Symmetries and conservation laws in Hořava gravity

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Hořava gravity has been proposed as a renormalizable quantum gravity without the ghost problem through anisotropic scaling dimensions which break Lorentz symmetry in UV. In the Hamiltonian formalism, due to the Lorentz-violating terms, the constraint structure looks quite different from that of general relativity (GR) but we have recently found that "*there exists the case where we can recover the same number of degrees of freedom as in GR*", in a rather general setup. In this paper, we study its Lagrangian perspectives and examine the full diffeomorphism (*Diff*) symmetry and its associated conservation laws in Hořava gravity. Surprisingly, we find that the *full Diff* symmetry in the action can also be recovered when a certain condition, called "supercondition," which *superselects* the Lorentz-symmetric sector in Hořava gravity, is satisfied. This indicates that the broken Lorentz symmetry, known as "foliation-preserving" *Diff*, is just an *apparent* symmetry of the Hořava gravity action and rather its "*full action symmetry can be as large as the Diff in GR*." The supercondition exactly corresponds to the *tertiary* constraint in Hamiltonian formalism which is the second-class constraint and provides a nontrivial realization of the Lorentz symmetry otherwise being absent apparently. From the recovered Lorentz symmetry in the action, we obtain the conservation laws with the Noether currents as in covariant theories. The general formula for the conserved Noether charges reproduces the mass of four-dimensional static black holes with an *arbitrary* cosmological constant in Hořava gravity, and is independent of ambiguities associated with the choice of asymptotic boundaries. We also discuss several challenging problems, including its implications to Hamiltonian formalism, black hole thermodynamics, radiations from colliding black holes.

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I. INTRODUCTION

The renormalizable gravity without the ghost problem has been studied by considering a gravity action *à la* Hořava, Lifshitz, and DeWitt (HLD) [1–3] in $(D+1)$ dimensions (up to some boundary terms),

$$S=\int_{\mathcal{M}} dtd^Dx\sqrt{g}N\left\{\frac{2}{\kappa^2}(K^{ij}K_{ij}-\lambda K^2)-\mathcal{V}[g^{ij},R^i{}_{jkl},\nabla_i]\right\} \tag{1}$$

with the higher-spatial-derivative potential $\mathcal{V}$, satisfying $[\mathcal{V}]\le D+z$ for the (power-counting) renormalizability,[1]

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[1]There seems to exist a widespread belief that the original Hořava action (1) is *not* renormalizable due to absence of $a_i = N^{-1}\nabla_i N$-dependent terms, like a_ia^i term, which can be produced by *scalar*-matter-induced loop corrections [4] (see also [5,6] for different results, i.e., "vanishing" coefficient [5] and "different" (nonzero) coefficient value [6]). However, the relevant coefficients should be matter dependent generally and so the effective action results may or may not affect the genuine gravity sector, depending on the physics in the matter sector. For example, one might consider a *cancellation* of all matter contributions from fundamental scalars, fermions, and gauge bosons for the a_ia^i term which can be quite possible by introducing, for example, "supersymmetry" (we thank Frank Saueressig for a discussion on this point). So, it is still an open problem whether the a_ia^i term can be nonzero when all matter contributions are considered. Moreover, in the "linear-perturbation" analysis of the purely gravity sector, there has been also claimed that the original Hořava action (1) is inconsistent due to the singularity (strong-coupling problem) of the $\lambda\to 1$ limit [7]. However, the singularity problem does not occur in the "fully nonlinear" (constraint) analysis (for example, see the discussion No. 3 in [8]), analogous to the Vainshtein mechanism in massive gravity [9]. Furthermore, the inconsistency (in the flat Minkowski background) disappears in a more realistic "time-dependent" background with a *inflaton* scalar field, even at the linear perturbation level (for example, see the discussion No. 2 in [10]). On the other hand, phenomenologically, it is questionable whether the gravity theory with the a_ia^i term can be a *sensible* theory both astro-physically and cosmologically due to its important deformations of the Newtonian gravity or GR at large distances (see [11] for a confirmation of GR on large scales) or effects on our currently accelerating universe which is based on the standard cosmology with a cosmological constant [12].

while keeping only the second-time-derivatives with the terms of $K^{ij}K_{ij}, K^2$ in the kinetic part, through the *anisotropic* scaling dimensions, $[\mathbf{x}] = -1, [t] = -z$ for the dynamical critical exponent $z > 1$. Here,

$$K_{ij} \equiv \frac{1}{2N}(\dot{g}_{ij} - \nabla_i N_j - \nabla_j N_i) \tag{2}$$

is the extrinsic curvature [the overdot $(\dot{})$ denotes the time derivative $\partial_t \equiv \partial_0 = ()_{,0}$] and $R^i{}_{jkl}, \nabla_i$ are the Riemann tensor, the spatial covariant derivative for D-dimensional spatial metric g_{ij} on the hypersurface with the ADM decomposition

$$ds^2 = -N^2dt^2 + g_{ij}(dx^i + N^i dt)(dx^j + N^j dt). \tag{3}$$

The peculiar property of the HLD action (1) is that the Lorentz, i.e., *diffeomorphism* (*Diff*) symmetry in general relativity (GR) is broken into the "foliation-preserving" diffeomorphism ($Diff_{\mathcal{F}}$) from either the DeWitt's λ parameter in IR ($\lambda \neq 1$) [2], or the higher-derivative terms in UV ($z \geq D$) for power-counting renormalizability [1,3,13]. However, it has been unclear how to *canonically* describe the sudden change (reduction) of symmetry beyond the GR limit, $\lambda \to 1, -\mathcal{V} \to \Lambda + (2/\kappa^2)R$, by tracing the missing (Lorentz) symmetry all the way down to UV.

On the other hand, we have recently found that the constraint structure in the Hamiltonian formalism looks quite different due to Lorentz-violating terms but "*there exits the case where we can recover the same number of degrees of freedom as in GR [8], at the fully nonlinear level*." This appears to mismatch with the *apparently* broken symmetry in the action and so it suggests some *unbroken/enhanced* symmetry in the Lagrangian formalism in order to be consistent with the Hamiltonian formalism.

In this paper, in order to clarify this problem, we examine the full action symmetry and its associated conservation laws in Hořava gravity. Surprisingly, we find that the *full Diff* symmetry can be recovered in the Hořava gravity action when the "supercondition" of $\mathcal{I}_0 \equiv \nabla_i \Omega^i = 0$ is satisfied, which exactly corresponds to the *tertiary* constraint in Hamiltonian formalism which is the second-class constraint. This provides a nontrivial realization of the *Diff* symmetry otherwise being absent apparently. From the recovered Lorentz symmetry in the action on the *superselected* sector, which is still (considered as) an off-shell condition, we obtain the conservation laws with the Noether currents as in covariant theories. The general formula for the conserved Noether charge reproduces the mass of four-dimensional static black hole with an *arbitrary* cosmological constant in Hořava gravity, and is independent of ambiguities associated with the choice of asymptotic boundaries.

The organization of the paper is as follows. In Sec. II, we consider the *Diff* symmetries of the Hořava gravity in comparison with GR and introduce the *supercondition* to consider a *superselected* sector which recovers the Lorentz symmetry in "off-shell". For the superselected sector, we obtain the *covariant* form of Noether currents. In Sec. III, we consider four-dimensional static black solutions and confirm our general mass formula. In Sec. IV, we discuss several challenging problems, including its implications to Hamiltonian formalism, black hole thermodynamics, radiations from colliding black holes. In Appendix A, we describe the computational details and Appendix B, we summarize the complete set of constraints in Hamiltonian formalism.

II. *Diff* SYMMETRIES AND CONSERVATION LAWS

To this end, we start by considering the potential $\mathcal{V}[g^{ij}, R^i{}_{jkl}]$, which is an arbitrary function of metric g_{ij} and curvature invariants only, *eg.*,

$$-\mathcal{V} = \Lambda + \xi R + \alpha R^n + \beta(R_{ij}R^{ij})^s + \gamma(R^i{}_{jkl}R_i{}^{jkl})^r + \cdots, \tag{4}$$

but without (covariant) derivatives, for simplicity. In order that the construction of a renormalizable action (1) is *not* spoiled by the mixing of space and time (derivatives) in the general coordinate transformations, which could produce ghosts of higher-time derivatives from the higher-spatial-derivative terms in the potential, we need to further constrain the allowed coordinate transformations into the foliation-preserving diffeomorphism ($Diff_{\mathcal{F}}$) [3],

$$\delta_\xi t = -\xi^0(t), \qquad \delta_\xi x^i = -\xi^i(t, \mathbf{x}), \tag{5}$$

$$\delta_\xi N = (N\xi^0)_{,0} + \xi^k \nabla_k N, \tag{6}$$

$$\delta_\xi N_i = \xi^0{}_{,0} N_i + \xi^j{}_{,0} g_{ij} + \nabla_i \xi^j N_j + N_{i,0}\xi^0 + \nabla_j N_i \xi^j, \tag{7}$$

$$\delta_\xi g_{ij} = \nabla_i \xi^k g_{kj} + \nabla_j \xi^k g_{ki} + g_{ij,0}\xi^0. \tag{8}$$

Then, from some straightforward computations [3], one can show that the standard action (1) is invariant under $Diff_{\mathcal{F}}$ (6)—(8)

$$\delta_\xi S = \int dt d^D x \{\partial_t[\xi^0(t)\mathcal{L}] + \partial_i[\xi^i(\mathbf{x}, t)\mathcal{L}]\}, \tag{9}$$

which reflects the scalar density nature of the Lagrangian density $\mathcal{L}$, defined by $S \equiv \int dt d^D x \mathcal{L}$. Here, each term in the potential as well as in the kinetic part of (1) is "separately" invariant for an arbitrary λ[2] and all the other parameters in the potential $\mathcal{V}$ so that the higher-time-derivative terms, as

[2]For the case $\lambda = 1/D$, a separate consideration is needed [3]. We will briefly discuss about this in Sec. IV, the discussion No. 4.

well as the mixing terms between the time and spatial-derivative terms, from the (Lorentz) transformation of the potential with higher-spatial derivative terms would not occur.

If we consider Einstein-Hilbert (EH) action in GR with $\lambda = 1$ and $-\mathcal{V} = \Lambda + (2/\kappa^2)R$, then there is an "*accidental*" symmetry enhancement which mixes each term in the action [14] so that we can recover the full *Diff* [2] $[\xi^\mu \equiv (\xi^0, \xi^i)]$,

$$\delta_\xi S_{\rm EH} = \int dtd^D x \partial_\mu [\xi^\mu (t, \mathbf{x}) \mathcal{L}_{\rm EH}] \tag{10}$$

with

$$\delta_\xi t = -\xi^0(t, \mathbf{x}), \qquad \delta_\xi x^i = -\xi^i(t, \mathbf{x}), \tag{11}$$

$$\delta_\xi N = (N\xi^0)_{,0} - N\nabla_i \xi^0 g^{ij} N_j + \xi^k \nabla_k N, \tag{12}$$

$$\delta_\xi N_i = \xi^0{}_{,0} N_i + \xi^j{}_{,0} g_{ij} + \nabla_i \xi^0 (g^{kl} N_k N_l - N^2) + \nabla_i \xi^j N_j + N_{i,0}\xi^0 + \nabla_j N_i \xi^j, \tag{13}$$

$$\delta_\xi g_{ij} = \nabla_i \xi^0 N_j + \nabla_j \xi^0 N_i + \nabla_i \xi^k g_{kj} + \nabla_j \xi^k g_{ki} + g_{ij,0}\xi^0. \tag{14}$$

This shows a *sudden reduction* of the *Diff* symmetry beyond the GR limit in Hořava gravity but it is not quite satisfactory due to lack of canonical understanding of missing (Lorentz) symmetry. On the other hand, in the Hamiltonian formalism, the symmetries of an action are revealed in the existence of constraints between the field variables and their conjugate momenta, which being the *canonical* generators of the symmetry transformations. In the EH case, there are $2(D+1)$ *first-class* constraints which generate the full *Diff* transformations (12)–(14) so that we have $(D+1)(D-2)/2$ physical graviton (transverse traceless) modes. Whereas, in HLD case with the action (1), the constraint structure is quite different, having the *second-class* constraints also due to Lorentz-violating terms, but we have recently found [8] that

> "there exits the case (called Case A) where the same number of degrees of freedom can be recovered as in GR, at the "fully nonlinear" level".

This may suggest that, even though its "apparent" symmetry is just the Lorentz-violating $Diff_{\mathcal{F}}$, *the "full" symmetry of HLD action in the Lagrangian formalism can be as large as Diff in GR*, in order to be consistent with the Hamiltonian analysis.

In order to examine this, which may fill the gap in those two sharply different symmetries in (5)–(8) and (11)–(14), we study the full *Diff* with an arbitrary $\xi^\mu(\mathbf{x}, t)$ to see if the Hořava gravity action (1) can be invariant in a nontrivial way, just beyond the *apparent* symmetry of $Diff_{\mathcal{F}}$. To this end in a canonical way, we start by considering the variation of the action (1) with the potential $\mathcal{V}[g^{ij}, R^i{}_{jkl}]$, under the arbitrary variations of the ADM variables,

$$\delta S = \int dtd^D x[-\mathcal{H}\delta N - \mathcal{H}^i \delta N_i + \mathbf{E}^{ij}\delta g_{ij} + \partial_\mu \Theta^\mu (\delta N_i, \delta g_{ij})], \tag{15}$$

where we have the bulk terms with

$$\mathcal{H} \equiv -\frac{\delta S}{\delta N} = \sqrt{g}\left[\left(\frac{2}{\kappa^2}\right)(K_{ij}K^{ij} - \lambda K^2) + \mathcal{V}\right], \tag{16}$$

$$\mathcal{H}^i \equiv -\frac{\delta S}{\delta N_i} = -2\sqrt{g}\left(\frac{2}{\kappa^2}\right)\nabla_j (K^{ij} - \lambda g^{ij} K), \tag{17}$$

$$\mathbf{E}^{ij} \equiv \frac{\delta S}{\delta g_{ij}} = E^{ij}_{(0)} - \sqrt{g}\left[N P^{iklm} R^j{}_{klm} + \frac{1}{2} N g^{ij} \mathcal{V}[g^{ij}, R^i{}_{jkl}] - 2\nabla_k \nabla_l (N P^{iklj})\right], \tag{18}$$

and the boundary terms $\Theta^\mu(\delta N_i, \delta g_{ij})$ with

$$\Theta^0 \equiv \sqrt{g}\left(\frac{2}{\kappa^2}\right)(K^{ij} - \lambda g^{ij} K)\delta g_{ij}, \tag{19}$$

$$\Theta^i \equiv \sqrt{g}\left(\frac{2}{\kappa^2}\right)(2N^l G^{ijkm} K_{km}\delta g_{jl} - N^i G^{ljmn} K_{mn} \delta g_{jl} - 2G^{kjil} K_{kj}\delta N_l) + 2\sqrt{g} P^{jkil} N \nabla_k \delta g_{lj} - 2\sqrt{g}\delta g_{lj} \nabla_k (P^{jikl} N), \tag{20}$$

where $G^{ijkm} \equiv \delta^{ijkm} - \lambda g^{ij} g^{km}$ is the DeWitt metric [2]. Here, the tensor

$$P_i{}^{jkl} \equiv \left(\frac{\partial \mathcal{L}}{\partial R^i{}_{jkl}}\right)_{g^{mn}} = -\left(\frac{\partial \mathcal{V}}{\partial R^i{}_{jkl}}\right)_{g^{mn}}, \tag{21}$$

by treating g^{ij} and $R^i{}_{jkl}$ are independent fields [15], has the same symmetries in the indices as those of Riemann tensor $R^i{}_{jkl}$ (see Appendix A for the explicit forms for $\mathbf{E}^{ij}$ and Θ^i).

Plugging *Diff* transformations (12)–(14) into the arbitrary variation (15) and doing some straightforward computations, we obtain the action transformation [16],

$$\delta_\xi S = \int dtd^D x[-\mathcal{H}\delta_\xi N - \mathcal{H}^i \delta_\xi N_i + \mathbf{E}^{ij}\delta_\xi g_{ij} + \partial_\mu \Theta^\mu (\delta_\xi N_i, \delta_\xi g_{ij})], \tag{22}$$

$$= \int dtd^D x[\xi^0 \mathcal{I}_0 + \xi^i \mathcal{I}_i + \partial_\mu \Psi^\mu (\delta_\xi N_i, \delta_\xi g_{ij})], \tag{23}$$

where

$$\mathcal{I}_0 \equiv N\dot{\mathcal{H}} - \nabla_m(NN^m\mathcal{H}) + N_i\dot{\mathcal{H}}^i + \nabla_m[\mathcal{H}^m(g^{jl}N_jN_l - N^2)] + \dot{g}_{ij}\mathbf{E}^{ij} - 2\nabla_m(N_i\mathbf{E}^{mi}), \tag{24}$$

$$\mathcal{I}_i \equiv (g_{ij}\mathcal{H}^j)_{,0} + \nabla_m(\mathcal{H}^mN_i) - \mathcal{H}\nabla_iN - \mathcal{H}^j\nabla_iN_j - 2g_{ij}\nabla_m\mathbf{E}^{jm}, \tag{25}$$

$$\Psi^0 \equiv -\xi^0(N\mathcal{H} + N_i\mathcal{H}^i) - \xi^j g_{ij}\mathcal{H}^i + \Theta^0, \tag{26}$$

$$\Psi^i \equiv \xi^0[NN^i\mathcal{H} - \mathcal{H}^i(g^{lj}N_lN_j - N^2) + 2N_j\mathbf{E}^{ij}] + \xi^j(-N_j\mathcal{H}^i + 2g_{jl}\mathbf{E}^{il}) + \Theta^i. \tag{27}$$

Here, it is important to note that the formal expressions of (24)–(27) are generally valid for arbitrary potential $\mathcal{V}[g^{ij}, R^i{}_{jkl}]$ though the explicit expressions of $\mathcal{H}, \mathcal{H}^i$, $\mathbf{E}^{ij}, \Theta^\mu, \Psi^\mu$ may depend on the potential form.

Now, by expressing the time-derivatives in terms of extrinsic curvature K_{ij} in (2) and using the definitions (16)–(18), *but without using the dynamical equations of motion* $\mathbf{E}^{ij} = 0$, *nor the constraints* $\mathcal{H} \approx 0, \mathcal{H}^i \approx 0$, we obtain

$$\mathcal{I}_0 = \nabla_i\left\{2N^2\left[\nabla_j\pi^{ij} + \left(\frac{\kappa^2}{2}\right)\left(\frac{2\lambda}{\lambda D - 1}(\pi\nabla_l P^{kl}{}_k{}^i - P^{kl}{}_k{}^i\nabla_l\pi) + 2P_{jkl}{}^i\nabla^k\pi^{jl} - 2\pi^{jl}\nabla^k P_{jkl}{}^i\right)\right]\right\} \equiv \nabla_i\Omega^i, \tag{28}$$

$$\mathcal{I}_i = 0, \tag{29}$$

$$\Psi^0 = \xi^0\mathcal{L} + \partial_i\mathcal{U}^{0i}, \tag{30}$$

$$\Psi^i = \xi^i\mathcal{L} + \mathbf{\Sigma}^i + \partial_0\mathcal{U}^{i0} + \partial_j\mathcal{U}^{ij}, \tag{31}$$

where $\mathbf{\Sigma}^i$ and $\mathcal{U}^{\mu\nu} = -\mathcal{U}^{\mu\nu}$, which is called the "super-potential" [17,18], are given by

$$\mathbf{\Sigma}^i = 2N^2\left[\left(\frac{\kappa^2}{2}\right)\left(\frac{2\lambda}{\lambda D - 1}(P^{li}{}_{lk}\nabla^k(\xi^0\pi) - \xi^0\pi\nabla^k P^{li}{}_{lk}) + 2\xi^0\pi^{jl}\nabla^k P_{jkl}{}^i - 2P_{jkl}{}^i\nabla^k(\xi^0\pi^{jl})\right) + \pi^{ij}\nabla_j\xi^0 - \xi^0\nabla_j\pi^{ij}\right], \tag{32}$$

$$\mathcal{U}^{0i} = -\mathcal{U}^{i0} = 2\sqrt{g}(\xi^0N_j + \xi_j)\left(\frac{2}{\kappa^2}\right)(K^{ij} - \lambda g^{ij}K), \tag{33}$$

$$\mathcal{U}^{ij} = -\mathcal{U}^{ji} \tag{34}$$

[see Appendix A for the computational details of (28)–(31) and the explicit forms for $\mathcal{U}^{ij}$ of (34)]. Here, we note that $\mathcal{I}_i = 0$ identically, which is the analog of the spatial component of the (contracted) Bianchi identity $\widehat{\nabla}_\mu G^{\mu i} = 0$ in GR, for the Einstein tensor $G^{\mu\nu} = \hat{R}^{\mu\nu} - (1/2)\hat{g}^{\mu\nu}\hat{R}$ with the $(D+1)$-dimensional Ricci tensor $\hat{R}^{\mu\nu}$, Ricci scalar $\hat{R}$ [19] and covariant derivative $\widehat{\nabla}_\mu$: In the Hamiltonian formalism, it is due to the first-class nature of the momentum constraint $\mathcal{H}_i \approx 0$, the generator of the spatial $Diff$, even in HLD gravity [8]. In $\mathcal{I}_0$, we have replaced the extrinsic curvature with the canonical momenta $\pi_{ij} = (2/\kappa^2)\sqrt{g}(K_{ij} - \lambda K g_{ij})$ in order to compare it with the Hamiltonian analysis.

In GR case, we have $\mathbf{\Sigma}^i = 0$ and $\mathcal{I}_0 = 0$, which corresponds to the time-component of Bianchi identity, $\widehat{\nabla}_\mu G^{\mu 0} = 0$ and hence obtain the usual Lorentz invariance of (10) from (23). For the $Diff_{\mathcal{F}}$ in Hořava gravity case, the parameter $\xi^0(t) \equiv \tilde{\xi}^0(t)$ for the $\xi^0\mathcal{I}_0$ in (23) can be factorized out from the space integration and the $\xi^0\mathcal{I}_0$ term turns into a (spatial) boundary term, but it is "exactly canceled" by another boundary term $\nabla_i\mathbf{\Sigma}^i$ so that we can obtain the invariance of (9) under $Diff_{\mathcal{F}}$.

For the full *Diff* with an arbitrary $\xi^0(t, \mathbf{x})$, on the other hand, the $\mathcal{I}_0$ term, which is *nonzero* [see the explicit expression in (A10)] unless we consider $\lambda = 1$ and vanishing of all the higher-derivatives terms (as in the GR case), cannot be removed from the bulk terms anymore and may result the noninvariance of the HLD action generally, confirming the usual belief of its generic Lorentz violation. Therefore, the only way of obtaining the full *Diff* invariance for the HLD action, *if it exits*, would be to consider a "supercondition,"

$$\mathcal{I}_0 \equiv \nabla_i\Omega^i = 0, \tag{35}$$

which *superselects* the Lorentz-invariant sector in HLD gravity. We note also, from (17) and (28), that the supercondition can be written as

$$\mathcal{I}_0 = \Omega - \nabla_i(N^2\mathcal{H}^i) = 0, \tag{36}$$

where

$$\Omega \equiv \nabla_i (N^2 C^i) \tag{37}$$

with

$$C^i \equiv \kappa^2 \left(\frac{2\lambda}{\lambda D - 1} (\pi \nabla_l P^{kl}{}_k{}^i - P^{kl}{}_k{}^i \nabla_l \pi) + 2P_{jkl}{}^i \nabla^k \pi^{jl} - 2\pi^{jl} \nabla^k P_{jkl}{}^i \right). \tag{38}$$

Then, it is remarkable that (37) reduces to the *tertiary* constraint $\Omega \approx 0$ in Hamiltonian formalism [8], using the dynamical equations of motion,[3] i.e., in *on-shell*, from $\dot{\mathcal{H}} \approx \Omega/N \approx 0$, with the Hamiltonian constraint $\mathcal{H} \approx 0$ and the momentum constraint $\mathcal{H}^i \equiv -2\nabla_j \pi^{ij} \approx 0$ (see Appendix B for a summary of the complete set of constraints). In other words, the supercondition (35) in our Lagrangian formalism of HLD gravity is *on-shell* equivalent to the tertiary constraint in the Hamiltonian formalism. On the other hand, in GR, the tertiary constraint is trivial, up to the Hamiltonian and momentum constraints, $\mathcal{H} \approx 0, \mathcal{H}^i \approx 0$, which are the secondary constraints.[4] The intimate relation to constraints in the Hamiltonian formalism may support for our introduction of the supercondition (35) in the Lagrangian formalism. However, it is important to note that, in our Lagrangian formalism, the supercondition is *assumed* to be valid even *off-shell*,[5] i.e., prior to considering the "classical geometry dynamics" which extremizes the action. In this way, the *full Diff* can be maintained all the way down to UV, even beyond the GR limit. Here, the supercondition $\mathcal{I}_0 = 0$ in HLD gravity corresponds to the temporal component of Bianchi identity, $\widehat{\nabla}_\mu G^{\mu 0} = 0$ in GR.

[3] $\dot{g}_{ij} = \{g_{ij}, H_C\}, \dot{\pi}^{ij} = \{\pi^{ij}, H_C\}$ with the canonical Hamiltonian $H_C = \int dx^D \{N\mathcal{H} + N_i \mathcal{H}^i\}$.

[4] From this result, one can use (24) as an off-shell, Lagrangian definition of the *tertiary* constraints via the terms of $N\dot{\mathcal{H}}$ and $N_i \dot{\mathcal{H}}^i$, which are usually quite cumbersome in Hamiltonian formalism. Actually, using the new definition in this paper we have generalized the Hamiltonian analysis on tertiary constraint for $\dot{\mathcal{H}}$ in [8], where we have considered only an arbitrary function of R in the potential (4) (see also Appendix B). Moreover, one can find easily the exactly same constraint algebra for the Hamiltonian and momentum constraints as in the previous case [8], $\{\langle \eta\mathcal{H}\rangle, \langle \zeta\mathcal{H}\rangle\} = \langle (\eta\nabla_i \zeta - \zeta\nabla_i \eta) C^i\rangle, \{\langle \eta\mathcal{H}\rangle, \langle \zeta^i \mathcal{H}_i\rangle\} = -\langle \zeta^i \nabla_i \eta \mathcal{H}\rangle$, $\{\langle \eta^i \mathcal{H}_i\rangle, \langle \zeta^j \mathcal{H}_j\rangle\} = \langle (\eta^i \nabla_i \zeta^j - \zeta^i \nabla_i \eta^j)\mathcal{H}_j\rangle$, for the *Diff* parameters η, ζ, and the smeared constraints, $\langle \eta\mathcal{H}\rangle \equiv \int d^D x \eta \mathcal{H}$, *etc.*

[5] From the form of the Bianchi identity in GR, $\overline{\nabla}_\mu G^\mu{}_0 = \overline{\nabla}_0 G^0{}_0 + \overline{\nabla}_i G^i{}_0 = (\mathcal{H}_{\rm GR}/2\sqrt{g})_{,0} + \cdots + \nabla_i (\mathcal{H}^i/2N\sqrt{g}) = 0$ with $G^0{}_0 = \mathcal{H}_{\rm GR}/2\sqrt{g}, G^i{}_0 = \mathcal{H}^i/2\sqrt{g}N$ for the Hamiltonian constraint in GR, $\mathcal{H}_{\rm GR} \approx 0$, the supercondition suggests *formally* the same Bianchi identity but now with $G^0{}_0 \equiv \mathcal{H}/2\sqrt{g}$ in HLD gravity also. This provides a more fundamental *off-shell* reason for the appearance of the tertiary constraint via $\dot{\mathcal{H}}$.

Finally, from (15), (23), and with the help of the super condition $\mathcal{I}_0 = 0$ and the Bianchi identity $\mathcal{I}_i = 0$, we can obtain the Noether currents ($\Sigma^0 = 0$),

$$\begin{aligned} \mathcal{J}^\mu(\delta_\xi g) &\equiv \Theta^\mu(\delta_\xi N_i, \delta_\xi g_{ij}) - \Psi^\mu(\delta_\xi N_i, \delta_\xi g_{ij}) \\ &= \Theta^\mu - \xi^\mu \mathcal{L} - \mathbf{\Sigma}^\mu - \partial_\nu \mathcal{U}^{\mu\nu}, \end{aligned} \tag{39}$$

which satisfies

$$\partial_\mu \mathcal{J}^\mu(\delta_\xi g) = \mathcal{H}\delta_\xi N + \mathcal{H}^i \delta_\xi N_i - \mathbf{E}^{ij} \delta_\xi g_{ij}. \tag{40}$$

Note that the Noether currents satisfies the usual conservation laws *on-shell*, i.e., $\mathcal{H} = \mathcal{H}^i = \mathbf{E}^{ij} = 0$, for an arbitrary *Diff* transformation, but *off-shell* for Killing vectors ξ^μ, $\delta_\xi N = \delta_\xi N_i = \delta_\xi g_{ij} = 0$. The second term Σ^μ is due to the *apparent* noncovariance of the Horava action.[6] Moreover, the last part in the Noether current (39), corresponds to the identically conserved or *off-shell current* $\mathcal{J}^\mu_{\rm off} \equiv \partial_\nu \mathcal{U}^{\mu\nu}$ [17,18,21]. Then, the conserved charge passing through a hypersurface Σ is given by

$$Q(\xi) = \int_\Sigma d^D x \sqrt{g} n_\mu J^\mu(\delta_\xi g) \tag{41}$$

for the unit normal vector n^μ of Σ and the *covariantly* conserved charge $J^\mu = (\sqrt{g}N)^{-1} \mathcal{J}^\mu$, satisfying $\nabla_\mu J^\mu = 0$ [27] and the physically *measurable* charge can be obtained by subtracting the background charge $\bar{Q}(\xi) \equiv \int_\Sigma d^D x \sqrt{\bar{g}} \bar{n}_\mu \bar{J}^\mu(\delta_\xi \bar{g})$, *i.e.*, $Q(\xi)_{\rm phys} \equiv Q(\xi) - \bar{Q}(\xi)$ generally, where the bars denote the background quantities.

III. AN EXAMPLE: STATIC BLACK HOLES IN (3+1) DIMENSIONS

In order to check the general Noether charge formula (41), let us consider the static metric ansatz

$$ds^2 = -N^2(r) dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega_k^2 \tag{42}$$

by which the original Horava gravity action in $(3+1)$ dimensions [3] reduces to the case with the potential form of (4), due to the vanishing Cotton tensor, $C^{ij} \equiv \epsilon^{ikl} \nabla_k (R^j{}_l - \delta^j{}_l R/4) = 0$. Here, $d\Omega_k^2$ denotes the line element for two-dimensional surface with a constant scalar curvature, $R^{(2)} = 2k$ for spherical, plane, and hyperbolic topologies with $k = +1, 0, -1$, respectively.

Then, for the timelike Killing vector $\xi^\mu = (1, 0, 0, 0)$ and the normal vector $n_\mu = (-N, 0, 0, 0)$, the only nonvanishing contributions in the Noether charge (41) come from the second term in the current (39) and, after the angular integrations, is given by

[6] The noncovariance term appears also in Chern-Simons theories [20].

$$
\begin{aligned}
Q(\xi^0) &= \Omega_k \int_0^r dr r^2 \left(\frac{N}{\sqrt{f}}\right)\xi^0 \mathcal{L} \\
&= -\sigma \xi^0 \left(\frac{N}{\sqrt{f}}\right)\left[-\frac{\lambda}{r}(f-k)^2 + 2(\omega-\Lambda_W) r(f-k) - \Lambda_W^2 r^3\right] + \sigma \int_0^r dr \partial_r \left(\xi^0 \frac{N}{\sqrt{f}}\right) \\
&\quad \times \left[-\frac{\lambda}{r}(f-k)^2 + 2(\omega-\Lambda_W) r(f-k) - \Lambda_W^2 r^3\right] - \sigma(\lambda-1)\int_0^r dr \xi^0 \left(\frac{N}{\sqrt{f}}\right)\left[\frac{(f-k)^2}{r^2} + \frac{(\partial_r f)^2}{2}\right]
\end{aligned} \tag{43}
$$

in the usual parametrization,

$$
\xi \equiv \frac{\kappa^4 \mu^2 (\Lambda_W + \omega)}{8(1-3\lambda)}, \qquad \alpha \equiv \frac{\kappa^2 \mu^2 (1-4\lambda)}{32(1-3\lambda)}, \qquad \beta \equiv \frac{\kappa^2 \mu^2}{8}, \qquad \gamma \equiv 0, \qquad \Lambda \equiv -\frac{2\kappa^2 \mu^2 \Lambda_W^2}{8(1-3\lambda)} \tag{44}
$$

with $\sigma \equiv \Omega_k \kappa^2 \mu^2 / 8(3\lambda - 1)$, the IR-modification parameter ω, and D-dimensional cosmological constant parameter Λ_W [3,22–26]. Here, it is important that we need to (1) first, change the Lagrangian into the total derivatives (the first term of the second line in the above formula) plus the bulk terms (the remaining terms in the second and third lines), and then (2) second, compute the charge by plugging the known solutions: If we first plug the solutions into the Noether charge formula (43), we obtain the trivially vanishing charge because the Lagrangian in the charge formula is proportional to the Hamiltonian constraint $\mathcal{H} \approx 0$, which is solved by the solutions.[7]

Now, by plugging the general static vacuum solution for arbitrary cosmological constant parameter Λ_W and IR parameter ω with $\lambda = 1$ [24,26], whose *uniqueness* is guaranteed by the corresponding *Birkhoff's theorem* [28] (for more general cases, see the discussion No. 8 below),

$$
N^2 = f = k + (\omega - \Lambda_W) r^2 + \epsilon \sqrt{r[\omega(\omega - 2\Lambda_W) r^3 + \boldsymbol{\beta}]}, \tag{45}
$$

where $\epsilon = \pm 1$ and $\boldsymbol{\beta}$ is an integration constant,[8] we can obtain

$$
Q(\xi^0) = \sigma \boldsymbol{\beta}, \tag{46}
$$

which exactly agrees with the mass in the conventional Hamiltonian approach [25,30,31]. Note that the mass $\mathcal{M} \equiv Q(\xi^0)$ satisfies the first law of black hole thermodynamics

$$
\delta \mathcal{M} = T_H \delta \mathcal{S} \tag{47}
$$

with the black hole temperature T_H[9] and the entropy $\mathcal{S}$ with a logarithmic term, up to an arbitrary constant $\mathcal{S}_0$,

$$
T_H \equiv \frac{\hbar \kappa|_H}{2\pi} = \frac{\hbar(3\Lambda_W^2 r_H^4 + 2k(\omega - \Lambda_W) r_H^2 - k^2)}{8\pi r_H (k + (\omega - \Lambda_W) r_H^2)}, \tag{48}
$$

$$
\mathcal{S} = \frac{4\pi\sigma}{\hbar}((\omega - \Lambda_W) r_H^2 + 2k \ln r_H) + \mathcal{S}_0 \tag{49}
$$

for the surface gravity $\kappa_H = (1/2)\partial_r f|_{r_H}$ at the black hole horizon r_H.

Two remarkable properties of this result are as follows. First, the result (43) does not depend on the boundary $(D-1)$-hypersurface only if there is a timelike Killing vector inside the boundary. This means that the boundary needs not to be an asymptotic infinity even in asymptotically de Sitter space as well as in flat or anti–de Sitter space (for similar results in covariant theories, see [33,34]). Second, related to the first property, there are no divergences in anti–de Sitter space, and it is independent on the ambiguities associated with the choice of asymptotic boundary at $r \to \infty$ in de Sitter space [35]. So, for the asymptotically de Sitter black hole, the boundary can be any region between the outer black hole horizon r_+ and the cosmological horizon r_{++}, *i.e.* $r_+ < r < r_{++}$.

IV. CONCLUDING REMARKS

In conclusion, we have shown that the Lorentz symmetry, which is represented by *Diff* symmetry, is preserved on the superselected sector of $\mathcal{I}_0 \equiv \nabla_i \Omega^i = 0$ even in HLD gravity action where the explicit Lorentz violating terms are introduced for (power-counting) renormalizablity. This indicates that the full *Diff* symmetry of HLD action can be as large as the *Diff* in GR and, from the obtained full *Diff*

[7]This looks tricky but this kind of prescription seems to be essential to get the right answer (see also [27] for some related discussions). In particular, for $\lambda = 1$ and $(N/\sqrt{f}) =$ constant, our charge (43) agrees with the (generalized) Misner-Sharp mass [25].

[8]$\epsilon = -1(+1)$ represent an asymptotically flat or anti-de Sitter (de Sitter) with $\omega, \mu^2 > 0$ $(\omega, \mu^2 < 0)$ [26]. Here, we consider only the GR-*branch* solutions which have the GR limits in IR regime as in [22–24]. The other choices of the ϵ represent the *non-GR branch* solutions which do not have the GR limits and these are important for studying the genuine higher-derivative solutions [29].

[9]See [32], for an explicit computation of the Hawking radiation and temperature for *relativistic* matters, based on the quantum tunneling approaches.

symmetry, we find the conservation laws with the Noether currents as in covariant theories [18]. Several further remarks about challenging problems are in order.

(1) The supercondition $\mathcal{I}_0 \equiv \nabla_i \Omega^i = 0$ is similar to the Maxwell's equation $\nabla_i B^i = 0$ for the magnetic field B^i without (magnetic) monopoles. If we define the $(D-2)$-form "currents" for D-dimensional space with the component $J^{i \ldots n} = \epsilon^{i \ldots njk} \nabla_j \Omega_k$, it satisfies the (spatial) conservation laws $\nabla_i J^{i \ldots n} = 0$ as in the equations $j^i = \epsilon^{imn} \partial_m B_n$ for the magnetostatics with the electric currents j^i. Then, we can solve Ω^i in terms of the currents $J_{i \ldots n}$, which are the additional data for a complete specification of Ω^i. For example, if the supercondition $\nabla_i \Omega^i = 0$ and $\epsilon^{i \ldots njk} \nabla_j \Omega_k = J^{i \ldots n} = 0$ are satisfied for the *whole-space* region, i.e., without singularities, then $\Omega^i = 0$ would be the only solution and this would correspond to "Case A" in the Hamiltonian analysis of [8] where the degrees of freedom in HLD gravity are the same as in GR at the fully nonlinear level. Otherwise, Ω^i would be nonvanishing generally due to either (a) nontrivial tolopoly/cohomology, or (b) singularities, or from (c) nonvanishing current $J^{i \ldots n} \neq 0$. This latter case would correspond to "Case B" in [8] where an extra scalar graviton mode exists in Hamiltonian analysis of HLD gravity. It would be interesting to find the generic (formal) solution of Ω^i in curved space, corresponding to Biot-Savert's law in electromagnetism in Minkowski space-time.

(2) From the invariance of the action (23), we have obtained the tertiary constraints $\nabla_i (N^2 C^i) = 0$ in Hamiltonian formalism via $\dot{\mathcal{H}}$ and $\dot{\mathcal{H}}^i$ in the supercondition $\mathcal{I}_0 = 0$. As have been noted above, the action invariance does not necessarily mean the same degrees of freedom as in GR, which is manifestly Lorentz invariant. In Hamiltonian formalism, we need to find a complete set of constraints to completely specify the degrees of freedom. This implies that we need more consistency analysis in Lagrangian formalism, corresponding to the preservation of constraints in Hamiltonian formalism. We suspect that the higher-order invariance of $\delta_\xi \delta_\eta \ldots \delta_\zeta S = 0$ would be important in HLD gravity and needs to be considered in order to obtain the complete set of constraints, consistently with the Hamiltonian formalism.

(3) Our formulation about the gravity sector is self-contained and independent on the matter sector. If we now consider matter action S_m as well, which may have nonrelativistic higher-derivative terms also in accordance with the HLD gravity, the additional contributions to the action transformation are $\int dtd^D x [\xi^0 \sqrt{\hat{g}} \widehat{\nabla}^\mu T_{\mu 0} + \xi^i \sqrt{\hat{g}} \widehat{\nabla}^\mu T_{\mu i}]$ with the energy-momentum tensors for matters $T_{\mu\nu} = -\frac{2}{\sqrt{\hat{g}}} \frac{\delta S_m}{\delta \hat{g}^{\mu\nu}}$, for the $(D+1)$-dimensional metric $\hat{g}_{\mu\nu}$ and its associated covariant derivatives $\widehat{\nabla}^\mu$, together with the matter contributions to the boundary terms Θ^μ and Ψ^μ. But, from the supercondition $\mathcal{I}_0 = 0$ and the Bianchi identity $\mathcal{I}_i = 0$ in the gravity sector, which indicating their geometrical origin, the consistent theory with the full *Diff* is possible only for the covariantly-conserved matters, i.e., $\widehat{\nabla}^\mu T_{\mu\nu} = 0$, regardless higher-derivatives in HLD gravity. In other words, only the energy-conserving matters $\widehat{\nabla}^\mu T_{\mu 0} = 0$, as well as the momentum-conserving matters $\widehat{\nabla}^\mu T_{\mu i} = 0$, can be consistent with the HLD gravity. Actually, there seems to exist some evidence for the covariant conservation laws of matter's energy momentum tensors and of the effective energy-momentum tensors from the higher-derivative terms, separately, for spherically symmetric case [36].[10] It would be interesting to see whether the covariant form of the conservation laws holds generally, as another supercondition in matter sectors.

(4) For the special value of IR Lorentz-violation parameter $\lambda = 1/D$, the theory has *anisotropic* Weyl symmetry additionally [3] but a separate consideration is needed. Based on the Hamiltonian analysis [8], which gives the same degrees of freedom as in GR, its full symmetry would be also as large as that of GR, though its details of the symmetry are different. It would be interesting to clarify the full action symmetry also and its connection to the Case A.

(5) From the obtained Lorentz symmetry of HLD action for the superselected sector of $\nabla_i \Omega^i = 0$, one can consider the corresponding *Ward-like* identity $\langle \delta_\xi \mathcal{F} \rangle - (i/\hbar) \int \xi \langle \mathcal{F} \partial_\mu \mathcal{J}^\mu \rangle dtd^D x = 0$ for a *Diff* invariant observable $\mathcal{F}$, from the *Diff* invariance of the path-integral measure with the first and second-class constraints [37]. The proof of renormalizability for HLD gravity from the gravitational Ward-like identity would be a challenging problem.

(6) In our formulation, we have considered the arbitrary potential $\mathcal{V}[g^{ij}, R^i{}_{jkl}]$ without derivatives ∇_i. The inclusion of derivatives in the potential, i.e., $\mathcal{V}[g^{ij}, R^i{}_{jkl}, \nabla_i]$ would be more desirable to describe the most general systems as in the action (1). The computations are straightforward but, due to the

[10]One can *formally* write the Horava gravity's equations of motion into a *covariant* Einstein's equation $G^{\mu\nu} = 8\pi G T^{\mu\nu}_{\rm eff}$ by considering the higher-derivative contributions as the effective energy-momentum tensor $T^{\mu\nu}_{\rm eff}$. Then, from the usual (covariant) Bianchi identity on the Einstein tensor $\widehat{\nabla}_\mu G^{\mu\nu} = 0$, one can find the covariant conservation laws $\widehat{\nabla}_\mu T^{\mu\nu}_{\rm eff} = 0$ [36]. However, the geometric origin of this identity/property is still unknown.

complications, we have not yet succeeded in obtaining the canonical form of (23). However, we believe that the formulation itself should not depend on the existence of derivatives in the potential so that there should be no fundamental problem to get the corresponding canonical forms.

(7) The *Diff* invariance of HLD gravity sheds a new light on the very meaning of black hole entropy and its thermodynamical laws, due to the revival of Lorentz invariant concept of the event horizon, which has been essential to give an absolute meaning to black hole entropy as a *measure of observable ignorance* inside the event horizon as well as the role of the *universal horizons* in the presence of the (Lorentz-invariant) event horizons [38].[11] In particular, for the $k=1$ black holes in an asymptotically flat/AdS space, the logarithmic correction to the usual Bekenstein-Hawking entropy implies the "positive" *minimum* of horizon radius r_H for the *positive* black hole entropy $\mathcal{S}$, which is consistent with the existence of a positive minimum for the mass $\mathcal{M}$ where the Hawking temperature T_H vanishes [26].[12] Moreover, in that case, the black hole entropy $\mathcal{S}$ increases (the second law of black hole thermo dynamics) by $\Delta\mathcal{S} = (8\pi\sigma/\hbar)[(\omega - \Lambda_W)r_H + 2k/r_H]\Delta r_H = T_H\Delta\mathcal{M}$ for the increased mass $\Delta\mathcal{M} > 0$. From the associated increase of area[13] $A_H = 4\pi r_H^2$, one can compute the upper bound to the energy of the gravitational radiations when one black hole captures another. For the asymptotically flat black holes ($\Lambda_W = 0$) [23] with the area ($\boldsymbol{\beta} \equiv 4\omega M$)

$$A_H = 8\pi M^2[1 + (1 - (2\omega M^2)^{-1})^{1/2} - (4\omega M^2)^{-1}], \tag{50}$$

the increased area gives the inequality ($m \equiv \omega^{1/2}M$)

$$\begin{aligned} m_3^2[1 + (1 - (2m_3^2)^{-1})^{1/2}] \\ > m_1^2[1 + (1 - (2m_1^2)^{-1})^{1/2}] \\ + m_2^2[1 + (1 - (2m_2^2)^{-1})^{1/2}] - 1/4. \end{aligned} \tag{51}$$

Then, the energy emitted in gravitational or other form of radiations is $m_1 + m_2 - m_3$ and its efficiency $\epsilon \equiv (m_1 + m_2 - m_3)/(m_1 + m_2)$ is limited by (51). The highest limit on ϵ is $1 - 3/4\sqrt{2} \approx 0.47$ which occurs when $m_1 = m_2 = 2^{-1/2}$, which are the minimum values for positive Hawking temperature $T_H > 0$, and $m_3 = 3/4$. On the other hand, when particles or fields which do not have horizons, impinge on a single black hole, one finds

$$\begin{aligned} m_2^2[1 + (1 - (2m_2^2)^{-1})^{1/2}] \\ > m_1^2[1 + (1 - (2m_1^2)^{-1})^{1/2}]. \end{aligned} \tag{52}$$

Note that m_2 *cannot* be less than m_1. This means that *one cannot extract energy from a black hole* and there is *no analog of the Penrose process* for Kerr or charged black hole in GR [45,46]. This is basically due to fact that one cannot turn off the parameter ω arbitrarily, in contrast to the rotation parameter a in a Kerr black hole or the charge parameter e in a Reissner-Norström black hole even though they look similar in the black hole area formula (50) [47].[14]

These results are *quantitatively* (the lower highest limit $\epsilon = 1 - 2^{-1/2} \approx 0.29$ for two nonrotating initial black holes with the same masses [47]) and *qualitatively* (no energy extractions via an analog of the Penrose process [45,46]) different from GR black holes which could be tested experimentally in the near future.

(8) The ambiguities associated with the choice of asymptotic boundary at $r \to \infty$ is absent in the charge formula (43) when we consider $\lambda = 1$ solution (45), where the second and the third (bulk) terms in (43) vanish. Note that this case covers a wide range of static (vacuum) solutions with higher curvatures, including those in GR [34]. However, for $\lambda \neq 1$ generally [50] which is beyond the GR limit, we still seems to need asymptotic boundary at $r \to \infty$, even for the asymptotically de Sitter black hole, in order to obtain its "finite" physical mass. The intimate physical connection between the IR-Lorentz violation and the need of an infinite boundary (of our universe) is still unclear. However, for the asymptotically de Sitter black hole, the choice of the asymptotic boundary at $r \to \infty$ might not be

[11] In Lorentz-violating gravities, the thermodynamical properties, like the Hawking radiation have been long-standing issues and there have been some controversial results. In [39,40], it is argued of the radiations at the universal horizon but *none* at the Killing horizon. In [41], it is shown the opposite results in a more direct calculation, i.e., the radiations at the Killing horizon but none at the universal horizon, which seems to support for our new formulation and some earlier results on Hawking radiations for relativistic matters, like [32,42,43]. Recently, it has been clarified that the disagreements were due to the different choices observer's frames (or vacuum) [44].

[12] One can choose S_0 so that the two minimum horizon radii agree. This choice achieves the third law of black hole thermodynamics, i.e., $\mathcal{S} = 0$ at $T_H = 0$.

[13] This indicates the energy conditions, especially the weak energy conditions (WEC) and the null energy conditions (NEC) are not violated by the higher-derivative contributions to the (effective) energy-momentum tensors. For example, for the asymptotically flat case, i.e., $\Lambda_W = 0$, see [36].

[14] Actually, (52) is exactly the same as that of the Kerr black hole case with the identification $a = 2^{-1}$, whereas (51) as that of Reissner-Norström black hole with $e = 2^{-1}$ in [47] (see [48,49] for some earlier discussions on the similarity).

quite nonsensical because the possible communications between inside and outside of the cosmological horizon from the Lorentz violating effect.

(9) Based on our proposal of the off-shell *Diff* invariance which is not spoiled by covariant matter couplings, the manifestly *Diff* invariant formulation *can be possible* by the change of variables. Its explicit formulation will be interesting because it would be really curious how it differs from the usual covariant higher-curvature gravities. Actually, it reminds us about a covariant formulation of HLD gravity using the Stueckelberg's trick [51]. It would be important to see whether their formulation is equivalent to ours or not.

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APPENDIX A: COMPUTATIONAL DETAILS OF (28)–(31), AND EXPLICIT FORM OF $\mathcal{U}^{ij}$

Here, we describe the computational details of (28)–(31), and the explicit form of $\mathcal{U}^{ij}$.

First, in order to compute $\mathcal{I}_0$ in (28) from (24), without using the dynamical equations of motion $\mathbf{E}^{ij}=0$ nor the constraints $\mathcal{H}\approx 0, \mathcal{H}^i\approx 0$, we first consider the time derivative $\partial_t(\equiv(\dot{}))$ of the potential $\mathcal{V}$, which appears in the term $N\dot{\mathcal{H}}$ in (24),

$$\begin{aligned}\frac{d\mathcal{V}}{dt}&=\frac{\partial\mathcal{V}}{\partial g_{ij}}\partial_t g_{ij}+\frac{\partial\mathcal{V}}{\partial R^i{}_{jkl}}\partial_t R^i{}_{jkl}\\&=\dot{g}_{ij}P^{ilnp}R^j{}_{lnp}-P_i{}^{jkl}\dot{R}^i_{jkl},\end{aligned}\tag{A1}$$

where we have used [15]

$$\left(\frac{\partial\mathcal{V}}{\partial g^{ij}}\right)_{R^m{}_{nkl}}=-P_i{}^{lnp}R_{jlnp}.\tag{A2}$$

Expressing the time derivatives in terms of extrinsic curvature via its definition (2), we have the relation

$$\begin{aligned}\frac{d\mathcal{V}}{dt}&=(2NK_{ij}+\nabla_iN_j+\nabla_jN_i)P^{ilnp}R^j{}_{lnp}-P_i{}^{jkl}\dot{R}^i_{jkl}\\&=(2NK_{ij}+\nabla_iN_j+\nabla_jN_i)P^{ilnp}R^j{}_{lnp}\\&\quad-P_i{}^{jkl}(\nabla_kH^i{}_{jl}-\nabla_lH^i{}_{jk}),\end{aligned}\tag{A3}$$

where we have used a useful relation,

$$\dot{R}^i_{jkl}=\nabla_kH^i{}_{jl}-\nabla_lH^i{}_{jk}\tag{A4}$$

with

$$\begin{aligned}H^l_{ij}&\equiv\nabla_i(NK_j{}^l)+\nabla_j(NK_i{}^l)-\nabla^l(NK_{ij})\\&\quad+\nabla_{(i}\nabla_{j)}N^l-R^l{}_{(ij)}{}^mN_m.\end{aligned}\tag{A5}$$

With all these identities and (16)–(18), we can compute (24) as

$$\mathcal{I}_0=\nabla_i\left\{2N^2\left[\nabla_j\pi^{ij}+\frac{\kappa^2}{2}\left(\frac{2\lambda}{\lambda D-1}(\pi\nabla_lP^{kl}{}_k{}^i-P^{kl}{}_k{}^i\nabla_l\pi)+2P_{jkl}{}^i\nabla^k\pi^{jl}-2\pi^{jl}\nabla^kP_{jkl}{}^i\right)\right]\right\},\tag{A6}$$

$$\equiv\nabla_i\Omega^i,\tag{A7}$$

in terms of the canonical momenta $\pi_{ij}=(2/\kappa^2)\sqrt{g}(K_{ij}-\lambda Kg_{ij})$.

As an explicit example, if we consider the potential (4)

$$\begin{aligned}-\mathcal{V}[g^{ij},R^i{}_{jkl}]&=\Lambda+\xi R+\alpha R^n+\beta(R_{ij}R^{ij})^s+\gamma(R^i{}_{jkl}R_i{}^{jkl})^r\\&=\Lambda+\xi\delta^k_pg^{ql}R^p{}_{qkl}+\alpha(\delta^k_ig^{jl}R^i{}_{jkl})^n+\beta(\delta^k_iR^i{}_{jkl}\delta^q_pR^p{}_{mqn}g^{mj}g^{ln})^s+\gamma(R^i{}_{jkl}R_i{}^{jkl})^r,\end{aligned}\tag{A8}$$

$P_i{}^{jkl}$ is given by

$$P_i{}^{jkl}=\xi\delta_i^{[k}g^{l]j}+\alpha nR^{n-1}\delta_i^{[k}g^{l]j}+\beta s\zeta^{s-1}(\delta_i^{[k}R^{l]j}+g^{j[l}R^{k]}{}_i)+2\gamma r\rho^{r-1}R_i{}^{jkl},\tag{A9}$$

where we denote $\zeta\equiv R_{ij}R^{ij},\rho\equiv R_{ijkl}R^{ijkl}$.

By plugging (A9) into (A6), we obtain

$$
\begin{aligned}
\mathcal{I}_0 &\equiv \nabla_i\Omega^i = \nabla_i[\Omega^i_{(0)} + \tilde{\xi}\Omega^i_{(1)} + \tilde{\alpha}\Omega^i_{(2)} + \tilde{\beta}\Omega^i_{(3)} + \tilde{\gamma}\Omega^i_{(4)}],\\
\Omega^i_{(0)} &= 2N^2\nabla_j\pi^{ij},\\
\Omega^i_{(1)} &= 2N^2\left[\frac{(\lambda-1)}{(\lambda D-1)}\nabla^i\pi - \nabla_j\pi^{ij}\right],\\
\Omega^i_{(2)} &= 2nN^2\left[\frac{(\lambda-1)}{(\lambda D-1)}(R^{n-1}\nabla^i\pi - \pi\nabla^i R^{n-1}) - (R^{n-1}\nabla_j\pi^{ij} - \pi^{ij}\nabla_j R^{n-1})\right],\\
\Omega^i_{(3)} &= 2sN^2\left[\frac{(2\lambda-1)}{(\lambda D-1)}(\zeta^{s-1}R^{ij}\nabla_j\pi - \pi\nabla_j(\zeta^{s-1}R^{ij})) - \frac{\lambda}{(\lambda D-1)}(\zeta^{s-1}R\nabla^i\pi - \pi\nabla^i(\zeta^{s-1}R))\right.\\
&\quad\left. + (\zeta^{s-1}R^{jk}\nabla^i\pi_{jk} - \pi_{jk}\nabla^i(\zeta^{s-1}R^{jk})) - (\zeta^{s-1}R^{ij}\nabla_k\pi_j{}^k - \pi_j{}^k\nabla_k(\zeta^{s-1}R^{ij})) - (\zeta^{s-1}R^{jk}\nabla_k\pi^i{}_j - \pi^i{}_j\nabla_k(\zeta^{s-1}R^{jk}))\right],\\
\Omega^i_{(4)} &= 2rN^2\left[\frac{4\lambda}{\lambda D-1}(\pi\nabla_k(\rho^{r-1}R^{ik}) - \rho^{r-1}R^{ik}\nabla_k\pi) + 4\rho^{r-1}R_{jkl}{}^i\nabla^k\pi^{jl} - 4\pi^{jl}\nabla^k(\rho^{r-1}R_{jkl}{}^i)\right],
\end{aligned} \tag{A10}
$$

where $\zeta \equiv R_{ij}R^{ij}, \rho \equiv R_{ijkl}R^{ijkl}, (\tilde{\xi}, \tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}) \equiv (\kappa^2/2)(\xi, \alpha, \beta, \gamma)$. On the other hand, if we consider $\mathcal{I}_i$ in (25) similarly, one can find that it vanishes identically $\mathcal{I}_i \equiv 0$, which proves (29), as in GR or general covariant theories.

Similarly, if we consider Ψ^0 and Ψ^i in (26), (27), respectively, one can find that, from (19) and (20) as well as (16)–(18),

$$
\begin{aligned}
\Psi^0 &\equiv -\xi^0(N\mathcal{H} + N_i\mathcal{H}^i) - \xi^j g_{ij}\mathcal{H}^i + \Theta^0\\
&= \xi^0\mathcal{L} + \partial_i\left[2\sqrt{g}(\xi^0 N_j + \xi_j)\left(\frac{2}{\kappa^2}\right)G^{ijkl}K_{kl}\right]
\end{aligned} \tag{A11}
$$

and

$$
\begin{aligned}
\Psi^i &\equiv \xi^0[NN^i\mathcal{H} - \mathcal{H}^i(g^{jk}N_jN_k - N^2) + 2N_j\mathcal{H}^{ij}] + \xi^j(-N_j\mathcal{H}^i + 2g_{jl}\mathbf{E}^{il}) + \Theta^i\\
&= \xi^i\mathcal{L} + \mathbf{\Sigma}^i(\xi^0) - \partial_0\left[2\sqrt{g}(\xi^0 N_j + \xi_j)\left(\frac{2}{\kappa^2}\right)G^{ijkl}K_{kl}\right] + \partial_j[\mathcal{A}^{ij}(\xi^0) + \mathcal{B}^{ij}(\xi^m)],
\end{aligned} \tag{A12}
$$

where

$$
\mathbf{\Sigma}^i(\xi^0) = 2N^2\left[\left(\frac{\kappa^2}{2}\right)\left(\frac{2\lambda}{\lambda D-1}(P^{li}{}_{lk}\nabla^k(\xi^0\pi) - \xi^0\pi\nabla^k P^{li}{}_{lk}) + 2\xi^0\pi^{jl}\nabla^k P_{jkl}{}^i - 2P_{jkl}{}^i\nabla^k(\xi^0\pi^{jl})\right) + \pi^{ij}\nabla_j\xi^0 - \xi^0\nabla_j\pi^{ij}\right] \tag{A13}
$$

and $\mathcal{A}^{ij}$, $\mathcal{B}^{ij}$ are antisymmetric tensors as

$$
\begin{aligned}
\mathcal{A}^{ij}(\xi^0) &\equiv 2\sqrt{g}\left[\left(\frac{2}{\kappa^2}\right)2\xi^0 N_m N^{[j}G^{i]mkl}K_{kl} + P^{ijkl}(2\xi^0 N_l\nabla_k N + N\nabla_l(\xi^0 N_k)) + 4\xi^0 NN^l\nabla^k P^{[j}{}_{kl}{}^{i]}\right],\\
\mathcal{B}^{ij}(\xi^m) &\equiv 2\sqrt{g}\left[\left(\frac{2}{\kappa^2}\right)2\xi_m N^{[j}G^{i]mkl}K_{kl} + 4\xi^l\nabla^k(NP^{[j}{}_{kl}{}^{i]}) - 2NP^{[j}{}_{kl}{}^{i]}\nabla^k\xi^l\right].
\end{aligned} \tag{A14}
$$

Then, one can write $\Psi^\mu \equiv \xi^\mu\mathcal{L} + \Sigma^\mu(\xi^0) + \partial_\nu\mathcal{U}^{\mu\nu}$ with $\Sigma^0 \equiv 0$ and the "superpotential" $\mathcal{U}^{\mu\nu}$, which is anti-symmetric $\mathcal{U}^{\mu\nu} = -\mathcal{U}^{\mu\nu}$ [17,18] and given by

$$
\mathcal{U}^{0i} = -\mathcal{U}^{i0} \equiv 2\sqrt{g}(\xi^0 N_j + \xi_j)\left(\frac{2}{\kappa^2}\right)(G^{ijkl}K_{kl}), \tag{A15}
$$

$$\mathcal{U}^{ij} = -\mathcal{U}^{ji} \equiv \mathcal{A}^{ij}(\xi^0) + \mathcal{B}^{ij}(\xi^m), \tag{A16}$$

proving (30) and (31).

If we consider the potential (A8) with $P_i{}^{jkl}$ tensor (A9), as an explicit example, one can find $\mathbf{E}^{ij}$ in (18) as

$$\begin{aligned}
\mathbf{E}^{ij} &\equiv E^{ij}_{(0)} + \xi E^{ij}_{(1)} + \alpha E^{ij}_{(2)} + \beta E^{ij}_{(3)} + \gamma E^{ij}_{(4)},\\
E^{ij}_{(0)} &= \sqrt{g}\left(\frac{2}{\kappa^2}\right)\bigg[-N^i\nabla_k K^{jk} - N^j\nabla_k K^{ik} + K^{ik}\nabla^j N_k + K^{jk}\nabla^i N_k + N^k\nabla_k K^{ij} + 2NK^{ik}K^j{}_k - NKK^{ij} + \frac{1}{2}g^{ij}NK^{kl}K_{kl}\\
&\quad - g^{ik}g^{jl}\dot{K}_{kl}\bigg] + \lambda\sqrt{g}\bigg[\frac{1}{2}Ng^{ij}K^2 + N^j\nabla^i K + N^i\nabla^j K - g^{ij}N^k\nabla_k K - g^{ij}K^{kl}\dot{g}_{kl} + g^{ij}g^{kl}\dot{K}_{kl}\bigg],\\
E^{ij}_{(1)} &= \sqrt{g}\left[N\left(-R^{ij} + \frac{1}{2}Rg^{ij} + \frac{\Lambda}{\xi}g^{ij}\right) + (g^{il}g^{jk} - g^{ij}g^{kl})\nabla_l\nabla_k N\right],\\
E^{ij}_{(2)} &= \sqrt{g}\left[N\left(-nR^{n-1}R^{ij} + \frac{1}{2}R^n g^{ij}\right) + n(g^{il}g^{jk} - g^{ij}g^{kl})\nabla_l\nabla_k(NR^{n-1})\right],\\
E^{ij}_{(3)} &= \sqrt{g}\left[N\left(-2s\zeta^{s-1}R^{ik}R^j{}_k + \frac{1}{2}\zeta^s g^{ij}\right) + s(g^{ik}g^{jm}g^{ln} + g^{jk}g^{mi}g^{nl} - g^{kl}g^{mi}g^{nj} - g^{ij}g^{km}g^{ln})\nabla_l\nabla_k(N\zeta^{s-1}R_{mn})\right],\\
E^{ij}_{(4)} &= \sqrt{g}\left[N\left(-2r\rho^{r-1}R^{iklm}R^j{}_{klm} + \frac{1}{2}\rho^r g^{ij}\right) + 4r\nabla_k\nabla_l(\rho^{r-1}NR^{iklj})\right],
\end{aligned} \tag{A17}$$

and Θ^i in (20) as

$$\begin{aligned}
\Theta^i &\equiv \Theta^i_{(0)} + \xi\Theta^i_{(1)} + \alpha\Theta^i_{(2)} + \beta\Theta^i_{(3)} + \gamma\Theta^i_{(4)},\\
\Theta^i_{(0)} &= \sqrt{g}\left(\frac{2}{\kappa^2}\right)[2N^l G^{ijkm}K_{km}\delta g_{jl} - N^i G^{ljmn}K_{mn}\delta g_{jl} - 2G^{kjil}K_{kj}\delta N_l],\\
\Theta^i_{(1)} &= 2\sqrt{g}g^{j[k}g^{l]i}[N\nabla_k\delta g_{lj} - (\nabla_k N)\delta g_{lj}],\\
\Theta^i_{(2)} &= 4n\sqrt{g}[Ng^{i[l}g^{k]j}R^{n-1}\nabla_k\delta g_{lj} - \nabla_k(Ng^{i[l}g^{k]j}R^{n-1})\delta g_{lj}],\\
\Theta^i_{(3)} &= 4s\sqrt{g}[N\zeta^{s-1}g_{[l}{}^{[i}R^{j]}{}_{k]}\nabla^k\delta g^l{}_j - \nabla^k(N\zeta^{s-1}g_{[l}{}^{[i}R^{j]}{}_{k]})\delta g^l{}_j],\\
\Theta^i_{(4)} &= 4r\sqrt{g}\rho^{r-1}R^{jkil}N\nabla_k\delta g_{lj} - 4r\sqrt{g}\delta g_{lj}\nabla_k(\rho^{r-1}NR^{jikl}).
\end{aligned} \tag{A18}$$

For $\mathbf{\Sigma}^i$ in (32), we find

$$\begin{aligned}
\mathbf{\Sigma}^i(\xi^0) &\equiv \Sigma^i_{(0)} + \tilde{\xi}\Sigma^i_{(1)} + \tilde{\alpha}\Sigma^i_{(2)} + \tilde{\beta}\Sigma^i_{(3)} + \tilde{\gamma}\Sigma^i_{(4)},\\
\Sigma^i_{(0)} &= 2N^2\left(\frac{2}{\kappa^2}\right)[-\xi^0\nabla_i\pi^{ij} + \pi^i{}_j\nabla^j\xi^0],\\
\Sigma^i_{(1)} &= 2N^2[-\hat{\lambda}\nabla^i(\xi^0\pi) + \xi^0\nabla_j\pi^{ij} + \pi^i{}_j\nabla^j\xi^0],\\
\Sigma^i_{(2)} &= 2nN^2[-\hat{\lambda}(R^{n-1}\nabla^i(\xi^0\pi) - \xi^0\pi\nabla^i R^{n-1})],\\
\Sigma^i_{(3)} &= 2sN^2[\tilde{\lambda}(\zeta^{s-1}R\nabla^i(\xi^0\pi) - \xi^0\pi\nabla^i(R\zeta^{s-1})) - \bar{\lambda}(\zeta^{s-1}R^{ij}\nabla_j(\xi^0\pi) - \xi^0\pi\nabla_j(\zeta^{s-1}R^{ij})) + (\xi^0\pi^{jk}\nabla^i(\zeta^{s-1}R_{jk})\\
&\quad - \zeta^{s-1}R_{jk}\nabla^i(\xi^0\pi^{jk})) + (\zeta^{s-1}R_j{}^k\nabla^j(\xi^0\pi^i{}_k) - \xi^0\pi^i{}_k\nabla^j(\zeta^{s-1}R_j{}^k)) + (\zeta^{s-1}R^{ij}\nabla_k(\xi^0\pi_j{}^k) - \xi^0\pi_j{}^k\nabla_k(\zeta^{s-1}R^{ij}))],\\
\Sigma^i_{(4)} &= 2rN^2[4\tilde{\lambda}(\rho^{r-1}R^{ij}\nabla_j(\xi^0\pi) - \xi^0\pi\nabla_j(\rho^{r-1}R^{ij})) + 4(\rho^{r-1}R^i{}_{jkl}\nabla^l(\xi^0\pi^{jk}) - \xi^0\pi^{jk}\nabla^l(\rho^{r-1}R^i{}_{jkl}))],
\end{aligned} \tag{A19}$$

where $\hat{\lambda} \equiv (\lambda - 1)/(\lambda D - 1), \tilde{\lambda} \equiv \lambda/(\lambda D - 1), \bar{\lambda} \equiv \hat{\lambda} + \tilde{\lambda} = (2\lambda - 1)/(\lambda D - 1)$, and $\mathcal{A}^{ij}$, $\mathcal{B}^{ij}$ as

$$
\begin{aligned}
\mathcal{A}^{ij}(\xi^0) &\equiv \mathcal{A}^{ij}_{(0)} + \xi\mathcal{A}^{ij}_{(1)} + \alpha\mathcal{A}^{ij}_{(2)} + \beta\mathcal{A}^{ij}_{(3)} + \gamma\mathcal{A}^{ij}_{(4)},\\
\mathcal{A}^{ij}_{(0)} &= 2\sqrt{g}\left(\frac{2}{\kappa^2}\right)[2\xi^0 N_m N^{[j}G^{i]mkl}K_{kl}],\\
\mathcal{A}^{ij}_{(1)} &= 2\sqrt{g}[g^{i[k}g^{l]j}(2\xi^0 N_l\nabla_k N + N\nabla_l(\xi^0 N_k))],\\
\mathcal{A}^{ij}_{(2)} &= 2n\sqrt{g}[R^{n-1}g^{i[k}g^{l]j}(2\xi^0 N_l\nabla_k N + N\nabla_l(\xi^0 N_k)) + 4\xi^0 N N_l g^{i[l}g^{k]j}\nabla_k R^{n-1}],\\
\mathcal{A}^{ij}_{(3)} &= 2s\sqrt{g}[2s\zeta^{s-1}g^{[j}_{[l}R^{i]}_{k]}(2\xi^0 N^l\nabla^k N + N\nabla^l(\xi^0 N^k)) + 8\xi^0 N N^l g^{[i}_{[l}\nabla^{|k|}(\zeta^{s-1}R^{j]}_{k]})],\\
\mathcal{A}^{ij}_{(4)} &= 2r\sqrt{g}[8\xi^0 N N^l\nabla^k(\rho^{r-1}R^{[j}{}_{kl}{}^{i]}) + 2\rho^{r-1}R^{ijkl}(2\xi^0 N_l\nabla_k N + N\nabla_l(\xi^0 N_k))],
\end{aligned}
\tag{A20}
$$

and

$$
\begin{aligned}
\mathcal{B}^{ij}(\xi^m) &\equiv \mathcal{B}^{ij}_{(0)} + \xi\mathcal{B}^{ij}_{(1)} + \alpha\mathcal{B}^{ij}_{(2)} + \beta\mathcal{B}^{ij}_{(3)} + \gamma\mathcal{B}^{ij}_{(4)},\\
\mathcal{B}^{ij}_{(0)} &= 2\sqrt{g}\left(\frac{2}{\kappa^2}\right)[2\xi_m N^{[j}G^{i]mkl}K_{kl}],\\
\mathcal{B}^{ij}_{(1)} &= 2\sqrt{g}[2g^{l[i}g^{j]}{}_k(N\nabla^k\xi_l - 2\xi_l\nabla^k N)],\\
\mathcal{B}^{ij}_{(2)} &= 2n\sqrt{g}[2g^{l[i}g^{j]}{}_k R^{n-1}N\nabla^k\xi_l + 4\xi_l\nabla^k(g^{l[j}g^{i]}{}_k R^{n-1}N)],\\
\mathcal{B}^{ij}_{(3)} &= 2s\sqrt{g}[4\zeta^{s-1}g^{[j[k}R^{i]l]}N\nabla_k\xi_l + 8\xi_l\nabla_k(\zeta^{s-1}g^{[j[k}R^{i]l]}N)],\\
\mathcal{B}^{ij}_{(4)} &= 2r\sqrt{g}[4\rho^{r-1}R^{k[ij]l}N\nabla_k\xi_l + 8\xi_l\nabla_k(\rho^{r-1}R^{k[ji]l}N)].
\end{aligned}
\tag{A21}
$$

APPENDIX B: A SUMMARY OF THE COMPLETE SET OF CONSTRAINTS OBTAINED IN [8] ($\lambda \neq 1/D$)

Here, we summarize the complete set of constraints for the Hamiltonian formalism, obtained in [8] ($\lambda \neq 1/D$) (see [8] for the detailed computations and the case of $\lambda = 1/D$). In [8], we have considered the HLD action (1) with a potential $\mathcal{V}(R)$, which is an arbitrary function of curvature scalar R, or more explicitly, $-\mathcal{V} = \Lambda + \xi R + \alpha R^n$, for the computational simplicity. Then, from the *primary* constraints

$$\pi_N \equiv \delta S/\delta\dot{N} \approx 0, \qquad \pi^i \equiv \delta S/\delta\dot{N}_i \approx 0 \tag{B1}$$

and their preservation $\dot{\Phi}^\mu = \{\Phi^\mu, H_C\} \approx 0$ [$\Phi^\mu \equiv (\pi_N, \pi^i)$] with the canonical Hamiltonian (up to boundary terms)

$$H_C = \int_{\Sigma_t} d^D x\{N\mathcal{H} + N_i\mathcal{H}^i\}, \tag{B2}$$

one obtain the *secondary* constraints

$$\mathcal{H} \approx 0, \qquad \mathcal{H}^i \approx 0, \tag{B3}$$

which are the Hamiltonian and momentum constraints, for $\mathcal{H}$ and $\mathcal{H}^i$ in (16) and (17), respectively. Here, the weak equality '$\approx$' means that the constraint equations are used after calculating the Poisson brackets. Up to now, the constraint analysis looks parallel with GR but now with the modified expression of $\mathcal{H}$ as (16). However, we note that $\mathcal{H}^i$ has still the same expression as in GR in terms of canonical momenta $\pi^{ij} = (2/\kappa^2)\sqrt{g}(K^{ij} - \lambda K g^{ij})$, though different from in terms of g^{ij} and $\dot{g}^{ij}$, or K^{ij}. This fact indicates the same role of $\mathcal{H}^i$ as in GR in Hamiltonian formalism, due to the fundamental role of π^{ij}.

The fundamentally different constraint analysis in HLD gravity than in GR starts from the different constraint algebra

$$\{\mathcal{H}(x), \mathcal{H}(y)\} = C^i(x)\nabla^x_i\delta^D(x-y) - C^i(y)\nabla^y_i\delta^D(x-y), \tag{B4}$$

$$\{\mathcal{H}(x), \mathcal{H}_i(y)\} = -\mathcal{H}(y)\nabla^y_i\delta^D(x-y), \tag{B5}$$

$$\{\mathcal{H}_i(x), \mathcal{H}_j(y)\} = \mathcal{H}_i(y)\nabla^x_j\delta^D(x-y) + \mathcal{H}_j(x)\nabla^x_i\delta^D(x-y), \tag{B6}$$

where C^i is the same quantity in (38) but with $\beta = \gamma = 0$, as in $\mathcal{H}$ (16). Then, the preservation of the secondary constraints with the *extended* Hamiltonian $H_E = H_C + \int_{\Sigma_t} d^D x(u_\mu\Phi^\mu)$, with the Lagrange multipliers u_μ due to the arbitrariness from the primary constraints,

$$
\begin{aligned}
\dot{\mathcal{H}} &\equiv \{\mathcal{H}, H_E\}\\
&= \frac{1}{N}\nabla_i(N^2 C^i) + \nabla_i(N^i\mathcal{H}) \approx 0,
\end{aligned}
\tag{B7}
$$

$$\dot{\mathcal{H}}_i \equiv \{\mathcal{H}_i, H_E\} = \mathcal{H}\nabla_i N + \nabla_j(N^j\mathcal{H}_i) + \mathcal{H}_j\nabla_i N^j \approx 0 \quad \text{(B8)}$$

produce the *tertiary* constraint,

$$\Omega \equiv \nabla_i(N^2C^i) + N\nabla_i(N^i\mathcal{H}^t) \approx 0, \quad \text{(B9)}$$

excluding the trivial case of $N = 0$ for all space-time.

One more step of preserving the tertiary constraint gives

$$\dot{\Omega} \equiv \{\Omega, H_E\} \approx \{\Omega, H_C\} + 2C^iN^2\nabla_i\left(\frac{u_t}{N}\right) \approx 0. \quad \text{(B10)}$$

Now, the remaining further analysis depends on whether $C^i \approx 0$ or $C^i \not\approx 0$.

1. Case $C^i \approx 0$

In this case, the multiplier u_t is not determined in (B10) but we need one more step with a further constraint $\Xi \equiv \{\Omega, H_C\}$ (see [8] for the explicit expression and the more details). Then, the full set of constraints is given by $\chi_A \equiv (\pi_N, \mathcal{H}, \Omega, \Xi) \approx 0, \Gamma_B \equiv (\pi^i, \mathcal{H}_i) \approx 0$. Here, the constraints $\chi_A \approx 0$ are the *second-class* constraints with the constraint algebra,

$$\begin{aligned}
&\{\pi_N(x), \mathcal{H}(y)\} = 0,\\
&\{\pi_N(x), \Omega(y)\} \approx -2\nabla_i^y(NC^i(y)\delta^D(x-y)) \approx 0,\\
&\{\pi_N(x), \tilde{\Xi}(y)\} = \Delta(x-y),\\
&\{\mathcal{H}(x), \mathcal{H}(y)\} = C^i(x)\nabla_i^x\delta^D(x-y) - C^i(y)\nabla_i^y\delta^D(x-y) \approx 0,\\
&\{\mathcal{H}(x), \tilde{\Omega}(y)\} \approx \{\pi_N(x), \Xi^i(y)\}, \text{etc.}, \quad \text{(B11)}
\end{aligned}$$

whose determinant $det(\{\chi_A, \chi_B\})$ is nonvanishing generally.

On the other hand, the constraints $\Gamma_A \equiv (\pi^i, \mathcal{H}_i) \approx 0$ are the *first-class* constraints with the vanishing determinant $det(\{\Gamma_A, \Gamma_B\}) = 0$. Then, the number of dynamical degrees of freedom in the "configuration" space is given by

$$s = \frac{1}{2}(P - 2N_1 - N_2) = \frac{1}{2}(D+1)(D-2), \quad \text{(B12)}$$

where $P = (D+1)(D+2)$ is the number of canonical variables in "phase" space $(N, \pi_N, N_i, \pi^i, g_{ij}, \pi_{ij})$, $N_1 = 2D$ is the number of the first-class constraints $(\pi^i, \mathcal{H}_i) \approx 0$, and $N_2 =$ "4" is the number of the second-class constraints $(\pi_N, \mathcal{H}, \Omega, \Xi) \approx 0$. It is remarkable that the *2 first-class* constraints $(\pi_N, \mathcal{H}) \approx 0$ in GR transform into the *4 second-class* constraints $(\pi_N, \mathcal{H}, \Omega, \Xi) \approx 0$ in the Case A of HLD gravity, maintaining the same dynamical degrees of freedom s.

2. Case $C^i \not\approx 0$

In this case, the multiplier u_t is now determined in (B10) and there is no further constraint. Then, in contrast to Case A, there are the second-class constraints $\chi_A \equiv (\pi_N, \mathcal{H}, \Omega) \approx 0$, whose determinant $\det(\{\chi_A, \chi_B\})$ is generally nonvanishing, whereas the first-class constraints $\Gamma_A \equiv (\pi^i, \mathcal{H}_i)$ are the same as in the Case A. So, the number of dynamical degrees of freedom is

$$s = \frac{1}{2}[(D+1)(D+2) - 2 \times 2D - \text{“3”}] = \frac{1}{2}(D+1)(D-2) + \frac{1}{2}, \quad \text{(B13)}$$

with $N_1 = 2D$ and $N_2 =$ "3", which shows *one-extra* degree of freedom in phase space, in addition to the usual $(D+1)(D-2)$ graviton (transverse traceless) modes in GR or the Case A of HLD gravity in arbitrary $(D+1)$ dimensions. This result supports the previous case-by-case results [28,31] but in a more generic setup.

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Correction: Errors in Eqs. (5), (11), and (41) and in an inline equation in the third sentence following Eq. (40) have been fixed. The text following Eq. (41) has been modified to add missing text and correct an error in an inline equation.