Analytical results for binary dynamics at the first post-Newtonian order in Einstein-Cartan theory with the Weyssenhoff fluid

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The quantum spin effects inside matter can be modeled via the Weyssenhoff fluid, which permits us to unearth a formal analogy between general relativity and Einstein-Cartan theory at the first post-Newtonian order. In this framework, we provide some analytical formulas pertaining to the dynamics of binary systems having the spins aligned perpendicular to the orbital plane. We derive the expressions of the relative orbit and the coordinate time, which in turn allow us to determine the gravitational waveform, and the energy- and angular-momentum fluxes. The potentialities of our results are presented in two astrophysical applications, where we compute (i) the quantum spin contributions to the energy flux and gravitational waveform during the inspiral phase and (ii) the macroscopic angular momentum of one of the bodies starting from the time-averaged energy flux and the knowledge of a few timing parameters.

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I. INTRODUCTION

When Einstein laid down the basis for general relativity (GR) in 1916, quantum mechanics had not yet been formalized. This means that quantum concepts like the spin have no geometrical counterpart in GR, which thus configures as a purely classical theory. The revolutionary ideas underlying this framework soon became a source of inspiration for several authors. In particular, in 1920 Cartan developed an extension of GR, now referred to as Einstein-Cartan (EC) theory [1], where the most general metric-compatible affine connection was taken into account. This geometric formulation was then revisited by Kibble and Sciama in the 1960s, who devised it within the gauge theory of the Poincaré group [2,3]. It was then realized that the torsion tensor $S^{\lambda}{}_{\mu\nu}$, i.e., the antisymmetric part of the affine connection, is associated with the intrinsic quantum spin of matter [4–6].

One of the chief differences between the EC model and GR resides in their geometrical foundations, as the former is framed in the Riemann-Cartan environment, whereas the latter is in the Riemannian arena [4]. This explains why the EC pattern naturally fits the gauge paradigm, whereas if this is applied to Einstein gravity, we end up with the teleparallel equivalent of GR (TEGR), i.e., the gauge theory of the translation group [7,8]. Furthermore, EC field equations can be derived from a Palatini action over a Riemann-Cartan geometry, where the torsion is independent of the metric. In this way, the principle of least action yields ($\kappa \coloneqq 8\pi G/c^4$) [4,9]

$$G^{\alpha\beta} = \kappa \mathbb{T}^{\alpha\beta}, \tag{1a}$$

$$S_{\mu\nu}{}^{\lambda} + \delta^{\lambda}_{\mu} S_{\nu\rho}{}^{\rho} - \delta^{\lambda}_{\nu} S_{\mu\rho}{}^{\rho} = \kappa \tau_{\mu\nu}{}^{\lambda}, \tag{1b}$$

where $G^{\alpha\beta}$ is the EC tensor. From the above equations it is clear that both mass and spin represent the source of gravitation since $\mathbb{T}^{\alpha\beta}$ and $\tau_{\mu\nu}{}^{\lambda}$ denote the canonical stress-energy- and spin-angular-momentum tensors, respectively. They are linked via the relation $\mathbb{T}^{\alpha\beta} = T^{\alpha\beta} + (\nabla_\gamma + 2S_{\gamma\mu}{}^{\mu}) \times (\tau^{\alpha\beta\gamma} - \tau^{\beta\gamma\alpha} + \tau^{\gamma\alpha\beta})$, $T^{\alpha\beta}$ being the metric stress-energy tensor. A peculiar aspect of EC theory is that Eq. (1b) is an algebraic equation. As a consequence, torsion does not propagate, and hence it is confined only to the region occupied by matter. If one exploits Eq. (1b), then EC field equations can be recast in the GR-like form [4]

$$\hat{G}^{\alpha\beta} = \kappa \Theta^{\alpha\beta}. \tag{2}$$

Here, $\hat{G}^{\alpha\beta}$ is the (symmetric) Einstein tensor, which differs from $G^{\alpha\beta}$ due to the torsion contributions. Moreover, $\Theta^{\alpha\beta} \coloneqq T^{\alpha\beta} + \kappa \mathcal{S}^{\alpha\beta}$ denotes the combined energy-momentum tensor, where $\mathcal{S}^{\alpha\beta}$ is what we dub the torsional energy-momentum tensor, due to its (quadratic) dependence on $\tau_{\mu\nu}{}^{\lambda}$. It is worth noticing that Eq. (2) does not imply that EC

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theory is a trivial generalization of GR. In fact, the formulation (2) is useful from a practical point of view, but it does not entail that we have abandoned the Riemann-Cartan arena. In other words, GR pertaining to bodies endowed with angular momentum and EC theory are not equivalent. Indeed, in GR macroscopic rotations engender the modification of the stress-energy tensor and not of the geometry, while in the EC model both are affected by the presence of the quantum spin.

In EC theory, matter-field dynamics can be derived from the generalized conservation laws [4,9,10]

$$(\nabla_\nu + 2S_{\nu\alpha}{}^{\alpha})\mathbb{T}_\mu{}^\nu = 2\mathbb{T}_\lambda{}^\nu S_{\mu\nu}{}^\lambda - \tau_{\nu\rho}{}^\sigma R_{\mu\sigma}{}^{\nu\rho}, \tag{3a}$$

$$2(\nabla_\lambda + 2S_{\lambda\alpha}{}^{\alpha})\tau_{\mu\nu}{}^\lambda = \mathbb{T}_{\mu\nu} - \mathbb{T}_{\nu\mu}, \tag{3b}$$

where both the covariant derivative and the Riemann tensor include torsion contributions. The above relations are a consequence of Eq. (1). In particular, Eq. (3a) follows from the contracted Bianchi identity framed in Riemann-Cartan geometry, while Eq. (3b) originates from the antisymmetric part of $G^{\alpha\beta}$ [9].

The first physical features of the equations of motion can be figured out by considering a test particle. In the EC framework, test-body trajectories are neither geodesics nor autoparallel curves already at the pole-particle approximation, which yields, in fact, a set of Mathisson-Papapetrou-Dixon-like equations for the translational dynamics. These explicitly exhibit the contributions of the torsion tensor and contain a coupling term between the quantum spin of the object and the curvature of the spacetime [9,11]. On the other hand, the standard Mathisson-Papapetrou-Dixon equations of GR refer to the macroscopic angular momentum of the body and are worked out in the pole-dipole approximation [12].

A more advanced scenario is represented by the dynamics of a self-gravitating system, which is fundamental in gravitational-wave (GW) theory. In our research program, we have studied the GW generation problem with the Blanchet-Damour formalism in EC theory by first considering a source shaped by the Weyssenhoff fluid [13,14]. This is characterized by the tensors [14,15]

$$\mathbb{T}^{\alpha\beta} = p^\alpha u^\beta + (u^\alpha u^\beta/c^2 + g^{\alpha\beta})P, \tag{4a}$$

$$\tau_{\alpha\beta}{}^\gamma = s_{\alpha\beta} u^\gamma, \tag{4b}$$

where p^α, u^α, P, and $s_{\alpha\beta}$ are the four-momentum, four-velocity, pressure, and spin density tensor of the fluid, respectively. The conservation laws (3) yield a generalized Euler equation and precessional motion showing significant deviations from the GR expectations. We have studied both the translational and the rotational fluid evolution via the post-Newtonian (PN) approximation scheme and by adopting the Frenkel condition $s_{\alpha\beta}u^\beta = 0$ [14]. Then, we studied compact binaries, which represent the main candidates for GWs in astrophysics. These can be formally described by applying the point-particle limit to the continuous fluid distribution. In this way, we have derived the equations of motion and the radiative multipole moments of a spinning PN two-body system [14,16,17].

Our investigation regarding binary systems has revealed some remarkable novel results holding at the first post-Newtonian (1PN) level, which we summarize as follows:

(1) Both the translational and the rotational equations of motion formally resemble those of GR up to a normalization factor in the spin vector [16,17].
(2) The effacing principle is valid. This conclusion has been achieved after a careful investigation of the inner-structure-dependent integrals occurring in the dynamical equations, which have been verified to give no contribution [16,17]. Moreover, the zero-range spin interaction, which represents a distinct feature of the EC model, is absent.
(3) There exists a formal agreement between the radiative multipole moments of GR and EC theory. This will be proved explicitly in this paper.

Therefore, despite the profound differences between the GR and EC frameworks, we have discovered, at 1PN order, some *a priori* unpredictable formal similarities between the GR treatment of bodies endowed with a macroscopic angular momentum and the EC characterization of spinning objects modeled through the Weyssenhoff fluid. However, it should be stressed that had we chosen an alternative fluid description, we might have attained distinct results. A scheme of our findings is given in Fig. 1.

Building upon the cited achievements, a wide variety of astrophysical applications involving both the dynamics and the ensuing radiative phenomena of either spinning compact binaries or weakly self-gravitating spinning binaries framed either in GR or the EC model is expected. The advantageous osmosis between these two theories also offers a great opportunity to transfer a series of methodologies and results from one setting to the other. In this same vein, we aim to provide a set of analytical formulas regarding the orbital motion and the underlying coordinate time of binary systems whose spins are supposed to be aligned perpendicular to the orbital plane. These findings are extremely convenient for speeding up the calculations and avoiding numerical prescriptions and hence can be employed in a timely manner to evaluate the two-body gravitational waveforms and fluxes.

The paper is organized as follows. After considering the dynamics and the radiative multipole moments of binary systems in Sec. II, we derive the analytical formulas describing their relative orbit and coordinate time in Sec. III; in Sec. IV, we provide two applications, which involve the quantum spin and the macroscopic angular momentum of the bodies; finally, in Sec. V, we draw the conclusions and outline future perspectives.

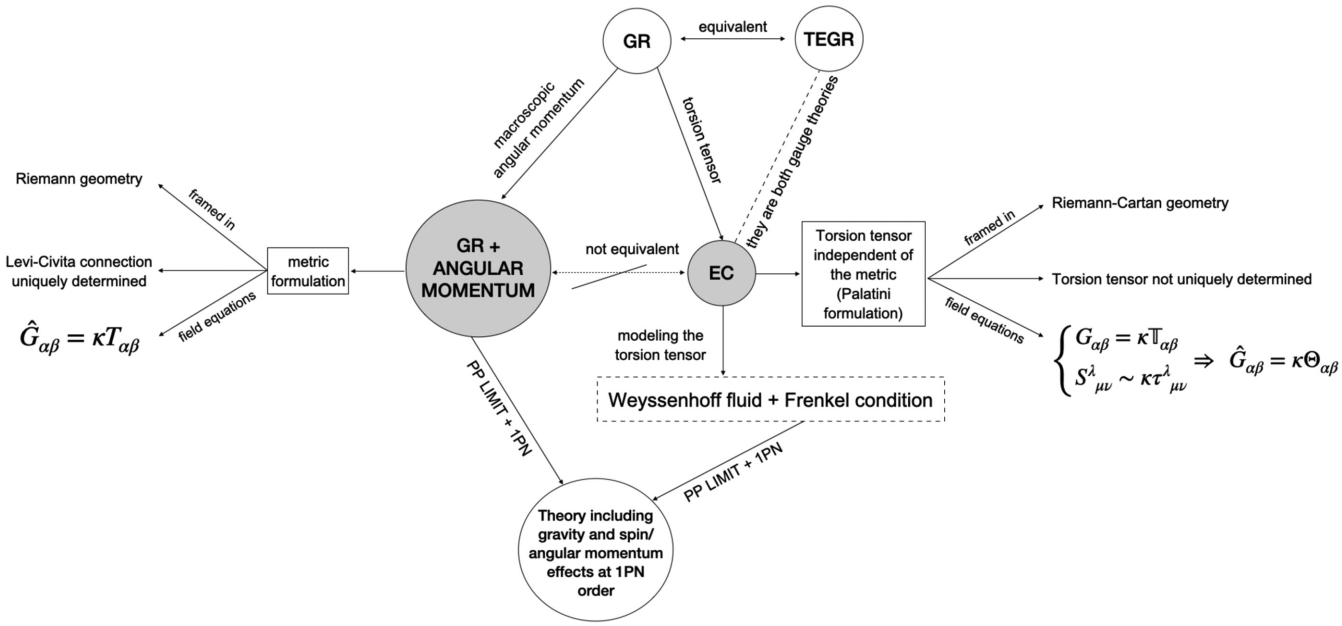

FIG. 1. Scheme showing differences and formal analogies between the GR description of bodies endowed with macroscopic angular momentum and the EC treatment of objects having a quantum spin. These two frameworks are not equivalent because GR is a metric theory of gravity, whereas the EC model follows a Palatini formulation. However, if we consider the Weyssenhoff fluid and the Frenkel condition to model the quantum spin effects in the EC framework, we discover, after having applied the point-particle limit (referred to as the PP limit), that the two theories share some common facets at the 1PN level.

A. Notations and conventions

Greek indices take values 0, 1, 2, 3, while lowercase Latin ones take 1, 2, 3. The spacetime coordinates are $x^\mu = (ct, \boldsymbol{x})$. Four-vectors are written as $a^\mu = (a^0, \boldsymbol{a})$, and $\boldsymbol{a}\cdot\boldsymbol{b} \coloneqq \delta_{lk} a^l b^k$, $|\boldsymbol{a}| \equiv a \coloneqq (\boldsymbol{a}\cdot\boldsymbol{a})^{1/2}$, and $(\boldsymbol{a}\times\boldsymbol{b})^i \coloneqq \varepsilon_{ilk} a^l b^k$, where ε_{kli} is the total antisymmetric Levi-Civita symbol. The symmetric-trace-free (STF) projection of a tensor $A^{ij\ldots k}$ is indicated with $A^{\langle ij\ldots k\rangle}$. Superscripts (l) denote l successive time derivatives, and $L = i_1 i_2 \ldots i_l$ denotes a multi-index consisting of l spatial indices.

II. BINARY-SYSTEM EQUATIONS OF MOTION AND MULTIPOLE MOMENTS

In this section, we set out the dynamics and the radiative multipole moments of binary systems, which can be obtained by applying the point-particle limit to a continuous fluid model [18]. This procedure relies on the assumption that the fluid can be decomposed in a collection of $N = 2$ mutually well-separated components, each of them representing a body (of course, this approach can be easily generalized to any $N \geq 2$). This methodology entails the introduction of some center-of-mass variables which permit one to substitute the fine-grained description of the system based on a number of fluid variables (such as the density and the pressure) with a coarse-grained picture. We have pursued this scheme in Refs. [14,16,17], where we have exploited the semiclassical Weyssenhoff model of a neutral spinning perfect fluid in EC theory and the PN formalism.

The section is organized as follows. We start with the 1PN equations of motion for binary systems and the resulting first integrals (see Secs. II A and II B). Subsequently, after having displayed the general formulas of the waveform and the fluxes (cf. Sec. II C), we derive the underlying radiative multipole moments and show their formal analogy with their GR counterpart in Sec. II D.

Henceforth, the bodies are labeled by capital letters $A, B = 1, 2$.

A. Equations of motion

We consider a binary system composed of two spinning, weakly self-gravitating, slowly moving, and widely separated companions with masses $m_1 \geq m_2$, total mass $M = m_1 + m_2$, reduced mass $\mu = \frac{m_1 m_2}{M}$, and symmetric mass ratio $\nu = \frac{\mu}{M}$. Given a harmonic coordinate system x^μ, let $\boldsymbol{r}_A$ be the position vector, $\boldsymbol{v}_A = \frac{\mathrm{d}\boldsymbol{r}_A}{\mathrm{d}t}$ the velocity, and $\boldsymbol{s}_A$ the spin of the objects. The latter is defined by

$$\varepsilon_{jki} s^i_A(t) \coloneqq \int_A \mathrm{d}^3\boldsymbol{x}\, s_{jk}, \tag{5}$$

where $s_{\mu\nu}$ is the spin density tensor [cf. Eq. (4b)]. The above equation reflects the well-known fact that in the EC model the quantum spin $\boldsymbol{s}_A$ is related to a geometrical feature of the spacetime, i.e., the torsion tensor. Furthermore, it makes clear the difference between $\boldsymbol{s}_A$ and the macroscopic angular-momentum vector $\hat{\boldsymbol{s}}_A$ adopted in GR. In particular, the former cannot be written in terms of kinematical quantities, unlike the latter, which is defined through an integral involving the density of the fluid and the cross product between the position and the velocity vectors of a fluid element relative to the center of mass (see Ref. [18] for more details).

The motion of the binary system can be conveniently described by choosing an orthogonal reference frame centered in the barycenter, which, without loss of generality, is supposed to be static. In this frame, after having defined the relative vectors $\boldsymbol{R} := \boldsymbol{r}_1 - \boldsymbol{r}_2$ and $\boldsymbol{V} := \frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{R}$, we find that the 1PN translational dynamics is ruled by the relative acceleration

$$\boldsymbol{A} := \frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{V} = \boldsymbol{A}_{\mathrm{GR}} + \boldsymbol{A}_{\mathrm{EC}} + \mathrm{O}(c^{-4}), \tag{6}$$

where [16]

$$\boldsymbol{A}_{\mathrm{GR}} = -\frac{GM}{R^2}\boldsymbol{N} + \frac{GM}{c^2R^2}\left\{\left[2(2+\nu)\frac{GM}{R} + \frac{3\nu}{2}(\boldsymbol{N}\cdot\boldsymbol{V})^2 - (1+3\nu)V^2\right]\boldsymbol{N} + 2(2-\nu)(\boldsymbol{N}\cdot\boldsymbol{V})\boldsymbol{V}\right\}, \tag{7a}$$

$$\boldsymbol{A}_{\mathrm{EC}} = \frac{-4G}{c^2R^3}\left[\boldsymbol{V}\times\left(2\boldsymbol{s} + \frac{3}{2}\boldsymbol{\sigma}\right) - 3\boldsymbol{N}(\boldsymbol{N}\times\boldsymbol{V})\cdot(\boldsymbol{s}+\boldsymbol{\sigma}) - 3\boldsymbol{N}\times\left(\boldsymbol{s}+\frac{\boldsymbol{\sigma}}{2}\right)(\boldsymbol{N}\cdot\boldsymbol{V})\right] - \frac{12G}{c^2R^4\mu}\{\boldsymbol{s}_1(\boldsymbol{N}\cdot\boldsymbol{s}_2) + \boldsymbol{s}_2(\boldsymbol{N}\cdot\boldsymbol{s}_1) + \boldsymbol{N}[\boldsymbol{s}_1\cdot\boldsymbol{s}_2 - 5(\boldsymbol{N}\cdot\boldsymbol{s}_1)(\boldsymbol{N}\cdot\boldsymbol{s}_2)]\}, \tag{7b}$$

with $\boldsymbol{N} = \boldsymbol{R}/R$ and

$$\boldsymbol{s} := \boldsymbol{s}_1 + \boldsymbol{s}_2, \qquad \boldsymbol{\sigma} := \frac{m_2}{m_1}\boldsymbol{s}_1 + \frac{m_1}{m_2}\boldsymbol{s}_2. \tag{8}$$

The 1PN motion is determined by Eqs. (6) and (7) jointly with the conservation law $\mathrm{d}\boldsymbol{s}_A/\mathrm{d}t = \mathrm{O}(c^{-2})$ (see Ref. [16] for further details).

B. First integrals

As shown in Ref. [17], the 1PN dynamics of the binary system follows from an acceleration-dependent Lagrangian, which permits us to determine the expressions of the related conserved energy and angular momenta. The total specific energy E can be written as

$$E = E_{\mathrm{GR}} + E_{\mathrm{EC}} + \mathrm{O}(c^{-4}), \tag{9}$$

where

$$E_{\mathrm{GR}} = \left(\frac{V^2}{2} - \frac{GM}{R}\right) + \frac{1}{c^2}\left\{\frac{GM}{2R}\left[\frac{GM}{R} + \nu(\boldsymbol{N}\cdot\boldsymbol{V})^2 + (\nu+3)V^2\right] + \frac{3}{8}(1-3\nu)V^4\right\}, \tag{10a}$$

$$E_{\mathrm{EC}} = \frac{2G}{c^2R^2}\left\{(\boldsymbol{N}\times\boldsymbol{V})\cdot\boldsymbol{\sigma} + \frac{2}{\mu R}[3(\boldsymbol{N}\cdot\boldsymbol{s}_1)(\boldsymbol{N}\cdot\boldsymbol{s}_2) - \boldsymbol{s}_1\cdot\boldsymbol{s}_2]\right\}, \tag{10b}$$

while the total specific angular momentum $\boldsymbol{J}$ reads as

$$\boldsymbol{J} = \boldsymbol{L}_{\mathrm{GR}} + \boldsymbol{L}_{\mathrm{EC}} + \frac{\bar{\boldsymbol{s}}}{\mu} + \mathrm{O}(c^{-4}), \tag{11}$$

with

$$\boldsymbol{L}_{\mathrm{GR}} = \boldsymbol{L}_{\mathrm{N}}\left\{1 + \frac{1}{c^2}\left[\frac{GM}{R}(\nu+3) + \frac{(1-3\nu)}{2}V^2\right]\right\}, \tag{12a}$$

$$\boldsymbol{L}_{\mathrm{EC}} = \frac{2}{c^2M}\left\{\frac{GM}{R}\boldsymbol{N}\times[\boldsymbol{N}\times(\boldsymbol{\sigma}+2\boldsymbol{s})] - \frac{1}{2}\boldsymbol{V}\times(\boldsymbol{V}\times\boldsymbol{\sigma})\right\}. \tag{12b}$$

In the above equations, the Newtonian specific angular momentum is

$$\boldsymbol{L}_{\mathrm{N}} = \boldsymbol{R}\times\boldsymbol{V}, \tag{13}$$

and we have introduced the total spin vector

$$\bar{\boldsymbol{s}} := \bar{\boldsymbol{s}}_1 + \bar{\boldsymbol{s}}_2, \tag{14}$$

where $\bar{\boldsymbol{s}}_A$ is the refined spin which keeps its magnitude constant during the motion (i.e., it satisfies $\bar{\boldsymbol{s}}_A\cdot\mathrm{d}\bar{\boldsymbol{s}}_A/\mathrm{d}t = 0$). It is defined as [17]

$$\bar{\boldsymbol{s}}_A := \boldsymbol{s}_A + \frac{1}{c^2}\left[\frac{Gm_B}{R}\boldsymbol{s}_A + \frac{1}{2}(\boldsymbol{s}_A\cdot\boldsymbol{V}_A)\boldsymbol{V}_A\right] + \mathrm{O}(c^{-4}), \quad (A\neq B). \tag{15}$$

As noted in Refs. [16,17], both the translational and the rotational 1PN GR dynamics pertaining to weakly self-gravitating binary systems having a macroscopic angular-momentum vector $\hat{\boldsymbol{s}}_A$ are formally recovered if the substitution

$$\boldsymbol{s}_A \to \frac{1}{2}\hat{\boldsymbol{s}}_A, \tag{16}$$

is applied to the 1PN motion in EC theory. As will be clear from our forthcoming analysis, in the case of maximally rotating compact objects, the above relation should be slightly modified according to

$$\boldsymbol{s}_A \to \frac{1}{2c}\hat{\boldsymbol{s}}_A, \tag{17}$$

where, following the usual GR conventions [19], $\hat{\boldsymbol{s}}_A$ now has the dimensions of an angular momentum multiplied by c. Note that, in order to ease the notations, we always use the symbol $\hat{\boldsymbol{s}}_A$, although the physical dimensions of the macroscopic angular momentum are different in Eqs. (16) and (17). Indeed, this should not create any confusion, as it will be clear from the context which relation we are referring to.

C. Gravitational waveform and fluxes

Let us indicate with $I_L^{\rm rad}$ and $J_L^{\rm rad}$ the STF mass-type and current-type radiative multipole moments of order l, respectively. Bearing in mind that, at the 1PN level, there is no difference between the harmonic and the radiative coordinates [20,21], the 1PN-accurate asymptotic waveform $\mathscr{H}_{ij}^{\rm TT}$ reads as [13,14,22]

$$\begin{aligned}\mathscr{H}_{ij}^{\rm TT}(x^\mu) = \frac{2G}{c^4|\boldsymbol{x}|}\mathscr{P}_{ijkl}(\boldsymbol{n})\Bigg\{&\overset{(2)}{I}{}^{\rm rad}_{kl}(u)\\ &+\frac{1}{c}\left[\frac{1}{3}n_a\overset{(3)}{I}{}^{\rm rad}_{kla}(u)+\frac{4}{3}n_b\epsilon_{ab(k}\overset{(2)}{J}{}^{\rm rad}_{l)a}(u)\right]\\ &+\frac{1}{c^2}\left[\frac{1}{12}n_an_b\overset{(4)}{I}{}^{\rm rad}_{klab}(u)\right.\\ &\left.+\frac{1}{2}n_bn_c\epsilon_{ab(k}\overset{(3)}{J}{}^{\rm rad}_{l)ac}(u)\right]+\mathrm{O}(c^{-3})\Bigg\},\end{aligned} \tag{18}$$

where $u = t - |\boldsymbol{x}|/c$, $\boldsymbol{n} = \boldsymbol{x}/|\boldsymbol{x}|$, and $\mathscr{P}_{ijkl}(\boldsymbol{n})$ is the transverse-traceless (TT) projection operator onto the plane orthogonal to $\boldsymbol{n}$. Moreover, the total radiated power $\mathcal{F}$ (also dubbed energy flux or gravitational luminosity) and the angular-momentum flux $\mathcal{G}_i$ read as [22]

$$\begin{aligned}\mathcal{F}(t) = \frac{G}{c^5}\Bigg\{&\frac{1}{5}\overset{(3)}{I}{}^{\rm rad}_{ij}\overset{(3)}{I}{}^{\rm rad}_{ij}+\frac{1}{c^2}\left[\frac{1}{189}\overset{(4)}{I}{}^{\rm rad}_{ijk}\overset{(4)}{I}{}^{\rm rad}_{ijk}\right.\\ &\left.+\frac{16}{45}\overset{(3)}{J}{}^{\rm rad}_{ij}\overset{(3)}{J}{}^{\rm rad}_{ij}\right]+\mathrm{O}(c^{-4})\Bigg\},\end{aligned} \tag{19a}$$

$$\begin{aligned}\mathcal{G}_i(t) = \frac{G}{c^5}\varepsilon_{ijk}\Bigg\{&\frac{2}{5}\overset{(2)}{I}{}^{\rm rad}_{jl}\overset{(3)}{I}{}^{\rm rad}_{kl}+\frac{1}{c^2}\left[\frac{1}{63}\overset{(3)}{I}{}^{\rm rad}_{jlp}\overset{(4)}{I}{}^{\rm rad}_{klp}\right.\\ &\left.+\frac{32}{45}\overset{(2)}{J}{}^{\rm rad}_{jl}\overset{(3)}{J}{}^{\rm rad}_{kl}\right]+\mathrm{O}(c^{-4})\Bigg\}.\end{aligned} \tag{19b}$$

Note that the linear-momentum flux involves a Newtonian formula (see Ref. [22] for more details) and hence will not be considered in this paper.

D. Radiative multipole moments

The general form, with the required PN accuracy, of the radiative multipole moments occurring in the formulas of Sec. II C was first obtained in Ref. [13], where we solved the GW generation problem in EC theory via the Blanchet-Damour formalism.[1] Then, the expressions valid in the case of spinning binary systems have been derived in Ref. [14]. Starting from these results and adopting a mass-centered coordinate system, we find that, after some manipulations, the mass-type radiative moments can be written as

$$\begin{aligned}I_{ij}^{\rm rad} = \mu R_{\langle ij\rangle}&\left[1+\frac{29}{42c^2}(1-3\nu)V^2-\frac{(5-8\nu)}{7c^2}\frac{GM}{R}\right]\\ &+\frac{\mu(1-3\nu)}{21c^2}\left[11R^2V_{\langle ij\rangle}-12(\boldsymbol{R}\cdot\boldsymbol{V})R_{\langle i}V_{j\rangle}\right]\\ &+\frac{8\nu}{3c^2}\left[2(\boldsymbol{V}\times\boldsymbol{\sigma})^{\langle i}R^{j\rangle}-(\boldsymbol{R}\times\boldsymbol{\sigma})^{\langle i}V^{j\rangle}\right]\\ &+\mathrm{O}(c^{-3}),\end{aligned} \tag{20}$$

$$I_{ijk}^{\rm rad} = -\mu\sqrt{1-4\nu}R_{\langle ijk\rangle}+\mathrm{O}(c^{-2}), \tag{21}$$

$$I_{ijkl}^{\rm rad} = \mu(1-3\nu)R_{\langle ijkl\rangle}+\mathrm{O}(c^{-2}), \tag{22}$$

while the current-type radiative moments are

$$\begin{aligned}J_{ij}^{\rm rad} = &-\mu\sqrt{1-4\nu}\epsilon_{kl\langle i}R_{j\rangle k}V_l\\ &+3\mu\left(\frac{s_1^{\langle i}R^{j\rangle}}{m_1}-\frac{s_2^{\langle i}R^{j\rangle}}{m_2}\right)+\mathrm{O}(c^{-2}),\end{aligned} \tag{23}$$

$$J_{ijk}^{\rm rad} = \mu(1-3\nu)R_{\langle ij}\epsilon_{k\rangle lp}R_lV_p+4\nu R^{\langle i}R^j\sigma^{k\rangle}+\mathrm{O}(c^{-2}). \tag{24}$$

Notice that in Eq. (20) we have exploited the equations of motion (6) jointly with the fact that the spin vector is conserved modulo $\mathrm{O}(c^{-2})$ corrections.

At this stage, some comments are in order. First, Eqs. (20), (23), and (24) agree formally with their corresponding GR moments [21,23,24] if the quantum spin $\boldsymbol{s}_A$ is replaced by the macroscopic angular-momentum vector $\hat{\boldsymbol{s}}_A$ following the scheme already introduced in Eqs. (16) and (17). This is a crucial consistency check since, as pointed out before, the same conclusion is valid in the context of the 1PN dynamics of spinning binaries. Moreover, this is a remarkable result if we recall that Eqs. (20)–(24) have been obtained by applying the point-particle procedure to the radiative moments pertaining to a Weyssenhoff fluid in EC theory, i.e., a generalization of the usual perfect fluid adopted in GR. Furthermore, we see that in our framework, Eqs. (20)–(24) are immediately consistent with the Frenkel

[1] The radiative moments $I_L^{\rm rad}$, $J_L^{\rm rad}$ are related to the STF radiative multipole moments U_L, V_L employed in Ref. [13] by the relations $U_L \coloneqq \overset{(l)}{I}{}^{\rm rad}_L$, $V_L \coloneqq \overset{(l)}{J}{}^{\rm rad}_L$.

spin supplementary condition. On the other hand, in the context of GR the radiative moments follow the center-of-mass definition stemming from the Frenkel constraint only if a suitable transformation is invoked [see Eq. (13) in Ref. [23] and Appendix A in Ref. [24] for further details].

III. ANALYTICAL FORMULAS

In this section, we investigate the dynamical and radiative features of binary systems analytically. We suppose that the spins of the companions and the orbital angular momentum $\boldsymbol{L} \coloneqq \boldsymbol{L}_{\rm GR} + \boldsymbol{L}_{\rm EC}$ are aligned, namely [cf. Eqs. (11)–(15)],

$$\bar{\boldsymbol{s}} \cdot \boldsymbol{R} = 0, \qquad \bar{\boldsymbol{s}} \cdot \boldsymbol{V} = 0, \qquad \bar{\boldsymbol{s}}_1 \times \bar{\boldsymbol{s}}_2 = \boldsymbol{0}. \tag{25}$$

In this setting, there is no spin precession, as $\mathrm{d}\bar{\boldsymbol{s}}_A/\mathrm{d}t = \mathrm{O}(c^{-4})$ [cf. Eqs. (27) and (28) in Ref. [17]], and the dynamics takes place in a fixed plane [see Eqs. (6) and (7)]. Thus, we can set the barycentric frame (x, y, z) in such a way that the motion occurs in the plane (x, y), where we introduce polar coordinates (R, θ), θ being the angle between $\boldsymbol{R}$ and the x axis measured counterclockwise. This also means that we can write $\bar{\boldsymbol{s}}_1 = (0, 0, \bar{s}_{1z})$ and $\bar{\boldsymbol{s}}_2 = (0, 0, \bar{s}_{2z})$.

In our hypotheses, the orbital angular momentum $\boldsymbol{L}$ and the total spin vector $\bar{\boldsymbol{s}}$ are *separately* conserved modulo $\mathrm{O}(c^{-4})$ corrections [see Eq. (11)]. In other words, we have one additional first integral with respect to the most general case. We can take advantage of this situation, as the number of conserved quantities and degrees of freedom of the system coincide. Therefore, the conserved $\bar{\boldsymbol{s}}$ resolves the rotational motion as it gives the only nonvanishing component of the spins, whereas the constants E and L determine the translational motion, as they can be used to work out the functions $R(t)$ and $\theta(t)$. In this configuration, we are able to parametrize the relative orbit *á la* Damour-Deruelle (see Sec. III A). Here, the coordinate time plays a key role, and its analytical expression is worked out in Sec. III B. Such formulas are essential to evaluate the energy- and angular-momentum fluxes. These are given, for generic spin orientations, in Sec. III C, where we also derive some new contributions, which, to the best of our knowledge, have not been transcribed in the literature.

A. Polar equation of the relative orbit

Starting from the expressions of E and L, and employing polar coordinates, we find that the 1PN relative dynamics of the companions is described by

$$\left(\frac{\mathrm{d}R}{\mathrm{d}t}\right)^2 = \mathcal{A} + \frac{2\mathcal{B}}{R} + \frac{\mathcal{C}}{R^2} + \frac{\mathcal{D}}{R^3} + \mathrm{O}(c^{-4}), \tag{26a}$$

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = \frac{\mathcal{H}}{R^2} + \frac{\mathcal{I}}{R^3} + \mathrm{O}(c^{-4}), \tag{26b}$$

with

$$\mathcal{A} = A, \tag{27a}$$

$$\mathcal{B} = B, \tag{27b}$$

$$\mathcal{C} = C + \frac{1}{c^2}\frac{4LE}{M}\sigma_z, \tag{27c}$$

$$\mathcal{D} = D + \frac{4G}{c^2}\left[2\frac{s_{1z}s_{2z}}{\mu} - L(2\sigma_z + s_z)\right], \tag{27d}$$

$$\mathcal{H} = H - \frac{1}{c^2}\frac{2E}{M}\sigma_z, \tag{27e}$$

$$\mathcal{I} = I + \frac{4G}{c^2}s_z, \tag{27f}$$

where $s_z = s_{1z} + s_{2z}$, $\sigma_z = \frac{m_2}{m_1}s_{1z} + \frac{m_1}{m_2}s_{2z}$, $\mathfrak{s}_z \coloneqq s_{1z}s_{2z}/\mu$ [recall that $\bar{\boldsymbol{s}}_A = \boldsymbol{s}_A + \mathrm{O}(c^{-2})$, cf. Eq. (15)]. The parameters A, B, C, D, H, I can be found in the Damour and Deruelle paper [see Eq. (2.17) in Ref. [25]] with the caveat that the energy E and the orbital angular momentum L must be read off from our Eqs. (9)–(12).

Equation (26) assumes the same functional form as in GR [see Eqs. (2.15) and (2.16) in Ref. [25]]. We can then employ the strategy pursued by Damour and Deruelle, which uses conchoidal transformations in order to map Eq. (26) to an auxiliary Newtonian-like form. It should be clear that, in our case, these transformations formally mirror those of GR, the only difference being the occurrence of (some of) the coefficients displayed in Eq. (27). Therefore, we can parametrize the radial and the angular 1PN motion in the EC model as

$$n(t - t_0) = u - e_t \sin u + \mathrm{O}(c^{-4}), \tag{28a}$$

$$R(t) = a_R(1 - e_R\cos u) + \mathrm{O}(c^{-4}), \tag{28b}$$

$$\theta(t) = \theta_0 + 2K\arctan\left[\left(\frac{1+e_\theta}{1-e_\theta}\right)\tan\frac{u}{2}\right] + \mathrm{O}(c^{-4}), \tag{28c}$$

where the 1PN quantities $\{n, a_R, e_R, e_t, e_\theta, K\}$ are

$$n = \frac{(-\mathcal{A})^{\frac{3}{2}}}{\mathcal{B}}, \tag{29a}$$

$$a_R = -\frac{\mathcal{B}}{\mathcal{A}} - \frac{\mathcal{D}}{2L^2}, \tag{29b}$$

$$e_t = \left[1 - \frac{\mathcal{A}}{\mathcal{B}^2}\left(\mathcal{C} + \frac{\mathcal{B}\mathcal{D}}{L^2}\right)\right]^{1/2}, \tag{29c}$$

$$e_R = \left(1 - \frac{\mathcal{AD}}{2\mathcal{B}L^2}\right) e_t, \tag{29d}$$

$$e_\theta = \left(1 - \frac{\mathcal{AD}}{\mathcal{B}L^2} - \frac{\mathcal{AI}}{\mathcal{BH}}\right) e_t, \tag{29e}$$

$$K = 1 + \frac{1}{c^2}\frac{3G^2M^2}{L^2}. \tag{29f}$$

Starting from the above relations, it is easy to show that the polar equation of the 1PN relative orbit is

$$R(\theta) = \frac{e_R}{e_\theta} a_R \frac{1 - e_\theta^2}{1 + e_\theta \cos(\frac{\theta - \theta_0}{K})} + a_R\left(1 - \frac{e_R}{e_\theta}\right) + \mathrm{O}(c^{-4}). \tag{30}$$

B. Coordinate time

Having obtained the orbital radius (30), we can now derive the analytical formula of $t(\theta)$, which expresses the coordinate time as a function of the polar angle. This can be done by generalizing the strategy developed in Ref. [26], which relies on first determining a main discontinuous function $f(\theta)$, which is then made smooth via the introduction of the accumulation function $F_n(\theta)$.

In order to build up the differential equation for $t(\theta)$, we split the energy E, the orbital angular momentum L, and the radius R as follows: $E = E_0 + \frac{1}{c^2}E_1 + \mathrm{O}(c^{-4})$, $L = L_0 + \frac{1}{c^2}L_1 + \mathrm{O}(c^{-4})$, and $R = R_0 + \frac{1}{c^2}R_1 + \mathrm{O}(c^{-4})$. The expressions of E_0, E_1, L_0, and L_1 can be promptly read off from Eqs. (10) and (12), whereas for R_0 and R_1 we make use of Eq. (30). Since EC theory is the same as GR at the 0PN level, R_0 reads as [cf. Eq. (5a) in Ref. [26]]

$$R_0 = \frac{1}{B_1 + B_2\cos(\tilde{K}\tilde{\theta})}, \tag{31}$$

where $\tilde{\theta} = \theta - \theta_0$, $B_1 > B_2 \geq 0$, and

$$B_1 = \frac{1}{h_0^2 GM}, \tag{32a}$$

$$B_2 = e_0 B_1, \tag{32b}$$

$$\tilde{K} = 1 - \frac{1}{c^2}\frac{3}{h_0^2} + \frac{2G}{c^2 h_0 L_0^2}\left[4s_z + 3\sigma_z - 6\frac{\mathfrak{s}_z}{L_0}\right], \tag{32c}$$

$$h_0 = \frac{L_0}{GM}, \tag{32d}$$

$$e_0 = \sqrt{1 + 2E_0 h_0^2}, \qquad (0 \leq e_0 < 1), \tag{32e}$$

where e_0 is the Newtonian eccentricity and $\tilde{K}$ the 1PN expansion of $1/K$ [cf. Eq. (29f)]. This last factor is responsible for the orbit precession, which is also influenced by the presence of the spin. The term R_1 can be written as the sum of the GR and EC contributions, i.e., $R_1 = R_1^{\mathrm{GR}} + R_1^{\mathrm{EC}}$. We find [cf. Eq. (5b) in Ref. [26]]

$$R_1^{\mathrm{GR}} = A_0 + A_1 R_0 + A_2 R_0^2 \cos(\tilde{K}\tilde{\theta}), \tag{33a}$$

$$R_1^{\mathrm{EC}} = R_0^2[W_1 + W_2\cos(\tilde{K}\tilde{\theta}) + W_3\cos(\tilde{K}\tilde{\theta})], \tag{33b}$$

where

$$A_0 = \frac{G\mu}{2}, \tag{34a}$$

$$A_1 = 2E_0\left(\frac{\nu}{2} - 2\right) + W_0, \tag{34b}$$

$$A_2 = -\frac{E_0^2}{e_0 GM}\left(\frac{E_1}{E_0^2} + \frac{2W_0}{E_0} + \frac{\nu - 15}{2}\right), \tag{34c}$$

$$W_1 = \frac{4(e_0^2 + 3)s_z + (e_0^2 + 8)\sigma_z}{G^2M^3h_0^5} - \frac{6(e_0^2 + 2)\mathfrak{s}_z}{G^3h_0^6M^4}, \tag{34d}$$

$$W_2 = \frac{4(3e_0^2 + 1)(GMh_0 s_z - \mathfrak{s}_z)}{G^3M^4h_0^6 e_0} + \frac{(3 + 6e_0^2 - e_0^4)GMh_0\sigma_z}{e_0 G^3M^4h_0^6}, \tag{34e}$$

$$W_3 = \frac{e_0^2(2\mathfrak{s}_z - Gh_0M\sigma_z)}{G^3M^4h_0^6}, \tag{34f}$$

and $W_0 = L_1/L_0 - 3/h_0^2$.

If we write the 1PN differential equation for $t(\theta)$ in the form [see Eq. (26b)]

$$\mathrm{d}t = \frac{\mathrm{d}\theta}{\frac{\mathcal{H}}{R^2} + \frac{\mathcal{I}}{R^3}} + \mathrm{O}(c^{-4}) = \mathrm{d}t_{\mathrm{GR}} + \mathrm{d}t_{\mathrm{EC}} + \mathrm{O}(c^{-4}), \tag{35}$$

then the substitution of the splittings shown above for E, L, and R gives [cf. Eq. (8) in Ref. [26]]

$$\mathrm{d}t_{\mathrm{GR}} = \frac{R_0^2}{L_0}\left\{1 + \frac{1}{c^2}\left[E_0(1 - 3\nu) + \frac{2R_1}{R_0} - \frac{4GM}{R_0}(\nu - 2) - \frac{L_1}{L_0}\right]\right\}\mathrm{d}\theta, \tag{36a}$$

$$\mathrm{d}t_{\mathrm{EC}} = \frac{2R_0(E_0R_0\sigma_z - 2GMs_z)}{c^2L_0^2M}\mathrm{d}\theta. \tag{36b}$$

At this stage, we can use the explicit formulas of R_0 and R_1 [cf. Eqs. (31) and (33)] to obtain the complete expression of the differential equation for $t(\theta)$, which can be easily integrated, yielding the result

$$f(\theta) = \frac{1}{c^2 L_0^2 \tilde{K}} \left\{ (C_1 R_0 + C_2) R_0 \sin(\tilde{K}\tilde{\theta}) + C_0 \arctan\left[\sqrt{\frac{B_1 - B_2}{B_1 + B_2}} \tan\left(\frac{\tilde{K}\tilde{\theta}}{2}\right) \right] \right\}, \tag{37}$$

where

$$C_0 = \frac{2L_0}{(B_1^2 - B_2^2)^{5/2}} \left\{ 2(B_1^2 - B_2^2)^2 [A_0 - G(\nu - 2)M] + B_1(B_1^2 - B_2^2)\left[(2A_1 + c^2 - 3E_0\nu + E_0) - \frac{L_1}{L_0} \right] + 3B_2[B_2 W_3 - B_1(A_2 + W_2)] + W_1(2B_1^2 + B_2^2) \right\}, \tag{38a}$$

$$C_1 = \frac{L_0}{B_2(B_1^2 - B_2^2)} [B_1 B_2 (A_2 + W_2) - 2B_1^2 W_3 + B_2^2 (W_3 - W_1)], \tag{38b}$$

$$C_2 = \frac{L_0}{B_2(B_1^2 - B_2^2)^2} \left\{ B_2^2 (B_2^2 - B_1^2) \left[(2A_1 + c^2 - 3E_0\nu + E_0) - \frac{L_1}{L_0} \right] + A_2 B_2 (B_1^2 + 2B_2^2) + (2B_1^3 W_3 + B_1^2 B_2 W_2 - B_1 B_2^2 (3W_1 + 5W_3) + 2B_2^3 W_2) \right\}. \tag{38c}$$

The function $f(\theta)$ is discontinuous on intervals lying outside $[0, 2\pi]$. Therefore, in order to regularly connect its different curve branches, we make use of the accumulation function $F_n(\theta)$ [26], which reads as

$$F_n(\theta) = \begin{cases} 0 & \text{if } \tilde{\theta} \in [0, P_\theta] \\ 2nf(P_\theta) & \text{if } \tilde{\theta} \in [P_\theta(2n+1), P_\theta(2n+2)], \end{cases} \tag{39}$$

where $P_\theta = \pi/\tilde{K}$ is the characteristic period and $n \in \mathbb{N}$. For a generic θ, the related value of n can be calculated considering $q = [(\tilde{\theta} - P_\theta)/P_\theta]$, where $[\cdot]$ stands for the integer part of a number. Thus, if q is an even number, then $n = (q+2)/2$; on the other hand, if q is an odd number, then $n = (q+1)/2$. Therefore, we can conclude that the correct *analytical* form of $t(\theta)$ is

$$t(\theta) = f(\theta) + F_n(\theta) + \mathrm{O}(c^{-4}). \tag{40}$$

In Fig. 2, we show the agreement between the numerical solution of Eq. (35) and the analytical expression (40). Here, we have considered two neutron stars (NSs), whose quantum spins are modeled as follows:

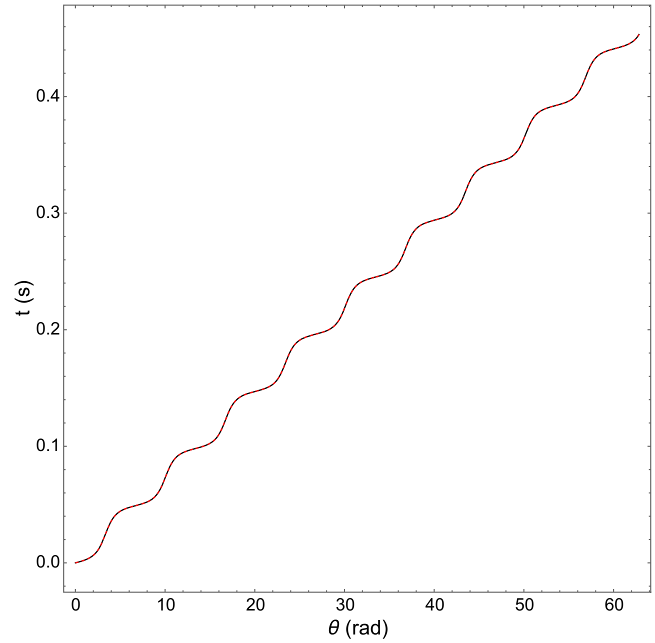

FIG. 2. Function $t(\theta)$ with $\theta \in [0, 20\pi]$ for a binary NS system having the following parameters: $m_1 = 1.60 M_\odot$, $m_2 = 1.17 M_\odot$, $\theta_0 = 0$, $E_0 = -6.80 \times 10^{14}\ \mathrm{m^2\,s^{-2}}$, $E_1 = 1.07 \times 10^{30}$, $L_0 = 8.62 \times 10^{12}\ \mathrm{m^2\,s^{-1}}$, $L_1 = 2.57 \times 10^{28}$ s, $s_{1z} = 1.21 \times 10^{57}\hbar$, $s_{2z} = 4.73 \times 10^{56}\hbar$. The black continuous line represents the numerical solution, whereas the red dashed line is the analytical expression (40).

$$s_{Az} = \mathcal{N}\hbar \frac{4\pi}{3} \left(\frac{6Gm_A}{c^2} \right)^3, \tag{41}$$

where $\mathcal{N} = 10^{44}\ \mathrm{m^{-3}}$ is estimated as the inverse of the nucleon volume [13,14].

The analytical expression of $t(\theta)$ is extremely useful for speeding up the computations in several astrophysical applications such as the following (see Ref. [26] and references therein): pulsar timing software such as TEMPO2; coherent pulsar search algorithms; and GW astronomy, where it can be used to match the observational data with theoretical templates.

C. Energy- and angular-momentum fluxes

The analytic formulas presented in the previous sections play a fundamental role in the evaluation of the energy- and angular-momentum fluxes. Let us start with the luminosity (19a), which, for generic spin directions, can be written as

$$\mathcal{F} = \frac{8}{15} \frac{G^3 \mu^2 M^2}{c^5 R^4} \left[\mathcal{F}_{\mathrm{N}} + \frac{1}{c^2} (\mathcal{F}_{\mathrm{1PN}} + \mathcal{F}_{\mathrm{SO}} + \mathcal{F}_{\mathrm{SS}} + \mathcal{F}_{\mathrm{SS'}}) + \mathrm{O}(c^{-3}) \right]. \tag{42}$$

Here, the GR contributions are [27]

$$\mathcal{F}_{\mathrm{N}} = 12V^2 - 11(\boldsymbol{N}\cdot\boldsymbol{V})^2, \tag{43a}$$

$$\begin{aligned}\mathcal{F}_{1\mathrm{PN}} = {}& \frac{GM}{7R}\left[(734-30\nu)(\boldsymbol{N}\cdot\boldsymbol{V})^2 + (4-16\nu)\frac{GM}{R}\right] \\ &+ \frac{V^2}{14}\left[\frac{80GM}{R}(\nu-17) + \left(\frac{785}{2}-426\nu\right)V^2 \right.\\ &\left. + (1392\nu-1487)(\boldsymbol{N}\cdot\boldsymbol{V})^2\right] + \frac{1}{7}\left(\frac{2061}{4}-465\nu\right) \\ &\times (\boldsymbol{N}\cdot\boldsymbol{V})^4,\end{aligned} \tag{43b}$$

whereas for the EC corrections we find

$$\begin{aligned}\mathcal{F}_{\mathrm{SO}} = {}& \frac{2}{MR^2}\boldsymbol{L}_{\mathrm{N}}\cdot\left\{\boldsymbol{s}\left[78(\boldsymbol{N}\cdot\boldsymbol{V})^2 - \frac{8GM}{R} - 80V^2\right]\right. \\ &\left. + \frac{(\boldsymbol{\chi}_2-\boldsymbol{\chi}_1)}{(m_1-m_2)^{-1}}\left[51(\boldsymbol{N}\cdot\boldsymbol{V})^2 + \frac{4GM}{R} - 43V^2\right]\right\},\end{aligned} \tag{44a}$$

$$\begin{aligned}\mathcal{F}_{\mathrm{SS}} = {}& \frac{2}{R^2}\{3(\boldsymbol{\chi}_1\cdot\boldsymbol{\chi}_2)[47V^2 - 55(\boldsymbol{N}\cdot\boldsymbol{V})^2] \\ &- 3(\boldsymbol{N}\cdot\boldsymbol{\chi}_1)(\boldsymbol{N}\cdot\boldsymbol{\chi}_2)[168V^2 - 269(\boldsymbol{N}\cdot\boldsymbol{V})^2] \\ &- 17(\boldsymbol{N}\cdot\boldsymbol{V})[(\boldsymbol{N}\cdot\boldsymbol{\chi}_2)(\boldsymbol{V}\cdot\boldsymbol{\chi}_1) \\ &+ (\boldsymbol{N}\cdot\boldsymbol{\chi}_1)(\boldsymbol{N}\cdot\boldsymbol{\chi}_2)] + 71(\boldsymbol{V}\cdot\boldsymbol{\chi}_1)(\boldsymbol{V}\cdot\boldsymbol{\chi}_2)\},\end{aligned} \tag{44b}$$

$$\begin{aligned}\mathcal{F}_{\mathrm{SS}'} = {}& \frac{1}{R^2}\sum_A\{3(\boldsymbol{\chi}_A\cdot\boldsymbol{\chi}_A)[3(\boldsymbol{N}\cdot\boldsymbol{V})^2 + V^2] \\ &+ [3(\boldsymbol{N}\cdot\boldsymbol{V})(\boldsymbol{\chi}_A\cdot\boldsymbol{N}) - (\boldsymbol{\chi}_A\cdot\boldsymbol{V})]^2\},\end{aligned} \tag{44c}$$

where we have defined

$$\boldsymbol{\chi}_A \coloneqq \frac{\boldsymbol{s}_A}{m_A}. \tag{45}$$

Some fundamental remarks on the above relations should be given. First of all, we note that Eq. (44c) contains a spin-spin interaction that goes like s_A^2, which stems from $\overset{(3)}{J}{}^{\mathrm{rad}}_{ij}\overset{(3)}{J}{}^{\mathrm{rad}}_{ij}$ (while similar corrections due to $\overset{(3)}{I}{}^{\mathrm{rad}}_{ij}\overset{(3)}{I}{}^{\mathrm{rad}}_{ij}$ will arise at higher PN orders). Such terms also occur in GR, but they are not reported in Refs. [23,24].[2] Therefore, to the best of our knowledge, formula (44c) is displayed for the first time in this paper. Moreover, we stress again that Eq. (42) is formally analogous to the GR flux [23,24,29] if we consider either the substitution (16) (in the case of weakly self-gravitating bodies) or Eq. (17) (for compact objects). Note that in the EC model we consider that the quantities $\mathcal{F}_{\mathrm{SO}}, \mathcal{F}_{\mathrm{SS}}$, and $\mathcal{F}_{\mathrm{SS}'}$ related to compact binaries show up at 1PN level, whereas, as explained in Ref. [29], in GR they are either of order $\mathrm{O}(c^{-3})$ ($\mathcal{F}_{\mathrm{SO}}$) or $\mathrm{O}(c^{-4})$ ($\mathcal{F}_{\mathrm{SS}}$ and $\mathcal{F}_{\mathrm{SS}'}$). This is due to the factor $\mathcal{N}$, for which we have provided a first assessment in Eq. (41) [see also Eq. (47) below]. In fact, the form of $\mathcal{N}$ can change depending on the chosen matter model, and this can influence the formal PN structure of s_{Az}. For a more detailed discussion, see Sec. III C 1.

The spin-spin corrections proportional to s_A^2 also appear in the angular-momentum flux (19b) and, to the best of our knowledge, are not presented in the literature. We find that they are given by

$$\boldsymbol{\mathcal{G}}_{\mathrm{SS}'} = \frac{8}{5}\frac{G^3\mu^2M^2}{c^7R^6}\sum_A[2\boldsymbol{L}_{\mathrm{N}}(\boldsymbol{\chi}_A\cdot\boldsymbol{\chi}_A) - \boldsymbol{\chi}_A(\boldsymbol{\chi}_A\cdot\boldsymbol{L}_{\mathrm{N}})]. \tag{46}$$

The remaining contributions to $\boldsymbol{\mathcal{G}}$ being formally the same as in GR (modulo the multiplicative factor in the spin) can be read from Ref. [24]. Similarly, the gravitational waveform (18) agrees formally with that of GR and hence will not be written explicitly here (see Ref. [24] for details).

1. Digression on the formal analogy between general relativity and Einstein-Cartan theory

In this section, we clarify some fundamental aspects of the formal analogy between EC and GR frameworks.

First of all, this parallelism is formally valid at 1PN order if we consider well-separated fluid bodies having weak self-gravity, as confirmed by Eq. (16), where both $\boldsymbol{s}_A$ and $\hat{\boldsymbol{s}}_A$ are $\mathrm{O}(c^0)$ quantities. In this setup, in fact, both the spin-orbit and spin-spin contributions arise in GR at the 1PN level in both the dynamics and the radiation field (see Refs. [18,24] for more details).

On the other hand, when strongly self-gravitating and maximally rotating compact objects are taken into account, the usual procedure exploited in GR consists in using a variable $\hat{\boldsymbol{s}}_A$ of Newtonian order having the dimensions of an angular momentum multiplied by c [see Eq. (1.1) in Ref. [19] and Eq. (17)]. Therefore, the leading spin-orbit and the spin-spin GR couplings pertaining to compact bodies are of order $\mathrm{O}(c^{-3})$ and $\mathrm{O}(c^{-4})$, respectively. However, due to the different nature of the vectors $\boldsymbol{s}_A$ and $\hat{\boldsymbol{s}}_A$, we cannot stick to the conventions employed in GR, where $\hat{\boldsymbol{s}}_A$ has a precise and fixed form, while $\boldsymbol{s}_A$ can be described by a plethora of models. For this reason, we have proposed in EC theory a first estimate for the spin vector in Eq. (41) [see also Eq. (47) below], where $\boldsymbol{s}_A$ is naively of formal 3PN order. In this way, the involved PN orders are shifted, as in the EC framework the leading spin-orbit and spin-spin corrections show up formally at the 4PN and 7PN levels, respectively. Nevertheless, the formal correspondence with GR is still recovered by means of Eq. (17).

[2] Kidder has informed us in a private communication that he has calculated these terms in GR. However, the final result was not published since he deemed that the contributions proportional to s_A^2 should be combined with the corrections due to the quadrupole-moment tensor induced by the oblateness of the bodies (see Ref. [28] for more details).

However, some remarks are necessary to better explain these points. First of all, the spin vector can be calculated, in general, starting from Eq. (5), where we recall $s_{ij} = \mathrm{O}(c^0)$ [13,14]. This means that the functional form of $\boldsymbol{s}_A$, along with its ensuing PN structure, depends on the adopted matter model. In other words, the way the integral (5) is performed is influenced by the form of s_{ij}.

For example, the scheme we have conceived in Eq. (41) [and Eq. (47) below] is derived by assuming that s_{ij} is constant throughout the body. This brings into play the volume of the compact object, which involves either the event horizon in the black hole (BH) case or some gravitational radii for NSs. We stress that this is a rough calculation because in more realistic models we should divide the volume of the body in regions of different densities and allow for a nonconstant s_{ij}. Moreover, we should not forget the presence of the constant $\mathcal{N}$ in Eq. (41), which is a novel feature with respect to GR. This quantity can indeed be seen as a sort of compensation variable for counterbalancing the factor c^{-6} occurring in Eq. (41). For this reason, we have regarded both spin-orbit and spin-spin corrections as 1PN effects in the EC framework in the case of compact binaries as well.

This last point allows us to discuss another fundamental facet of our studies. In the context of compact objects, even if we consider a new model for the spin vector $\boldsymbol{s}_A$ having a different PN structure with respect to $\hat{\boldsymbol{s}}_A$, the formal analogy between GR and EC theory still holds, as we just need to consider Eq. (17), which shows that two frameworks are formally equivalent up to a multiplicative factor and some powers of c in the spin.

IV. APPLICATIONS

In this section, we apply our findings to two astrophysical situations. In Sec. IV A, we calculate the quantum spin corrections to the energy flux and the waveform of both binary NS and BH systems. In Sec. IV B, we propose a method to estimate the unknown macroscopic angular momentum of one of the bodies hosted in a binary system which exploits the measurement of some observed quantities and the analytical expression of the time-averaged energy flux. The main motivation behind this last application relies on showing that it is possible to share methodologies and results between the formally equivalent GR and EC theories. Furthermore, we will see that the role fulfilled by the spin-spin term (44c) occurring in the energy flux is crucial.

Like before, the spins and the angular momenta are supposed to be aligned perpendicular to the orbital plane.

A. Quantum spin contributions to the energy flux and the gravitational waveform

In Ref. [14], we provided a first estimate of the spin contributions to the energy flux and the gravitational waveform of both binary NSs and BHs. Our analysis was not complete because the 1PN dynamics in EC theory was not at our disposal at that time. Therefore, we decided to set up a hybrid approach, where the bodies were supposed to follow a GR motion parametrized by the standard Damour-Deruelle solution. Now, thanks to the results of this paper, we have all the ingredients for calculating the correct order of magnitude of the spin corrections to the gravitational signal. As already noted in Ref. [14], the new contributions to the 1PN-accurate formulas of $\mathcal{F}$ and $\mathscr{H}_{ij}^{\mathrm{TT}}$ will come only from the time derivatives of the radiative mass quadrupole moment I_{ij}^{rad}, as the derivatives of the other moments are unchanged.

In this treatment, we neglect any GW backreaction effect on the source dynamics since this hypothesis is not too restrictive as it applies to some known astrophysical GW sources (see, e.g., Refs. [30,31]). Moreover, the quantum spin has the following expression [cf. Eq. (41)] [13,14]

$$s_{Az} = \begin{cases} \mathcal{N}\hbar \frac{4\pi}{3}\left(\frac{6Gm_A}{c^2}\right)^3 & \text{for NSs,} \\ \mathcal{N}\hbar \frac{4\pi}{3}\left(\frac{2Gm_A}{c^2}\right)^3 & \text{for BHs.} \end{cases} \tag{47}$$

In order to fulfill our goal, we define

$$\mathcal{E}_{\mathcal{F}}(t) \coloneqq \left|\frac{\mathcal{F}_{\mathrm{EC}}(t)}{\mathcal{F}_{\mathrm{GR}}(t)}\right|, \qquad \mathcal{E}_{\mathscr{H}}(t) \coloneqq \left|\frac{\mathscr{H}_{11}^{\mathrm{EC}}(t)}{\mathscr{H}_{11}^{\mathrm{GR}}(t)}\right|. \tag{48}$$

Here, $\mathcal{F}_{\mathrm{GR}} \coloneqq \mathcal{F}_{\mathrm{N}} + c^{-2}\mathcal{F}_{\mathrm{1PN}}$ and $\mathcal{F}_{\mathrm{EC}} \coloneqq c^{-2}(\mathcal{F}_{\mathrm{SO}} + \mathcal{F}_{\mathrm{SS}} + \mathcal{F}_{\mathrm{SS'}})$ [cf. Eq. (42)]; similarly, we have defined $\mathscr{H}_{11}^{\mathrm{TT}} \coloneqq \mathscr{H}_{11}^{\mathrm{GR}} + \mathscr{H}_{11}^{\mathrm{EC}}$, where $\mathscr{H}_{11}^{\mathrm{GR}}$ contains the GR contribution, while $\mathscr{H}_{11}^{\mathrm{EC}}$ involves only the EC terms.

In the hybrid scheme of Ref. [14], we found for binary NSs ($2.2M_\odot \le M \le 4.32M_\odot$), $\mathcal{E}_{\mathcal{F}} \sim \mathcal{E}_{\mathscr{H}} \sim 10^{-23}$, whereas for binary BHs ($6M_\odot \lesssim M \lesssim 10^{10}M_\odot$), we found $\mathcal{E}_{\mathcal{F}} \sim \mathcal{E}_{\mathscr{H}} \sim 10^{-13}$–$10^{-23}$. Now, exploiting the appropriate dynamics (30) and the formula (40) of the coordinate time to speed up the calculations, we obtain for binary NSs $\mathcal{E}_{\mathcal{F}} \sim \mathcal{E}_{\mathscr{H}} \sim 10^{-21}$, while for binary BHs $\mathcal{E}_{\mathcal{F}} \sim \mathcal{E}_{\mathscr{H}} \sim 10^{-11}$–$10^{-21}$. Therefore, in the full description, the spin contributions are 2 orders of magnitude larger than those obtained with the hybrid approach. This means that we can confirm the validity of our former results. In particular, in agreement with the predictions of the EC model [14], spin effects become more prominent when the companions get closer, as the gravitational field intensity increases. However, also with this new estimate, the EC spin corrections featuring the inspiral stage can hardly be observed with the actual and near-future GW devices. Even so, it should be noticed that our framework relies on the simple configuration (47), while more sophisticated models, like those addressing the dense matter equation of state of NSs, might be detectable.

B. Determining the macroscopic angular momentum of a companion star in a binary system

In astrophysics, there exist binary systems composed of a primary compact object (e.g., a pulsar) and a companion star (e.g., a white dwarf), where we know the macroscopic angular momentum $\hat{\boldsymbol{s}}_1$ of the former but not of the latter, which we denote by $\hat{\boldsymbol{s}}_2$ (see, e.g., Refs. [32,33] for some examples). Therefore, we propose a strategy to determine $\hat{s}_{2z}$ which exploits our analytical developments along with the measurement of the following observables: the masses of the bodies, their orbital separation a, the orbital period P_b and its modulation in time $\dot{P}_\mathrm{b}$, the Newtonian eccentricity e_0, and the rotation frequency f_1 of the primary body. We recall that $\hat{s}_A$ can be calculated via the moment of inertia $\mathcal{I}_A$ and the angular velocity $2\pi f_A$ of the body A as $\hat{s}_A = \mathcal{I}_A 2\pi f_A$. We will suppose that the object A is a sphere of radius $\mathcal{R}_A$, so that $\mathcal{I}_A = \frac{2}{5} m_A \mathcal{R}_A^2$.

The crucial point of our method is that $\dot{P}_\mathrm{b}/P_\mathrm{b}$ satisfies the following relation [see Eq. (4.23) in Ref. [27] for a comparison]

$$\frac{\dot{P}_\mathrm{b}}{P_\mathrm{b}} = \frac{3}{2\mu E}\left[1 - \frac{(\nu - 15)}{6}\frac{E}{c^2}\right]\langle \mathcal{F} \rangle + \mathrm{O}(c^{-10}), \tag{49}$$

where $\langle \mathcal{F} \rangle = \frac{1}{P_\mathrm{b}} \int_0^{P_\mathrm{b}} \mathcal{F}(t)\mathrm{d}t$ is the time average of the flux. In the above equation, E and $\mathcal{F}$ can be obtained directly from Eqs. (9) and (42), provided that we take into account the relation (17) valid for maximally rotating compact objects.

To show how our strategy works, we consider an explicit example represented by the massive pulsar PSR J0348 + 0432, which is hosted in a relativistic compact binary, where we discover that the companion star is a white dwarf [34]. This astrophysical system is gaining a lot of attention since it permits one to test gravity in the strong-field regime and allows one to evaluate the orbital decay due to the GW emission. The timing parameters of PSR J0348 + 0432 are estimated with 1σ uncertainty by TEMPO2 and are listed in Table I. Plugging the input data into Eq. (49), we obtain a quadratic equation in $\hat{s}_{2z}$, which admits two real solutions having opposite sign. In our hypotheses, we choose the positive root, and hence we get $\hat{s}_{2z} = 2.25 \times 10^{38}$ J s, which corresponds to the rotation frequency $f_2 = 0.13$ Hz.

At this stage, some comments are in order. First of all, we note that the quadratic character of the equation for $\hat{s}_{2z}$ is due to the novel spin-spin correction (44c). Moreover, as pointed out before, the GR contributions to the flux due to the macroscopic angular momentum occur either at 1.5PN or 2PN order when compact objects are investigated [19]. Therefore, our treatment should involve, in general, the corrections appearing beyond the 1PN level which do not depend on $\hat{\boldsymbol{s}}_A$. However, these do not significantly alter our estimate of $\hat{s}_{2z}$; in addition, the proposed method is still valid since the missing PN terms do not change the nature of the algebraic equation to be solved to get $\hat{s}_{2z}$.

TABLE I. List of input parameters (taken from Ref. [34]) and output values (last two rows) of the binary system formed by the pulsar PSR J0348 + 0432 (labeled as body 1) and the companion white dwarf (labeled as body 2).

Parameters	Units	Values
m_1	$M_\odot$	2.01
m_2	$M_\odot$	0.17
$\mathcal{R}_1$	km	17.92
$\mathcal{R}_2$	km	45.61
f_1	Hz	25.56
$\hat{s}_{1z}$	10^{40} J s	8.31
a	10^6 km	0.83
e_0		2.01×10^{-6}
P_b	d	0.10
$\dot{P}_\mathrm{b}$	10^{-12} s s^{-1}	−0.27
$\hat{s}_{2z}$	10^{38} J s	2.25
f_2	Hz	0.13

V. CONCLUSIONS

Among the many proposed generalizations of GR, the EC model deals with the microphysical quantum realm and naturally fits the gauge paradigm. In this context, we have worked out the point-particle limit of the Weyssenhoff fluid, and we have discovered that, at the 1PN level, both the dynamical equations and the radiative multipole moments of weakly self-gravitating binary systems formally agree with the corresponding formulas framed in GR, if the substitution (16) is applied. The formal equivalence between the GR and EC frameworks is not spoiled even if we consider compact objects, as can be recovered via Eq. (17). These results are not trivial if we take into account the distinct nature of GR and EC models and the different structure of the equations employed to derive them. Consider, for example, the case of the dynamics. The equations of motion of the Weyssenhoff fluid stem from the generalized conservation laws (3), which lead to a generalized Euler equation and precession motion. These involve novel terms depending on torsion that are not present in GR. Even so, we have found that the effacing principle also holds in EC theory, and the peculiar spin-spin contact interaction of gravitational origin does not contribute, at least at 1PN order.

Driven by the formal 1PN analogy between the GR and EC frameworks, in this paper we have derived some 1PN-accurate analytical formulas pertaining to the dynamics and the radiative phenomena of binary systems. In Sec. III, we have exploited the Damour-Deruelle approach to work out the relative motion of two companions having spins and orbital angular momentum aligned. Then, we have proposed a method to determine the function $t(\theta)$ of the coordinate time in terms of the polar angle θ, which relies on the introduction of a suitably defined accumulation function. This result is crucial to speed up the evaluation of those quantities requiring the knowledge of the dynamical aspects of a binary system, such as the energy- and the angular-momentum fluxes, which have been displayed in Sec. III C. Here, we have also shown the spin-spin couplings (44c) proportional to s_A^2, which, to the best of our knowledge, are not reported in the literature.

Two applications of our analytical treatment have been given in Sec. IV. In the first one, we have improved the treatment of Ref. [14], and we have calculated the quantum spin contributions to the waveform and the energy flux of a binary BH and NS system. We have found that the obtained EC spin effects featuring the inspiral stage are too weak to be observed with the actual and near-future GW apparatuses. However, potentially detectable results can be achieved during the merger stage, where the gravitational interaction is more prominent. In the second framework, we have proposed a strategy to infer the angular momentum of a body which can be useful in several astrophysical settings where it is not possible to measure the rotation frequency of both objects comprising a binary system. Our scheme requires the value of some observables, such as the orbital period modulation, the masses of the two companions, their separation, and the orbital eccentricity. Plugging these parameters into Eq. (49), the unknown angular momentum is obtained by solving an algebraic second-degree equation.

The importance of our findings consists in the fact that they can be exploited both in the GR and EC frameworks thanks to their formal resemblance. In general, having analytical relations and a mathematical methodology to obtain them can be extremely advantageous in astrophysical contexts. In this viewpoint, our formula of $t(\theta)$ is useful for two main reasons: (i) it can be employed for fitting the observational data, without resorting to numerical routines, as well as gathering accurate results in acceptable times; and (ii) the adopted strategy can be extended to higher PN orders once a Damour-Deruelle-like solution is found.

Our programmatic research activity has revealed some features of EC theory which, for the time being, cannot be detected. However, there are some points of our approach which need to be improved. and this strongly spurs us to continue our inquiry. Indeed, GWs continue to be one of the most promising tools to inquire about new physics and test possible deviations from GR in favor of modified frameworks [35–37]. In addition, the examination of theoretical models that are better able to describe the spin distribution inside compact objects should be developed. Besides these practical aspects, the related PN calculations can unearth new and unexpected theoretical results, apart from those highlighted in this paper.

This article can open up some interesting future perspectives. First of all, the 1PN relationship between the GR and EC models and its connection to the matter field modeling the spin effects should be further explored. In fact, it should be understood whether the link between the two theories still holds if a different fluid is considered. Moreover, the results obtained in this paper can be exploited to analyze radiation-reaction forces affecting the evolution of binary systems framed in the EC framework. Lastly, it would be interesting to extend our approaches and methodologies to cosmology as well since it represents a natural arena for the EC pattern as witnessed by the recent literature (see, e.g., Refs. [38–40]). These topics deserve a careful investigation in separate papers.

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