 0$ , it turns out that in the natural extension of the coordinate system used by Bondi and Sachs the formulas for energy flux, energy, and the like turn out to be remarkably simple, and furthermore reduce for  $\Lambda = 0$  to theirs. For this reason, we will go right away into the analysis in that particular coordinate system.

<sup>1</sup>This is clear from the nonconformal flatness of the boundary metric.

### A. Asymptotic behavior of the metric

In the coordinate system  $(u, r, \theta, \phi)$  originally introduced by Bondi [1] and generalized later to the non-axisymmetric case by Sachs [2,3], the line element reads

$$ds^2 = e^{2\beta} \frac{V}{r} du^2 - 2e^{2\beta} dudr + r^2 g_{AB} (dx^A - U^A du)(dx^B - U^B du) \quad (1)$$

with  $-\infty < u < \infty$  and  $0 < r < \infty$ . The  $x^A$  are coordinates on the two-sphere, which we choose here to be the standard spherical one;  $x^A = (\theta, \phi)$  with  $0 \leq \theta \leq \pi$  and  $0 \leq \phi < 2\pi$ .<sup>2</sup> The coordinate  $u$  is null because when  $du = 0$  and  $dx^A = 0$ , one has  $ds^2 = 0$ . Radiation is “observed” as  $r \rightarrow \infty$ . In this limit, one approaches the future boundary—often denoted by  $\mathcal{I}$ . These coordinates nicely encode that the topology of  $\mathcal{I}$  is  $\mathbb{R} \times \mathbb{S}^2$ , which is the relevant setting for studying gravitational radiation emitted by compact sources.

The functions  $\beta$ ,  $V$ ,  $g_{AB}$ , and  $U^A$  depend on  $x^A$ ,  $u$ , and  $r$ . The procedure is to expand the metric components in powers of  $r^{-1}$ , demand reasonable boundary conditions and impose Einstein’s equations order by order in  $r$ . The latter step does not restrict the dependence on  $u$  and  $x^A$ , but leads to relationships between different coefficients in the expansion. We will omit the details of this calculation and state the result to the order needed for the determination of possible asymptotic “charges,” and their fluxes.

One finds

$$\beta = -\frac{1}{32r^2} C^{AB} C_{AB} + \frac{1}{128r^4} ((C^{AB} C_{AB})^2 - 12C^{AB} E_{AB}) + \dots, \quad (2a)$$

$$V = \frac{\Lambda r^3}{3} - D_A U^A_{(0)} r^2 - \left(1 + \frac{\Lambda}{16} C^{AB} C_{AB}\right) r + 2M + \dots \quad (2b)$$

$$U^A = U^A_{(0)} - \frac{1}{2r^2} D_B C^{AB} - \frac{2}{3r^3} \left(N^A - \frac{1}{2} C^{AB} D^C C_{BC}\right) + \dots, \quad (2c)$$

$$g_{AB} = \gamma_{AB} + \frac{C_{AB}}{r} + \frac{C^{CD} C_{CD} \gamma_{AB}}{4r^2} + \frac{E_{AB}}{r^3} + \dots, \quad (2d)$$

$$\det g_{AB} = \sin^2 \theta. \quad (2e)$$

These expressions depend on  $\Lambda$  explicitly in Eq. (2b) and through  $U^A_{(0)}(\Lambda)$ , which vanishes for  $\Lambda = 0$ , but depends implicitly on it according to Eq. (4) below. When  $\Lambda = 0$ ,

<sup>2</sup>Strictly speaking, one of course needs two charts to cover the two-sphere.

TABLE I. Key results summarized. Cases  $\Lambda = 0$  and  $\Lambda > 0$  compared and contrasted.

	$\Lambda = 0$	$\Lambda > 0$
Asymptotic region	Future null $\mathcal{I}$	Future spacelike $\mathcal{I}$
Topology asymptotic region	$\mathbb{R} \times \mathbb{S}^2$	$\mathbb{R} \times \mathbb{S}^2$ ( $= \mathbb{S}^3$ with two points removed) to describe radiation emitted by bounded sources
Conformal completion of infinity possible?	Yes	Yes
Coordinates	$ds^2 = e^{2\beta} \frac{V}{r} du^2 - 2e^{2\beta} du dr + r^2 g_{AB} (dx^A - U^A du)(dx^B - U^B du),$ $\det g_{AB} = \sin^2 \theta$	$ds^2 = e^{2\beta} \frac{V}{r} du^2 - 2e^{2\beta} du dr + r^2 g_{AB} (dx^A - U^A du)(dx^B - U^B du),$ $\det g_{AB} = \sin^2 \theta$
Falloff	(Bondi gauge with the Sachs condition) $\beta = O(r^{-2}),$ $U^A = -\frac{1}{2r^2} D_B C^{AB}$ $-\frac{2}{3r^3} (N^A - \frac{1}{2} C^{AB} D^C C_{BC}) + \dots,$ $V = -r + 2M + \dots,$ $g_{AB} = \gamma_{AB} + \frac{C_{AB}}{r} + \frac{\gamma_{AB} C_{CD} C^{CD}}{4r^2} + \frac{E_{AB}}{r^3} + \dots$	(Bondi gauge with the Sachs condition) $\beta = O(r^{-2}),$ $U^A = U_{(0)}^A - \frac{1}{2r^2} D_B C^{AB}$ $-\frac{2}{3r^3} (N^A - \frac{1}{2} C^{AB} D^C C_{BC}) + \dots,$ $V = \frac{\Lambda r^3}{3} - D_A U_{(0)}^A r^2 - \left(1 + \frac{\Lambda}{16} C_{AB} C^{AB}\right) r + 2M + \dots,$ $g_{AB} = \gamma_{AB} + \frac{C_{AB}}{r} + \frac{\gamma_{AB} C_{CD} C^{CD}}{4r^2} + \frac{E_{AB}}{r^3} + \dots,$ $D_A U_{(0)B} + D_B U_{(0)A} - \gamma_{AB} D_C U_{(0)}^C = \frac{\Lambda}{3} C_{AB}$
Radiation field vanishes at infinity	Yes ( $U_{(0)}^A = 0$ )	No ( $U_{(0)}^A \neq 0$ )
Imprint on the metric of the most general wave	Symmetric traceless tensor $C_{AB}$ arbitrary functions of the retarded time and the angles (generic graviton)	Symmetric traceless tensor $C_{AB}$ arbitrary functions of the retarded time and the angles (generic graviton)
Symmetry	Infinite-dimensional (“BMS”) Lie algebra: $so(3, 1) +$ “supertranslations”	Four-dimensional Lie algebra: $so(3) \oplus \mathbb{R}$
Energy (“Bondi mass”)	$E = \frac{1}{4\pi G} \oint d^2 S M$	$E = \frac{1}{4\pi G} \oint d^2 S M$
Angular momentum	$\vec{J} = \frac{1}{8\pi G} \oint d^2 S \hat{r} \epsilon^{AB} D_A N_B$	$\vec{J} = \frac{1}{8\pi G} \oint d^2 S \hat{r} \epsilon^{AB} D_A N_B$
Angular momentum ambiguity	Yes (angular momentum not invariant under supertranslations)	No (there are no supertranslations)
Energy flux	$\frac{dE}{du} = -\frac{1}{32\pi G} \oint d^2 S N_{AB} N^{AB},$ with $N_{AB} := \dot{C}_{AB}$	$\frac{dE}{du} = -\frac{1}{32\pi G} \oint d^2 S \left[ N_{AB}^{(\Lambda)} N^{(\Lambda)AB} + \frac{2\Lambda}{3} C^{AB} C_{AB} - \frac{\Lambda}{6} C^{AB} D^2 C_{AB} + \frac{7\Lambda^2}{144} (C^{AB} C_{AB})^2 - \frac{\Lambda^2}{3} C^{AB} E_{AB} + (4M + D_A D_B C^{AB}) (D_C U_{(0)}^C) \right],$ with $N_{AB}^{(\Lambda)} := \dot{C}_{AB} + \mathcal{L}_{U_{(0)}} C_{AB} - \frac{1}{2} (D_C U_{(0)}^C) C_{AB} - \frac{\Lambda}{6} \gamma_{AB} C_{CD} C^{CD}$
Inputs to arrive at a formula for the mass and its variation (energy flux)	Equations of motion (asymptotic form of the solution should include the generic graviton) Mass should reduce to known expressions when there is no radiation Energy flux should be negative or zero	Equations of motion (asymptotic form of the solution should include the generic graviton) Mass should reduce to known expressions when there is no radiation Action principle should be well-defined

they reduce to those of Sachs. Here  $D_A$  is the covariant derivative with respect to the metric of the unit two-sphere  $\gamma_{AB}$ . The indices  $A, B$  are lowered and raised with the metric  $\gamma_{AB}$ . The symmetric tensors  $C_{AB}$  and  $E_{AB}$  are traceless;  $\gamma^{AB} C_{AB} = \gamma^{AB} E_{AB} = 0$ .

Besides Eq. (2), there are two further restrictions on the coefficients which are of decisive importance in the analysis. They are as follows: (i) The zeroth-order term in  $g_{AB}$  is required to be the standard line element on the unit two-sphere,

$$\gamma_{AB} dx^A dx^B = d\theta^2 + \sin^2 \theta d\phi^2. \quad (3)$$

This additional demand, imposed by Bondi, which does not follow from Einstein's equations and it is not a mere restriction on the coordinate system, turns out to be of enormous consequence; it will guarantee later on that no divergent quantities appear in the analysis of a problem that has no physical singularities. In contradistinction, Eq. (2e) can be imposed to all orders by a change of coordinates  $r = f(r', \theta, \phi)$ ; and (ii) Besides the relations between the coefficients in Eq. (2), Einstein's equations imply

$$2D_{(A} U_{B)}^{(0)} - \gamma_{AB} D_C U_{(0)}^C = \frac{\Lambda}{3} C_{AB}. \quad (4)$$

Equation (4) exhibits the key difference in the imprint of the gravitational wave on the metric for  $\Lambda = 0$  versus  $\Lambda \neq 0$ . In fact, as we will see, the tensor  $C_{AB}$  describes the field of the wave when it is  $u$ -dependent, and we see from (4) that when  $\Lambda = 0$ , the waves do not affect the metric to the lowest order. However, when  $\Lambda \neq 0$  the wave affects the metric even to the lowest order through the shift vector  $U_{(0)}^A$ .

Note that the particular solution to Eq. (4) exclusively exhibits modes with  $\ell \geq 2$  that are inherited from the tensor  $C_{AB}$ . The information of the gravitational wave is exclusively contained within these modes. On the other hand, the solution of the homogeneous equation, specifically the conformal Killing equation on the two-sphere, only has  $\ell = 1$  modes and are independent of the wave degrees of freedom. These latter modes represent the freedom in selecting the frame at infinity and can be set to zero without loss of generality.

### 1. Remark

The fact that no regularization is needed at any step in the present work and that, in particular, all the charges are finite follows from allowing a generic  $U_{(0)}^A \neq 0$ . Had we imposed  $U_{(0)}^A = 0$ , we would have been forced to let  $\gamma_{AB}$  be a generic metric, but divergences would appear.

## B. Asymptotic symmetries for $\Lambda = 0$ and $\Lambda > 0$ compared and contrasted

### 1. Mass for $\Lambda = 0$

When the cosmological constant vanishes, Bondi proposed that the integral over a two-sphere of the coefficient  $M(u, \theta, \phi)$  appearing in Eq. (2b)

$$E = \frac{1}{4\pi G} \oint d^2 S M, \quad (5)$$

is the total energy of the system (with  $d^2 S = \sin \theta d\theta d\phi$ ). To validate this guess, he observed first that for the static Schwarzschild solution,  $M$  was indeed the Schwarzschild mass. Then he moved on to investigate dynamical cases

with gravitational waves, when the integral of  $M$  over a large sphere was expected to diminish as a function of  $u$  due to an energy flux emitted by a source within the sphere and going out to infinity (the coordinate  $u$  is a retarded coordinate because the sign of the  $dudr$  term in the line element is negative). This crucial test was satisfied because one can verify, from Einstein's equations, that

$$\frac{dE}{du} = -\frac{1}{32\pi G} \oint d^2 S N_{AB} N^{AB} < 0 \quad (\Lambda = 0). \quad (6)$$

The mass expression in Eq. (5) has later also been derived using other methods such as the Landau-Lifschitz approach based on a pseudo-tensor (see e.g., [8]) and covariant phase-space methods (see e.g., [9,10]).

### 2. Angular momentum for $\Lambda = 0$

If one were to attempt guessing an expression for the angular momentum, one would naturally focus on the shift  $N_A$  because it carries the imprint of being stationary (versus static). One would need a two-form to integrate over the sphere constructed out of this shift. The simplest candidate is its exterior derivative. So, one would write

$$\vec{J} = \frac{1}{8\pi G} \oint d^2 S \hat{r} \epsilon^{AB} D_A N_B. \quad (7)$$

The first test would be to check if this formula gives the right value for the angular momentum of the Kerr-de Sitter solution [which can be brought to satisfy the boundary conditions in Eq. (2), see Sec. V C]. If one does so, one finds that indeed the test is passed. One does not expect the angular momentum flux to have a definite sign so that test is not available, but a complete analysis of the asymptotically defined symmetries confirms its validity. The vector  $N^A$  is referred to as ‘‘angular momentum aspect’’.<sup>3</sup>

### 3. Symmetry for $\Lambda = 0$

In order to prove that Eqs. (5) and (7) are the energy and the angular momentum, one needs to show that they generate time translations and spatial rotations at infinity when acting on phase space. That proof, and much more, was given by Sachs who, in a brilliant analysis did two things: (i) He discovered, extending previous work of Bondi, Metzner, and Van der Burg, that the asymptotically defined symmetry is enormously larger than the expected Poincaré group, and that the commutators of its Killing vectors form an infinite-dimensional Lie algebra now called the Bondi-Metzner-Sachs algebra [2]; and (ii) He postulated a commutation rule for the two independent

<sup>3</sup>Beware, conventions differ on the exact definition of the angular-momentum aspect; some authors shift  $N^A$  by terms proportional to  $C_{AB}$  and its derivatives, and/or multiply it by a numerical factor.

components of the news  $C_{AB}$  and showed that, with just that,  $M(\theta, \phi)$  and the Lorentz generators  $J_{\mu\nu}$  that he also constructed, generate the symmetry algebra [3]. In particular, the zero mode (5) generates time translations. By guessing the commutation rule, Sachs did not need to use the action principle, but just the equations of motion. Later developments have permitted to recover the canonical generators of the Bondi-Metzner-Sachs algebra from the action principle [9,11,12].

#### 4. Symmetry for $\Lambda > 0$

For  $\Lambda = 0$ , besides the energy and angular momentum, one has boosts  $\vec{K}$  and infinitely many supertranslation generators  $M(\theta, \phi)$ , with spherical modes  $\ell \geq 1$ . The situation is dramatically different for  $\Lambda \neq 0$ , in which case only  $E$  and  $\vec{J}$  are present. The complete asymptotic symmetry algebra consists just of time translations and spatial rotations, and the expressions for the generators are the same as for  $\Lambda = 0$ . This is why we have brought them out especially above.

### III. REGGE-TEITELBOIM ANALYSIS OF THE SYMMETRIES FOR $\Lambda \neq 0$

#### A. Preservation of the asymptotic behavior of the metric

Since  $\Lambda$  does not appear explicitly in the asymptotic form (1) of the metric, the form of the asymptotic Killing vectors for  $\Lambda \neq 0$  is the same as the one given by Sachs for  $\Lambda = 0$  [Eqs. (III5)–(III7) in [3]], that is

$$\xi^u = T(u, x^A), \quad (8a)$$

$$\xi^r = -\frac{r}{2}(D_A X^A + D_A I^A - U^A D_A T), \quad (8b)$$

$$\xi^A = X^A(u, x^A) + I^A(u, r, x^A) \quad (8c)$$

$$\text{with } I^A = -(D_B T) \int_r^\infty dr' \left( \frac{e^{2\beta}}{r'^2} g^{AB} \right). \quad (8d)$$

The preservation of Eq. (4) under the action of the asymptotic Killing vectors implies that  $X^A$  must obey the following differential equation

$$2D_{(A} X_{B)} - \gamma_{AB} D_C X^C = 2U_{(A}^{(0)} D_{B)} T - \gamma_{AB} U_{(0)}^C D_C T. \quad (9)$$

The preservation of the falloff of the metric also requires that the parameters  $T$  and  $X^A$  obey the following first-order differential equations in time

$$\dot{T} = \frac{1}{2} D_A X^A - \frac{3}{2} U_{(0)}^A D_A T, \quad (10)$$

$$\dot{X}^A = \dot{T} U_{(0)}^A - U_{(0)}^A U_{(0)}^B D_B T - \frac{\Lambda}{3} D^A T. \quad (11)$$

In particular, Eq. (10) is obtained from the preservation of the decay of the  $g_{ur}$  component, and Eq. (11) from the  $g_{uA}$  component. Equations (9)–(11) constrain the algebra to three rotations and the time translation as we will see in the next subsection.

#### B. Symmetry algebra

The symmetry algebra is determined from  $T$  and  $X^A$  satisfying Eqs. (9)–(11). Equation (9) constraints  $T$  tremendously: from a generic function of  $u, \theta, \phi$  to a function of  $u$  only. Equation (10) then further requires that  $T$  is time independent, so that  $T$  can only be a constant. Using this, we find that there are only three independent solutions for  $X^A$  describing exactly the three rotations on the sphere. We will now show in detail how this comes about.

To analyze Eq. (9) it is useful to introduce  $Y^A$ ,

$$Y^A = X^A - U_{(0)}^A T, \quad (12)$$

which explicitly separates a “frame rotation” at infinity. In which case, we get

$$2D_{(A} Y_{B)} - \gamma_{AB} D_C Y^C = -\frac{\Lambda}{3} T C_{AB}. \quad (13)$$

This equation has the same form as the one obeyed by the zero-order shift [Eq. (4)], except for a negative sign—which is just a matter of convention in the definition of  $Y^A$ —and the appearance of the factor  $T$  on the right-hand side. Decomposing  $Y^A$  into vector spherical harmonics, we see that the left-hand side of Eq. (13) contains no  $\ell = 1$  modes as these are in the kernel of the conformal Killing operator. Therefore, the right-hand side cannot contain any  $\ell = 1$  modes. Decomposing  $T$  and  $C_{AB}$  into spin-weighted spherical harmonics

$$T = \sum_{\ell, m} T_{\ell m}(u)_0 Y_{\ell m}, \quad (14)$$

$$C_{AB} = \sum_{\ell, m} C_{\ell m}^E(u) (-_2 Y_{\ell m} m_A m_B + {}_2 Y_{\ell m} \bar{m}_A \bar{m}_B) - i C_{\ell m}^B(u) (-_2 Y_{\ell m} m_A m_B - {}_2 Y_{\ell m} \bar{m}_A \bar{m}_B), \quad (15)$$

where  $m_A, \bar{m}_A$  are complex null vectors on the two-sphere satisfying  $m^A \bar{m}_A = 1$ , we find that if we project onto  $\bar{m}^A \bar{m}^B$ , their product can be written as

$$T \bar{m}^A \bar{m}^B C_{AB} = \sum_{\ell, m} C_{\ell m -2} Y_{\ell m}. \quad (16)$$

So we need to determine what the constraints on  $T_{\ell m}$  are such that  $C_{\ell m}$  does not contain any  $\ell = 1$  modes. We find that



$$\begin{aligned} \mathcal{C}_{\ell m} &= \sum_{\ell', m', \ell'', m''} T_{\ell' m'} (C_{\ell'' m''}^E - i C_{\ell'' m''}^B) \\ &\times \int d^2 S_0 Y_{\ell' m'} Y_{\ell'' m''} \bar{Y}_{\ell m} \end{aligned} \quad (17)$$

$$\begin{aligned} &= \sum_{\ell', m', \ell'', m''} \sqrt{\frac{(2\ell+1)(2\ell'+1)(2\ell''+1)}{4\pi}} \\ &\times (-1)^m T_{\ell' m'} (C_{\ell'' m''}^E - i C_{\ell'' m''}^B) \\ &\times \begin{pmatrix} \ell & \ell' & \ell'' \\ m & m' & m'' \end{pmatrix} \begin{pmatrix} \ell & \ell' & \ell'' \\ -2 & 0 & 2 \end{pmatrix}, \end{aligned} \quad (18)$$

where in going from Eq. (17) to (18), we used that  ${}_s \bar{Y}_{\ell m} = (-1)^{m+s} {}_{-s} Y_{\ell m}$  and that the integral over three spin-weighted spherical harmonics is given by the product of two  $3j$  symbols; see Eq. (34.3.22) in Ref. [13]. Spin-weighted spherical harmonics  ${}_s Y_{\ell m}$  are not defined for  $|s| > \ell$  so  $C_{\ell m}^E/C_{\ell m}^B$  does not have any modes with  $\ell = 0$  or 1. Hence,  $\mathcal{C}_{\ell m}$  contains no  $\ell = 1$  modes only if  $T_{\ell m}$  is nonzero for  $\ell = 0$ , because  $C_{\ell m}^E/C_{\ell m}^B$  is generically nonzero for  $\ell \geq 2$  and the  $3j$  symbols are nonzero when  $|\ell' + \ell''| \leq \ell \leq \ell' + \ell''$ .

So far, we have seen that  $T(u, \theta, \phi) = T_0(u)$  and  $X^A$  satisfies the conformal Killing equation. Substituting this into Eq. (10), we obtain

$$\dot{T}_0 = \frac{1}{2} D_A X^A. \quad (19)$$

The only consistent solution is if both sides of the equation vanish independently. Hence, we find that  $T_0$  is  $u$ -independent and  $X^A$  is also divergence-free. Finally, from Eq. (11), we obtain that  $X^A$  is time independent. Therefore, we find that  $T$  and  $X^A$  are

$$T = T_0 \quad \text{and} \quad X^A = \epsilon^{AB} D_B (\vec{\Omega} \cdot \hat{r}), \quad (20)$$

for constant  $T_0$  and  $\vec{\Omega}$ . Substituting this back into the form of the asymptotic Killing vector fields, we obtain

$$\xi^u = T_0, \quad (21a)$$

$$\xi^r = 0, \quad (21b)$$

$$\xi^A = \epsilon^{AB} D_B (\vec{\Omega} \cdot \hat{r}). \quad (21c)$$

One immediately recognizes this as the  $\mathbb{R} \oplus so(3)$  algebra. This result generalizes the findings in [4], where it was shown that the asymptotic symmetry group is exactly the four-dimensional group of time translations and rotations when  $\mathcal{I}$  has  $\mathbb{R} \times \mathbb{S}^2$  topology and the induced metric at  $\mathcal{I}$  is conformally flat. The requirement of conformal flatness, which severely restricted the allowed gravitational radiation

by essentially cut the degrees of freedom of the gravitational field in half, can be lifted.

## IV. REVISITING REGGE-TEITELBOIM FOR $\Lambda > 0$ AND RADIATION AT FUTURE INFINITY

### A. Charges

The variation of the charge is obtained using the covariant approach of Barnich and Brandt [14], which—as they proved—is equivalent to the standard Regge-Teitelboim analysis [15].<sup>4</sup> In particular, if  $h_{\mu\nu} := \delta g_{\mu\nu}$  corresponds to the functional variation of the spacetime metric, then the general expression for the variation of the charge is given by

$$\begin{aligned} \delta_\xi Q &= \frac{1}{16\pi G} \oint_{\mathcal{I}} (d^2 x)_{\mu\nu} \left[ \xi^\nu \nabla^\mu h - \xi^\nu \nabla_\sigma h^{\mu\sigma} + \xi_\sigma \nabla^\nu h^{\mu\sigma} \right. \\ &\left. + \frac{1}{2} h \nabla^\nu \xi^\mu + \frac{1}{2} h^{\nu\sigma} (\nabla^\mu \xi_\sigma - \nabla_\sigma \xi^\mu) - (\mu \leftrightarrow \nu) \right], \end{aligned} \quad (22)$$

where  $\xi^\mu$  is the asymptotic Killing vector,  $h := g^{\mu\nu} h_{\mu\nu}$  and the volume form is

$$(d^2 x)_{\mu\nu} := \frac{1}{4} \epsilon_{\mu\nu\lambda\sigma} dx^\lambda \wedge dx^\sigma. \quad (23)$$

Applying this to our setup, we find

$$\begin{aligned} \delta_\xi Q &= \frac{1}{16\pi G} \oint_{\mathcal{I}} d^2 S \left[ T \delta(4M) + \frac{T}{2} N_{AB}^{(\Lambda)} \delta C^{AB} \right. \\ &\left. + X^A \delta \left( 2N_A + \frac{1}{16} D_A (C_{BD} C^{BD}) \right) \right. \\ &\left. - T U_{(0)}^A \delta \left( 2N_A + \frac{1}{16} D_A (C_{BD} C^{BD}) \right) \right], \end{aligned} \quad (24)$$

where the tensor  $N_{AB}^{(\Lambda)}$  is defined by

$$\begin{aligned} N_{AB}^{(\Lambda)} &:= \dot{C}_{AB} + \mathcal{L}_{U_{(0)}} C_{AB} - \frac{1}{2} (D_C U_{(0)}^C) C_{AB} \\ &- \frac{\Lambda}{6} \gamma_{AB} C_{CD} C^{CD}. \end{aligned} \quad (25)$$

$N_{AB}^{(\Lambda)}$  generalizes the Bondi news tensor when the cosmological constant is nonzero. This expression acquires a similar structure as the one obtained in [9] for the asymptotically flat case with  $M$  playing the role of the ‘‘Bondi mass aspect’’ and  $N^A$  that of the ‘‘angular-momentum aspect.’’ However, there are some differences that come from the presence of a nonzero cosmological constant. Apart from the

<sup>4</sup>Note that it is also equivalent to the Wald-Zoupas method [16,17] for an appropriate choice of boundary terms (see e.g., [18]), as well as to the one of Abott *et al.* [19–21].

correction coming from  $\Lambda$  in the “news tensor”  $N_{AB}^{(\Lambda)}$  in Eq. (25), there is an additional nonintegrable term proportional to  $U_{(0)}^A$  that vanishes in the limit when  $\Lambda \rightarrow 0$ .

It is worth emphasizing that the variation of the charge is finite in the limit when  $r \rightarrow \infty$ , without the need of any *ad hoc* regularization procedure. The only potentially divergent terms were those proportional to  $r$ , which after some appropriate integration by parts on the sphere acquire the form

$$\begin{aligned} \delta_\xi Q_{\text{div}} = & -\frac{r}{32\pi G} \oint d^2S \left( 2D_{(A}X_{B)} - \gamma_{AB}D_C X^C \right. \\ & - 2U_{(A}^{(0)}D_{B)}T + \gamma_{AB}U_{(0)}^C D_C T \\ & \left. - T \left[ 2D_{(A}U_{B)}^{(0)} - \gamma_{AB}D_C U_{(0)}^C - \frac{\Lambda}{3}C_{AB} \right] \right) \delta C^{AB}, \end{aligned} \quad (26)$$

which, by virtue of Eqs. (4) and (9), vanishes identically.

Thus, if one assumes that  $\delta T_0 = \delta \vec{\Omega} = 0$ , then the integrable part (in the functional sense) of the variation of the charge, takes the form

$$Q^{\text{int}}[T_0, \vec{\Omega}] = T_0 E + \vec{\Omega} \cdot \vec{J}, \quad (27)$$

where the energy  $E$  and angular momentum  $\vec{J}$  are

$$E = \frac{1}{4\pi G} \oint d^2S M, \quad (28)$$

$$\vec{J} = \frac{1}{8\pi G} \oint d^2S \hat{r} \epsilon^{AB} D_A N_B. \quad (29)$$

Note that the term proportional to  $T$  in the last line of Eq. (24) does not contribute to the mass, because it does not contain any  $\ell = 0$  modes [this can again be seen from an analysis of the  $3j$  symbols and noting that  $U_A^{(0)}$  is only nonzero for  $\ell \geq 2$ ]. As we will show in Sec. V C, these expressions give the expected results for the mass and angular momentum for the Kerr-de Sitter geometry, and allows to extend to notion of energy and angular momentum to the case when gravitational waves are present.

## B. Fluxes

The fluxes of energy and angular momentum can be directly obtained by taking the time derivative of Eqs. (28) and (29) in conjunction with Einstein’s equation. In particular, Einstein’s equations yield the evolution of  $M$  and  $N^A$ , respectively. The resulting expressions are rather long but manageable:

$$\begin{aligned} \dot{M} = & \frac{1}{4}D_A D_B N_{(\Lambda)}^{AB} - \frac{1}{8}N_{AB}^{(\Lambda)} N_{(\Lambda)}^{AB} + \frac{\Lambda}{96}C^{AB}D^2C_{AB} - \frac{\Lambda}{12}C^{AB}C_{AB} - \frac{\Lambda}{6}D_A N^A - \frac{\Lambda}{96}(D_C C_{AB})(D^C C^{AB}) \\ & + \frac{\Lambda^2}{24}C^{AB}E_{AB} - \frac{7\Lambda^2}{1152}(C^{AB}C_{AB})^2 - U_{(0)}^A D_A M - \frac{3}{2}M D_A U_{(0)}^A - \frac{1}{8}C^{AB}D_A D_B D_C U_{(0)}^C \end{aligned} \quad (30)$$

and

$$\begin{aligned} \dot{N}^A = & D^A M + \frac{1}{4}D^A D^B D^C C_{BC} - \frac{1}{4}D_B D^2 C^{AB} + \frac{5}{16}C^{AB}D^C N_{BC}^{(\Lambda)} - \frac{3}{16}C_{BC}D^B N_{(\Lambda)}^{AC} - \frac{\Lambda}{2}D_B E^{AB} \\ & - \frac{1}{2}N_{(\Lambda)}^{AB}D^C C_{BC} + \frac{1}{16}N_{(\Lambda)}^{BC}D^A C_{BC} + D_B C^{AB} + \frac{5\Lambda}{32}C_{BD}C^{CD}D_C C^{AB} + \frac{7\Lambda}{48}C^{AB}C^{CD}D_B C_{CD} \\ & - U_{(0)}^B D_B N^A + N^B D_B U_{(0)}^A - 2N^A D_C U_{(0)}^C - \frac{1}{64}U_{(0)}^A (C_{BD}C^{BD}) - \frac{1}{64}(D^2 U_{(0)}^A)C_{BD}C^{BD} \\ & + \frac{1}{32}D^A (D_C U_{(0)}^C) (C_{BD}C^{BD}). \end{aligned} \quad (31)$$

The energy flux is given by

$$\begin{aligned} \frac{dE}{du} = & -\frac{1}{32\pi G} \oint d^2S \left\{ N_{AB}^{(\Lambda)} N_{(\Lambda)}^{AB} + \frac{2\Lambda}{3}C^{AB}C_{AB} - \frac{\Lambda}{6}C^{AB}D^2C_{AB} + \frac{7\Lambda^2}{144}(C^{AB}C_{AB})^2 - \frac{\Lambda^2}{3}C^{AB}E_{AB} \right. \\ & \left. + (4M + D_A D_B C^{AB})(D_C U_{(0)}^C) \right\}. \end{aligned} \quad (32)$$

The first term on the right-hand side has the same form as the one that contributes to the loss of energy in the asymptotically flat case. However, there are now also corrections coming from the presence of the cosmological constant which are up to fourth order in the fields. These higher-order terms are characteristic of the full-nonlinear theory and cannot be seen in the linearized

approximation. In Sec. V D 1, we will show that when the higher-order terms are neglected, the total amount of energy radiated in a certain interval of time precisely coincide with the one reported in [22–24]. An important difference with the asymptotically flat case is that the flux of energy is not manifestly negative. This was also observed for the case of homogeneous gravitational perturbations on a de Sitter background in [22]. Moreover, this can also occur for Maxwell fields on a de Sitter background [22] and thus seems a rather generic feature of spacetimes with  $\Lambda > 0$ . This is likely due to the fact that there is no global timelike Killing vector field in de Sitter spacetime. However, as was pointed out in [25], and as we will show in Sec. V D, in the case of quadrupolar radiation in the linearized theory, the flux of energy is manifestly negative.

Analogously, the flux of angular momentum takes the form

$$\frac{d\vec{J}}{du} = \frac{1}{8\pi G} \oint d^2S \hat{r} \epsilon^{AB} D_A \dot{N}_B, \quad (33)$$

where  $\dot{N}_B$  is given by Eq. (31). Due to the cosmological constant there is no angular-momentum ambiguity, because there are no Abelian supertranslations as is the case with  $\Lambda = 0$ .

The flux of energy and angular momentum in Eqs. (32) and (33) can alternatively be obtained from the nonintegrable part of the variation of the charge in Eq. (24) following the prescription in [9] (see also [12,26]). We have verified this explicitly for the energy flux.

### C. No radiation condition

In the Bondi-Sachs coordinates we introduced in Sec. II, the presence of a gravitational wave manifests itself through the tensor  $C_{AB}$  [and, of course,  $U_{(0)}^A$  given its direct link to  $C_{AB}$  through Eq. (4)]. In particular, whenever  $C_{AB}$  is time dependent, there is gravitational radiation. This is the case for all the examples in Sec. V. This interpretation is further supported by the expressions for flux of energy and angular momentum in Eqs. (32) and (33). Whenever the spacetime under consideration is stationary, these fluxes vanish trivially as  $\dot{M}$  and  $\dot{N}_A$  are both zero. When  $C_{AB}$  is zero, the flux formulas are also both trivially zero. The scenario in which  $C_{AB}$  is nonzero, but time independent is subtle. From the expressions for the fluxes, it is not evident that they will both be zero. Although we have no formal proof, we have some evidence that this is indeed the case. First, we have two nontrivial examples in which this expectation is borne out. In particular, time-independent linearized quadrupolar waves and linearized Robinson-Trautman solutions. We will discuss the general time-dependent cases in detail in Secs. V D and V E, respectively. When we restrict those solutions to  $C_{AB}$  nonzero but  $u$  independent, the integrand for the energy flux is nonzero in both cases, but the resulting integral

vanishes [specifically, the first two lines cancel with the last line in Eq. (32)]. This is evident from the final expressions in Eqs. (65) and (75), which vanish if  $\partial_u C_{AB} = 0$ . The cancellation is nontrivial and all terms conspire for this to happen. The second line of evidence is by analogy with a related scenario in the asymptotically flat case. For asymptotically flat spacetimes, the presence of radiation also manifests itself through a  $u$ -dependent  $C_{AB}$ , or equivalently, a nonzero Bondi news tensor  $N_{AB} = \partial_u C_{AB}$ . Thus, if  $C_{AB}(u, \theta, \phi) = u \bar{C}_{AB}(\theta, \phi)$ ,  $N_{AB}$  is nonzero and one would classify this spacetime as radiative. However, the resulting energy flux in this case would be constant as  $N_{AB}$  is time-independent and the flux only depends on  $N_{AB}$ . As a result, the total energy radiated would be infinite. This is unphysical and therefore not considered to be a viable solution, and one typically (implicitly) does not include such solutions. We expect a similar scenario to hold here; if  $C_{AB}$  is a nonzero, but  $u$ -independent solution, either the flux will be zero, or it will be constant and as a result the energy radiated will be infinite. The latter is not physical and we do not include such solutions in our solution space.

## V. APPLICATION TO SPECIAL CASES

In this section, we show explicitly that the falloff conditions in Eq. (2) accommodate a wide range of physically interesting solutions to Einstein's equation. First, we discuss the de Sitter spacetime itself before moving on to two black hole solutions in the presence of a positive cosmological constant: the nonrotating Schwarzschild-de Sitter spacetime and the rotating Kerr-de Sitter spacetime. Next, we discuss linearized solutions to Einstein's equations with  $\Lambda > 0$  representing gravitational radiation emitted by a compact source. Finally, we describe a simple model of gravitational radiation with a single degree of freedom known as the Robinson-Trautman spacetime.

### A. de Sitter spacetime

The full de Sitter spacetime is not an example of the class of spacetimes we have defined. This is not problematic, as the goal of this paper is to describe radiation generated by compact sources in the presence of  $\Lambda$  in which case not the complete de Sitter spacetime, but the Poincaré patch of de Sitter spacetime with an additional point at  $\mathcal{I}$  removed is relevant. The removal of this additional point is natural as it represents the intersection of the future boundary with the source generating radiation (see also Sec. II in Ref. [25]). As a result, the future boundary has topology  $\mathbb{R} \times \mathbb{S}^2$  and is naturally coordinatized by  $(u, r, \theta, \phi)$ ,

$$ds^2 = -\left(1 - \frac{\Lambda r^2}{3}\right) du^2 - 2du dr + r^2(d\theta^2 + \sin^2\theta d\phi^2). \quad (34)$$



The behavior of these coordinates is illustrated in the conformal diagram in Fig. 1. The time translation vector field  $\frac{\partial}{\partial u}$  and the three rotational Killing vector fields are not only asymptotic symmetries, but symmetries of the entire spacetime. Translations and inverted translations, which are symmetries of the full de Sitter spacetime, do not leave  $i^0$  and  $i^+$  invariant and are therefore not permissible (for a more extensive discussion, see [4]).

### B. Schwarzschild-de Sitter spacetime

The simplest prototype for describing nondynamical isolated gravitating systems in the presence of a cosmological constant is undoubtedly the Schwarzschild-de Sitter spacetime. This spacetime describes a nonrotating black hole with  $\Lambda > 0$ . We consider the metric in Eddington-Finkelstein coordinates  $(u, r, \theta, \phi)$ ,

$$ds^2 = -\left(1 - \frac{2m}{r} - \frac{r^2}{l^2}\right) du^2 - 2du dr + r^2(d\theta^2 + \sin^2\theta d\phi^2), \quad (35)$$

where  $\Lambda = 3l^{-2}$ . The coordinate ranges are  $u \in (-\infty, \infty)$ ,  $r \in (0, \infty)$ ,  $\theta \in [0, \pi)$  and  $\phi \in [0, 2\pi)$ . While these coordinates do not provide a global chart of the spacetime, they suffice to cover the asymptotic region near  $\mathcal{I}$ . In terms of the asymptotic expansions of the metric in Eq. (2), this metric has  $M(u, \theta, \phi) = m$  and all other coefficients zero. In particular,  $C_{AB}$  and  $U_{(0)}^A$  are both zero so that there is no gravitational radiation.

This metric has four Killing vector fields; one Killing vector field generates time translations and the other three describe the spherical symmetry of the spacetime. A well-known property of (global) Killing vector fields is that every Killing field of the physical spacetime admits an extension to the boundary and is tangential to it. This is

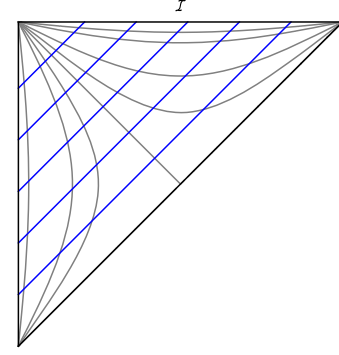


FIG. 1. Conformal diagram of de Sitter spacetime, where each point represents a two-sphere (except for the points at  $r = 0$  which are proper points). The gray lines denote surfaces of constant  $r$ , and the blue lines surfaces of constant  $u$  [see Eq. (34)].  $r = 0$  is the vertical line on the left, and  $r \rightarrow \infty$  at  $\mathcal{I}$ . The diagonal constant  $r$  surface corresponds to the cosmological horizon  $r = \ell$ .

also the case for the above Killing vector fields, which coincide exactly with the asymptotic symmetry vector fields. All charges and fluxes vanish except for the mass, which is

$$E_{\text{Sch-dS}} = \frac{m}{G}. \quad (36)$$

### C. Kerr-de Sitter spacetime

Kerr-de Sitter spacetimes are stationary, vacuum solutions to Einstein's equations describing rotating black holes in the presence of a positive cosmological constant. Let us consider the Kerr-de Sitter metric in standard Boyer-Lindquist coordinates  $(T, R, \Theta, \Phi)$  [27],

$$ds^2 = -2a\sin^2\Theta \left( \frac{2mR}{a^2\cos^2\Theta + R^2} + \frac{a^2 + R^2}{l^2} \right) dT d\Phi - \left( 1 - \frac{2mR}{a^2\cos^2\Theta + R^2} - \frac{a^2\sin^2\Theta + R^2}{l^2} \right) dT^2 + \sin^2\Theta \left( \frac{2a^2mR\sin^2\Theta}{a^2\cos^2\Theta + R^2} + (a^2 + R^2) \left( 1 + \frac{a^2}{l^2} \right) \right) d\Phi^2 + (a^2\cos^2\Theta + R^2) \left( \frac{dR^2}{R^2 - (a^2 + R^2)\frac{R^2}{l^2} - 2mR + a^2} + \frac{d\Theta^2}{1 + \frac{a^2\cos^2\Theta}{l^2}} \right), \quad (37)$$

where the parameter  $a$  is related to the amount of rotation of this rotating black hole. In the limit,  $a \rightarrow 0$  one recovers the Schwarzschild-de Sitter metric in static coordinates. Note that these Boyer-Lindquist coordinates are “twisted” at  $\mathcal{I}$ ; for instance, surfaces of constant  $T, R$  describe deformed spheres (consequently, the range of  $\Theta, \Phi$  is not the standard range for coordinates on the sphere). Inspired by the coordinate transformation used in [6] to undo this twisting, we perform the following asymptotic change of coordinates<sup>5</sup>

<sup>5</sup>The Kerr-de Sitter solution in Bondi coordinates was also written in [28].

$$\begin{aligned}
T &= u + l \operatorname{arctanh}\left(\frac{r}{l}\right) - \frac{ml^4 \left(1 - \frac{a^2 \sin^2 \theta}{2l^2}\right)}{2 \left(1 + \frac{a^2 \sin^2 \theta}{l^2}\right)^{5/2}} \frac{1}{r^4} + \dots, \\
R &= r \sqrt{1 + \frac{a^2 \sin^2 \theta}{l^2}} - \frac{\left(1 + \frac{a^2}{l^2}\right) a^2 \sin^2 \theta}{2 \left(1 + \frac{a^2 \sin^2 \theta}{l^2}\right)^{3/2}} \frac{1}{r} - \frac{ma^2 \sin^2 \theta}{2 \left(1 + \frac{a^2 \sin^2 \theta}{l^2}\right)^2} \frac{1}{r^2} + \dots, \\
\Theta &= \arccos\left(\frac{\cos \theta}{\sqrt{1 + \frac{a^2 \sin^2 \theta}{l^2}}}\right) - \frac{a^2 \sin(2\theta) \sqrt{1 + \frac{a^2}{l^2}}}{4 \left(1 + \frac{a^2 \sin^2 \theta}{l^2}\right)^2} \frac{1}{r^2} + \frac{3a^4 \sin(2\theta) \left(1 - 2 \sin^2 \theta \left(1 + \frac{a^2}{2l^2}\right)\right) \sqrt{1 + \frac{a^2}{l^2}}}{16 \left(1 + \frac{a^2 \sin^2 \theta}{l^2}\right)^4} \frac{1}{r^4} + \dots, \\
\Phi &= \frac{1}{1 + \frac{a^2}{l^2}} \left( \phi + \frac{a \left(u + l \operatorname{arctanh}\left(\frac{r}{l}\right)\right)}{l^2} \right) + \frac{ma^3 \sin^2 \theta}{4 \left(1 + \frac{a^2}{l^2}\right) \left(1 + \frac{a^2 \sin^2 \theta}{l^2}\right)^{5/2}} \frac{1}{r^4} + \dots
\end{aligned}$$

The leading terms of the Kerr-de Sitter metric near  $\mathcal{I}$  fit within our asymptotic conditions.<sup>6</sup> The metric on  $\mathcal{I}$  with  $u = \text{constant}$  is the unit two-sphere with  $\theta, \phi$  having their standard range, i.e.,  $\theta \in [0, \pi)$ ,  $\phi \in [0, 2\pi)$ . Moreover,  $U_{(0)}^A$  and  $C_{AB}$  are both equal to zero, which is consistent with the fact that there is no gravitational radiation in this spacetime. The mass and angular-momentum aspect are given by

$$M = m \frac{\left(1 - \frac{a^2 \sin^2 \theta}{2l^2}\right)}{\left(1 + \frac{a^2 \sin^2 \theta}{l^2}\right)^{5/2}}, \quad (38)$$

$$N^\theta = 0, \quad (39)$$

$$N^\phi = -\frac{3am}{\left(1 + \frac{a^2 \sin^2 \theta}{l^2}\right)^{5/2}}. \quad (40)$$

We also find that  $E_{AB}$  is

$$E_{\theta\theta} = -\frac{ma^2 \sin^2 \theta}{\left(1 + \frac{a^2 \sin^2 \theta}{l^2}\right)^{5/2}}, \quad (41)$$

$$E_{\phi\phi} = \frac{ma^2 \sin^4 \theta}{\left(1 + \frac{a^2 \sin^2 \theta}{l^2}\right)^{5/2}}, \quad (42)$$

$$E_{\theta\phi} = 0, \quad (43)$$

so that  $E_{AB}$  is traceless with respect to the unit two-sphere metric, as it should.

The mass and the angular momentum can be directly computed from the expressions for the charges in Eqs. (28)

<sup>6</sup>Note that the solution is not in the Bondi gauge everywhere, but its asymptotic form to the orders needed is. Indeed  $g_{rr} = O(r^{-6})$  and  $g_{rA} = O(r^{-4})$ .

and (29) (which also define the normalization of the Killing vectors here). They are given by

$$E_{\text{Kerr-dS}} = \frac{m}{G} \left(1 + \frac{a^2}{l^2}\right)^{-2}, \quad J_z = -aE_{\text{Kerr-dS}}. \quad (44)$$

These results coincide with the charges obtained using Hamiltonian methods by Marolf and Kelly in [29].<sup>7</sup> Moreover, these expression also precisely coincide with the ones obtained for Kerr-anti-de Sitter spacetimes after replacing  $l \rightarrow il$  [6]. Since this spacetime is stationary, the fluxes are trivially zero, which we verified by direct computation.

## D. Linearized solutions in de Sitter spacetime

### 1. Linearized charges and fluxes

The expressions for the charges and fluxes simplify drastically in the linearized context. Here, we will briefly comment on the linearized setting and explicitly connect the resulting flux of energy radiated across  $\mathcal{I}$  to existing results in the literature.

Let us consider the linearized gravitational field in retarded null coordinates  $(u, r, x^A)$  around the de Sitter background metric

$$d\bar{s}^2 = -\left(1 - \frac{\Lambda r^2}{3}\right) du^2 - 2dudr + r^2(d\theta^2 + \sin^2 d\phi^2). \quad (45)$$

<sup>7</sup>These final expressions also agree with the gravitational charges defined in terms of the electric part of the Weyl tensor in [4] despite the fact that the mass and angular momentum there refer to a differently normed Killing vector field. This is due to the fact that in [4], the  $\Theta, \Phi$  coordinates were assumed to have the standard range on the two-sphere, which is not the case. If this is corrected, the results here and in [4] differ exactly by the expected scaling with the Killing vector field.

The spacetime metric is then written as

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}, \quad (46)$$

where  $h_{\mu\nu}$  is kept only up to first order. Quantities that have dimensions of length are compared with an external fixed length scale. The linearization expression (46) is valid everywhere, not just asymptotically. The falloff of the metric in the linearized theory can be directly obtained from our asymptotic conditions in Eq. (2) by neglecting the terms that are quadratic in the fields. The asymptotic form of the metric then reads

$$\beta = \mathcal{O}\left(\frac{1}{r^5}\right), \quad (47a)$$

$$V = \frac{\Lambda r^3}{3} - D_A U_{(0)}^A r^2 - r + 2M + \dots, \quad (47b)$$

$$U^A = \left( U_{(0)}^A - \frac{D_B C^{AB}}{2r^2} - \frac{2N^A}{3r^3} + \dots \right), \quad (47c)$$

$$g_{AB} = \gamma_{AB} + \left( \frac{C_{AB}}{r} + \frac{E_{AB}}{r^3} + \dots \right), \quad (47d)$$

where  $U_{(0)}^A$  obeys Eq. (4). Note that  $U_A^{(0)}$ , which is of order zero in the asymptotic expansion in which the reference length is  $r$ , becomes of first order in the linearized theory, in which the reference length is a fixed distance. So, both expansions do not coincide at large distances. The time derivatives of  $M$  and  $N^A$  reduce to

$$\dot{M} = \frac{1}{2} D_A D_B N_{(\Lambda)}^{AB} - \frac{\Lambda}{6} D_A N^A, \quad (48)$$

$$\begin{aligned} \dot{N}^A &= D^A M + \frac{1}{4} D^A D^B D^C C_{BC} - \frac{1}{4} D_B D^2 C^{AB} \\ &\quad - \frac{\Lambda}{2} D_B E^{AB} + D_B C^{AB}, \end{aligned} \quad (49)$$

while the linearized version of the News tensor now takes the form

$$N_{AB}^{(\Lambda)} = \dot{C}_{AB}. \quad (50)$$

In the linearized limit, the symmetry algebra of the de Sitter background metric can be naturally used (see Sec. VA). The radiation rates in the linear theory can be obtained from the corresponding expressions in the non-linear theory by dropping the cubic and the quartic terms in the fields. In the case of the energy, we have

$$\begin{aligned} \frac{dE}{du} &= -\frac{1}{32\pi G} \oint d^2S \left\{ N_{AB}^{(\Lambda)} N^{(\Lambda)AB} + \frac{2\Lambda}{3} C^{AB} C_{AB} \right. \\ &\quad - \frac{\Lambda}{6} C^{AB} D^2 C_{AB} - \frac{\Lambda^2}{3} C^{AB} E_{AB} \\ &\quad \left. + (4M + D_A D_B C^{AB}) (D_C U_{(0)}^C) \right\}. \end{aligned} \quad (51)$$

Remarkably, if one calculates the total flux of energy radiated in a finite interval of time  $\Delta u$  (assuming appropriate falloff near the edges of  $\Delta u$ ), one obtains perfect agreement with the results obtained independently by Chrúsciel *et al.* in [23], and by Kolanowski and Lewandowski in [24]. It also coincides with the expression found by Ashtekar *et al.* in a different set of coordinates [22]. This can be seen as follows. If we reexpress  $C_{AB}$  in the terms with explicit  $\Lambda$  dependence by using Eq. (4), we obtain

$$\begin{aligned} \frac{dE}{du} &= -\frac{1}{32\pi G} \oint d^2S \left( \dot{C}_{AB} \dot{C}^{AB} - 2\Lambda D^B U_{(0)}^A E_{AB} \right. \\ &\quad \left. + 4D^B U_{(0)}^A C_{AB} - D^B U_{(0)}^A D^2 C_{AB} \right. \\ &\quad \left. + (4M + D_A D_B C^{AB}) (D_C U_{(0)}^C) \right). \end{aligned}$$

After an integration by parts on the sphere, one obtains

$$\begin{aligned} \frac{dE}{du} &= -\frac{1}{32\pi G} \oint d^2S \left( \dot{C}_{AB} \dot{C}^{AB} + U_{(0)}^A [2\Lambda D^B E_{AB} \right. \\ &\quad \left. - 4D^B C_{AB} + D^B D^2 C_{AB} - D_A (4M + D_B D_C C^{BC})] \right). \end{aligned}$$

Using the linearized equation of motion for  $N^A$  in (49), we can write this compactly as

$$\frac{dE}{du} = \frac{1}{8\pi G} \oint d^2S \left( -\frac{1}{4} \dot{C}_{AB} \dot{C}^{AB} + U_{(0)}^A \dot{N}_A \right).$$

So that the total amount of energy radiated in the interval of time  $\Delta u$  is given by

$$E|_{\Delta u} = \frac{1}{8\pi G} \int_{\Delta u} du \oint d^2S \left( -\frac{1}{4} \dot{C}_{AB} \dot{C}^{AB} + U_{(0)}^A \dot{N}_A \right).$$

Assuming that there is no flux of radiation outside this interval of time, we can integrate by parts in the null time and discard the corresponding boundary terms, so that we can write

$$E|_{\Delta u} = -\frac{1}{8\pi G} \int_{\Delta u} du \oint d^2S \left( \frac{1}{4} \dot{C}_{AB} \dot{C}^{AB} + \dot{U}_{(0)}^A N_A \right).$$

Now, by virtue of our asymptotic conditions,

$$h_{uA} = -\frac{1}{r^2} U_A.$$

Therefore, if we consider an asymptotic expansion of the form

$$h_{uA} = h_{uA}^{(2)} r^2 + h_{uA}^{(1)} r + h_{uA}^{(0)} + \frac{h_{uA}^{(-1)}}{r} + \dots,$$

we obtain

$$h_{uA}^{(2)} = -U_{(0)A}, \quad h_{uA}^{(-1)} = \frac{2}{3} N_A.$$

Thus, in terms of these variables, the total amount of energy radiated in the interval of time  $\Delta u$  acquires the following form:

$$E|_{\Delta u} = -\frac{1}{32\pi G} \int_{\Delta u} du \oint d^2S \left( \dot{C}_{AB} \dot{C}^{AB} - 6\dot{h}_{uA}^{(2)} \dot{h}_u^{(-1)A} \right).$$

This expression precisely coincides with the ones found in [23,24].

## 2. Explicit solutions for quadrupolar modes

Using the set-up in the previous subsection, we will now study explicit solutions to the linearized Einstein's equation. The strategy is to solve for the homogeneous solution of Einstein's equation, which corresponds in a partial wave expansion to the gravitational field away from the source generating the gravitational waves (which is assumed to be bounded). In particular, the homogeneous solutions corresponding to a fixed  $\ell$  in the spherical harmonic expansion, should necessarily be generated by a source with multipole moment  $\ell$ . Even though the homogeneous solution is only valid outside the source, in principle we could match this solution with the “inner” solution using matched asymptotic expansions [30]. The matching with an interior solution is beyond the scope of this paper, which focuses on the solution far away from the source. The waves here could be considered a generalization of the “Teukolsky waves” from flat spacetime to de Sitter spacetime—apart from the different gauge choice that we implement [31].

Since the background spacetime is spherically symmetric, it is convenient to use similar techniques as Regge and Wheeler did when solving for linearized perturbations off Schwarzschild (although we will not implement the Regge-Wheeler gauge) [32]. In particular, we will use separation of variables and for the angular part of the perturbations, we introduce scalar, vector and tensor spherical harmonics. Following the notation in [33], we note that the scalar harmonics are the usual spherical-harmonic functions  $Y_{\ell m}(x^A)$  satisfying the eigenvalue equation  $D^2 Y_{\ell m} = -\ell(\ell+1)Y_{\ell m}$ . There are two types of vector harmonics: even-parity  $Y_A^{\ell m}$  (also known as

electric) and odd-parity  $X_A^{\ell m}$  (also known as magnetic), which are related to the scalar harmonics through the covariant derivative operator compatible with  $\gamma_{AB}$

$$Y_A^{\ell m} := D_A Y_{\ell m}; \quad (52)$$

$$X_A^{\ell m} := -\epsilon_A^B D_B Y_{\ell m}. \quad (53)$$

The even- and odd-parity harmonics are orthogonal in the sense that  $\int d^2S \bar{Y}_{\ell m}^A X_A^{\ell' m'} = 0$ . The tensor harmonics also come in the same two types:

$$Y_{AB}^{\ell m} := D_{(A} Y_{B)}^{\ell m} - \frac{1}{2} \gamma_{AB} D_C Y_C^{\ell m}; \quad (54)$$

$$X_{AB}^{\ell m} := -\epsilon_{(A}^C D_{B)} Y_C^{\ell m}. \quad (55)$$

These operators are traceless, i.e.,  $\gamma^{AB} Y_{AB}^{\ell m} = 0 = \gamma^{AB} X_{AB}^{\ell m}$  and orthogonal in the same sense as the vector harmonics are. The separation of variables takes the form

$$h_{uu} = \sum_{\ell, m} f_{\ell m}(u, r) Y_{\ell m}, \quad (56a)$$

$$h_{ur} = \sum_{\ell, m} h_{\ell m}(u, r) Y_{\ell m}, \quad (56b)$$

$$h_{uA} = \sum_{\ell, m} F_1^{\ell m}(u, r) Y_A^{\ell m} + G_1^{\ell m}(u, r) X_A^{\ell m}, \quad (56c)$$

$$h_{AB} = \sum_{\ell, m} F_2^{\ell m}(u, r) Y_{AB}^{\ell m} + G_2^{\ell m}(u, r) X_{AB}^{\ell m}, \quad (56d)$$

where the sum here is restricted to  $\ell \geq 2$  and  $m$  ranges from  $-\ell$  to  $\ell$ . We neglect the  $\ell = 0$  and  $\ell = 1$  multipoles, which are nonradiative and require a special treatment. We have also set  $h_{rr} = h_{rA} = \gamma^{AB} h_{AB} = 0$  to ensure that the linearized metric satisfies the required falloff in Eq. (47). This gauge choice can always be made (in fact, there is some residual gauge freedom left that we will use in the analysis below).

The even and odd-parity modes remain decoupled in the linearized Einstein's equation and so are all the  $\ell, m$  modes in the spherical decomposition. We will restrict ourselves in this section to the  $\ell = 2$  modes; the structure of the solution is very similar for higher- $\ell$  modes and we will briefly comment on the form of the general solution at the end. Solving for the simpler, odd-parity sector first, we find the following retarded solution:

$$G_1^{\ell=2}(u, r) = \sum_{m=-2}^2 \left[ \frac{1}{2} \dot{C}_2^m \frac{r^2}{l^4} - \left( l^{-2} \dot{C}_1^m - \ddot{C}_1^m \right) + \frac{2}{r} \dot{C}_1^m + \frac{3}{2r^2} \dot{C}_1^m \right], \quad (57)$$

$$G_2^{\ell=2}(u, r) = \sum_{m=-2}^2 \left[ \left( C_2^m + C_1^m - \ddot{C}_1^m l^2 \right) \frac{r^2}{l^4} + r \left( l^{-2} \dot{C}_1^m - \ddot{C}_1^m \right) + \frac{1}{r} \dot{C}_1^m \right], \quad (58)$$

with  $C_1^m, C_2^m$  dimensions of length squared. The term proportional to  $r^2$  in  $G_2^{\ell=2,m}$  spoils the falloff behavior of the angular part of the metric in Eq. (2). This is, however, easily remedied by realizing that the solution is not completely gauge fixed and with the residual gauge freedom this part of the metric can be gauged away. With an appropriate gauge choice, we set<sup>8</sup>

$$C_2^m \equiv \ddot{C}_1^m l^2 - C_1^m. \quad (59)$$

This gauge choice is further preserved by the residual gauge freedom generated by  $\chi^A = \epsilon^{AB} D_B (\vec{\Omega} \cdot \hat{r})$ . With this gauge choice, and introducing  $B_m = \dot{C}_1^m$ , we finally obtain the following odd-parity solutions for the quadrupolar modes with  $\ell = 2$

$$h_{uu}^{\text{odd}} = 0, \quad (60a)$$

$$h_{ur}^{\text{odd}} = 0, \quad (60b)$$

$$h_{uA}^{\text{odd}} = \sum_{m=-2}^2 \left[ \frac{1}{2} \left( \ddot{B}_m - l^{-2} B_m \right) \frac{r^2}{l^2} + \left( \dot{B}_m - l^{-2} B_m \right) + \frac{2}{r} \dot{B}_m + \frac{3}{2r^2} B_m \right] X_A^{2m}, \quad (60c)$$

$$h_{AB}^{\text{odd}} = \sum_{m=-2}^2 \left[ r \left( \ddot{B}_m - l^{-2} B_m \right) - \frac{1}{r} B_m \right] X_{AB}^{2m}, \quad (60d)$$

where  $B_m = B_m(u)$  is an arbitrary function of the retarded time  $u$  with dimensions of length. Note that the leading order of the angular metric is independent of the wave, but that  $C_{AB} \neq 0$  and  $U_{(0)}^A \neq 0$  [with the constraint relating these metric coefficients in Eq. (4) satisfied].

The solution for general  $\ell$  is more complicated, but has these general features,

$$G_1^\ell(u, r) = \sum_{m=-2}^2 \left[ \frac{1}{2} \dot{C}_2^{\ell m} \frac{r^2}{l^4} + \sum_{i=0}^{\ell} a_i^{(\ell)}(r) C_i^{\ell m} \right], \quad (61)$$

<sup>8</sup>By the Stewart-Walker lemma, the linearized Weyl tensor is gauge-invariant and a straightforward computation shows that the linearized Weyl tensor is independent of  $C_2^m$ . Therefore, the  $C_2^m$  solution is pure gauge and contains no physical degrees of freedom. This interpretation of the solution is consistent with the gauge choice made in Eq. (59).

$$G_2^\ell(u, r) = \sum_{m=-2}^2 \left[ C_2^{\ell m} \frac{r^2}{l^4} + \sum_{i=0}^{\ell} \begin{cases} b_i^{(\ell)}(r) C_i^{\ell m} & \text{if } \ell \text{ even} \\ b_i^{(\ell)}(r) C_i^{\ell m} & \text{if } \ell \text{ odd} \end{cases} \right], \quad (62)$$

with the  $C$ -coefficients depending on  $u$  only, the factor  $(i)$  on top of these coefficients indicate its  $i$ th derivative with respect to  $u$  and  $a_i^{(\ell)}, b_i^{(\ell)}$  are polynomials in  $r$  (and its inverse powers) with the highest power being  $r^2$ . Note that the term proportional to  $C_2$  is in fact independent of  $\ell$  and can always be gauged away. As a result, even though generically  $G_2$  contains terms proportional to  $r^2$  which could spoil the desired falloff, these terms can always be set to zero by a clever gauge choice for  $C_2$ —similar to the case with  $\ell = 2$ . Hence, linearized solutions with odd-parity satisfy the desired falloff conditions for any  $\ell \geq 2$ .

The analysis for the even-parity sector mimicks that of the odd-parity sector, but is more involved as more terms are nonzero. Nonetheless, also in this case one can gauge fix the solution to obtain a linearized solution that satisfies the falloff conditions prescribed in Eq. (47). Specifically, the retarded  $\ell = 2$  even-parity solutions for  $h_{\mu\nu}$  takes the form

$$h_{uu}^{\text{even}} = \sum_{m=-2}^2 \left[ 3 \left( \ddot{A}_m - 4l^{-2} A_m \right) \frac{r}{l^2} + 6 \left( \dot{A}_m - l^{-2} A_m \right) \frac{1}{r} + \frac{6}{r^2} \dot{A}_m + \frac{3}{r^3} A_m \right] Y_{2m}, \quad (63a)$$

$$h_{ur}^{\text{even}} = 0, \quad (63b)$$

$$h_{uA}^{\text{even}} = \sum_{m=-2}^2 \left[ \left( 2l^{-2} A_m - \frac{1}{2} \ddot{A}_m \right) \frac{r^2}{l^2} + \left( 4l^{-2} A_m - \dot{A}_m \right) + \frac{2}{r} \dot{A}_m + \frac{3}{2r^2} A_m \right] Y_A^{2m}, \quad (63c)$$

$$h_{AB}^{\text{even}} = \sum_{m=-2}^2 \left[ r \left( \ddot{A}_m - 4l^{-2} A_m \right) + \frac{1}{r} A_m \right] Y_{AB}^{2m}, \quad (63d)$$

where  $A_m = A_m(u)$  is an arbitrary function of the retarded time  $u$  and dimensions of length. Also, similar to the odd-parity sector, we have set  $h_{rr} = h_{rA} = \gamma^{AB} h_{AB} = 0$ . Note that there is backreaction on the “background” metric as  $r \rightarrow \infty$  through the leading term of  $h_{uA}$ , that is,  $U_{(0)}^A \neq 0$ . The backreaction onto the leading-order part is unique to  $\Lambda \neq 0$ . In the limit  $l \rightarrow \infty$ , this backreaction vanishes. This is immediately clear from the limit of the even- and odd-parity solutions:



$$\lim_{l \rightarrow \infty} h_{uu} = \sum_{m=-2}^2 \left[ \frac{6\ddot{A}_m}{r} + \frac{6\dot{A}_m}{r^2} + \frac{3A_m}{r^3} \right] Y_{2m}, \quad (64a)$$

$$\lim_{l \rightarrow \infty} h_{ur} = 0, \quad (64b)$$

$$\begin{aligned} \lim_{l \rightarrow \infty} h_{uA} = & \sum_{m=-2}^2 \left( \left[ -\ddot{A}_m + \frac{2\dot{A}_m}{r} + \frac{3A_m}{2r^2} \right] Y_A^{2m} \right. \\ & \left. + \left[ \ddot{B}_m + \frac{2\dot{B}_m}{r} + \frac{3B_m}{2r^2} \right] X_A^{2m} \right), \end{aligned} \quad (64c)$$

$$\lim_{l \rightarrow \infty} h_{AB} = \sum_{m=-2}^2 \left( \left[ \frac{\ddot{A}_m}{r} + \frac{A_m}{r^3} \right] r^2 Y_{AB}^{2m} + \left[ \frac{\ddot{B}_m}{r} - \frac{B_m}{r^3} \right] r^2 X_{AB}^{2m} \right), \quad (64d)$$

where  $A_m$  and  $B_m$  reduce to the standard quadrupole moments on flat spacetime.

Connecting these results with the Bondi-Sachs expansions, we find that the linear part of the metric coefficients is given by

$$M = \sum_{m=-2}^2 3 \left( \ddot{A}_m - l^{-2} A_m \right) Y^{2m}, \quad (65a)$$

$$U_A^{(0)} = \frac{1}{l^2} \sum_{m=-2}^2 \left( 2l^{-2} A_m - \frac{1}{2} \ddot{A}_m \right) Y_A^{2m} + \frac{1}{2} \left( \ddot{B}_m - l^{-2} B_m \right) X_A^{2m}, \quad (65b)$$

$$N_A = \sum_{m=-2}^2 -3\dot{A}_m Y_A^{2m} + 3\dot{B}_m X_A^{2m}, \quad (65c)$$

$$C_{AB} = \sum_{m=-2}^2 \left( \ddot{A}_m - 4l^{-2} A_m \right) Y_{AB}^{2m} + \left( \ddot{B}_m - l^{-2} B_m \right) X_{AB}^{2m}, \quad (65d)$$

$$E_{AB} = \sum_{m=-2}^2 A_m Y_{AB}^{2m} - B_m X_{AB}^{2m}, \quad (65e)$$

and all other coefficients vanishing or determined by lower-order terms.

The radiation rate at the linearized level in Eq. (51) reduces after some further simplifications to

$$\begin{aligned} \frac{dE}{du} = & -\frac{3}{8\pi G} \sum_{m=-2}^2 \left[ \left| \ddot{A}_m - 4l^{-2} A_m \right|^2 \right. \\ & - 3l^{-2} \ddot{A}_m^* \left( \ddot{A}_m - 4l^{-2} A_m \right) + \left| \ddot{B}_m - l^{-2} B_m \right|^2 \\ & \left. + 3l^{-2} \ddot{B}_m^* \left( \ddot{B}_m - l^{-2} B_m \right) \right], \end{aligned} \quad (66)$$

where the star indicates complex conjugation. If we consider the total energy radiated during some large time interval  $\Delta u$ , where we assume that at far past and at far future the system will not radiate so that we can remove the boundary terms in time, then

$$\begin{aligned} E_{\Delta u} = & -\frac{3}{8\pi G} \int_{-\infty}^{\infty} du \sum_{m=-2}^2 \left[ \left( \ddot{A}_m - \frac{\dot{A}_m}{l^2} \right) \right. \\ & \left. \left( \ddot{A}_m^* - \frac{4\dot{A}_m^*}{l^2} \right) + \left( \ddot{B}_m - \frac{\dot{B}_m}{l^2} \right) \left( \ddot{B}_m^* - \frac{4\dot{B}_m^*}{l^2} \right) \right]. \end{aligned} \quad (67)$$

In particular, after integration by parts, we have

$$\begin{aligned} E_{\Delta u} = & -\frac{3}{8\pi G} \int_{-\infty}^{\infty} du \sum_{m=-2}^2 \left[ \left| \ddot{A}_m \right|^2 + \frac{5|\dot{A}_m|^2}{l^2} + \frac{4|\dot{A}_m|^2}{l^4} \right. \\ & \left. + \left| \ddot{B}_m \right|^2 + \frac{5|\dot{B}_m|^2}{l^2} + \frac{4|\dot{B}_m|^2}{l^4} \right]. \end{aligned} \quad (68)$$

This flux is manifestly negative. Therefore, the total energy always decreases for a source characterized by a quadrupole. The flat spacetime limit yields the expected result for a quadrupolar source  $A, B$

$$\lim_{l \rightarrow \infty} E_{\Delta u} = -\frac{3}{8\pi G} \int_{-\infty}^{\infty} du \sum_{m=-2}^2 \left[ |\ddot{A}_m|^2 + |\ddot{B}_m|^2 \right]. \quad (69)$$

Note that for  $\Lambda < 0$ , i.e.  $l \rightarrow il$ , the energy flux is nonzero so that the boundary is not reflective (as is typically imposed). In fact, the energy flux can have an arbitrary sign depending on the values of  $A$ , its time derivatives and  $l$ .

### E. Robinson-Trautman spacetime

The Robinson-Trautman spacetime is an exact solution of Einstein equations that describes the backreaction of a nonlinear gravitational wave on a Schwarzschild spacetime. The Robinson-Trautman solution is dynamical; it models gravitational radiation expanding from a radiating object. Since ultimately, we are interested in describing gravitational radiation emitted by compact sources in the presence of a cosmological constant, this example is of particular interest for our analysis. The original Robinson-Trautman solution contained no cosmological constant [34], but it was soon realized that the solution easily accommodates for a nonzero cosmological constant. This class of spacetimes is the most general radiative vacuum solution admitting a geodesic, shear-free and twist-free null congruence of diverging rays. It has been shown that starting with arbitrary, smooth initial data at some retarded time  $u = u_0$ , the cosmological Robinson-Trautman solutions converge exponentially fast to a Schwarzschild-de Sitter solution at large retarded times ( $u \rightarrow \infty$ ). Thus, these solutions also belong to the class of

solutions discussed in this paper. In this section, we will show this explicitly by providing the form of this solution in Bondi-Sachs-like coordinates.

The line element of the Robinson-Trautman solution with a positive cosmological constant is given by

$$ds^2 = -2H(u, r, \theta, \phi)du^2 - 2dudr + \frac{r^2}{P^2(u, \theta, \phi)}(d\theta^2 + \sin^2\theta d\phi^2), \quad (70)$$

with

$$2H(u, r, \theta, \phi) = -\frac{r^2}{\ell^2} - \frac{2r\dot{P}}{P} + \frac{1}{2}\mathcal{R}_h - \frac{2m(u)}{r}.$$

Here  $P(u, \theta, \phi)$  is an arbitrary function of the retarded time and the angles, and contains the information of the gravitational wave. According to Einstein's equations, the following equation governs the time evolution of  $m(u)$ ,

$$\dot{m} = 3m\frac{\dot{P}}{P} - \frac{1}{8}\Delta_h\mathcal{R}_h. \quad (71)$$

The Laplacian  $\Delta_h$  is defined with respect to the metric  $h_{AB} = P^{-2}(u, \theta, \phi)(d\theta^2 + \sin^2\theta d\phi^2)$ . In the particular case when  $P = 1$  (no radiation), the Schwarzschild de Sitter solution is recovered.

The Robinson-Trautman solution, as written in Eq. (70), does not fit immediately within the asymptotic conditions in Eq. (2). The reason is the presence of the function  $P^{-2}$  appearing in front of the metric of the two-sphere. In order to accommodate the solution one must perform an appropriate change of coordinates. In general, the implementation of this change of coordinates is technically a very hard task. However, a simplified analysis can be achieved by considering the Robinson-Trautman metric with axial symmetry. In addition, and for clarity to the reader, in this section we will only consider a linearized version of the solution. The nonlinear analysis will be discussed in Appendix B.

Assuming for simplicity axial symmetry, the linearized Robinson-Trautman solution expanded around a Schwarzschild-de Sitter background is obtained by expressing the function  $P = P(u, \theta)$  as follows:

$$P = 1 + \epsilon p(u, \theta).$$

Here  $\epsilon$  is a small parameter that controls the linearized expansion. In this approximation, the leading order of Eq. (71) becomes

$$\dot{m} = \frac{1}{4}\epsilon \left[ 12m\dot{p} + p^{(4)} - \cot^2\theta p'' + 2\cot\theta p''' + \cot\theta(\csc^2\theta + 2)p' \right] + O(\epsilon^2). \quad (72)$$

Here the prime denotes derivatives with respect to  $\theta$ . In order to accommodate the solution within our asymptotic

conditions, one can implement the following change of coordinates to linear order in  $\epsilon$

$$\begin{aligned} u &\rightarrow u - \epsilon f(u, \theta) + O(r^{-3}), \\ r &\rightarrow r(1 + \epsilon p(u, \theta)) - \frac{1}{2}\epsilon D^2 f(u, \theta) + O(r^{-2}), \\ \theta &\rightarrow \theta + \epsilon \frac{f'(u, \theta)}{r} + O(r^{-3}), \\ \phi &\rightarrow \phi, \end{aligned}$$

where

$$p = \dot{f}. \quad (73)$$

Thus, one finds

$$M = m[1 - \epsilon(3\dot{f} + f\dot{m})], \quad (74a)$$

$$C_{\theta\theta} = \epsilon(f'' - \cot\theta f'), \quad C_{\phi\phi} = -\epsilon \sin^2\theta(f'' - \cot\theta f'), \quad (74b)$$

$$C_{\theta\phi} = 0, \quad (74c)$$

$$U_{(0)}^\theta = \frac{\epsilon}{\ell^2} f', \quad U_{(0)}^\phi = 0, \quad (74d)$$

$$N^\theta = -3\epsilon m f', \quad N^\phi = 0, \quad (74e)$$

$$E_{AB} = 0. \quad (74f)$$

Decomposing  $f$  into spherical harmonics  $f = \sum_l f_l Y_{l0}$ , we can also compactly write  $C_{AB}$  as  $C_{AB} = 2\epsilon \sum_l f_l Y_{AB}^{l0}$ . In addition, at linear order the News tensor in Eq. (25) takes the simple form  $N_{AB}^{(\Lambda)} = \dot{C}_{AB}$ .

To obtain the flux of energy one can replace (74) in Eq. (32), while retaining only the terms up to order  $\epsilon^2$ . After some integrations by parts one finally obtains

$$\frac{dE}{du} = -\frac{\epsilon^2}{16\pi G} \oint d^2S \left[ (\dot{f}'' - \cot\theta \dot{f}')^2 + \frac{3m}{\ell^2} \partial_u (f'^2) \right]. \quad (75)$$

## VI. COMPARISON WITH ALTERNATIVE APPROACHES

Given the observational evidence for an accelerated expansion of our Universe and the recent gravitational wave observations, the challenge of understanding gravitational waves in the presence of a positive cosmological constant  $\Lambda$  has received considerable attention in recent years. At the linearized level, most of the previous results in the literature agree with each other. As we will see below, our results are also in agreement with them.

However, the situation is drastically different in the full nonlinear theory. Different methods and/or different boundary conditions are employed—some of which even require regularization; the results in general do not agree. We describe below some of these approaches without any pretense of being exhaustive.

### A. Linearized gravity

An important starting point is a thorough understanding of weak gravitational waves on a de Sitter background. There are two key issues predominantly studied within this context: (1) a mathematically sound and physically sensible notion of energy and its flux; and (2) finding explicit solutions for gravitational waves generated by a compact source and their link to time-changing quadrupole moments, thereby generalizing the well-known flat result.<sup>9</sup>

There are various notions of energy and its flux in the literature that mostly distinguish themselves by the method used to derive it (as a consequence, these notions typically are equivalent up to boundary terms), where in spacetime the energy (flux) is evaluated (mostly on  $\mathcal{I}$  or across the cosmological horizon) and by the class of linearized solutions for which the energy is defined. For instance, in [24], the energy flux across  $\mathcal{I}$  is derived using the same symplectic methods as in the earlier work in [22], but for a slightly larger class of linearized solutions. In [36], the authors use the Wald-Zoupas prescription to define energy (and angular momentum). In all these cases, the resulting energy flux is finite and the result gauge invariant.

On the other hand, the energy flux obtained in [23] by direct use of Noether currents is not finite (note also earlier work [37]). In that paper, they remedy this issue by isolating the terms which would lead to infinite energy and introducing a “renormalized canonical energy.” Their argument for the plausibility of this procedure is based on the observation that the diverging terms have dynamics of their own, which evolves independently from the remaining part of the canonical energy.

Yet another approach, which applies only for sources supporting the short wavelength approximation, as it relies on the Isaacson effective stress-tensor, matches the results in [25] if one identifies the transverse-traceless gauge with a certain projection operation.<sup>10</sup>

<sup>9</sup>The gravitational memory effect in de Sitter spacetimes has been investigated in [35], however, this paper focused on the cosmological horizon and is therefore more difficult to relate to the results in this paper that exclusively apply near  $\mathcal{I}$ .

<sup>10</sup>For linearized solutions on Minkowski spacetime, this is a well-defined and consistent procedure for the leading order components of the gravitational field. This is shown explicitly in [38], in which the first notion is referred to as “TT” gauge and the second as “tt” gauge. However, it is not clear that the two notions are also equivalent for the leading order fields in de Sitter spacetime.

In [39], the authors employ the symplectic current density of the covariant phase space to show that the integrand in the energy flux expression on the cosmological horizon is same as that on  $\mathcal{I}$ . This result is interesting as it suggests that at the linearized level propagation of energy flux is along null rays in de Sitter spacetime, despite the fact that gravitational waves themselves have tail terms due to backscattering off the background curvature.

The second key issue investigated is the link between time variation of some compact source generating gravitational waves and the resulting gravitational waves themselves. This was investigated in [25] by solving the linearized Einstein’s equation on de Sitter background sourced by a (first-order) stress-energy tensor. To study the limit to  $\mathcal{I}$ , the authors introduce a late-time approximation in addition to the commonly used post-Newtonian approximation. This allowed them to express the leading terms of the gravitational waves in terms of the quadrupole moments of sources. Moreover, the energy carried away by this gravitational waves was studied using Hamiltonian methods on the covariant phase space of the linearized solutions introduced in their earlier paper [22]. This showed that despite the fact that in principle the energy for linearized perturbations on de Sitter spacetime can be negative (note that this is not in contrast with the finiteness discussed in the previous paragraph), the energy of gravitational waves emitted by compact objects is always positive. This is also consistent with our results in Sec. V D. The quadrupolar solutions in [25] were also reinterpreted in [40], by writing the solutions in Bondi-Sachs-type coordinates different from the ones introduced in this paper. The authors showed that the quadrupolar solutions can be accommodated by a nonzero shear for the leading-order part of  $g_{AB}$ . This is different but not in contradiction with the results in this paper, which show that the radiative solution contributes to the sub-leading part of  $g_{AB}$  and to  $U_{(0)}^A$ , while the leading-order part of  $g_{AB}$  is equal to  $\gamma_{AB}$ . This is a gauge choice. Other papers relating the source dynamics modeled by some compact stress-energy tensor to the gravitational wave and the energy have relied on the short wave approximation [41–43]. These results are consistent with the results in [22].

The gravitational memory effect, which describes the permanent displacement of test masses after a gravitational wave has passed, has only sparsely been analyzed in de Sitter spacetimes. This interesting physical effect was studied near the cosmological horizon of de Sitter spacetime in [35,44], and a “linear” memory effect has been linked to the tail of the de Sitter Green’s function in [45]. However, the connection to asymptotic symmetries—which exist for the memory effect in asymptotically flat spacetimes—have not been studied. We intend to explore this in upcoming research.

## B. Full nonlinear theory

Early investigations of the asymptotic structure of asymptotically de Sitter spacetimes in full nonlinear general relativity such as [46–48] imposed too stringent boundary conditions by demanding conformal flatness of the induced three-dimensional metric on  $\mathcal{I}$ . As a result, these early investigations concluded that the asymptotic symmetry group is the full de Sitter group. However, as was shown in [4], imposing conformal flatness ruled out many physically relevant spacetimes as they enforced a vanishing flux of radiation across  $\mathcal{I}$  (and consequently all charges are strictly conserved, see also [49]). For a more explicit link between some of the results here and in those papers, see Appendix A.

The observation that demanding the asymptotic symmetry group to be the de Sitter one ruled out gravitational waves sparked new interest in this challenging problem. It led the authors in [50] to consider Bondi-Sachs-type coordinates for asymptotically de Sitter spacetimes, which are not conformally flat at  $\mathcal{I}$ . A nice property of their coordinates is that the Weyl tensor has peeling behavior near  $\mathcal{I}$  [51]. While these authors also rely on the Bondi framework, their falloff conditions on the metric coefficients are different from those considered here. In particular, the authors did not fix  $g_{AB}$  to be equal to the unit two-sphere but instead allowed for a nonzero shear at leading order. Their shear contains all the information about gravitational radiation. However, based on our analysis, using their falloff conditions the variation of the charge is infinite. This makes these falloff conditions not as attractive. Moreover, the analysis in those papers was restricted to axisymmetric spacetimes and limited to the study of Einstein’s equations; these papers did not study gravitational charges and fluxes.

Subsequently, various authors used the Newman-Penrose formalism to define and study asymptotically de Sitter spacetimes [52–54]. The two earlier papers by Saw used a special choice of null foliation, thereby excluding the Robinson-Trautman spacetime with a positive cosmological constant as part of their allowed class of spacetimes. The class of null foliations was generalized in [54] by Mao to accommodate for the Robinson-Trautman spacetime. A nice feature of the falloff conditions on the spin coefficients and Weyl scalars in those papers is that they have a well-defined flat limit. However, Mao finds that the asymptotic symmetry algebra consists of all diffeomorphisms on the two-sphere and translations in the  $u$ -direction. In the limit  $l \rightarrow \infty$ , the asymptotic symmetry algebra becomes the algebra of all diffeomorphisms on the two-sphere and supertranslations known as the extended BMS algebra [55] instead of the BMS algebra. Gravitational charges and fluxes were not studied in these papers.

Another set of recent papers on this topic uses similar techniques to those used here [56,57]. These authors find that the asymptotic symmetries form a Lie algebroid instead of a Lie algebra, as they used different falloff

conditions on the metric. Their asymptotic symmetries consist of infinite-dimensional “non-Abelian supertranslations” and superrotations, and like in [54] reduces in the limit  $l \rightarrow \infty$  to the extended BMS algebra. These boundary conditions were used in [58] to define a Bondi news-like tensor using a Newman-Penrose tetrad.

Inspired by the dictionary between Bondi and Fefferman-Graham gauges [59], the authors used earlier results in Fefferman-Graham gauges to define a new class of asymptotically de Sitter spacetimes. In their follow-up work [60], in order to obtain finite charges and fluxes, these authors introduce a holographic renormalization procedure while all charges and fluxes are naturally finite in this paper and do not require any *ad hoc* regularization. The latter work also states more clearly that their interest is in spacetimes with compact spatial slices, as opposed to this work.

Other work has focused on studying the possible isometries of asymptotically de Sitter spacetimes. One of the key results is that the asymptotic symmetry algebra they find is maximally four-dimensional [61], which in spirit agrees with our work.

Research in a different direction focused on the question of how to identify the presence of gravitational radiation in the presence of  $\Lambda$  using geometric tools only and without referring to a specific coordinate system [62]. The criterion proposed is based on value of super-Poynting vector at  $\mathcal{I}$ ; if it vanishes, there is no gravitational radiation across  $\mathcal{I}$  while if it is nonzero, there is gravitational radiation across  $\mathcal{I}$ . This criterion is straightforward to check as the super-Poynting vector is the commutator of the leading order electric  $\mathcal{E}_{ab}$  and magnetic  $\mathcal{B}_{ab}$  part of the Weyl tensor. When the cosmological constant vanishes, this criterion is equivalent to the standard “identification” method of gravitational radiation at null infinity through the means of the (non)vanishing of the Bondi news tensor [63]. When  $\mathcal{B}_{ab}$  vanishes on  $\mathcal{I}$ , the super-Poynting vector also vanishes and this criterion implies that there is no radiation. However, the vanishing of  $\mathcal{B}_{ab}$  is not a necessary condition. In particular, certain Kerr-de Sitter generalized spacetimes have nonvanishing  $\mathcal{B}_{ab}$  on  $\mathcal{I}$  yet their super-Poynting vector vanishes. Here we find that there is gravitational radiation whenever  $C_{AB}$  [and hence  $U_{(0)}^A$ ] is nonzero. When these are zero,  $\mathcal{B}_{ab}$  vanishes. Therefore, our criterion to establish the presence of gravitational radiation seems to be stricter. In other words, based on the super-Poynting vector criterion a spacetime may be labeled as nonradiating, while based on  $C_{AB}$  it would be considered radiating. It is therefore not too surprising that the authors in follow-up work found that the asymptotic symmetry algebra for the spacetimes they considered are infinite dimensional [64–66].

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### APPENDIX A: LINK WITH GEOMETRIC APPROACH

The goal of this appendix is to show how the metric in Eq. (1) with the falloff conditions specified in Eq. (2) is related to earlier results obtained using purely geometric techniques in [4,22,25,67]. In particular, here we show that the falloff conditions are such that the class of spacetimes described here are not strongly asymptotically de Sitter spacetimes. This is as desired, because strongly asymptotically de Sitter spacetimes have no radiative fluxes across  $\mathcal{I}$ .

From the coordinates and expression for the physical metric in the main body of this paper, we can construct Bondi-Sachs-type coordinates near  $\mathcal{I}$  for the conformally completed spacetime  $\tilde{g}_{\mu\nu} = \Omega^2 g_{\mu\nu}$  with  $r = \Omega^{-1}$ . In particular, we straightforwardly obtain

$$\tilde{g}_{\mu\nu} dx^\mu dx^\nu \hat{=} U W du^2 + \gamma_{AB} (dx^A - U^A du)(dx^B - U^B du), \quad (\text{A1})$$

where the falloff of the coefficients directly follows from that in Eq. (2)

$$\beta = -\frac{1}{32} C^{AB} C_{AB} \Omega^2 + \frac{1}{128} \left( (C^{AB} C_{AB})^2 - 12 C^{AB} E_{AB} \right) \Omega^4 + \dots, \quad (\text{A2a})$$

$$W = \frac{\Lambda}{3} - D_A U^A_{(0)} \Omega - \left( 1 + \frac{\Lambda}{16} C^{AB} C_{AB} \right) \Omega^2 + 2M \Omega^3 + \dots \quad (\text{A2b})$$

$$U^A = U^A_{(0)} - \frac{1}{2} D_B C^{AB} \Omega^2 - \frac{2}{3} \left( N^A - \frac{1}{2} C^{AB} D^C C_{BC} \right) \Omega^3 + \dots, \quad (\text{A2c})$$

$$q_{AB} = \gamma_{AB} + C_{AB} \Omega + \frac{C^{CD} C_{CD} \gamma_{AB}}{4r^2} \Omega^2 + E_{AB} \Omega^3 + \dots \quad (\text{A2d})$$

The surfaces of constant  $u$  are outgoing null surfaces for both the unphysical and the physical spacetime, since conformal transformations do not change the properties of null geodesics. With this choice of the falloff conditions and conformal factor, the divergence of  $\tilde{\nabla}_\mu \Omega$  is nonzero. This is different from the typical choice made in the asymptotically flat context, where one often chooses  $\Omega$  such that one is in a conformal divergence-free frame because for asymptotically flat spacetimes this choice simplifies intermediate results significantly.

In [4], it was shown that strongly asymptotically de Sitter spacetimes have a conformally flat metric at  $\mathcal{I}$  and all fluxes vanish across  $\mathcal{I}$ . It is clear from Eq. (A1) that the class of spacetimes considered in this paper do not belong to strongly asymptotically de Sitter spacetimes. This can also be seen by studying the Weyl tensor. In particular, conformal flatness of the metric at  $\mathcal{I}$  is equivalent to the vanishing of the next-to-leading-order magnetic part of the Weyl tensor at  $\mathcal{I}$ . Given the above expression for the metric, we can compute the Weyl tensor explicitly. We find that, as expected, the leading order part of the Weyl tensor vanishes on  $\mathcal{I}$ , that is,  $C_{abcd} \hat{=} 0$  [4]. The next-to-leading-order part is nonzero and we will decompose it into its electric and magnetic part,

$$\mathcal{E}_{ab} := l^2 \Omega^{-1} C_{abcd} n^c n^d, \quad (\text{A3})$$

$$\mathcal{B}_{ab} := l^2 \Omega^{-1*} C_{abcd} n^c n^d = \frac{l^2}{2} \epsilon_{ac}{}^{mn} C_{mnbcd} n^c n^d, \quad (\text{A4})$$

where both  $\mathcal{E}_{ab}$  and  $\mathcal{B}_{ab}$  are symmetric, traceless and orthogonal to  $n^a$ , which here is equal to  $n^a \partial_a = \partial/\partial u$ . The resulting expressions are rather long, so here we only show the part linear in the metric coefficients,



$$\begin{aligned}
\mathcal{E}_{ab}dx^a dx^b|_{\text{lin}} \hat{=} & -\frac{1}{l^2}V^{(3)}(du + l^2 d\Omega)^2 + \left(\frac{3}{2l^2}U_A^{(3)} + \frac{l^2}{2}\partial_u[D^2 + 1]U_A^{(0)}\right)(du + l^2 d\Omega)dx^A \\
& -\frac{l^2}{2}\left(\frac{3}{l^4}q_{AB}^{(3)} - \frac{1}{l^2}V^{(3)}\gamma_{AB} + [2l^2\partial_u^2 + D^2 - 4]\left[D_{(A}U_{B)}^{(0)} - \frac{1}{2}\gamma_{AB}D_C U_{(0)}^C\right]\right. \\
& \left. - \left[D_{(A}D_{B)} - \frac{1}{2}\gamma_{AB}D^2\right]D_C U_{(0)}^C\right)dx^A dx^B
\end{aligned} \tag{A5}$$

and

$$\begin{aligned}
\mathcal{B}_{ab}dx^a dx^b|_{\text{lin}} \hat{=} & \frac{1}{2}[D^2 + 2]\epsilon^{AB}D_A U_B^{(0)}(du + l^2 d\Omega)^2 \frac{l^2}{2}\partial_u[D^2 + 1]\epsilon_A{}^B U_B^{(0)}(du + l^2 d\Omega)dx^A \\
& -\frac{l^2}{2}\left(\frac{1}{2}[D^2 + 2](\epsilon^{EF}D_E U_F^{(0)})\gamma_{AB} - 2l^2\partial_u^2\left[D_{(A}(\epsilon_B)^C U_C^{(0)}) - \frac{1}{2}\gamma_{AB}D_C(\epsilon^{CE}U_E^{(0)})\right]\right. \\
& \left. - \left[D_{(A}D_{B)} - \frac{1}{2}\gamma_{AB}D^2\right]\epsilon^{EF}(D_E U_F^{(0)})\right)dx^A dx^B.
\end{aligned} \tag{A6}$$

It is evident that  $\mathcal{B}_{ab}$  is nonzero and consequently the class of spacetimes considered in this paper are not strongly asymptotically de Sitter as desired, because strongly asymptotically spacetimes remove half the permissible data and have no fluxes of energy across  $\mathcal{I}$ .

Using the explicit linearized solutions in Sec. **VD**, we find that for quadrupolar gravitational waves, the electric and magnetic part of the Weyl tensor are

$$\begin{aligned}
\mathcal{E}_{ab}dx^a dx^b \hat{=} & \sum_{m=-2}^2 \left\{ 6l^{-2}(l^{-2}A_m - \ddot{A}_m)Y_{2m}du^2 + \left(\partial_u(l^{-2}A_m - \ddot{A}_m)Y_A^{2m} - \partial_u(4l^{-2}B_m - \ddot{B}_m)X_A^{2m}\right)dudx^A \right. \\
& \left. + \left(\left[-\frac{3}{2}(l^{-2}A_m - \ddot{A}_m) + \frac{1}{2}\partial_u^2(l^{-2}A_m - \ddot{A}_m)\right]Y_{AB}^{2m} - 3(l^{-2}A_m - \ddot{A}_m)Y_{2m}S_{AB} - \frac{1}{2}\partial_u^2(4B_m - l^2\ddot{B}_m)X_{AB}^{2m}\right)dx^A dx^B \right\}
\end{aligned} \tag{A7}$$

and

$$\begin{aligned}
\mathcal{B}_{ab}dx^a dx^b \hat{=} & \sum_{m=-2}^2 \left\{ 6l^{-2}(l^{-2}B_m - \ddot{B}_m)Y_{2m}du^2 - (\partial_u(4l^{-2}A_m - \ddot{A}_m)Y_A^{2m} - \partial_u(l^{-2}B_m - \ddot{B}_m)X_A^{2m})dudx^A \right. \\
& \left. + \left(\left[-\frac{3}{2}(l^{-2}B_m - \ddot{B}_m) + \frac{1}{2}\partial_u^2(l^{-2}B_m - \ddot{B}_m)\right]Y_{AB}^{2m} - 3(l^{-2}B_m - \ddot{B}_m)Y_{2m}S_{AB} + \frac{1}{2}\partial_u^2(4A_m - l^2\ddot{A}_m)X_{AB}^{2m}\right)dx^A dx^B \right\}.
\end{aligned} \tag{A8}$$

We have explicitly verified that these expressions are symmetric, transverse  $\mathcal{E}_{ab}n^b \hat{=} 0 \hat{=} \mathcal{B}_{ab}n^b$  and traceless  $q^{ab}\mathcal{E}_{ab} \hat{=} 0 \hat{=} q^{ab}\mathcal{B}_{ab}$ . In taking the limit  $l \rightarrow \infty$ , one needs to be careful to rescale  $\mathcal{E}_{ab}$  and  $\mathcal{B}_{ab}$ ; otherwise, due to the overall factor of  $l^2$  in the definition in Eqs. (A3) and (A4), this limit trivially diverges. The flat limit is

$$\lim_{l \rightarrow \infty} l^{-2} \mathcal{E}_{ab} dx^a dx^b \hat{=} \sum_{m=-2}^2 -2\partial_u^A B_m X_{AB}^{2m} dx^A dx^B, \tag{A9}$$

$$\lim_{l \rightarrow \infty} l^{-2} \mathcal{B}_{ab} dx^a dx^b \hat{=} \sum_{m=-2}^2 2\partial_u^A A_m X_{AB}^{2m} dx^A dx^B. \tag{A10}$$

Note that the parity-even solution, which is sometimes also called an electric solution, contributes to the magnetic part of the Weyl tensor and vice versa. There is no contradiction here, as the names electric and magnetic refer to very different notions.

In this linearized limit one can also explicitly show that the  $\ell = 1$  modes in  $U_A^{(0)}$  do not contribute to  $\mathcal{E}_{ab}$  nor to  $\mathcal{B}_{ab}$ .<sup>11</sup> This further supports the interpretation of those modes as nonradiative.

<sup>11</sup>To show this, first decompose  $U_A^{(0)}$  into an ‘‘electric’’ and ‘‘magnetic’’ part:  $U_A^{(0)} = D_A f + \epsilon_A{}^B D_B g$  and use that if  $f$  and  $g$  are  $\ell = 1$  modes  $D_A D_B f = -\gamma_{AB} f$  and similarly for  $g$ .

## APPENDIX B: CHANGE OF COORDINATES FOR THE NONLINEAR ROBINSON-TRAUTMAN SOLUTION

In this appendix we show that the non-linear Robinson-Trautman solution can be accommodated within the asymptotic conditions in Eq. (2). Starting from the solution in Eq. (70) one can perform the following asymptotic change of coordinates

$$\begin{aligned} u &\rightarrow f(u, \theta) + \frac{\ell^2 \csc \theta (\sin \theta h' - \sin h)}{P} \frac{1}{r} + \frac{U_2}{r^2} + \dots, \\ r &\rightarrow \sin \theta \csc h P r + R_0 + \frac{R_1}{r} + \dots, \\ \theta &\rightarrow h(u, \theta) - \frac{P f'}{r} - \frac{1}{2} \ell^2 \left( h'' + \cot \theta h' - \frac{1}{2} \csc^2 \theta \sin 2h \right) \frac{1}{r^2} + \dots, \\ \phi &\rightarrow \phi, \end{aligned}$$

with

$$\begin{aligned} U_2 &= \frac{\ell^4 (\csc \theta \sin h - h')}{4 f' P^3} [P(-2h'' - 2 \cot \theta h' + \csc^2 \theta \sin(2h)) \\ &\quad - 2(\csc \theta \sin h - h')(f'(\partial_f P) + (\partial_h P)(h' + \csc \theta \sin h))], \\ R_0 &= \frac{1}{2} \ell \sin \theta \csc h \left( \frac{\ell(-h'' - \cot \theta h' + \frac{1}{2} \csc^2 \theta \sin 2h)}{f'} - \frac{f' P(\partial_h P)}{\ell} + \frac{2\ell(\partial_f P)(h' - \csc \theta \sin h)}{P} \right), \\ R_1 &= \frac{\ell^4 \csc^3 \theta \sin^3 h}{16 f'^2 P^5} \{ -2P^2(\partial_h P)(h' \sin \theta \csc h - 1)^2 (h' \sin \theta \csc h + 1)(8f' \sin \theta \csc h(\partial_f P) \\ &\quad + 3(\partial_h P)(h' \sin \theta \csc h + 1)) + 4P^3(\sin^2 \theta h'^2 \csc^2 h - 1)[4 \sin \theta f' \csc h(\partial_f \partial_h P)(h' \sin \theta \csc h - 1) \\ &\quad + 2\partial_h^2 P(h'^2 \sin^2 \theta \csc^2 h - 1) + (\partial_h P)(\sin \theta \csc^2 h(\sin \theta h'' + \cos \theta h') - \cot h)] \\ &\quad - 8\ell^2 P(\partial_f^2 P)(h' \sin \theta \csc h - 1)^3 (h' \sin \theta \csc h + 1) + 8\ell^2 (\partial_f P)^2 (h' \sin \theta \csc h - 1)^3 (h' \sin \theta \csc h + 1) \\ &\quad + \csc^2 h P^4 [2h''^2 \sin^4 \theta \csc^2 h + 4h'' \sin^2 \theta (h' \sin \theta \csc^2 h(\cos \theta - 2h' \sin \theta \cot h) + \cot h) \\ &\quad + h'(h' \sin^2 \theta \csc^2 h(8h' \sin \theta (h' \sin \theta \csc^2 h - \cos \theta \cot h) + 4 \cos(2h) + \cos(2\theta) - 11) + 2 \sin(2\theta) \cot h) \\ &\quad - 3 \cos(2h) + 5] \}, \end{aligned}$$

and  $P = P(f(u, \theta), h(u, \theta))$ . The functions  $f$  and  $h$  are required to satisfy the following conditions

$$h'^2 + \frac{P^2 f'^2}{\ell^2} = \csc^2 \theta \sin^2 h, \quad (\text{B1})$$

$$P(\dot{f}h' - f'h) = \csc^2 \theta \sin^2 h. \quad (\text{B2})$$

Note that in the linear approximation, Eq. (B1) indicates that  $h = \theta$ , while Eq. (B2) implies Eq. (73), i.e.,  $\dot{f} = -\omega$ . In the flat limit ( $\ell \rightarrow \infty$ ), Eq. (B1) implies that  $h = \theta$ , while according to Eq. (B2) one has  $\dot{f} = P^{-1}$ . This condition is the same as that obtained in [68].

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