Consistent mass formulas for higher even-dimensional Taub-NUT spacetimes and their AdS counterparts

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Currently, there is a great deal of interest in the seeking of consistent thermodynamics of the Lorentzian Taub-Newman-Unti-Tamburino (NUT) spacetimes. Despite a lot of "satisfactory" efforts that have been made, all of these activities have been restricted to the four-dimensional cases, with the higher even-dimensional cases remaining unexplored. The aim of this article is to fill the gap for the first time. To the end of this subject, we first adopt our own idea that "The NUT charge is a thermodynamical multi-hair" to investigate the consistent thermodynamics of $D = 6$, 8, 10 Lorentzian Taub-NUT spacetimes without a cosmological constant. Similarly to the $D = 4$ cases, as in our previous works, we find that the first law and Bekenstein-Smarr mass formulas are perfectly satisfied if we still assign the secondary hair $J_n = Mn$ as a conserved charge in both mass formulas. Turning to the cases with a nonzero cosmological constant, our treatment continues to work very well and all the results can be fairly generalized to the corresponding Taub-NUT anti–de Sitter spacetimes in higher even dimensions, although we do not know how to define and introduce a similar higher-dimensional version of the dual (magnetic) mass that is well known in four dimensions. Based upon the preceding results, we will also derive the reduced version of the mass formulas when the secondary hair J_n is viewed as a redundant thermodynamic variable.

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I. INTRODUCTION

Taub-Newman-Unti-Tamburino (NUT) solutions [1,2] have long been a source of insight into gravitational thermodynamics. The solutions possess a number of undesirable properties that, while at first sight are highly pathological, actually result in important clarifications in our understanding of black hole thermodynamics, such as the geometrical interpretation of entropy. Recently, there has been a resurgence of great interest in exploring the consistent thermodynamics of the Lorentzian Taub-NUT spacetimes [3–23]. In our opinion, these current investigations of the first law of the NUT-charged spacetimes can be categorized into three different schemes: (I) Retaining the mass unmodified and introducing new global-like charges (secondary hairs) together with their conjugate potentials [3,4]; (II) Keeping the mass unchanged and including new nonglobal Misner charges and their conjugate variables [6–11]; and (III) Only modifying the mass by taking account for the contribution of new nonglobal charges [16,17]. Note that in Ref. [23], the thermodynamic mass that enters into the first law of the four-dimensional Taub-NUT spacetime is the horizon mass [24]. Besides these, there is less interest [6,15] to consider the entropy as the Noether charge [25] that includes the horizon area and the contribution from the Misner strings. However, all of the above-mentioned efforts are only restricted to four-dimensional cases, leaving thermodynamics of the Lorentzian Taub-NUT spacetimes in higher even-dimensions unexplored, which motivates the subject of the present article.

In our previous papers [3–5], we have advocated a new idea that "The NUT charge is a thermodynamical multi-hair" and put forward a simple, systematic way to study the consistent thermodynamics of almost all of the four-dimensional (dyonic) NUT-charged spacetimes. It should be emphasized that, unlike all other attempts [6–17,22,23], our scheme only relies on deriving first a new meaningful Christodoulou-Ruffini-type squared-mass formula [26,27] satisfied by the four-dimensional (dyonic) NUT-charged spacetimes, and the only needed input in this derivation is to introduce the secondary hairs ($J_n = Mn$, $Q_n = qn$ and $P_n = pn$) as new conserved charges. Then the consistent thermodynamic first law and Bekenstein-Smarr mass formulas of these NUT-charged spacetimes can be deduced via some simple and purely algebraic manipulations from

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this squared-mass formula, which can hardly be given by the other papers as mentioned above. Subsequently, the usual Bekenstein-Hawking one-quarter area-entropy relation can be naturally restored for the generic NUT-charged spacetime (and all its extensions) without imposing any constraint condition and with no need to assume ahead that the one-quarter area-entropy relation should hold true. The advantage of our proposal that the NUT charge acts as a thermodynamical multi-hair is that it cannot only explicate the rotationlike and electromagnetic chargelike characters, but it can also simultaneously explain many other exotic properties. What is more, our consistent mass formulas [3,4] are unique, and all expressions for thermodynamical quantities are exceedingly simple and succinct. This is in contrast to all other works where not only can the consistent first law of the NUTty dyonic spacetimes have the electric-type, magnetic-type, mixed-type versions [7,10], and even many other ones [11], but also the expressions of the related thermodynamical variables are quite complicated.

In addition, it should be emphasized that the introduction of the secondary hair $J_n = Mn$ in our previous works [3–5] is not merely based upon thermodynamical reasons, but also comes from many other considerations. For instance, our secondary hair $J_n = Mn \equiv M_5$ must endow the character of a global conserved charge, which exactly corresponds to the mass of the five-dimensional gravitational magnetic monopole [28], so that it can be naturally included into the first law and Bekenstein-Smarr mass formula. On the other hand, it cannot only help to explain the gyromagnetic ratio of a Kerr-NUT-type spacetime [29], but also accounts for the quantization condition for a gravitational monopole [30–32]. What is more, it has been later shown in Ref. [33] that only when considering the secondary hair $J_n = Mn$ as a independent charge can the area (or entropy) products of the NUT-charged spacetimes be subject to the universal rules [34], and the mass be expressed as a sum of the surface energy, the rotational energy, and the electromagnetic energy [35].

In this work, we will continue to apply our proposal that "The NUT charge is a thermodynamical multi-hair" to investigate consistent thermodynamics of the $D = 6$, 8, 10 Lorentzian Taub-NUT spacetimes without and with a cosmological constant. Our paper is organized as follows. In Sec. II, we start with the construction of a novel Christodoulou-Ruffini-like squared-mass formula of the six-dimensional Lorentzian Taub-NUT solution by additionally including only one secondary hair $J_n = Mn$, as was done in Refs. [3,4]. Using this squared-mass formula, which can be thought of as representing a hypersurface embedded into one more high-dimensional thermodynamical state space, both the differential and integral mass formulas can be deduced through a simple mathematical manipulation. Then, the procedure is extended to the six-dimensional Lorentzian Taub-NUT-anti–de Sitter (AdS) case. In Sec. III, we proceed to discuss the cases of the eight-dimensional Lorentzian Taub-NUT and Taub-NUT-AdS spacetimes, respectively. Then, in Sec. IV, we extend to investigate the cases of the ten-dimensional Taub-NUT spacetime and its AdS extension. We find that our scheme in the $D = 6$, 8, 10 cases works successfully as in the four-dimensional case [3], and summarize in Sec. V the main results for the generic $(2k+2)$-dimensional Taub-NUT-AdS spacetimes. In Sec. VI, we will turn to consider the secondary hair J_n as a redundant thermodynamic variable and derive the corresponding mass formulas for all $(2k+2)$ dimensions when J_n is not considered as a independent thermodynamic variable. Finally, we present our conclusions and outlooks in Sec. VII. In the Appendix, we briefly present the main results of our extensions to the cases of the $(2k+2)$-dimensional multi-NUTty spacetimes without a cosmological constant.

II. SIX-DIMENSIONAL TAUB-NUT SPACETIME

As shown in Ref. [36] for the six-dimensional Taub-NUT spacetime, there are two different choices for the base space, namely, $S^2 \times S^2$ and $\mathbb{CP}^2$. We start our investigation of the mass formulas in the case of the $S^2 \times S^2$ base space, but the same procedure is also applicable to the case of the $\mathbb{CP}^2$ base space. Using $S^2 \times S^2$ as a base space, the metric of the six-dimensional Lorentzian Taub-NUT solution has the form

$$ds_6^2 = -f(r)\left(dt + 2n\sum_{i=1}^{2}\cos\theta_i d\phi_i\right)^2 + \frac{dr^2}{f(r)} + (r^2+n^2)\sum_{i=1}^{2}(d\theta_i^2 + \sin^2\theta_i d\phi_i^2), \quad (1)$$

where

$$f(r) = \frac{r^4 + 6n^2r^2 - 3n^4 - 6mr}{3(r^2+n^2)^2},$$

in which m and n are the mass parameter and the NUT charge parameter, respectively.

Our aim is to derive various mass formulas and to discuss consistent thermodynamics of the six-dimensional Lorentzian Taub-NUT spacetime. To begin with, let us present some known quantities that can be evaluated via the standard method. First, the area and the surface gravity at the horizon are easily computed as

$$A_h = 16\pi^2(r_h^2+n^2)^2 = 16\pi^2\mathcal{A}_h, \quad \kappa = \frac{1}{2}f'(r_h) = \frac{1}{2r_h}, \quad (2)$$

in which a reduced horizon area $\mathcal{A}_h = (r_h^2+n^2)^2$ is introduced just for briefness, and r_h represents the greatest root of the horizon equation: $r_h^4 + 6n^2r_h^2 - 3n^4 - 6mr_h = 0$.

As for the global conserved charges (M and N), the Komar mass is divergent, while the Abbott-Deser (AD) mass [37] is finite. The AD mass M associated to the Killing vector ∂_t and the NUT charge N read

$$M = 8\pi m, \qquad N = 8\pi n. \tag{3}$$

In addition to the above global conserved charges (M, N) which act as the primary hairs, below just as we did in the four-dimensional cases [3–5], we will also simply introduce an extra secondary hair $J_n \simeq mn$ into the mass formula, which appears in the following asymptotic expansions of the metric components $g_{t\phi_1}$ and $g_{t\phi_2}$ at infinity:

$$g_{tt} \simeq -\frac{1}{3} - \frac{4n^2}{3r^2} + \frac{2m}{r^3} + \mathcal{O}(r^{-4}),$$
$$g_{t\phi_i} \simeq 2ng_{tt}\cos\theta_i, \quad i = 1, 2. \tag{4}$$

It should be mentioned that the hairs (M, N, J_n) that appeared in the first law and the Bekenstein-Smarr mass formula are the lowest three moments of the multipole moments, $m^i n^j$ or $(m + In)^k$ where $i \geq 0$, $j \geq 0$, $k \geq 0$ are non-negative integers. In particular, J_n is the only introduced secondary hair so that all the thermodynamical expressions of the solutions will not be complicated in our work.

A. Consistent mass formulas of the six-dimensional Taub-NUT spacetime

In order to establish the first law which is reasonable and consistent in both physical and mathematical senses, we employ the algebraic approach suggested in Refs. [3,4,38] to construct a meaningful Christodoulou-Ruffini-type squared-mass formula. First, via reexpressing $r_h = \sqrt{\mathcal{A}_h^{1/2} - n^2}$ in terms of the reduced horizon area and substituting it into the equation $(r_h^4 + 6n^2r_h^2 - 3n^4)^2 = 36m^2r_h^2$, we get the following identity:

$$m^2 = \frac{1}{36\sqrt{\mathcal{A}_h}}\left(\mathcal{A}_h + 4n^2\sqrt{\mathcal{A}_h} - 8n^4\right)^2 + \frac{m^2n^2}{\sqrt{\mathcal{A}_h}}, \tag{5}$$

which can be alternatively converted to a quartic polynomial of $\mathcal{A}_h$:

$$(\mathcal{A}_h^2 + 36m^2n^2 + 64n^8)^2 = 16(9m^2 + 16n^6 - 2n^2\mathcal{A}_h)^2\mathcal{A}_h.$$

Next, in addition to the conserved charges M and N given in Eq. (3), only one extra input that we need is to introduce the secondary hair $J_n = Mn = 8\pi mn$ as a thermodynamic independent variable. Then after substituting $m = M/(8\pi)$, $n = N/(8\pi)$ and $\mathcal{A} = 8\pi\mathcal{A}_h$ into Eq. (5), one can arrive at an useful identity

$$M^2 = \frac{\sqrt{2\pi}}{18\sqrt{\mathcal{A}}}\left(\mathcal{A} + \frac{N^2}{8\pi^2}\sqrt{2\pi\mathcal{A}} - \frac{N^4}{64\pi^3}\right)^2 + \frac{2\sqrt{2\pi}}{\sqrt{\mathcal{A}}}J_n^2, \tag{6}$$

which is our new Christodoulou-Ruffini-like squared-mass formula for the six-dimensional Taub-NUT spacetime. Alternatively, the above equation (6) can be converted to a quartic polynomial of the area $\mathcal{A} = \mathcal{A}(M, N, J_n)$:

$$\left(\mathcal{A}^2 + 36J_n^2 + \frac{N^8}{4096\pi^6}\right)^2 = \frac{\mathcal{A}}{8\pi^3}\left(N^2\mathcal{A} - 36\pi M^2 - \frac{N^6}{64\pi^3}\right)^2.$$

At this step, it would be stressed that Eq. (5) can be thought of as representing a hypersurface in the three-dimensional thermodynamical state space, whose variables $(m, n, \mathcal{A}_h)$ exactly match with the numbers of the solution parameters that appeared in the structure function $f(r)$. After introducing an extra hair J_n, which is nothing but a kind of higher-dimensional embedding trick, it becomes a hypersurface in the four-dimensional state space, as specified by Eq. (6), which now has four independent variables $(M, N, J_n, \mathcal{A})$. Our below discussions will be based upon this higher one-dimensional thermodynamical state space.

Having finished the above task, we are now in a position to obtain the differential and integral mass formulas for the six-dimensional Taub-NUT spacetime. Since the secondary hair J_n will be treated as an independent variable,[1] the above squared-mass formula (6) can be regarded formally as a basic functional relation: $M = M(\mathcal{A}, N, J_n)$. As was done in Refs. [3–5,39–41], differentiating it with respect to the thermodynamical variables $(\mathcal{A}, N, J_n)$ yields their conjugate quantities, and subsequently we can arrive at the differential and integral mass formulas with the conjugate thermodynamic potentials given by the ordinary Maxwell relations.

For instance, differentiating the squared-mass formula (6) with respect to $\mathcal{A}$ yields one-quarter of the surface gravity

$$\kappa = 4\frac{\partial M}{\partial \mathcal{A}}\bigg|_{(N,J_n)} = \frac{1}{2r_h}, \tag{7}$$

which is exactly the same one as given in Eq. (2). Similarly, by differentiating the squared-mass formula (6) with respect to the NUT charge N and the secondary hair J_n, then their conjugate gravitomagnetic potential ψ_h and quasiangular momentum ω_h can be derived, respectively, as follows:

$$\psi_h = \frac{\partial M}{\partial N}\bigg|_{(\mathcal{A},J_n)} = \frac{4nr_h(r_h^2 - 3n^2)}{3(r_h^2 + n^2)}, \tag{8}$$

[1] However, one may object to this viewpoint. A treatment without viewing it as a independent thermodynamic variable in the mass formulas for all $(2k + 2)$ dimensions is presented in Sec. VI.

$$\omega_h = \frac{\partial M}{\partial J_n}\bigg|_{(\mathcal{A},N)} = \frac{n}{r_h^2 + n^2}. \quad (9)$$

Now, one can check that both the differential and integral mass formulas are completely fulfilled

$$dM = (\kappa/4)d\mathcal{A} + \omega_h dJ_n + \psi_h dN, \quad (10)$$

$$3M = \kappa\mathcal{A} + 4\omega_h J_n + \psi_h N, \quad (11)$$

among all the aforementioned thermodynamical conjugate pairs. Comparing these mass formulas [(10) and (11)] with the standard ones, it is highly urged that the following familiar identifications be made:

$$S = \frac{A_h}{4} = \frac{\pi}{2}\mathcal{A} = 4\pi^2(r_h^2 + n^2)^2, \quad T = \frac{\kappa}{2\pi} = \frac{1}{4\pi r_h}, \quad (12)$$

which naturally recovers the famous Bekenstein-Hawking one-quarter area-entropy relation of the six-dimensional Taub-NUT spacetime, completely similar to the $D = 4$ cases.

B. Extension to the Taub-NUT-AdS$_6$ spacetime

Now we will extend the above discussion to explore the Lorentzian Taub-NUT-AdS$_6$ spacetime with a nonzero negative cosmological constant. The metric is still given by Eq. (1), but now we have

$$f(r) = \frac{1}{3(r^2+n^2)^2}[r^4 + 6n^2r^2 - 3n^4 - 6mr + 3g^2(r^6 + 5n^2r^4 + 15n^4r^2 - 5n^6)],$$

where $l = 1/g$ is the cosmological scale.

First, we will employ the conformal completion method [42] to calculate the conserved mass M of the Taub-NUT-AdS$_6$ solution. This conformal Ashtekar-Magnon-Das (AMD) mass can be evaluated via the integral in terms of the conformal Weyl tensor over the spatial conformal boundary at infinity. The Taub-NUT-AdS$_6$ spacetime is asymptotically local AdS, and admits an asymptotic boundary five-metric that approaches to

$$\begin{aligned} d\bar{s}_5^2 &= \lim_{r\to\infty}\frac{ds_6^2}{r^2} \\ &= -g^2\left(dt + 2n\sum_{i=1}^{2}\cos\theta_i d\phi_i\right)^2 \\ &\quad + \sum_{i=1}^{2}(d\theta_i^2 + \sin^2\theta_i d\phi_i^2), \end{aligned} \quad (13)$$

with which one can define a normal vector: $\hat{n}^a = -gr^2\delta_r^a$.

Note that the five-volume form of the conformal boundary AdS metric (13) is simply given by

$$\mathbb{V}_5 = g\sin\theta_1\sin\theta_2 dt\wedge d\theta_1\wedge d\theta_2\wedge d\phi_1\wedge d\phi_2, \quad (14)$$

then, using the inner-product rule, $\langle\partial_\mu, dx^\mu\rangle = \delta_\mu^\nu$, we can obtain the area vector $d\Sigma_t = \langle\partial_t, \mathbb{V}_5\rangle = g\sin\theta_1\sin\theta_2 d\theta_1\wedge d\theta_2\wedge d\phi_1\wedge d\phi_2$, from which we can get its only nonvanishing component:

$$dS_t = g\prod_{i=1}^{2}\sin\theta_i d\theta_i d\phi_i. \quad (15)$$

Since the conserved charge associated with a unit Killing vector ξ^ν is defined as

$$\mathcal{Q}[\xi] = \frac{1}{24\pi g^3}\int (r^3 C^t_{a\nu b}\hat{n}^a\hat{n}^b\xi^\nu dS_t)\Big|_{r\to\infty}, \quad (16)$$

where $C^t_{a\nu b}$ is the Weyl conformal tensor, we can easily obtain the conserved charge associated with the timelike vector ∂_t as

$$\mathcal{Q}[\partial_t] = 8\pi m - \frac{8\pi}{3}(1 + 6g^2n^2)n^2r + \mathcal{O}(r^{-1}), \quad (17)$$

which is clearly divergent at spatial infinity. Therefore, in order to obtain a finite expression for the conformal mass $M = 8\pi m$, one must subtract the divergence due to the contribution from the massless (pure NUT) background spacetime. So we see that while the conformal completion method can get a finite expression for the NUT-AdS$_4$ spacetime, it fails to do so in the higher even-dimensional NUT-charged AdS spacetimes [43]. This situation is very much similar to the Komar integral which can achieve a finite value for the four-dimensional RN-NUT spacetime while it cannot obtain a finite one for higher even-dimensional NUT-charged spacetimes.

On the other hand, the Abbott-Deser method [37] is a reference background subtraction approach, which fairly gives a finite AD mass [44]. Incidentally, one can also use the counterterm method [45–48] to obtain the same result for the mass, but also to get a finite expression for the Euclidean action at the same time. However we shall not adopt this method here due to its involved computations.

Unfortunately, since it is unclear to us how to define a dual (magnetic) mass in the higher-dimensional spacetime, we will not consider the dual mass here and hereafter. The NUT charge will be simply taken as $N = 8\pi n$ just like the case without a cosmological constant.

Next, the surface gravity at the horizon that is specified by the largest root of the equation $f(r_h) = 0$ can be evaluated as

$$\kappa = \frac{1}{2}f'(r_h) = \frac{1 + 5g^2(r_h^2 + n^2)}{2r_h}, \quad (18)$$

while the horizon area reads $A_h = 16\pi^2\mathcal{A}_h$, in which the reduced horizon area is still denoted as $\mathcal{A}_h = (r_h^2 + n^2)^2$.

Now we would like to derive a novel Christodoulou-Ruffini-like squared-mass formula like the case without a cosmological constant. Accordingly, inserting $r_h = \sqrt{\mathcal{A}_h^{1/2} - n^2}$ into the equation $[r_h^4 + 6n^2r_h^2 - 3n^4 + 3g^2(r_h^6 + 5n^2r_h^4 + 15n^4r_h^2 - 5n^6)]^2 = 36m^2r_h^2$ will yield

$$m^2 = \frac{1}{36\sqrt{\mathcal{A}_h}}[(1 + 6g^2n^2)(\mathcal{A}_h + 4n^2\sqrt{\mathcal{A}_h} - 8n^4) + 3g^2\mathcal{A}_h^{3/2}]^2 + \frac{m^2n^2}{\sqrt{\mathcal{A}_h}}, \tag{19}$$

which can be converted to a sextic polynomial of $\mathcal{A}_h$:

$$[9g^4\mathcal{A}_h^3 + (1 + 6g^2n^2)(1 + 30g^2n^2)\mathcal{A}_h^2 + 64n^8(1 + 6g^2n^2)^2 + 36m^2n^2]^2 = 4[18m^2 + (1 + 6g^2n^2)(3g^2\mathcal{A}_h + 24g^2n^4 + 4n^2)(8n^4 - \mathcal{A}_h)]^2\mathcal{A}_h. \tag{20}$$

Finally, plugging $m = M/(8\pi)$, $n = N/(8\pi)$, $\mathcal{A} = 8\pi\mathcal{A}_h$, and $g^2 = 4\pi P/5$ into Eq. (19), where $P = (D-1)(D-2)g^2/(16\pi)$ is the generalized pressure [49], and also introducing a secondary hair, $J_n = Mn$, as before, then after a little algebra we obtain a useful identity

$$M^2 = \frac{\sqrt{2\pi}}{18\sqrt{\mathcal{A}}}\left[\left(1 + \frac{3N^2}{40\pi}P\right)\left(\mathcal{A} + \frac{N^2}{8\pi^2}\sqrt{2\pi\mathcal{A}} - \frac{N^4}{64\pi^3}\right) + \frac{3}{10\pi}(2\pi\mathcal{A})^{3/2}P\right]^2 + \frac{2\sqrt{2\pi}}{\sqrt{\mathcal{A}}}J_n^2, \tag{21}$$

which is nothing but the Christodoulou-Ruffini-like squared-mass formula for the six-dimensional Taub-NUT-AdS spacetime. Equation (21) consistently reduces to Eq. (6), which is obtained in the case of the six-dimensional Taub-NUT spacetime when the generalized pressure P is turned off.

Like the case without a cosmological constant, Eq. (19) represents a hypersurface in the four-dimensional state space with four free variables $(m, n, g, \mathcal{A}_n)$. After introducing an extra hair J_n, it is embedded into a five-dimensional thermodynamic state space defined by Eq. (21), which is our starting point of the following prescription.

The differentiation of the squared-mass formula (21) leads to the first law

$$dM = (\kappa/4)d\mathcal{A} + \omega_h dJ_n + \psi_h dN + VdP, \tag{22}$$

where

$$\kappa = 4\frac{\partial M}{\partial \mathcal{A}}\bigg|_{(N,J_n,P)} = \frac{1 + 5g^2(r_h^2 + n^2)}{2r_h},$$

$$\omega_h = \frac{\partial M}{\partial J_n}\bigg|_{(\mathcal{A},N,P)} = \frac{n}{r_h^2 + n^2},$$

$$\psi_h = \frac{\partial M}{\partial N}\bigg|_{(\mathcal{A},J_n,P)} = \frac{2nr_h[2r_h^2 - 6n^2 + 3g^2(r_h^4 + 10n^2r_h^2 - 15n^4)]}{3(r_h^2 + n^2)},$$

$$V = \frac{\partial M}{\partial P}\bigg|_{(\mathcal{A},N,J_n)} = \frac{16\pi^2 r_h(r_h^6 + 5n^2r_h^4 + 15n^4r_h^2 - 5n^6)}{5(r_h^2 + n^2)}.$$

When the NUT charge parameter n vanishes, the thermodynamic volume reduces to $V = 16\pi^2 r_h^5/5$.

Utilizing all the expressions obtained above, one can directly verify that the Bekenstein-Smarr mass formula,

$$3M = \kappa\mathcal{A} + 4\omega_h J_n + \psi_h N - 2VP, \tag{23}$$

is completely satisfied also. It is naturally suggested to identify $S = A_h/4 = 4\pi^2\mathcal{A}_h$ and $T = \kappa/(2\pi)$, so that the solution acts like a genuine black hole without breaking the classical one-quarter area-entropy relation.

In the remaining two sections, we will adopt the same strategy to deal with the eight- and ten-dimensional NUT-charged (AdS) spacetimes, respectively. The interpretation of our squared-mass formulas in both dimensions is essentially the same one as was done just in the six-dimensional case, and will not be repeated once more again.

III. EIGHT-DIMENSIONAL TAUB-NUT SPACETIME

In this section, we will extend the above discussion to the case of the eight-dimensional Taub-NUT spacetime, to which there are two different choices [36] for the base manifold, namely, $S^2 \times S^2 \times S^2$ and $S^2 \times \mathbb{CP}^2$. Likewise, in the six-dimensional case we will only consider the case where the base space is $S^2 \times S^2 \times S^2$, so that the metric owns a $U(1)$ fibration over $S^2 \times S^2 \times S^2$:

$$ds_8^2 = -f(r)\left(dt + 2n\sum_{i=1}^{3}\cos\theta_i d\phi_i\right)^2 + \frac{dr^2}{f(r)} + (r^2 + n^2)\sum_{i=1}^{3}(d\theta_i^2 + \sin^2\theta_i d\phi_i^2), \tag{24}$$

where

$$f(r) = \frac{r^6 + 5n^2r^4 + 15n^4r^2 - 5n^6 - 10mr}{5(r^2 + n^2)^3}.$$

At the horizon that is the largest root of $f(r_h) = 0$, the area and the surface gravity can be evaluated via the standard method as

$$A_h = 64\pi^3(r_h^2+n^2)^3 = 64\pi^3\mathcal{A}_h, \qquad \kappa = \frac{1}{2}f'(r_h) = \frac{1}{2r_h}, \tag{25}$$

where we now denote the reduced horizon area: $\mathcal{A}_h = (r_h^2+n^2)^3$.

Similar to the six-dimensional case, the AD mass and the NUT charge can be computed as

$$M = 48\pi^2 m, \qquad N = 48\pi^2 n. \tag{26}$$

A. Consistent mass formulas of the eight-dimensional Taub-NUT spacetime

To derive our squared-mass formula, we will adopt the same trick as we did in the last section, so we first express the positive root $r_h = \sqrt{\mathcal{A}_h^{1/3} - n^2}$ in terms of the reduced horizon area and substitute it into the equation $(r_h^6 + 5n^2r_h^4 + 15n^4r_h^2 - 5n^6)^2 = 100m^2r_h^2$. After some algebraic computations, one can obtain the following useful identity:

$$m^2 = \frac{1}{100\mathcal{A}_h^{1/3}}(\mathcal{A}_h + 2n^2\mathcal{A}_h^{2/3} + 8n^4\mathcal{A}_h^{1/3} - 16n^6)^2 + \frac{m^2n^2}{\mathcal{A}_h^{1/3}}, \tag{27}$$

which can also be converted into a polynomial of $\mathcal{A}_h$ after eliminating the fractional powers. Because of its complexity, we shall omit it here.

Subsequently, after inserting $m = M/(48\pi^2)$, $n = N/(48\pi^2)$, and $\mathcal{A} = 48\pi^2\mathcal{A}_h$ into Eq. (27) and including only one secondary hair, $J_n = Mn$, as before, we can obtain a novel squared-mass formula:

$$M^2 = \frac{(6\pi^2)^{1/3}}{50\mathcal{A}^{1/3}}\left[\mathcal{A} + \frac{N^2(6\pi^2\mathcal{A}^2)^{1/3}}{576\pi^4} + \frac{N^4(36\pi\mathcal{A})^{1/3}}{165888\pi^7} - \frac{N^6}{15925248\pi^{10}}\right]^2 + \frac{2(6\pi^2)^{1/3}}{\mathcal{A}^{1/3}}J_n^2. \tag{28}$$

Now we employ a similar procedure as manipulated in the previous section, i.e., viewing the secondary hair $J_n = Mn$ as an independent thermodynamical variable, then performing the partial derivative of the above squared-mass formula (28) with respect to one of its thermodynamical quantities $(\mathcal{A}, N, J_n)$ and simultaneously fixing the remaining ones, respectively, and this will lead to their corresponding conjugate quantities.

First, differentiating the squared-mass formula (28) with respect to $\mathcal{A}$ yields one-sixth of the surface gravity:

$$\kappa = 6\frac{\partial M}{\partial \mathcal{A}}\bigg|_{(N,J_n)} = \frac{1}{2r_h}, \tag{29}$$

which coincides with the one given in Eq. (25). Next, the potential ψ_h and the quasiangular momentum ω_h, which are conjugate to N and J_n, respectively, are given by

$$\psi_h = \frac{\partial M}{\partial N}\bigg|_{(\mathcal{A},J_n)} = \frac{2nr_h(r_h^4 + 10n^2r_h^2 - 15n^4)}{5(r_h^2+n^2)}, \tag{30}$$

$$\omega_h = \frac{\partial M}{\partial J_n}\bigg|_{(\mathcal{A},N)} = \frac{n}{r_h^2+n^2}. \tag{31}$$

Using all the above thermodynamical conjugate pairs, it is easy to check that both differential and integral mass formulas are completely obeyed

$$dM = (\kappa/6)d\mathcal{A} + \omega_h dJ_n + \psi_h dN, \tag{32}$$

$$5M = \kappa\mathcal{A} + 6\omega_h J_n + \psi_h N. \tag{33}$$

Then it is natural to recognize

$$S = \frac{A_h}{4} = \frac{\pi}{3}\mathcal{A} = 16\pi^3(r^2+n^2)^3, \qquad T = \frac{\kappa}{2\pi} = \frac{1}{4\pi r_h}, \tag{34}$$

so that the eight-dimensional Taub-NUT solution behaves like a genuine black hole without violating the beautiful one-quarter area-entropy law. Here we do not require in advance that the first law should be obeyed in order to obtain the consistent thermodynamical relations, rather it is just a very natural by-product of the purely algebraic deduction.

B. Extension to the Taub-NUT-AdS$_8$ spacetime

In this subsection, we would like to deal with the Lorentzian Taub-NUT-AdS$_8$ spacetime with a nonzero cosmological constant. The metric is still given by Eq. (24), but now

$$f(r) = \frac{1}{5(r^2+n^2)^3}[r^6 + 5n^2r^4 + 15n^4r^2 - 5n^6 - 10mr + g^2(5r^8 + 28n^2r^6 + 70n^4r^4 + 140n^6r^2 - 35n^8)],$$

in which $l = 1/g$ is the cosmological scale.

First, one can evaluate the AD mass for this spacetime as

$$M = 48\pi^2 m. \tag{35}$$

Next, we want to compute some thermodynamic quantities at the Killing horizon that is determined by $f(r_h) = 0$. At the horizon, the surface gravity can be obtained via the standard method as

$$\kappa = \frac{1}{2} f'(r_h) = \frac{1 + 7g^2(r_h^2 + n^2)}{2r_h}, \tag{36}$$

while the horizon area is $A_h = 64\pi^3 \mathcal{A}_h$, with the reduced horizon area still being denoted as $\mathcal{A}_h = (r_h^2 + n^2)^3$.

Then we substitute $r_h = \sqrt{\mathcal{A}_h^{1/3} - n^2}$ into the equation $[r_h^6 + 5n^2 r_h^4 + 15n^4 r_h^2 - 5n^6 + g^2(5r_h^8 + 28n^2 r_h^6 + 70n^4 r_h^4 + 140n^6 r_h^2 - 35n^8)]^2 = 100m^2 r_h^2$ to get an identity:

$$m^2 = \frac{1}{100\mathcal{A}_h^{1/3}} [(1 + 8g^2 n^2)(\mathcal{A}_h + 2n^2 \mathcal{A}_h^{2/3} + 8n^4 \mathcal{A}_h^{1/3} - 16n^6) + 5g^2 \mathcal{A}_h^{4/3}]^2 + \frac{m^2 n^2}{\mathcal{A}_h^{1/3}}. \tag{37}$$

Supposing that only the secondary hair $J_n = Mn$ is needed to be included as before, then after inserting $m = M/(48\pi^2)$, $n = N/(48\pi^2)$, $\mathcal{A} = 48\pi^2 \mathcal{A}_h$, and $g^2 = 8\pi P/21$ into Eq. (37), one can arrive at the following squared-mass formula:

$$M^2 = \frac{(6\pi^2)^{1/3}}{50\mathcal{A}^{1/3}} \left\{ \left(1 + \frac{N^2}{756\pi^3} P\right) \left[\mathcal{A} + \frac{N^2 (6\pi^2 \mathcal{A}^2)^{1/3}}{576\pi^4} + \frac{N^4 (36\pi \mathcal{A})^{1/3}}{165888\pi^7} - \frac{N^6}{15925248\pi^{10}}\right]^2 + \frac{10}{63} (36\pi \mathcal{A}^4)^{1/3} P \right\}^2 + \frac{2(6\pi^2)^{1/3}}{\mathcal{A}^{1/3}} J_n^2, \tag{38}$$

in which P is the generalized pressure. We point out that the squared-mass formula (38) consistently reduces to Eq. (28) when the cosmological constant vanishes.

Similar to the strategy that we used in the last subsection, one can view the mass as an implicit function, $M = M(\mathcal{A}, N, J_n, P)$, and then differentiating the squared-mass formula (38) with respect to its variables leads to a new reasonable differential mass formula

$$dM = (\kappa/6) d\mathcal{A} + \omega_h dJ_n + \psi_h dN + V dP, \tag{39}$$

where

$$\kappa = 6 \frac{\partial M}{\partial \mathcal{A}}\bigg|_{(N, J_n, P)} = \frac{1 + 7g^2(r_h^2 + n^2)}{2r_h},$$

$$\omega_h = \frac{\partial M}{\partial J_n}\bigg|_{(\mathcal{A}, N, P)} = \frac{n}{r_h^2 + n^2},$$

$$\psi_h = \frac{\partial M}{\partial N}\bigg|_{(\mathcal{A}, J_n, P)} = \frac{2nr_h}{5(r_h^2 + n^2)} [r_h^4 + 10n^2 r_h^2 - 15n^4 + 4g^2(r_h^6 + 7n^2 r_h^4 + 35n^4 r_h^2 - 35n^6)],$$

$$V = \frac{\partial M}{\partial P}\bigg|_{(\mathcal{A}, N, J_n)} = \frac{64\pi^3 r_h (5r_h^8 + 28n^2 r_h^6 + 70n^4 r_h^4 + 140n^6 r_h^2 - 35n^8)}{35(r_h^2 + n^2)}.$$

At the same time, one can check that the integral mass formula,

$$5M = \kappa \mathcal{A} + 6\omega_h J_n + \psi_h N - 2VP, \tag{40}$$

is also automatically satisfied.

The consistency of the above thermodynamic relations suggests that one should restore the well-known Bekenstein-Hawking area-entropy relation $S = A_h/4 = 16\pi^3 \mathcal{A}_h$ and Hawking-Gibbons temperature $T = \kappa/(2\pi)$, which means that the eight-dimensional Taub-NUT-AdS spacetime should be regarded as a generic black hole.

It is worth noting that the thermodynamic quantities of the base space of $S^2 \times \mathbb{CP}^2$ are the same ones as those in the case of $S^2 \times S^2 \times S^2$ base space, because the expression of the radial function $f(r)$ remains unchanged, and we will not repeat them here.

IV. TEN-DIMENSIONAL TAUB-NUT SPACETIME

Finally, we will turn to consider the ten-dimensional Taub-NUT spacetime and its AdS counterpart. As shown in Ref. [36] for the ten-dimensional Taub-NUT spacetime, there are three different choices for the base manifold, namely, $S^2 \times S^2 \times S^2 \times S^2$, $S^2 \times S^2 \times \mathbb{CP}^2$, and $\mathbb{CP}^2 \times \mathbb{CP}^2$. We will only consider the case in which the metric possesses a $U(1)$ fibration over $S^2 \times S^2 \times S^2 \times S^2$:

$$ds_{10}^2 = -f(r)\left(dt + 2n \sum_{i=1}^{4} \cos\theta_i d\phi_i\right)^2 + \frac{dr^2}{f(r)} + (r^2 + n^2) \sum_{i=1}^{4} (d\theta_i^2 + \sin^2\theta_i d\phi_i^2), \tag{41}$$

where

$$f(r) = \frac{5r^8 + 28n^2 r^6 + 70n^4 r^4 + 140n^6 r^2 - 35n^8 - 70mr}{35(r^2 + n^2)^4}.$$

At the horizon which is defined by the largest root of $f(r_h) = 0$, the horizon area and the surface gravity can be obtained as

$$A_h = 256\pi^4 (r_h^2 + n^2)^4 = 256\pi^4 \mathcal{A}_h, \qquad \kappa = \frac{1}{2} f'(r_h) = \frac{1}{2r_h}, \tag{42}$$

where the reduced area is denoted as $\mathcal{A}_h = (r_h^2 + n^2)^4$.

The expressions of the AD mass and the NUT charge can be similarly calculated as

$$M = 256\pi^3 m, \qquad N = 256\pi^3 n. \tag{43}$$

A. Consistent mass formulas of the ten-dimensional Taub-NUT spacetime

Adopting the same strategy as did before, we insert $r_h = \sqrt{\mathcal{A}_h^{1/4} - n^2}$ into the equation: $(5r_h^8 + 28n^2r_h^6 + 70n^4r_h^4 + 140n^6r_h^2 - 35n^8)^2 = 4900m^2r_h^2$, and after some computations, we can get an useful identity:

$$m^2 = \frac{1}{4900\mathcal{A}_h^{1/4}}(5\mathcal{A}_h + 8n^2\mathcal{A}_h^{3/4} + 16n^4\mathcal{A}_h^{1/2} + 64n^6\mathcal{A}_h^{1/4} - 128n^8)^2 + \frac{m^2n^2}{\mathcal{A}_h^{1/4}}. \tag{44}$$

After substituting $m = M/(256\pi^3)$, $n = N/(256\pi^3)$, $\mathcal{A} = 256\pi^3\mathcal{A}_h$, and the secondary hair $J_n = Mn$ into Eq. (44), one can obtain the following squared-mass formula:

$$M^2 = \frac{\pi^{3/4}}{49\mathcal{A}^{1/4}}\left[\mathcal{A} + \frac{N^2(\pi\mathcal{A})^{3/4}}{10240\pi^6} + \frac{N^4\sqrt{\pi\mathcal{A}}}{83886080\pi^{11}} + \frac{N^6(\pi\mathcal{A})^{1/4}}{343597383680\pi^{16}} - \frac{N^8}{2814749767106560\pi^{21}}\right]^2 + \frac{4\pi^{3/4}}{\mathcal{A}^{1/4}}J_n^2. \tag{45}$$

In the following, the differential and integral mass formulas for the ten-dimensional Taub-NUT spacetime will be derived under the assumption that the entire set of thermodynamic quantities is the mass M, the NUT charge N, and the secondary hair $J_n = Mn$, which will also be viewed as an independent variable. Differentiating the squared-mass formula (45) with respect to $\mathcal{A}$ yields one-eighth of the surface gravity:

$$\kappa = 8\frac{\partial M}{\partial \mathcal{A}}\bigg|_{(N,J_n)} = \frac{1}{2r_h}, \tag{46}$$

which is in accordance with the one given in Eq. (42). The gravitomagnetic potential ψ_h and the quasiangular momentum ω_h, which are conjugate to N and J_n, respectively, can be computed as

$$\psi_h = \frac{\partial M}{\partial N}\bigg|_{(\mathcal{A},J_n)} = \frac{8nr_h(r_h^6 + 7n^2r_h^4 + 35n^4r_h^2 - 35n^6)}{35(r_h^2 + n^2)}, \tag{47}$$

$$\omega_h = \frac{\partial M}{\partial J_n}\bigg|_{(\mathcal{A},N)} = \frac{n}{r_h^2 + n^2}. \tag{48}$$

One can readily verify that both the differential and integral mass formulas

$$dM = (\kappa/8)d\mathcal{A} + \omega_h dJ_n + \psi_h dN, \tag{49}$$

$$7M = \kappa\mathcal{A} + 8\omega_h J_n + \psi_h N, \tag{50}$$

are fully obeyed by using all the thermodynamical conjugate pairs given above. It is natural to identify

$$S = \frac{A_h}{4} = \frac{\pi}{4}\mathcal{A} = 64\pi^4(r_h^2 + n^2)^4, \quad T = \frac{\kappa}{2\pi} = \frac{1}{4\pi r_h}, \tag{51}$$

so that the ten-dimensional Taub-NUT solution acts like a true black hole without violating the beautiful one-quarter area-entropy relation. Here, we do not require ahead that the first law be obeyed to achieve consistent thermodynamical connections, rather, it is a very natural by-product of purely algebraic deduction.

B. Extension to the Taub-NUT-AdS$_{10}$ spacetime

Finally we would like to tackle the Lorentzian Taub-NUT-AdS$_{10}$ spacetime with a nonzero cosmological constant. The metric is still given by Eq. (41), and now we have

$$f(r) = \frac{1}{35(r^2 + n^2)^4}[5r^8 + 28n^2r^6 + 70n^4r^4 + 140n^6r^2 - 35n^8 - 70mr + 5g^2(7r^{10} + 45n^2r^8 + 126n^4r^6 + 210n^6r^4 + 315n^8r^2 - 63n^{10})],$$

where $l = 1/g$ is the cosmological scale.

Similar to the low dimensional case, one can compute the AD mass for this spacetime as

$$M = 256\pi^3 m. \tag{52}$$

Below, we will evaluate some thermodynamic quantities related to the Killing horizon which is specified by $f(r_h) = 0$. The surface gravity at the horizon is easily obtained via the standard method as

$$\kappa = \frac{1}{2}f'(r_h) = \frac{1 + 9g^2(r_h^2 + n^2)}{2r_h}, \tag{53}$$

and the event horizon area still reads $A_h = 256\pi^4\mathcal{A}_h$, in which the reduced horizon area is $\mathcal{A}_h = (r_h^2 + n^2)^4$.

Now it is a position to derive a novel squared-mass formula. Inserting $r_h = \sqrt{\mathcal{A}_h^{1/4} - n^2}$ into the equation $[5r_h^8 + 28n^2r_h^6 + 70n^4r_h^4 + 140n^6r_h^2 - 35n^8 + 5g^2(7r_h^{10} + 45n^2r_h^8 + 126n^4r_h^6 + 210n^6r_h^4 + 315n^8r_h^2 - 63n^{10})]^2 = 4900m^2r_h^2$, and after a little algebra, we can obtain a useful identity:

$$m^2 = \frac{1}{4900\mathcal{A}_h^{1/4}}[(1+10g^2n^2)(5\mathcal{A}_h + 8n^2\mathcal{A}_h^{3/4} + 16n^4\mathcal{A}_h^{1/2} + 64n^6\mathcal{A}_h^{1/4} - 128n^8) + 35g^2\mathcal{A}_h^{5/4}]^2 + \frac{m^2n^2}{\mathcal{A}_h^{1/4}}. \quad (54)$$

Then after plugging $m = M/(256\pi^3)$, $n = N/(256\pi^3)$, $\mathcal{A} = 256\pi^3\mathcal{A}_h$, and $g^2 = 2\pi P/9$ into Eq. (54), where P is the generalized pressure, and the secondary hair $J_n = Mn$, one can get the following identity:

$$M^2 = \frac{\pi^{3/4}}{49\mathcal{A}^{1/4}}\left\{\left(1+\frac{5N^2}{147456\pi^5}P\right)\left[\mathcal{A} + \frac{N^2(\pi\mathcal{A})^{3/4}}{10240\pi^6} + \frac{N^4\sqrt{\pi\mathcal{A}}}{83886080\pi^{11}} + \frac{N^6(\pi\mathcal{A})^{1/4}}{343597383680\pi^{16}} - \frac{N^8}{2814749767106560\pi^{21}}\right] + \frac{7}{18\pi}(\pi\mathcal{A})^{5/4}P\right\}^2 + \frac{4\pi^{3/4}}{\mathcal{A}^{1/4}}J_n^2, \quad (55)$$

which is the Christodoulou-Ruffini-like squared-mass formula for the ten-dimensional Taub-NUT-AdS spacetime. We again point out that this squared-mass formula consistently reduces to the one obtained in Eq. (45) when the generalized pressure P is turned off.

Now, as we did before, one can regard the mass M as an elementary function: $M = M(\mathcal{A}, N, J_n, P)$, and then after differentiating the squared-mass formula (55) with respect to its variables, one can obtain a reasonable differential mass formula:

$$dM = (\kappa/8)d\mathcal{A} + \omega_h dJ_n + \psi_h dN + VdP, \quad (56)$$

where

$$\kappa = 8\frac{\partial M}{\partial \mathcal{A}}\bigg|_{(N,J_n,P)} = \frac{1+9g^2(r_h^2+n^2)}{2r_h},$$

$$\omega_h = \frac{\partial M}{\partial J_n}\bigg|_{(\mathcal{A},N,P)} = \frac{n}{r_h^2+n^2},$$

$$\psi_h = \frac{\partial M}{\partial N}\bigg|_{(\mathcal{A},J_n,P)} = \frac{2nr_h}{35(r_h^2+n^2)}[4(r_h^6+7n^2r_h^4+35n^4r_h^2-35n^6) + 5g^2(5r_h^8+36n^2r_h^6+126n^4r_h^4+420n^6r_h^2-315n^8)],$$

$$V = \frac{\partial M}{\partial P}\bigg|_{(\mathcal{A},N,J_n)} = \frac{256\pi^4 r_h}{63(r_h^2+n^2)}(7r_h^{10}+45n^2r_h^8+126n^4r_h^6+210n^6r_h^4 + 315n^8r_h^2 - 63n^{10}).$$

In the meanwhile, one can easily verify that the Bekenstein-Smarr mass formula

$$7M = \kappa\mathcal{A} + 8\omega_h J_n + \psi_h N - 2VP, \quad (57)$$

is completely satisfied also.

Comparing our new mass formulas as displayed in Eqs. (56) and (57) with the familiar standard ones, it is strongly suggested that one should make the familiar identifications $S = A_h/4 = 64\pi^4\mathcal{A}_h$ and $T = \kappa/(2\pi)$, which restores the famous Bekenstein-Hawking one-quarter area-entropy relation of the ten-dimensional Taub-NUT-AdS spacetime in a very pleasing way, so that the solution behaves like a genuine black hole.

Here, we also point out that thermodynamic quantities in the cases of $S^2\times S^2\times\mathbb{CP}^2$ and $\mathbb{CP}^2\times\mathbb{CP}^2$ base space should be the same ones as those in the case of the $S^2\times S^2\times S^2\times S^2$ base manifold since the expression of the radial function $f(r)$ remains unchanged, so we will not present them.

V. SUMMARY: GENERAL $(2k+2)$-DIMENSIONAL CASES

To summarize, we have established the consistent thermodynamic first law and Bekenstein-Smarr mass formula for the generic $D = (2k+2)$ Lorentzian Taub-NUT (AdS) spacetimes whose metrics are compactly written as

$$ds_D^2 = -f(r)\left(dt + 2n\sum_{i=1}^k \cos\theta_i d\phi_i\right)^2 + \frac{dr^2}{f(r)} + (r^2+n^2)\sum_{i=1}^k (d\theta_i^2 + \sin^2\theta_i d\phi_i^2), \quad (58)$$

with the radial function being

$$f(r) = \left\{\int^r [1+(2k+1)g^2(x^2+n^2)]\frac{(x^2+n^2)^k}{x^2}dx - 2m\right\}\frac{r}{(r^2+n^2)^k}.$$

These higher even-dimensional Taub-NUT-AdS spacetimes are shown to be subject to the traditional forms of the first law and the Bekenstein-Smarr mass formula as follows:

$$dM = TdS + \omega_h dJ_n + \psi_h dN + VdP, \quad (59)$$

$$(D-3)M = (D-2)(TS + \omega_h J_n) + \psi_h N - 2VP, \quad (60)$$

provided that a new secondary hair $J_n = Mn$ is included just like in the case of their four-dimensional cousins [3,4].

The thermodynamical quantities that enter the above differential and integral mass formulas are given below

$$M = k(4\pi)^{k-1}m, \qquad N = k(4\pi)^{k-1}n,$$
$$J_n = k(4\pi)^{k-1}mn, \qquad S = \frac{1}{4}[4\pi(r_h^2+n^2)]^k,$$
$$T = \frac{f'(r_h)}{4\pi} = \frac{1+(2k+1)g^2(r_h^2+n^2)}{4\pi r_h},$$
$$\omega_h = \frac{n}{r_h^2+n^2}, \qquad P = \frac{k(2k+1)}{8\pi}g^2,$$
$$V = \frac{(4\pi)^k r_h^2}{r_h^2+n^2}\int^{r_h}\frac{(x^2+n^2)^{k+1}}{x^2}dx,$$
$$\psi_h = -\frac{1+(2k+1)g^2(r_h^2+n^2)}{2nr_h}(r_h^2+n^2)^k + \frac{(2k-1)r_h^2-n^2}{2n(r_h^2+n^2)}\int^{r_h}\frac{(x^2+n^2)^k}{x^2}dx + (2k+1)g^2\frac{(2k+1)r_h^2-n^2}{2n(r_h^2+n^2)}\int^{r_h}\frac{(x^2+n^2)^{k+1}}{x^2}dx.$$

By the way, the squared-mass formulas can be written as

$$M^2 = \frac{J_n^2}{4\pi(4S)^{1/k}} + \frac{k^2(4\pi)^{2k-1}}{4(4S)^{1/k}}\left\{g^2\frac{(4S)^{1+1/k}}{(4\pi)^{k+1}} + [1+2(k+1)g^2n^2]r_h\int^{r_h}\frac{(x^2+n^2)^k}{x^2}dx\right\}^2, \quad (61)$$

and the following identity must be used to verify that both mass formulas are indeed fulfilled:

$$m = \int^{r_h}[1+(2k+1)g^2(x^2+n^2)]\frac{(x^2+n^2)^k}{2x^2}dx. \quad (62)$$

Incidentally, we should point out that the four-dimensional NUT-charged case previously discussed in [3] without the inclusion of the dual mass can be enclosed as a special case in the above general expressions.

VI. THE EXTRA HAIR J_n AS A REDUNDANT VARIABLE

In the previous three sections (II–IV), which are summarized in Sec. V, by introducing an extra secondary hair $J_n = Mn$, which has been viewed as a independent thermodynamic variable just like the four-dimensional case [3,4], not only can the traditional thermodynamical first law and Bekenstein-Smarr mass formula be perfectly extended to the higher even-dimensional NUT-charged cases, their thermodynamical conjugate pairs can also be fairly subject to the common Maxwell relations.

However, one might object to our above measure adopted in the last sections (II–IV) and doubt that there exists a mathematical inconsistence in our preceding treatments, which would be a key flaw in that it fails to properly account for the number of independent parameters appearing in the solutions. In other words, there is a mismatch between the number of independent solution parameters and that of thermodynamical variables after introducing an extra secondary hair J_n, since it obviously enlarges by one between these numbers. This can be easily explained by counting the number of the solution parameter space and that of thermodynamical parameter space as follows. Note that in the usual NUT-less case, the horizon equation $f(r_h) = 0$ and the variant $\delta f(r_h) = 0$ with respect to its variables imply the Bekenstein-Smarr mass and the first law, respectively, and this is completely equivalent to deriving both mass formulas from the squared-mass formulas and no mismatch problem arises when both methods are used. Consider now the NUT-charged case, the equation $f(r_h) = 0$ means that its roots can be written as: $r_h = r_h(m, n, q, g)$, which in turn can be expressed as an entropy function: $S = S(M, N, Q, P)$. According to the traditional view, if no extra hair is included, then the entropy expression should be converted into a mass function: $M = M(S, N, Q, P)$, and nothing more is added by hand. However, different from the usual practice, we have advocated to include a new secondary hair J_n into the squared-mass formula in the above manipulations, which results in a function relation $M = M(S, J_n, N, Q, P)$ by enlarging one more parameter into the thermodynamical state space. This apparently leads to a conflict about the independent freedom of degree since there are only two free parameters among three thermodynamical variables (M, N, and J_n) due to the equality: $J_n = Mn = (4\pi)^{1-k}MN/k$.

To resolve the contradiction about the mismatch between the number of independent solution parameters and that of the thermodynamical variables, below we will provide a simple recipe to deal with this conflict by waiving the import of the secondary hair J_n, so that our preceding treatments would be viewed as a simpler intermediate step towards deriving the following reduced version of the mass formulas.

Consider $J_n = Mn$ as a redundant variable, that is to say, J_n is not an independent variable so that we abandon it to include J_n as a new hair. Previously, the impact of this constraint on the thermodynamical relations had already been addressed in the four-dimensional NUT-charged cases in our papers [3,4], but was ignored in the last sections for their higher-dimensional versions. Here we shall discuss this issue and derive the corresponding reduced mass formulas of the general $(2k+2)$-dimensional cases.

Now using $J_n = (4\pi)^{1-k}MN/k$, we can obtain the differentiation $k(4\pi)^{k-1}dJ_n = MdN + NdM$ by taking into account $N = k(4\pi)^{k-1}J_n/M$. With the help of these expressions, we can further eliminate J_n and dJ_n from the differential and integral mass formulas. Thus, the first law (59) and Bekenstein-Smarr mass formula (60) boil down to their nonstandard forms as follows:

$$\left[1 - \frac{N\omega_h}{k(4\pi)^{k-1}}\right]dM = TdS + \bar{\psi}_h dN + VdP, \quad (63)$$

$$(D-3)\left[1-\frac{N\omega_h}{k(4\pi)^{k-1}}\right]M = (D-2)TS + \bar{\psi}_h N - 2VP, \qquad (64)$$

where $\bar{\psi}_h = \psi_h + (4\pi)^{1-k}M\omega_h/k$.

It is easy to see that all of the thermodynamic quantities in the reduced mass formulas [(63) and (64)] cannot constitute the ordinary canonical conjugate pairs and do not obey the conventional Maxwell relations due to the presence of a prefactor $[1-(4\pi)^{1-k}N\omega_h/k]$ in front of dM and M. A similar situation previously appeared in the four-dimensional superentropic Kerr-Newman-AdS, ultraspinning Kerr-Sen-AdS, and ultraspinning dyonic Kerr-Sen-AdS black holes [39–41,50–53], where the chirality condition $J = Ml$ ($l = 1/g$ is the cosmological scales) reduces one of the numbers of independent thermodynamical parameters of their corresponding usual black holes after taking the $a \to l$ limit, so that the standard forms of usual thermodynamics are reduced to the nonstandard relations.

Finally, we can also note that the squared-mass formula is recast into

$$[k^2(4S)^{1/k} - (4\pi)^{1-2k}N^2]M^2 = \frac{k^4(4\pi)^{2k-1}}{4}\left\{g^2\frac{(4S)^{1+1/k}}{(4\pi)^{k+1}} + [1+2(k+1)g^2n^2]r_h\int^{r_h}\frac{(x^2+n^2)^k}{x^2}dx\right\}^2. \qquad (65)$$

VII. CONCLUSIONS AND OUTLOOKS

In our previous work [3,4], we have suggested from the thermodynamical perspective that the NUT charge behaves like a thermodynamical multi-hair in the mass formulas of the four-dimensional NUT-charged spacetimes, of which a great advantage is that not only will both the integral and differential mass formulas inherit the conventional forms in an elegant way, but also the thermodynamical quantities constitute the usual relations of common conjugate pairs. What is more, both the famous Bekenstein-Hawking one-quarter of area-entropy relation $S = A_h/4$ and the Hawking-Gibbons temperature formula $T = \kappa/(2\pi)$ can be naturally applied to all NUT-charged spacetimes. These are the most striking differences from other relevant attempts of the mainstream community [6–23]. On the other hand, the novelty of our proposal is that it not only aims to copy with thermodynamical aspect, but also takes account of other properties, such as the explanation of the gyromagnetic ratio and the quantization condition for a gravitational monopole. In particular, without considering the secondary hair $J_n = Mn$ as an independent charge, the universal rule of the area (entropy) products cannot be applied to the NUT-charged spacetimes [33].

In this paper, we have adopted the same strategy and successfully achieved the consistent first law and Bekenstein-Smarr mass formula for the six-, eight-, and ten-dimensional Lorentzian Taub-NUT (AdS) spacetimes. To date, our work is the only one to deal with thermodynamics of higher even-dimensional Lorentzian Taub-NUT (AdS) spacetimes. Similar to the cases of the four-dimensional Lorentzian Taub-NUT (AdS) solutions, as we did in our previous works [3,4], we also import only one secondary hair $J_n = Mn$ here. A key rudiment of this work is to deduce a reasonable Christodoulou-Ruffini-like squared-mass formula for each dimension, which represents a hypersurface in one more high-dimensional thermodynamical state space. From this squared-mass formula, the thermodynamical first law and Bekenstein-Smarr mass formula can be derived via simple differentiations with respect to its thermodynamic variables, and the resultant thermodynamical conjugate pairs meet their standard forms of the differential and integral mass formulas. After collecting all main results in a compact fashion for the generic $(2k+2)$-dimensional Lorentzian NUT-charged spacetimes, we then have dealt with the case when the secondary hair $J_n = Mn$ is not viewed as an independent variable so as to resolve a potential mathematical inconsistency behind in our preceding prescription. We should mention that all the results obtained in this paper resemble the cases of the four-dimensional Lorentzian Taub-NUT (AdS) spacetime; however, there is an exception in that the notion of a dual (magnetic) mass in higher dimensions is currently unclearly defined. Once an appropriate definition for it is proposed, our present work might be modified accordingly via the further inclusion of it.

Our study in this paper demonstrated that our idea "The NUT charge is a thermodynamical multi-hair" has a universal applicability, and our method is effective and systematical. A natural question is whether it is applicable to deal with the charged versions of the higher even-dimensional Taub-NUT spacetimes [54,55]. Preliminary research shows that only including one secondary hair $J_n = Mn$ is not sufficient to resolve the consistency of the first law and integral mass formula, so at least one more charge should be added into them. For more details, please see our recent work [56] about the electrically charged extension of the present paper. Another related issue is whether the present work can be extended to treat thermodynamics of the higher even-dimensional multi-NUTty spacetimes [57–59], since the solutions studied in this paper can be viewed as a special equal-NUT case of these more general spacetimes with multi-NUT parameters. The answer to this question is affirmative, please see the Appendix for the brief results in the cases without a cosmological constant. We hope to report the details of the related work soon.

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APPENDIX: CONSISTENT THERMODYNAMICS OF THE (2k + 2)-DIMENSIONAL MULTI-NUTty SPACETIMES

In this appendix, we will briefly give the main results of the consistent thermodynamics of the $D = (2k+2)$-dimensional Lorentzian multi-NUTty spacetimes without a cosmological constant. Using the base spaces $\prod_{i=1}^{k} \otimes S^2$, the line elements of these multi-NUTty spacetimes are written as [59]

$$ds_D^2 = -f(r)\left(dt + 2\sum_{i=1}^{k} n_i \cos\theta_i d\phi_i\right)^2 + \frac{dr^2}{f(r)} + \sum_{i=1}^{k}(r^2 + n_i^2)(d\theta_i^2 + \sin^2\theta_i d\phi_i^2), \tag{A1}$$

with the radial function being

$$f(r) = \frac{r}{\prod_{i=1}^{k}(r^2+n_i^2)}\left\{\int^r \frac{\prod_{i=1}^{k}(x^2+n_i^2)}{x^2}dx - 2m\right\}.$$

These multi-NUTty spacetimes obey the usual forms of the first law and the Bekenstein-Smarr mass formula as follows:

$$dM = TdS + \sum_{i=1}^{k}(\omega_i dJ_i + \psi_i dN_i), \tag{A2}$$

$$(2k-1)M = 2kTS + \sum_{i=1}^{k}(2k\omega_i J_i + \psi_i N_i), \tag{A3}$$

provided that we introduce k new secondary hairs: $J_i = Mn_i$.

The thermodynamical quantities that appear in the above differential and integral mass formulas are

$$M = k(4\pi)^{k-1}m, \quad N_i = k(4\pi)^{k-1}n_i, \quad J_i = k(4\pi)^{k-1}mn_i,$$

$$S = \frac{(4\pi)^k}{4}\prod_{i=1}^{k}(r_h^2+n_i^2), \quad T = \frac{1}{4\pi r_h}, \quad \omega_i = \frac{n_i}{r_h^2+n_i^2},$$

$$\psi_i = \frac{n_i}{k}\left(\sum_{p=1}^{k}\frac{r_h^2}{r_h^2+n_p^2}\right)\int^{r_h}\frac{\prod_{j=1}^{k}(x^2+n_j^2)}{x^2(x^2+n_i^2)}dx - \frac{n_i}{k(r_h^2+n_i^2)}\int^{r_h}\prod_{j=1}^{k}(x^2+n_j^2)\sum_{p=1}^{k}\frac{1}{x^2+n_p^2}dx,$$

where r_h is the largest root of the horizon equation: $f(r_h) = 0$. Incidentally, we would like to emphasize that throughout this article, all the integration constants in the integral expressions are set to zero.

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