Bañados-Silk-West effect with finite forces near different types of horizons: General classification of scenarios

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If two particles move toward a black hole and collide in the vicinity of the horizon, under certain conditions, their energy $E_{\rm c.m.}$ in the center of mass frame can grow unbounded. This is the Bañados-Silk-West (BSW) effect. Usually, this effect is considered for extremal horizons and geodesic (or electrogedesic) trajectories. We study this effect in a more general context, when both geometric and dynamic factors are taken into account. We consider generic axially symmetric rotating black holes. The near-horizon behavior of metric coefficients is determined by three numbers p, q, k that appear in the Taylor expansions for different types of a horizon. This includes nonextremal, extremal, and ultraextremal horizons. We also give general classification of possible trajectories that include so-called usual, subcritical, critical, and ultracritical ones depending on the near-horizon behavior of the radial component of the four-velocity. We assume that particles move not freely but under the action of some unspecified force. We find when the finiteness of a force and the BSW effect are compatible with each other. The BSW effect implies that one of two particles has fine-tuned parameters. We show that such a particle always requires an infinite proper time for reaching the horizon. Otherwise, either a force becomes infinite, or a horizon fails to be regular. This realizes the so-called principle of kinematic censorship that forbids literally infinite $E_{\rm c.m.}$ in any act of collision. The obtained general results are illustrated for the Kerr-Newman-(anti-)de Sitter metric used as an example. The description of diversity of trajectories suggested in our work can be of use also in other contexts, beyond the BSW effect. In particular, we find the relation between a force and the type of a trajectory.

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I. INTRODUCTION

The Bañados-Silk-West effect (BSW, after the names of its authors) [1] is one of the most interesting theoretical results in black hole physics during the last decade. It also revived interest to previous versions of high energy collisions near black holes [2–4]. Let two particles collide in the vicinity of a rotating black hole. Then, under certain conditions, an indefinitely growth of the energy in the center of mass frame $E_{\rm c.m.}$ becomes possible. This effect was found for (i) extremal horizons and (ii) free particles. Some objections against the BSW effect [5,6] were connected with failure of the factors (i), (ii), or both. However, it was shown later, that under some change of conditions, the BSW effect survives even for nonextremal black holes [7]. Moreover, it was shown earlier that the BSW effect arises due to the presence of the horizon as such, no matter how its explicit metric looks like [8]. In the present work, we make the next step and consider generic horizons including nonextremal, extremal, and utraextremal ones (more explicit definitions will be done in the text below). For spherically symmetric space-times, there exists their direct classification that enables us to distinguish between true regular horizons, lightlike singularities and so-called naked and truly naked horizons (see [9] and references therein). For generic axially symmetric rotating black holes, classification is much more complicated. The conditions that single out standard regular horizons (which we restrict ourselves by) were described in [10].

Also, the presence of a force can be, in principle, compatible with the BSW effect. For a particular case of extremal horizons, this was shown in [11]. Strong arguments in favor of this effect for nonextremal horizons were suggested in [12]. Moreover, sometimes it leads to another version of this effect that is absent without a force [13].

Instead of solving the equations of motion (that, as a rule, is practically impossible), we choose the near-horizon behavior of trajectories and find for each type of a horizon, when (i) the acceleration due to a force remains finite near

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the horizon, and, at the same time (ii) the BSW effect is allowed. For particles moving in the equatorial plane toward a black hole and experiencing finite forces, we build a general theory of the BSW effect. In doing so, we take into account factors connected with geometry (type of a horizon), kinematics (classification of trajectories), and dynamics (allowed behavior of a force).

One important aspect deserves separate attention. Although $E_{\text{c.m.}}$ can be made as large as one likes, if the BSW effect is present, its value must remain finite in each act of collision, so an infinite energy is forbidden. This statement is formulated as a separate principle of kinematic censorship [14]. As far as the BSW effect with free moving particles is concerned, it implies that if one of two colliding particles has fine-tuned parameters, then the proper time required to reach the extremal horizon is infinite [6]. Thus, collision occurs closely to the horizon but not exactly on it so that $E_{\text{c.m.}}$ remains finite, although arbitrarily large. We show how this principle manifests itself for more general types of horizons and the presence of a nonzero force.

The paper is organized as follows. In Sec. II, we write the general form of the metric under discussion and equations of particle motion under the action of a nonzero force. In Sec. III, we suggest classification of trajectories depending on the near-horizon behavior of the radial component of the four-velocity. In this way, we introduce notions of usual, subcritical, critical, and ultracritical particles. In Sec. IV, we establish main features of different types of trajectories in the vicinity of the black hole horizon. In Sec. V, we give the basic formulas for the gamma factor of relative motion of two particles relevant in the context of the BSW effect. We enumerate different possible combinations of types of both particles that produce the BSW effect. In Sec. VI, we list general expressions for the components of acceleration for equatorial particle motion. In Sec. VII, we establish the relations between the type of trajectory, acceleration, and characteristics of near-horizon metric. We derive the conditions when the corresponding force is finite for fine-tuned particles. In Sec. X, we derive, for completeness, similar conditions for usual particles, although this is irrelevant for the conditions of the BSW effect. In Sec. VIII, we collect our results about conditions when a force remains finite near the horizon for fine-tuned particles and different kinds of horizons. In Sec. IX, we prove the validity of the principle of kinematic censorship for the system under discussion. Then, in Sec. XI, we check the validity of our results using the Kerr-Newman-(anti-)de Sitter metric as an exactly solvable example. In Sec. XIII, we give the summary of the results obtained in this work.

II. METRIC AND EQUATIONS OF MOTION

We investigate the motion of particles in the background of a rotating black hole described in generalized Boyer-Lindquist coordinates by the metric

$$ds^2 = -N^2 dt^2 + g_{\varphi\varphi}(dt - \omega d\varphi)^2 + \frac{dr^2}{A} + g_{\theta\theta} d\theta^2. \tag{1}$$

All metric coefficients do not depend on t and φ. Positions of horizons are defined by the conditions $A(r_h) = N(r_h) = 0$, where r_h is the horizon radius.

The BSW phenomenon supposes that the energy $E_{\text{c.m.}}$ in center of mass frame of two colliding particles infinitely grows as the point of collision approaches the black hole horizon. For the extremal horizon, the parameters of one of particles (so-called critical) should be fine-tuned, the other particle being not fine-tuned (usual) [1,8]. Meanwhile, for more general types of the horizon, the situation can be more involved, as we will see it below. Choosing a general type of a trajectory, we relate it to the properties of the horizon and will see how the near-horizon behavior of acceleration looks like.

Let ξ^μ and η^μ be the Killing vectors responsible for time translation and rotation around the axis, respectively. Then, one can introduce the energy $E = -mu_\mu \xi^\mu = -mu_t$ and angular momentum $L = mu_\mu \eta^\mu = mu_\varphi$, where u^μ is the four-velocity, m being a particle's mass. It follows that along the particle trajectory, the derivative with respect to the proper time τ gives us

$$\frac{d\varepsilon}{d\tau} = -a_\mu \xi^\mu, \tag{2}$$

$$\frac{d\mathcal{L}}{d\tau} = a_\mu \eta^\mu, \tag{3}$$

where the four-acceleration

$$a_\mu = u_{\mu;\nu} u^\nu, \tag{4}$$

and the semicolon denotes covariant derivative, $\varepsilon = \frac{E}{m}$, $\mathcal{L} = \frac{L}{m}$. For a free particle, $a_\mu = 0$, and the energy and angular momentum are conserved.

Hereafter, we assume that the metric possesses a symmetry with respect to the equatorial plane $\theta = \frac{\pi}{2}$ and restrict ourselves by particle motion in this plane. Then, using the definitions of ε and $\mathcal{L}$ and the normalization condition $g_{\mu\nu} u^\mu u^\nu = -1$, one can find that

$$u^t = \frac{\mathcal{X}}{N^2}, \tag{5}$$

$$u^\varphi = \frac{\mathcal{L}}{g_{\varphi\varphi}} + \frac{\omega \mathcal{X}}{N^2}, \tag{6}$$

$$u^r = \sigma \frac{\sqrt{A}}{N} P, \tag{7}$$

$$P = \sqrt{\mathcal{X}^2 - N^2 \left(1 + \frac{\mathcal{L}^2}{g_{\varphi\varphi}}\right)}, \tag{8}$$

$$\mathcal{X} = \varepsilon - \omega\mathcal{L}, \tag{9}$$

where $\sigma = \pm 1$ depending on the direction of motion. As in our work, we restrict ourselves by motion within the equatorial plane, the component $u^\theta = 0$.

If $a_\mu \neq 0$, ε and $\mathcal{L}$ are not conserved, but Eqs. (5)–(7) hold true anyway.

It follows from Eqs. (5)–(7) that in coordinates (t, φ, r, θ),

$$u^\mu = \left(\frac{\mathcal{X}}{N^2}, \frac{\mathcal{L}}{g_\varphi} + \frac{\omega\mathcal{X}}{N^2}, \sigma\frac{\sqrt{A}}{N}P, 0\right). \tag{10}$$

In what follows, we will use, along with the coordinate components of vectors, also their tetrad components. It is convenient to introduce the tetrad attached to the so-called zero angular momentum observers (ZAMO) according to [15]. This tetrad reads in our coordinates

$$e^{(0)}_\mu = N(1,0,0,0), \qquad e^{(1)}_\mu = \sqrt{g_{\varphi\varphi}}(-\omega, 1, 0, 0), \tag{11}$$

$$e^{(2)}_\mu = \frac{1}{\sqrt{A}}(0,0,1,0), \qquad e^{(3)}_\mu = \sqrt{g_{\theta\theta}}(0,0,0,1). \tag{12}$$

We will use a letter "O" (orbital) to call them OZAMO to stress that for such an observer, $r = \text{const}$. Trajectories of this kind of observers are, in general, not geodesics in contrast to free-falling observers with a zero angular momentum (FZAMO).

III. FOUR-VELOCITIES AND CLASSIFICATION OF TRAJECTORIES

Hereafter, we will use the following classification of particles (trajectories) depending on the near-horizon behavior of u^r. Let $N \to 0$, $A \to 0$. Then, we call a particle usual if $|u^r| \approx \frac{\sqrt{A}}{N}$, subcritical if $|u^r|$ changes slower than $\sqrt{A}$ but faster than $\frac{\sqrt{A}}{N}$, critical if $|u^r| \approx \sqrt{A}$, and ultracritical if $|u^r|$ changes faster than $\sqrt{A}$.

The standard approach to investigation of particle trajectories consists in study, how the presence of external forces affects particle dynamics. Instead of solving this problem, we proceed in the opposite direction: We set the near-horizon trajectory, calculate acceleration, and elucidate when it is finite. Further, we select the trajectories that give simultaneously (i) finite acceleration and (ii) divergent $E_{\text{c.m.}}$.

Afterward, we are left with the angular component u^φ and the time one u^t. It is seen from (6) that in the rotational background (1), the angular component of velocity consists of two terms. The first one is due to the angular momentum and is related to rotation itself; the second term appears due to frame dragging. The second term is divergent near the horizon, so it is more natural to define the angular component of the four-velocity in the OZAMO frame (11):

$$u^\varphi_o = e^{(1)}_\mu u^\mu = \sqrt{g_{\varphi\varphi}}(u^\varphi - \omega u^t) = \frac{\mathcal{L}}{\sqrt{g_{\varphi\varphi}}}, \tag{13}$$

which is free from this divergence.

The time component u^t is given by Eq. (5) and can be also rewritten in another quite convenient form in terms of u^r and u^φ_o. It follows from the normalization condition and (7) that

$$u^t = \frac{1}{N}\sqrt{1 + \frac{(u^r)^2}{A} + (u^\varphi_o)^2}. \tag{14}$$

According to (5), $\mathcal{X} = u^t N^2$. Combining this with (14) and taking into account that $u^\varphi_o = O(1)$, we see that $\mathcal{X}_H \neq 0$ for usual particles and $\mathcal{X}_H = 0$ for other types (subcritical, critical, and ultracritical). Hereafter, subscript "H" denotes the quantities calculated on the horizon.

It also follows from our classification that near the horizon,

$$\begin{aligned} u^t &\approx \frac{1}{N^2} \text{ for usual particles,} \\ u^t &\approx \frac{u^r}{N\sqrt{A}} \text{ for subcritical ones,} \\ u^t &\approx \frac{1}{N} \text{ for critical and ultracritical.} \end{aligned} \tag{15}$$

Traditionally, the classification of the trajectories is based on the near-horizon behavior of $\mathcal{X}$, while properties of u^r are derived from this as consequences. Such an approach is convenient when dealing with usual and critical particles. However, as we are going to analyze more subtle details of trajectories and include into consideration subcritical and ultracritical ones, the reverse method (from properties of u^r to those of $\mathcal{X}$) is more convenient, as we will see it below. In principle, both approaches are equivalent to each other.

Using our classification and Eqs. (7) and (8), we can derive important consequences for the relation between u^r and $\mathcal{X}$ near the horizon. Namely, for usual and subcritical particles,

$$P \approx \mathcal{X} - \frac{N^2}{2\mathcal{X}}\left(1 + \frac{\mathcal{L}^2}{g_{\varphi\varphi}}\right)_H, \qquad |u^r| \approx \frac{\sqrt{A}}{N}\mathcal{X}, \tag{16}$$

and for critical ones,

$$\mathcal{X} \approx X_1 N, \;\; P \approx P_N N, \;\; |u^r| \approx P_1\sqrt{A} = \frac{P_1}{X_1}\frac{\sqrt{A}}{N}\mathcal{X}, \tag{17}$$

where X_1 and $P_1 = \sqrt{X_1^2 - (1 + \frac{\mathcal{L}^2}{g_{\varphi\varphi}})_H}$ are constants.

For ultracritical particles,

$$\mathcal{X} \approx \left(1+\frac{\mathcal{L}^2}{g_{\varphi\varphi}}\right)_H N, \qquad P \approx P_\delta N^{1+\delta}, \quad \delta > 0, \tag{18}$$

where P_δ is some another constant,

$$|u^r| \approx P_\delta \sqrt{A} N^\delta. \tag{19}$$

Thus, we see that for all particles, except from ultracritical ones, $|u^r|$ has the order $\frac{\sqrt{A}}{N}\mathcal{X}$. For ultracritical particles, $|u^r| \ll \frac{\sqrt{A}}{N}\mathcal{X}$.

IV. BEHAVIOR OF VELOCITY NEAR THE HORIZON

As the BSW effect happens near the horizon, we will focus on the behavior of accelerations and velocities in its vicinity. The situation depends strongly on the type of a horizon. The classification of the horizons is based on a character of the behavior of geometrical entities in a free-falling frame. Explicitly, it reveals itself in the type of the near-horizon expansion of the metric coefficients. Let us write them in a general form ($v = r - r_h$):

$$N^2 = \kappa_p v^p + o(v^p), \qquad A = A_q v^q + o(v^q), \tag{20}$$

$$\omega = \omega_H + \hat{\omega}_k v^k + \cdots + \hat{\omega}_{l-1} v^{l-1} + \omega_l(\theta) v^l + o(v^l), \tag{21}$$

$$g_a = g_{aH} + g_{a1} v + o(v). \tag{22}$$

Here, $a = \theta, \varphi$; the hat means that corresponding quantity does not depend on θ. It is assumed that p, q, k are some positive numbers. By definition, if $p = q = 1$, the horizon is nonextremal. If $p \geq 2$ and $q = 2$, it is extremal. For $q > 2$, it is called ultraextremal. For nonextremal horizons, the surface gravity is not equal to zero; for extremal and ultraextremal ones, it is zero. For more details, see [10].

We analyze behavior of accelerations for any type of horizon, so q, p, and k are arbitrary. According to the results, obtained in [10], the regularity of a horizon requires that

$$k \geq \left[\frac{p-q+3}{2}\right], \quad l \geq p, \tag{23}$$

where $[x]$ means integer part of x. In what follows, we tacitly assume that these and other conditions of regularity [10] are fulfilled.

These expansions allow us to obtain behavior of $\mathcal{X}$. To this end, we consider a general behavior of the radial velocity near the horizon in the form

$$u^r = (u^r)_c v^c + o(v^c). \tag{24}$$

Near the horizon,

$$\mathcal{X} \approx \mathcal{X}_s v^s, \tag{25}$$

where $s = 0$ for usual particles, and $s > 0$ in other cases.

It follows from (15) and (25) that for subcritical particles,

$$\mathcal{X} \approx \frac{N}{\sqrt{A}}|u^r| \to s = \frac{p-q}{2} + c. \tag{26}$$

For critical and ultracritical particles, $s = p/2$. However, for ultracritical particles, there exists another restriction. To see it, let us consider the radial component of the four-velocity. For the ultracritical particle, we require that

$$(u^r)^2 = \frac{A}{N^2}\left(\mathcal{X}^2 - N^2\left(1+\frac{\mathcal{L}^2}{g_{\varphi\varphi}}\right)\right) \sim v^{2c}, \tag{27}$$

where $c > \frac{q}{2}$.

Such a behavior of u^r can be obtained only if we impose additional restrictions on $\mathcal{X}^2$. It has to be equal to the second expression inside the radical (8) up to the corrections of a higher order:

$$\mathcal{X}^2 = N^2\left(1+\frac{\mathcal{L}^2}{g_{\varphi\varphi}}\right) \text{plus } v^{2c+p-q} \text{ terms.} \tag{28}$$

For the angular component of the four-velocity, we can write another near-horizon expansion:

$$u^\varphi_O = (u^\varphi_O)_H + (u^\varphi_O)_b v^b + o(v^b), \tag{29}$$

where $b > 0$.

Then, it follows from (13) that

$$\mathcal{L} \approx \sqrt{g_{\varphi\varphi}}(u^\varphi_O)_H + \sqrt{g_{\varphi\varphi}}(u^\varphi_O)_b v^b. \tag{30}$$

In a similar way, we can write

$$u^t \sim v^{-\beta}, \tag{31}$$

where for usual particles, $\beta = p$, for subcritical ones, $\beta = \frac{p+q}{2} - c$, and for critical and ultracritical, $b = \frac{p}{2}$.

It is convenient to summarize the above results in a Table I.

In this context, it is also interesting to discuss behavior of a proper time near the horizon. Using the definition of the radial component of four-velocity, we have

$$\tau = \int \frac{dr}{u^r} \sim \int v^{-c} dv. \tag{32}$$

Thus, we see that if $c = 1$, then $\tau \sim |\ln v|$, and if $c \neq 1$, $\tau \approx v^{-\alpha}$, where $\alpha = c - 1$. If $c = 1$, so $\alpha = 0$, the proper time diverges logarithmically, $\tau \sim |\ln v|$. This is the case

TABLE I. Characteristics of the near-horizon behavior of u^r, u^t, $\mathcal{X}$, and the proper time τ. Here, the proper time changes as $\tau \sim v^{-\alpha}$. The value $\alpha = 0$ means that the proper time logarithmically diverges $\tau \sim |\ln v|$.

	Type	c	β	s	$\alpha = c - 1$
1	Usual	$\frac{q-p}{2}$	p	0	$\frac{q-p-2}{2}$
2	Subcritical	$\frac{q-p}{2} < c < \frac{q}{2}$	$\frac{p+q}{2} - c$	$\frac{p-q}{2} + c$, $0 < s < \frac{p}{2}$	$\frac{q-p-2}{2} < \alpha < \frac{q-2}{2}$
3	Critical	$\frac{q}{2}$	$\frac{p}{2}$	$\frac{p}{2}$	$\frac{q-2}{2}$
4	Ultracritical	$c > \frac{q}{2}$	$\frac{p}{2}$	$\frac{p}{2}$	$\alpha > \frac{q-2}{2}$

considered in [1,6]. The case $c = 3/2$, $\alpha = \frac{1}{2}$ corresponds to so-called critical particles of class II considered for the Kerr metric in [16]. Similar solutions for the extremal Kerr-Newman metric are discussed in [17]. For equatorial motion, the proper time for fine-tuned particles in more general background is considered in [18].

The proper time is finite if $c < 1$. As for all trajectories that we are considering, $\frac{q-p}{2} \leq c$ (Table I), the proper time may be finite only if $q < p + 2$. Then it becomes possible for α to be negative.

If $q \geq p + 2$, the proper time diverges for all types of trajectories including the usual ones. It means that the region from infinity to the horizon is geodesically complete. Such objects are termed "remote horizons" in [9].

V. ENERGY OF COLLISION

As we mentioned above, we are mainly interested in the possibility of the BSW phenomenon, which is related to infinite growth of energy in the center of mass frame of two colliding particles. This energy is given by

$$E^2_{\text{c.m.}} = -m_1 m_2 u_1^\mu u_{2\mu} = m_1 m_2 \gamma, \tag{33}$$

where γ is the Lorentz gamma factor of relative motion. Substituting expressions for the four-velocity (10), we have

$$\gamma = \frac{\mathcal{X}_1 \mathcal{X}_2 - P_1 P_2}{N^2} - \frac{\mathcal{L}_1 \mathcal{L}_2}{g_{\varphi\varphi}}. \tag{34}$$

Hereafter, we assume that both particles move toward the horizon, so $\sigma_1 = \sigma_2 = -1$.

The second term is always regular, so we are interested in the near-horizon behavior of the first one. To this end, let us expand the expression for P (8):

$$\begin{aligned} P^2 &= \mathcal{X}^2 - N^2 \left(\frac{\mathcal{L}^2}{g_\varphi} + 1 \right) \\ &= (\mathcal{X}_s v^s + \mathcal{X}_{s+s'} v^{s+s'} + \cdots)^2 - \kappa_p v^p \left(\frac{L_H^2}{g_{\varphi H}} + 1 \right) + \cdots, \end{aligned} \tag{35}$$

where s' is some positive number. Now let us find how P behaves near horizon.

For usual and subcritical particles, $N \ll \mathcal{X}$, so we can expand the square root to obtain:

$$P = \mathcal{X} - \frac{N^2}{2\mathcal{X}} \left(\frac{\mathcal{L}^2}{g_{\varphi\varphi}} + 1 \right) + \cdots = \mathcal{X} + O(v^{p-s}). \tag{36}$$

In cases of critical and ultracritical particles, $\mathcal{X}$ and N decrease with the same rate, so we can write

$$\begin{aligned} P &= P_{p/2} v^{p/2} + \cdots \approx P_N N, \\ P_{p/2} &= \sqrt{\mathcal{X}^2_{p/2} - \kappa_p \left(\frac{\mathcal{L}_H^2}{g_{\varphi H}} + 1 \right)}, \\ P_N &= \frac{P_{p/2}}{\sqrt{\kappa_p}}, \end{aligned} \tag{37}$$

which agrees with (17) and (18). As we will further show, for our purposes, it is sufficient to keep the first term in this expansion.

Now let us analyze behavior of the gamma factor. Firstly, let us suppose that each of two particles is usual or subcritical. Using (36), we see that (34) becomes

$$\gamma = \frac{\mathcal{X}_1 z_2 + \mathcal{X}_2 z_1}{2} + O(1), \tag{38}$$

where $z \equiv \frac{1 + (\frac{\mathcal{L}^2}{g_{\varphi\varphi}})_H}{\mathcal{X}}$. Taking into account (25), we see that

$$\gamma \sim v^{-|s_1 - s_2|}. \tag{39}$$

Note that gamma factor is regular only if $s_1 = s_2$; in other cases, it diverges. This result was obtained earlier in [11] for the particular case of extremal horizons with $p = q = 2$ (see Sec. II E there). Meanwhile, now we see that this result is independent on the type of the horizon.

Let now particle 1 be critical or ultracritical and particle 2 be usual. Then,

$$\gamma \approx \frac{\mathcal{X}_2 (\mathcal{X}^{(1)}_{p/2} - P^{(1)}_{p/2})}{N \sqrt{\kappa_p}} = \frac{\mathcal{X}_2 (\mathcal{X}^{(1)}_{p/2} - P^{(1)}_{p/2})}{\kappa_p v^{p/2}}, \tag{40}$$

where $\mathcal{X}_2 = O(1)$.

TABLE II. The possibility of BSW phenomenon for different types of particles. D means that the gamma factor diverges, and R means that it is regular.

	First particle	Second particle	d	γ
1	Usual	Usual	0	R
2	Usual	Subcritical	s_2	D
3	Subcritical	Subcritical	$\lvert s_1 - s_2 \rvert$	D if $s_1 \neq s_2$ R if $s_1 = s_2$
4	Usual or subcritical	Critical or ultracritical	$p/2 - s_1$	D
5	Critical or ultracritical	Critical or ultracritical	0	R

If particle 1 is critical or ultracritical, while particle 2 is subcritical, in a similar way, we obtain (40) with $s > 0$ in (25), so

$$\gamma \approx \frac{(\mathcal{X}_2)_s (\mathcal{X}^{(1)}_{p/2} - P^{(1)}_{p/2})}{\kappa_p v^{p/2-s}}. \tag{41}$$

This expression is divergent since, according to Table I, $s < \frac{p}{2}$ for such particles.

If both particles are critical or ultracritical, then $\mathcal{X}_1 \approx P_1 \sim N$, $\mathcal{X}_2 \approx P_2 \sim N$. Thus, the gamma factor is regular.

We can generalize these results in Table II, where we also introduced quantity d that shows the rate of divergence of the gamma factor $\gamma \sim v^{-d}$. From Table II, we can deduce that the BSW phenomenon happens if, for a given type of horizon, it is possible to obtain two particles with different rates of decrease of $\mathcal{X}$. Firstly let us discuss this possibility for geodesic motion.

In the case of geodesic motion, acceleration is zero; thus, motion is defined only by two conserved quantities (energy and angular momentum) and by metric functions. In this case, ε and $\mathcal{L}$ in (9) are constants. It follows from (21) that

$$\mathcal{X} = \mathcal{X}_H - \omega_k \mathcal{L} v^k + o(v^k). \tag{42}$$

If a particle is fine-tuned, $\mathcal{X}_H = 0$, $\mathcal{L} = \frac{\varepsilon}{\omega_H} > 0$. This gives us that for fine-tuned particles $s = k$. This gives realization of the BSW phenomenon if the first particle is usual, while the second one is fine-tuned. It is worth noting that in this case, the relation $s = k \leq p/2$ has to hold. It comes from reality of the radial component of the four-velocity.

In a general case, when forces are present, the expansion for ε and $\mathcal{L}$ can, in principle, violate the equality $s = k$.

One reservation is in order. In some cases, the BSW process between a usual and fine-tuned particles fails because of impossibility for a fine-tuned one to reach the horizon. In particular, this happens for nonextremal horizons and geodesic particles. Then, the effect can be saved if one of particles is not fine-tuned exactly [7]. When the force is present, this is also compatible with the BSW effect [12]. A more general situation, with arbitrary p, q and the presence of a finite force, requires separate treatment. In this work, we put this issue aside and consider the "pure" BSW effect only, when one of particles is fine-tuned exactly.

VI. GENERAL EXPRESSIONS FOR ACCELERATION

As we will consider particle collisions under the presence of forces, for further analysis, we need to have explicit expressions for the components of acceleration. They are given in the present subsection. It follows from (4) that

$$a^\mu = u^\nu \partial_\nu u^\mu + \Gamma^\mu_{\nu\sigma} u^\nu u^\sigma, \tag{43}$$

where $\Gamma^\mu_{\nu k}$ are Christoffel symbols.

The tetrad components of acceleration: $a_o^{(a)} = a^\mu e_\mu^{(a)}$ can be found from (11):

$$a_o^{(t)} = N a^t, \quad a_o^{(r)} = \frac{a^r}{\sqrt{A}},$$
$$a_o^{(\varphi)} = \sqrt{g_{\varphi\varphi}}(a^\varphi - \omega a^t), \quad a_o^{(\theta)} = \sqrt{g_{\theta\theta}} a^\theta. \tag{44}$$

The scalar square of acceleration

$$a^2 = a^\mu a_\mu = (a_o^{(r)})^2 + (a_o^{(\theta)})^2 + (a_o^{(\varphi)})^2 - (a_o^{(t)})^2. \tag{45}$$

Calculating the Christoffel symbols, one can obtain under assumption of equatorial motion:

$$a_o^{(r)} = \frac{1}{\sqrt{A}} \Bigg\{ u^r \partial_r u^r - \frac{1}{2}\frac{\partial_r A}{A}(u^r)^2 - \frac{A}{2}\left(\mathcal{X}^2 \partial_r (N^{-2}) - \mathcal{L}^2 \partial_r (g_{\varphi\varphi}^{-1}) - 2\frac{\mathcal{X}\mathcal{L}}{N^2}\partial_r \omega\right)\Bigg\}, \tag{46}$$

$$a_o^{(t)} = N\left\{ u^r \partial_r u^t + u^r \left(\frac{\partial_r N^2}{N^2}\frac{\mathcal{X}}{N^2} + \frac{\mathcal{L}}{N^2}\partial_r \omega\right)\right\}, \tag{47}$$

$$a_o^{(\varphi)} = \sqrt{g_{\varphi\varphi}} \Bigg\{ u^r (\partial_r u^\varphi - \omega \partial_r u^t) - u^r \left(\frac{\mathcal{X}}{N^2}\partial_r \omega + \mathcal{L}\partial_r (g_{\varphi\varphi}^{-1})\right)\Bigg\}. \tag{48}$$

Component $a^\theta = 0$ because we consider equatorial motion with respect to which all metric functions are symmetric that causes cancellation of all terms in a^θ.

Expressions for $a_o^{(t)}$, $a_o^{(r)}$, and $a_o^{(\varphi)}$ may be simplified by the substitution of the expression for the four-velocity (10):

$$a_o^{(r)} = \mathcal{X}\frac{\sqrt{A}}{N^2}\left(\partial_r\mathcal{X} + \mathcal{L}\partial_r\omega - \frac{N^2}{\mathcal{X}}\frac{\mathcal{L}\partial_r\mathcal{L}}{g_{\varphi\varphi}}\right), \tag{49}$$

$$a_o^{(t)} = \frac{u^r}{N}(\partial_r\mathcal{X} + \mathcal{L}\partial_r\omega), \tag{50}$$

$$a_o^{(\varphi)} = \frac{u^r}{\sqrt{g_{\varphi\varphi}}}\partial_r\mathcal{L}. \tag{51}$$

Equations (50) and (51) agree with Eqs. (116) and (117) of [11].

Equivalently, we can write:

$$a_o^{(r)} = \mathcal{X}\frac{\sqrt{A}}{N^2}\left(\partial_r(\mathcal{X}+\mathcal{L}\omega) - \left(\omega + \frac{N^2}{\mathcal{X}}\frac{\mathcal{L}}{g_{\varphi\varphi}}\right)\partial_r\mathcal{L}\right), \tag{52}$$

$$a_o^{(t)} = \sigma P\frac{\sqrt{A}}{N^2}(\partial_r(\mathcal{X} + \mathcal{L}\omega) - \omega\partial_r\mathcal{L}), \tag{53}$$

$$a_o^{(\varphi)} = \sigma P\frac{\sqrt{A}\partial_r\mathcal{L}}{N\sqrt{g_{\varphi\varphi}}}. \tag{54}$$

VII. FINE-TUNED PARTICLES

We are interested in trajectories that are (i) compatible with finite acceleration near the horizon and (ii) lead to an indefinitely large growth of energy $E_{\mathrm{c.m.}}$ due to particle collision there. Property (ii) implies that the proper time required to reach the horizon is infinite for a fine-tuned (subcritical, critical, or ultracritical) particle so that it approaches the horizon only asymptotically and cannot cross it. (For the Kerr metric, this was noticed in [6]; a general proof will be given below in Sec. IX). Correspondingly, it is the OZAMO frame that is natural for them (see below for more detail) since a corresponding observer does not cross the horizon. Therefore, we can write asymptotic expansion for acceleration near the horizon in the form

$$\begin{aligned} a_o^{(t)} &= (a_o^{(t)})_{n_0}v^{n_0} + o(v^{n_0}),\\ a_o^{(r)} &= (a_o^{(r)})_{n_1}v^{n_1} + o(v^{n_1}),\\ a_o^{(\varphi)} &= (a_o^{(\varphi)})_{n_2}v^{n_2} + o(v^{n_2}), \end{aligned} \tag{55}$$

where n_0, n_1, n_2 should be non-negative. As we mentioned above, we will proceed in such a way: We set a near-horizon trajectory [equivalently, numbers c and b that appear in (24) and (30)] and calculate accelerations, thus finding n_0, n_1, and n_2. Requiring n_0, n_1, and n_2 to be non-negative, we find physically achievable trajectories that can produce the BSW effect. To realize this scheme, we have to establish several important restrictions on the parameters of our system.

One important reservation is in order. If we take into account (16)–(18), it follows from (52) and (53) that in the case of subcritical and critical particles, $n_0 = n_1$, while for ultracritical ones, $n_0 > n_1$. Thus, $n_0 \geq n_1$ and regularity of $a_o^{(r)}$ implies regularity of $a_o^{(t)}$, so it is sufficient to analyze $a_o^{(r)}$ and $a_o^{(\varphi)}$ only.

To find relation between n_0, n_1, n_2 and c, b, we express the four-velocity in terms of quantities $\mathcal{X}$ and $\mathcal{L}$ introduced above.

A. Relationship between acceleration and radial velocity near the horizon

In this subsection, we will find explicitly the asymptotic behavior of the expressions for accelerations listed above. Before proceeding further, we want to make some important reservations. We are interested in situations when the tetrad components of acceleration are finite in a relevant frame. By this frame, we imply a frame comoving with respect to a particle or any other one that moves with respect to it with a finite velocity giving a finite local Lorentz boost between them. For a usual particle, the role of such a frame is played by a frame attached to a free-falling observer (FZAMO, if for simplicity, we choose an observer with a zero angular momentum). However, in the OZAMO frame, its components may diverge since the Lorentz boost becomes singular. By contrast, the fine-tuned particles cannot cross a horizon. Therefore, it is the OZAMO frame that is natural for them so that the tetrad components of acceleration in the OZAMO frame should stay finite. {This general issue is discussed in more detail in Sec. III of [11]. In particular, see Eqs. (70) and (71) there}. It is the study of concrete near-horizon asymptotic expressions that we are now turning to.

Combining Eq. (25) with $s > 0$, (8) and (7) and using (24), we arrive at the following set of cases.

(i) $s < p/2$, subcritical particle: in this case, $u^r = \sigma\sqrt{\frac{A_q}{\kappa_p}}\mathcal{X}_s v^{s+(q-p)/2} + o(v^{s+(p-q)/2})$ that gives us relation $s = \frac{p-q}{2} + c$.

(ii) $s = p/2$, critical particle: in this case, $P = \sqrt{\mathcal{X}_{p/2}^2 - \kappa_p(\frac{\mathcal{L}_H^2}{g_\varphi} + 1)}\nu^{p/2} + o(u^{p/2})$, $u^r = \sigma\sqrt{\frac{A_q}{\kappa_p}}\sqrt{\mathcal{X}_{p/2}^2 - \kappa_p(\frac{\mathcal{L}_H^2}{g_\varphi} + 1)}\nu^{q/2} + o(u^{p/2})$.

(iii) $s = p/2$, ultracritical particle: in this case, $u^r = (u^r)_c v^c + \ldots$; thus, $P \approx \sqrt{\frac{\kappa_p}{A_q}}(u^r)_c v^{\frac{p-q}{2}+c}$.

The case $s > p/2$ is impossible, because P would become imaginary. Now let us compute components of acceleration for each s. We will write the asymptotics for P in the cases of subcritical and critical particles in the form $P \approx P_s v^s + o(v^s)$, where $P_s = \mathcal{X}_s$ for $s < p/2$ and $P_s = \sqrt{\mathcal{X}_{p/2}^2 - \kappa_p(\frac{\mathcal{L}_H^2}{g_\varphi} + 1)}$ for $s = p/2$. For the ultracritical particle, we will write $P \approx P_{\frac{p-q}{2}+c} v^{\frac{p-q}{2}+c}$, where $P_{\frac{p-q}{2}+c} = \sqrt{\frac{\kappa_p}{A_q}}(u^r)_c$.

Let us analyze behavior of radial acceleration. It follows from (46) that

$$a_o^{(r)} \approx \frac{1}{\sqrt{A_q}} \left\{ \frac{A_q}{\kappa_p} P_s^2 \left(s - \frac{p}{2} \right) v^s + \frac{A_q}{\kappa_p} \mathcal{X}_s^2 \frac{p}{2} v^s + A_q \frac{\mathcal{X}_s \mathcal{L}_H}{\kappa_p} k\omega_k v^k \right\} v^{s+q/2-p-1}. \quad (56)$$

Note that the term with $\mathcal{L}^2$ that is present in (46) is of higher order. To see this, let us consider parentheses in (46). The first term is of order of v^{2s-p-1} and, as $s \leq p/2$, this term is divergent. Meanwhile, the $\mathcal{L}^2$ term $= O(1)$ that proves the aforementioned statement.

The near-horizon behavior of (56) depends on what is bigger: s or k. Using Eq. (56), we can thus write:

$$n_1 = \min(s, k) + s + \frac{q}{2} - p - 1. \quad (57)$$

In accordance with reservations made above, now we are not interested in usual particles, so $s > 0$, whereas the case $s = 0$ is excluded from consideration.

The case when $a_o^{(r)}$ does not include $\partial_r \omega$ deserves separate attention. This may happen if ω is constant. As a matter of fact, such a metric is static. In this case, we can redefine angular coordinate $\tilde{\varphi} = \varphi - \omega t$ that will diagonalize metric (1), making it explicitly static. Then in (56), only two first terms survive, which gives us

$$n_1 = 2s + \frac{q}{2} - p - 1. \quad (58)$$

Hereafter, we denote this case as $k = 0$.

There is also a special case when coefficients in expansion of $\mathcal{X}$ and $\mathcal{L}$ are such that several terms in powers series (which are, generally speaking, potentially divergent) in the expression for acceleration cancel each other. Full cancellation happens, for example, for freely falling particles, for which $\mathcal{X} + \omega\mathcal{L} = \varepsilon$, where ε and $\mathcal{L}$ are constants that gives us zero acceleration. Then, $\partial_r(\mathcal{X} + \omega\mathcal{L}) = 0$ exactly. In a more general case, we can consider

$$\varepsilon = \mathcal{X} + \omega\mathcal{L} = \varepsilon_0 + \text{terms of } v^m \text{ order}, \quad m > k. \quad (59)$$

Using relation (52) for $a_o^{(r)}$, we see that the first term in brackets has the order v^{m-1}, while the second one has the order of v^{b-1}.

Thus, in this case,

$$n_1 = \min(m, b) + s + \frac{q}{2} - p - 1. \quad (60)$$

However, we will not pay much attention to this case further.

Now we want to rewrite all possible solutions for n_1 in terms of c. In the case of usual, subcritical, and critical particles, we can use relation $s = \frac{p-q}{2} + c$ that gives us

$$n_1 = \min\left(\frac{p-q}{2} + c, k\right) + c - 1 - \frac{p}{2} \quad \text{if } k \neq 0, \quad (61)$$

$$n_1 = 2c - 1 - \frac{q}{2} \quad \text{if } k = 0, \quad (62)$$

$$n_1 = \min(m, b) + c - \frac{p}{2} - 1 \quad \text{if } \mathcal{X} + \omega\mathcal{L} = \varepsilon_0 + O(v^m). \quad (63)$$

Thus, we found the expressions that include n_1, c, s, p, q, k. Our goal is to transform them to the form $c = c(n_1, p, q, k)$. Then we take into account the data from Table I in combination with the requirement $n_1 \geq 0$. This can give us restrictions on metric parameters relevant for different types of trajectories. Using the above formulas, we find for subcritical and critical particles

$$c = \begin{cases} \frac{2n_1+2+q}{4} & \text{if } n_1 \leq 2k + \frac{q-2-2p}{2}, k \neq 0 \\ n_1 + \frac{p}{2} + 1 - k & \text{if } n_1 > 2k + \frac{q-2-2p}{2}, k \neq 0 \\ \frac{2n_1+2+q}{4} & \text{if } k = 0 \\ n_1 + \frac{p}{2} + 1 - \min(m, b) & \text{if } \mathcal{X} + \omega\mathcal{L} = \varepsilon_0 + O(v^m) \text{ where } m > k, \end{cases} \quad (64)$$

where the relation $s = \frac{p-q}{2} + c$ was used.

The first solution in (64) exists for $n_1 \geq 0$ if

$$k \geq \frac{2p+2-q}{4} \tag{65}$$

only, while the second one exists for all k.

For ultracritical particles, we cannot use the aforementioned formula for s. In this case, $c > \frac{q}{2}$, $s = \frac{p}{2}$. Then, we obtain from (57) and (58):

$$c \text{ may be any value grater then } \frac{q}{2}, \tag{66}$$

$$n_1 = \min\left(\frac{p}{2}, k\right) + \frac{q-p}{2} - 1 \quad \text{if } k \neq 0, \tag{67}$$

$$n_1 = \frac{q-2}{2} \quad \text{if } k = 0. \tag{68}$$

The special case $\mathcal{X} + \omega\mathcal{L} = \varepsilon_0 + O(v^m)$ gives us, according to (60),

$$n_1 = \min(m, b) + \frac{q-p}{2} - 1. \tag{69}$$

We remind a reader that, according to what is said above, there is no need to require the regularity of the time component of acceleration. It is valid automatically if the radial one is regular.

Now, let us consider $a_o^{(\varphi)}$. It follows from (51) and (30) that for any type of particle,

$$a_o^{(\varphi)} \approx \mathcal{L}_h v^{c+b-1}, \tag{70}$$

where $\mathcal{L}_h$ is some constant, so

$$n_2 = c + b - 1 \rightarrow b = n_2 + 1 - c. \tag{71}$$

As we require $n_2 \geq 0$, this gives us the restriction $b \geq 1 - c$. According to Table I, this entails $b \geq 1 - \frac{q}{2}$ for critical and ultracritical trajectories.

One reservation is in order. The formulas under discussion include also the case when $a_o^{(\varphi)} = 0$, so $\mathcal{L} = \text{const}$. Then, formally, one can put $b \rightarrow \infty$. Correspondingly, it drops out from (64).

More information can be extracted from Table III.

TABLE III. Behavior of the angular component of acceleration.

	Type of trajectory	n_2
1	Usual	$\frac{q-p}{2} + b - 1$
2	Subcritical	$\frac{q-p}{2} + b - 1 < n_2 < b - 1 + \frac{q}{2}$
3	Critical	$\frac{q}{2} + b - 1$
4	Ultracritical	$n_2 > \frac{q}{2} + b - 1$

Now, we will analyze for which subspaces in the space of parameters (p, q, k) we can have near-horizon trajectories with non-negative n_0, n_1, and n_2 for all types of fine-tuned particles described in Table I.

B. Subcritical particles

Let us start with subcritical particles. According to Table I, $\frac{q-p}{2} < c < \frac{q}{2}$. If we use the first solution in (64), we get $\frac{q-2-2p}{2} < n_1 < \frac{q-2}{2}$. Now, let us find how it correlates with the condition of existence of the first solution in (64): $n_1 \leq 2k + \frac{q-2-2p}{2}$. For $k < \frac{p}{2}$, the condition $n_1 \leq 2k + \frac{q-2-2p}{2}$ is stronger than $n_1 < \frac{q-2}{2}$; for $k \geq \frac{p}{2}$, the condition $n_1 < \frac{q-2}{2}$ becomes stronger. The lower bound for n_1 is the same for all positive k. So, we can conclude that for the first solution in (64),

$$\frac{q-2-2p}{2} < n_1 \leq 2k + \frac{q-2-2p}{2} \quad \text{if } k < \frac{p}{2}, \tag{72}$$

$$\frac{q-2-2p}{2} < n_1 < \frac{q-2}{2} \quad \text{if } k \geq \frac{p}{2}. \tag{73}$$

For the second solution in (64), the condition $\frac{q-p}{2} < c < \frac{q}{2}$ entails $\frac{q}{2} - p - 1 + k < n_1 < \frac{q-p}{2} - 1 + k$. The condition $n_1 > 2k + \frac{q-2-2p}{2}$ is stronger than the lower bound. However, it is weaker than the upper one if $k < \frac{p}{2}$. Thus, we can conclude that for the second solution in (64),

$$2k + \frac{q-2-2p}{2} \leq n_1 < \frac{q-p}{2} - 1 + k \quad \text{if } k < \frac{p}{2}, \tag{74}$$

$$\text{Second solution is impossible if } k \geq \frac{p}{2}. \tag{75}$$

The special case $k = 0$ gives the condition $\frac{q-2-2p}{2} < n_1 < \frac{q-2}{2}$. This completes the analysis for the subcritical case.

C. Critical particles

The first solution in (64) gives us $c = \frac{q}{2} = \frac{2n_1+2+q}{4} \rightarrow n_1 = \frac{q-2}{2}$, so n_1 is non-negative if $q \geq 2$. The first solution exists if $n_1 \leq 2k + \frac{q-2-2p}{2}$, which gives us $k \geq \frac{p}{2}$. If we deal with the second solution, $n_1 + \frac{p}{2} + 1 - k = \frac{q}{2} \rightarrow n_1 = \frac{q-2}{2} + k - \frac{p}{2}$. The condition $n_1 \geq 0$ gives us $k \geq \frac{p-q}{2} + 1$. The second solution exists if $n_1 > 2k + \frac{q-2-2p}{2}$, which entails $k < \frac{p}{2}$. To conclude, we see that

$$\text{If } \frac{p-q}{2} + 1 \leq k < \frac{p}{2}, \qquad n_1 = \frac{q-2}{2} + k - \frac{p}{2}. \tag{76}$$

$$\text{If } k \geq \frac{p}{2} n_1 = \frac{q-2}{2}. \tag{77}$$

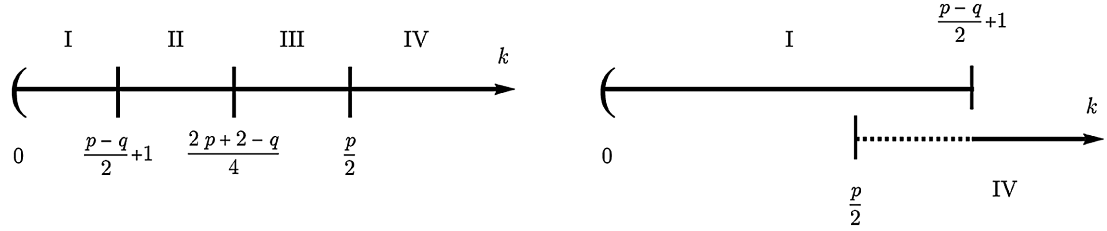

FIG. 1. Relevant regions of k. The left panel corresponds to $q > 2$. The right panel corresponds to $q \leq 2$, where regions II and III do not exist anymore and also points at regions' I and IV overlap. We define points on intersection to belong to region I (for explanation, see Sec. VII E).

The special case $k = 0$ gives the same result as the first solution in (64).

D. Ultracritical particles

When we turn to ultracritical trajectories, the situation is somewhat different. In this case, c is independent of n_1: c may take any value greater than $\frac{q}{2}$, while $n_1 = \frac{q-2}{2}$ if $k \geq \frac{p}{2}$, $n_1 = k + \frac{q-p}{2} - 1$ if $k < \frac{p}{2}$ [see (66)]. It would seem that these relations are the same as in the case of a critical trajectory (because forces are the same). However, as we discussed in Sec. IV, for ultracritical particles, an additional condition (28) has to hold.

E. Classification of different k regions

In the above consideration, we discussed the conditions of existence of different types of fine-tuned particles. Meanwhile, we are also interested in more subtle details that concern the relations between types of solutions enumerated in the lines of Eq. (64) and those of trajectories. To this end, it is convenient to systematize the obtained results and collect them in an unified scheme. In a natural way, the space of parameters (p, q, k) is split depending on a kind of particle (subcritical, critical, ultracritical) to different regions in which the radial component of acceleration is finite. Introducing the values of k that correspond to the borders of such regions, we obtain four different regions for k

$$0 < k < \frac{p-q}{2} + 1, \quad \text{region I,} \tag{78}$$

$$\frac{p-q}{2} + 1 \leq k < \frac{p+1-q/2}{2}, \quad \text{region II,} \tag{79}$$

$$\frac{p+1-q/2}{2} \leq k < \frac{p}{2}, \quad \text{region III,} \tag{80}$$

$$k \geq p/2, \quad \text{region IV.} \tag{81}$$

They are presented on Fig. 1. Here, according to (65), $k = \frac{2p+2-q}{4}$ is the minimum possible value for which the first solution in (64) exists, and $k = \frac{p}{2}$ is the maximum possible value for which subcritical particles given by the second solution in (64) can exist according to (75).

In region IV ($k \geq \frac{p}{2}$), we have corresponding conditions for different trajectories. As follows from (73), if $\frac{q-2-2p}{2} < n_1 < \frac{q-2}{2}$, we have a subcritical particle. As follows from (77), if $n_1 = \frac{q-2}{2}$, a particle is critical, while if $n_1 = \frac{q-2}{2}$ and the condition (28) holds, it is ultracritical.

In region III, we note that $\frac{2+2p-q}{4} \leq k < \frac{p}{2}$, and the conditions somewhat change. Following (72) and (74), we see that the first solution in (64) gives a subcritical particle only for $\frac{q-2-2p}{2} < n_1 \leq 2k + \frac{q-2-2p}{2}$. The second solution in (64) gives a subcritical particle only for $2k + \frac{q-2-2p}{2} < n_1 < \frac{q-p}{2} - 1 + k$ [see (74)]. If we are interested in the type of trajectory only, then these two regions may be joined. This gives us that for all $\frac{q-2-2p}{2} < n_1 < \frac{q-p}{2} - 1 + k$, we have subcritical particles. Critical particles can be obtained only if $n_1 = \frac{q-2}{2} + k - \frac{p}{2}$ [see (76)]. In the ultracritical case, acceleration is the same as for the critical case ($n_1 = \frac{q-2}{2} + k - \frac{p}{2}$), but condition (28) has to hold.

Region II gives the same conditions as region III. However, the first solution in (64) is absent [as we noted in discussion after (64)]. Thus, we can get subcritical particles only for $2k + \frac{q-2-2p}{2} < n_1 < \frac{q-p}{2} - 1 + k$ in region II (where $c = n_1 + 1 + p/2 - k$). Relations for critical and ultracritical are the same as for region III.

In region I, all types of trajectories give negative n_1 because for all trajectories, $n_1 \leq k + \frac{q-p}{2} - 1$, but as $k < \frac{p-q}{2} + 1$ in this region, we see that $n_1 < 0$ for all types of trajectories.

All these results are summarized in Table IV and Fig. 1. We added also special cases there: $k = 0$. As we see, the restrictions on n_1 and/or their exact expressions look different in different k regions.

To complete the picture of all possible cases, we explicitly present tables for different horizons. In the case of a nonextremal horizon ($q < 2$), all trajectories experience infinite forces (see Table VI). In the case of extremal horizons ($q = 2$), a finite force acts only if particle is critical or ultracritical. In the second case, it is possible only if $k = 0$ or it lies in regions III and IV (see Table V). In the case of ultracritical horizon ($q > 2$),

TABLE IV. Classification of near-horizon trajectories for different k regions for $q > 2$ (ultraextremal horizon). The fourth solution in (64) is not presented in this table. Definitions of different k regions are given in Fig. 1.

	k Region	n_1 Range	c	Type of trajectory
		Stationary metric		
1	I	For any type of trajectory, n_1 is negative (forces diverge)		
2	II	$2k + \frac{q-2-2p}{2} < n_1 < \frac{q-p}{2} - 1 + k$	$n_1 + 1 + p/2 - k$	Subcritical
		$n_1 = \frac{q-p}{2} - 1 + k$	$q/2$	Critical
		$n_1 = \frac{q-p}{2} - 1 + k$ and (28)	Any $c > \frac{q}{2}$	Ultracritical
3	III	$\max(0, \frac{q-2-2p}{2}) < n_1 \leq 2k + \frac{q-2-2p}{2}$	$\frac{2n_1+2+q}{4}$	Subcritical
		$2k + \frac{q-2-2p}{2} < n_1 < \frac{q-p}{2} - 1 + k$	$n_1 + 1 + p/2 - k$	Subcritical
		$n_1 = \frac{q-p}{2} - 1 + k$	$q/2$	Critical
		$n_1 = \frac{q-p}{2} - 1 + k$ and (28)	Any $c > \frac{q}{2}$	Ultracritical
4	IV	$\max(0, \frac{q-2-2p}{2}) < n_1 < \frac{q-2}{2}$	$\frac{2n_1+2+q}{4}$	Subcritical
		$n_1 = \frac{q-2}{2}$	$q/2$	Critical
		$n_1 = \frac{q-2}{2}$ and (28)	Any $c > \frac{q}{2}$	Ultracritical
		Static metric		
5	$k = 0$	Same results as in IV for stationary metric		

TABLE V. Classification of near-horizon trajectories for different k regions for $q = 2$ (extemal horizon). Regions II and III in this case are absent, so they are not presented in this table. The fourth solution in (64) is also not presented in this table. Definitions of different k regions are given in Fig. 1.

	k Region	n_1 Range	c	Type of trajectory
		Stationary metric		
1	I	For any type of trajectory, n_1 is negative (forces diverge)		
2	IV	$n_1 = 0$	1	Critical
		$n_1 = 0$ and (28)	Any $c > 1$	Ultracritical
		Static metric		
3	$k = 0$	Same results as in IV for stationary metric		

the situation is more complicated, and all possible trajectories are listed in Table IV.

However, there are several possibilities for these regions to intersect or disappear. Let us start with region I. From definition, we see that it disappears for $q \geq p + 2$, and then the whole range consists only from regions II, III, and IV. Region II is absent for $q \leq 2$. However, if $q \geq 2p + 2$, the upper bound becomes negative, and then this region also disappears so that the whole range consists of III and IV regions. Region III disappears only if $q \leq 2$. In this case, only regions I and IV remain. Moreover, they start to intersect in this case. For definiteness, we prescribe the corresponding points on this intersection to region I. According to Table VI, this choice is not important because in both I and IV regions, n_1 is negative.

TABLE VI. Classification of near-horizon trajectories for different k regions for $q < 2$ (nonextremal horizon). Regions II and III in this case are absent, so they are not presented in this table. The fourth solution in (64) is also not presented in this table. Definitions of different k regions are given in Fig. 1.

	k Region	n_1 Range	c	Type of trajectory
		Stationary metric		
1	I and IV	For any type of trajectory, n_1 is negative (forces diverge)		
		Static metric		
2	$k = 0$	For any type of trajectory, n_1 is negative (forces diverge)		

All obtained above results are collected in Table VII and represented on Fig. 2. On this figure, we show the cross section of the three-parametric space: k, p, and q by plane $k = \text{const}$ (Fig. 2). Here, a blue color represents region I; green, region II; orange, region III; and gray, region IV.

F. Case $q = p$

This case has to be considered separately because of practical importance. In this case, if $0 < k < 1$, we are at

TABLE VII. Possible k regions depending on p and q. Definitions of different k regions are given in Fig. 1.

Condition for q	Possible k regions
$q \leq 2$	I and IV
$2 < q < p + 2$	I, II, III, and IV
$p + 2 \leq q < 2p + 2$	II, III, and IV
$q \geq 2p + 2$	III and IV

region I; if $1 \leq k < \frac{p+2}{4}$, in region II; if $\frac{p+2}{4} \leq k < \frac{p}{2}$, in region III; if $\frac{p}{2} \leq k$, in region IV. As in general case, in region I, the force diverges for all types of horizons. In region II, as we can see from Table IV, all three types of trajectories are possible. The same holds for region III, but depending on k, expressions for c may be different. Also, as in general case, regions II and III are absent if $p < 2$.

So, summarizing, we have such trajectories in regions II and III:

(i) If $0 < n_1 \leq 2k - 1 - \frac{p}{2}$, trajectory is subcritical, and $c = \frac{2n_1+2+p}{4}$,
(ii) If $2k - 1 - \frac{p}{2} < n_1 < k - 1$, trajectory is subcritical, and $c = n_1 + 1 + p/2 - k$,
(iii) If $n_1 = k - 1$, trajectory is critical,
(iv) If $n_1 = k - 1$ and condition (28) holds, trajectory is ultracritical.

In region IV, however, we have:

(i) If $0 < n_1 < \frac{p}{2} - 1$, trajectory is subcritical,
(ii) If $n_1 = \frac{p}{2} - 1$, trajectory is critical,
(iii) If $n_1 = \frac{p}{2} - 1$ and condition (28) holds, trajectory is ultracritical.

VIII. RESULTS FOR NONEXTREMAL, EXTREMAL AND ULTRAEXTREMAL HORIZONS

In previous sections, we have analyzed relations between a type of trajectory and behavior of force depending on characteristics of a horizon. From these results, we extract now information about the possibility to have finite forces near the horizon for each type of horizon separately. We also collect the cases when this is consistent with the BSW effect.

A. Nonextremal horizon

For a nonextremal horizon, $p = q = 1$. All results corresponding to this case may be found in Table VI. For subcritical, critical, and ultracritical particles, acceleration diverges independently of k.

B. Extremal horizon

For extremal horizons, $q = 2$, while p may take any value $p \geq 2$. All results corresponding to this case may be found in Table V. For subcritical, critical, and ultracritical particles, acceleration diverges for all $k < \frac{p}{2}$. However, if $k \geq \frac{p}{2}$ or $k = 0$ (static metric), finite acceleration becomes possible for critical and ultracritical particles. In doing so, the BSW effect is also possible in the scenario that corresponds to line 4 in Table II.

C. Ultraextremal horizon

In this case q, $p > 2$. All results corresponding to this case may be found in Table IV. In this case, for all $k < \frac{p-q}{2} + 1$, acceleration diverges for all subcritical, critical, and ultracritical trajectories. If $k \geq \frac{p-q}{2} + 1$ or $k = 0$ (static metric), subcritical, critical, and ultracritical

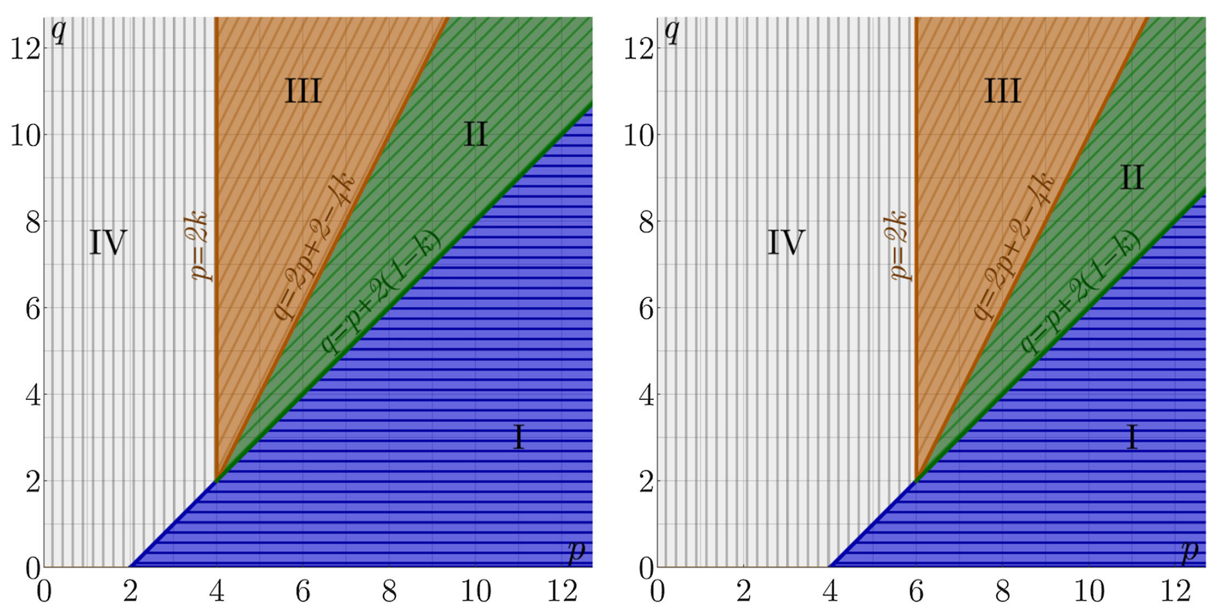

FIG. 2. Plot showing in which region lies k for all types of horizons, depending on p and q. The left panel is drawn for $k = 2$, and the right panel for $k = 3$. Different k regions are represented with different colors: IV (gray), III (orange), II (green), I (blue).

TABLE VIII. Classification of cases when forces are finite for different types of horizons and trajectories.

	Type of horizon	Type of trajectory	Region of k
1	Nonextremal	All types	Absent
2	Extremal	Subcritical Critical Ultracritical	Absent $k \geq p/2$ or $k = 0$
3	Ultraextremal	Subcritical Critical Ultracritical	$k \geq \frac{p-q}{2} + 1$ or $k = 0$

trajectories with finite accelerations can exist. The details of relations between behavior of the four-velocity and acceleration can be found in Table IV. In doing so, the BSW effect is described by lines 2, 3, and 4 in Table II.

The above results are summarized in Table VIII.

Thus, as long as we are interested in the existence of finite acceleration, it is sufficient to use the Table VIII under discussion only. However, previous tables give us not only the conditions of such existence but also much more detailed information about possible rates n_i that characterize the behavior of different components of acceleration.

IX. PARTICLES WITH FINITE PROPER TIME: KINEMATIC CENSORSHIP PRESERVED

In this section, we prove an interesting consequence of previous results: *all unusual trajectories with a finite proper time either experience infinite force, or the horizon fails to be regular.*

To prove this, we recall that the proper time is finite if $q < p + 2$ and the trajectory satisfies the condition

$$\frac{q-p}{2} \leq c < 1, \tag{82}$$

(see Sec. IV). Now our task is to find which values of n_1 can be obtained for such trajectories.

We start with the simplest case $q < 2$. As we concluded in Table VI, in this case, a force is infinite for all unusual trajectories.

Now let us move to the $q \geq 2$ case. The structure of possible solutions is more complicated, so we will analyze separately each of solutions obtained in (64) under assumption $q \geq 2$.

Let us start with the first solution in (64). From (82), one can obtain

$$\frac{q-2-2p}{2} \leq n_1 < \frac{2-q}{2}. \tag{83}$$

As $2 \leq q < p + 2$, both lower and upper bounds are negative, which gives us negative n_1. Also note that the third solution in (64) gives the same result.

From the second solution of (64), one can obtain

$$\frac{q-2-2p}{2} + k < n_1 < k - \frac{p}{2}. \tag{84}$$

It would seem that this can give positive n_1. However, we will show that this is impossible. Let us start with subcritical particles. For them, the second solution in (64) exists only if $k < \frac{p}{2}$ [see (74)]. As $q < p + 2$, both lower and upper bounds in (84) are negative. For critical and ultracritical particles, we have $c \geq \frac{q}{2}$. However, (82) requires $c < 1$, which leads to $q < 2$ for which case forces diverge, as is pointed out above.

The fourth solution gives us

$$\min(m, b) + \frac{q}{2} - p - 1 < n_1 < \min(m, b) - \frac{p}{2}. \tag{85}$$

By itself, this inequality can give us non-negative n_1, provided (59) holds with $m \geq p/2$ and $b \geq p/2$. However, this case is impossible because of another reason, not connected with a force. Namely, we can obtain a finite proper time for nonregular horizons only. Indeed, unusual trajectories can exist if $\varepsilon_0 = \omega_H L_H$ only, then it follows from (21) and (59) that $\mathcal{X} \sim -\hat{\omega}_k L_H v^k$. Thus, $s = k$. Meanwhile, a finite proper time can be obtained for $c < 1$ only. For subcritical particles (for which relation $c = \frac{q-p}{2} + k$ holds), this entails $k < \frac{p-q}{2} + 1$, but this is not consistent with the regularity condition (23).

In the case of critical and ultracritical particles, $c \geq \frac{q}{2}$. At the beginning of Sec. VII A, we noted that the requirement $P^2 \geq 0$ entails $s \leq \frac{p}{2}$, which implies $k \leq \frac{p}{2}$, so we can write $c \geq \frac{q-p}{2} + k$. As a proper time is finite only if $c < 1$, this also gives us violation of regularity condition (23).

This completes the proof of our initial statement.

X. USUAL PARTICLES

As we already said, we mainly investigate unusual particles since this is an essential ingredient of the BSW effect. Meanwhile, consideration of near-horizon trajectories of usual particles can be also of some interest beyond the context of the BSW effect. In general, the time and radial components of acceleration for such particles diverge near the horizon in the OZAMO frame (see Sec. VII A above). More precisely, the asymptotic form of the acceleration near the horizon takes the form {see Eqs. (70) and (71) in [11]}

$$a_o^{(r)} = \frac{\left(a_f^{(t)}\right)_0 - \left(a_f^{(r)}\right)_0}{N} + \left[\left(a_f^{(t)}\right)_1 - \left(a_f^{(r)}\right)_1\right] + \cdots, \tag{86}$$

$$a_o^{(t)} = -\frac{\left(a_f^{(t)}\right)_0 - \left(a_f^{(r)}\right)_0}{N} - \left[\left(a_f^{(t)}\right)_1 - \left(a_f^{(r)}\right)_1\right] + \cdots, \tag{87}$$

where

$$a_f^{(t,r)} = \left(a_f^{(r,t)}\right)_0 + \left(a_f^{(t,r)}\right)_1 N + \cdots. \tag{88}$$

Only in the exceptional case when $(a_f^{(t)})_0 = (a_f^{(r)})_0$, $a_o^{(t,r)}$ remains finite. It is worth noting that, as the four-acceleration is a spacelike vector, it follows from (45) that in this case, at least one of its angular components should be nonzero.

We can pose a question, when such a case can be realized. In our approach, this means the condition $n_i \geq 0$. For usual particles, for all types of horizons, the expressions for n_i are given by (90), (92), and (94) (for stationary case) and by (91), (93), and (94).

Let us start with the radial component of acceleration. Usual particles correspond to $s = 0$ or, equivalently, $c = \frac{q-p}{2}$. In this case, we cannot use (56), because v^s terms in (56) vanish. To analyze higher order terms that become dominant in this case, we use Eq. (49) and substitute expansion for $\mathcal{X}$ in the form

$$\mathcal{X} = \mathcal{X}_0 + \mathcal{X}_{s_1} v^{s_1} + o(v^{s_1}), \tag{89}$$

where $s_1 > 0$.

Using expression (49), we see that the first term in brackets is $\sim v^{s_1 - 1}$, and the second term is $\sim v^{k-1}$, while the third one is $\sim v^{p+b-1}$. This gives us

$$n_1 = \min(s_1, k, p + b) + \frac{q}{2} - p - 1 \quad \text{for stationary metric.} \tag{90}$$

In case of static metric $\omega = \text{const}$, and v^k terms in (49) are absent, which gives us

$$n_1 = \min(s_1, p + b) + \frac{q}{2} - p - 1 \quad \text{for static metric.} \tag{91}$$

Analyzing the time component, we are faced with the same issue of vanishing of v^s terms. Then, we need to find higher order terms in (50). Now, in contrast to (49), there is no term with $\partial_r \mathcal{L}$, so b drops out from the formulas, and the conditions analogous to (90) and (91) read

$$n_0 = \min(s_1, k) + \frac{q}{2} - p - 1 \quad \text{if for stationary metric,} \tag{92}$$

$$n_0 = s_1 + \frac{q}{2} - p - 1 \quad \text{for static metric.} \tag{93}$$

The angular component of acceleration can be obtained by substitution $c = \frac{q-p}{2}$ in (71), which gives us

$$n_2 = \frac{q - p}{2} + b - 1. \tag{94}$$

The condition of regularity requires

$$b \geq 1 - \frac{q - p}{2}. \tag{95}$$

It is worth stressing that the condition of regularity has now different status for (i) the time and radial components and (ii) the angular one. For the reason explained above, condition (i) singles out a special subclass of trajectories for which $(a_f^{(t)})_o = (a_f^{(r)})_o$. In the general case, (i) can be violated for usual particles. This is because of singular nature of local Lorentz transformation near the horizon between FZAMO and OZAMO. Meanwhile, this transformation does not touch upon the component $a_f^{(\varphi)}$. Therefore, condition (95) is mandatory for physically acceptable trajectories of usual particles.

XI. CHECKING RESULTS: ELECTROMAGNETIC FORCE

In this section, we check our results in the case when a force has electromagnetic nature using an exact solution of Einstein-Maxwell equations. To this end, we consider an electrically charged black hole. Axial symmetry requires that the vector potential has the form

$$\mathcal{A} = \mathcal{A}_t dt + \mathcal{A}_\varphi d\varphi. \tag{96}$$

As before, we consider axially symmetric metrics (1) and assume the same symmetry for the electromagnetic field. Then, we introduce generalized momenta P_μ in a standard way,

$$p_\mu = P_\mu - e\mathcal{A}_\mu, \tag{97}$$

where e is a particle's charge, p_μ being the kinematic momentum that obeys the normalization condition

$$-m^2 = g^{\mu\nu} p_\mu p_\nu. \tag{98}$$

Because of symmetry of this system, the quantities $\tilde{E} = -P_t$ and $\tilde{L} = P_\varphi$ remain constant. Thus, the normalization condition for the momentum gives us

$$-m^2 = -\frac{1}{N^2}(-\tilde{E} - e\mathcal{A}_t)^2 - \frac{2\omega}{N^2}(-\tilde{E} - e\mathcal{A}_t)(\tilde{L} - e\mathcal{A}_\varphi) - \left(\frac{\omega^2}{N^2} - \frac{1}{g_{\varphi\varphi}}\right)(\tilde{L} - e\mathcal{A}_\varphi)^2 + A(p_r)^2. \tag{99}$$

Introducing $E = \tilde{E} + eA_t$, $\mathcal{L} = \frac{\tilde{L}}{m} - \frac{e}{m} A_\varphi$ and $\mathcal{X} = \frac{E - \omega L}{m}$, we can express $u^r = A \frac{p_r}{m}$ in a form:

$$(u^r)^2 = \frac{A}{N^2}\left(\mathcal{X}^2 - N^2\left(1 + \frac{\mathcal{L}^2}{g_{\varphi\varphi}}\right)\right). \tag{100}$$

(i) If $\tilde{E} \neq \omega_H(\tilde{L} - eA_\varphi)_H - e(A_t)_H$, $\mathcal{X}_H \neq 0$, so $s = 0$ that gives us a usual trajectory.
(ii) If $\tilde{E} = \omega_H(\tilde{L} - eA_\varphi)_H - e(A_t)_H$, $\mathcal{X}_H = 0$, so $s > 0$ that gives us a unusual trajectory.

Now let us check what conditions we get in the case of the Kerr-Newman-(anti-)de Sitter space-time.

A. Motion in Kerr-Newman-(anti-)de-Sitter spacetime

To test our results in some specific case, we use Kerr-Newman-(anti-)de Sitter solution. In the Boyer-Lindquist coordinates, it has a form (see p. 209-210 in [19]):

$$ds^2 = -\frac{\Delta_r}{\Xi^2\rho^2}(dt - a\sin^2\theta d\varphi)^2 + \frac{\varrho^2}{\Delta_r}dr^2 + \frac{\varrho^2}{\Delta_\theta}d\theta^2 + \frac{\Delta_\theta \sin^2\theta}{\Xi^2\rho^2}\left(adt - (r^2+a^2)d\varphi\right)^2, \quad (101)$$

$$\rho^2 = r^2 + a^2\cos^2\theta, \quad (102)$$

$$\Delta_r = (r^2 + a^2)\left(1 - \frac{1}{3}\Lambda r^2\right) - 2Mr + Q^2, \quad (103)$$

$$\Delta_\theta = 1 + \frac{1}{3}\Lambda a^2\cos^2\theta, \quad (104)$$

$$\Xi = 1 + \frac{1}{3}\Lambda a^2, \quad (105)$$

with the vector potential

$$\mathcal{A} = -\frac{Qr}{\Xi\rho^2}(dt - a\sin^2\theta d\varphi) \Rightarrow \mathcal{A}_t = -\frac{Qr}{\Xi\rho^2},$$
$$\mathcal{A}_\varphi = -a\sin^2\theta A_t. \quad (106)$$

Here, Λ is a cosmological constant, and a is the Kerr parameter, with M being the mass of black hole and Q its charge. Using this solution, we can write

$$N^2 = \frac{\Delta_\theta\Delta_r\rho^2}{\Delta_\theta(r^2+a^2)^2 - \Delta_r a^2\sin^2\theta}, \quad (107)$$

$$\omega = \frac{\Delta_\theta a(r^2+a^2) - \Delta_r a}{\Delta_\theta(r^2+a^2)^2 - \Delta_r a^2\sin^2\theta}, \quad (108)$$

$$A = \frac{\Delta_r}{\rho^2}. \quad (109)$$

Now

$$\mathcal{X} = \tilde{\mathcal{X}} + \frac{e}{m}(1 - a\omega\sin^2\theta)\mathcal{A}_t, \quad (110)$$

where $\tilde{\mathcal{X}} = \frac{\tilde{E} - \omega\tilde{L}}{m}$.

The horizons correspond to zeros of Δ_r function. Then we see that near the horizon, $A \approx N^2 \approx \Delta_r$. The same behavior of A and N^2 near the horizon entails $q = p$, and this number is equal to degeneracy of a horizon. In what follows, we consider motion within the equatorial plane $\theta = \frac{\pi}{2}$ only.

1. Nonextremal horizon

Firstly let us consider a nonextremal horizon. In this case, $\Delta_r \approx \alpha v$, where α is a constant. Expanding (110) and taking into account that $\theta = \pi/2$, we have

$$\mathcal{X} = \tilde{\mathcal{X}}_H + \frac{e}{m}(\mathcal{A}_t)_H(1 - a\omega_H) + O(v). \quad (111)$$

If $\tilde{E} = \omega_H\tilde{L} + e(A_t)_H(1 - a\omega_H)$, then $\mathcal{X} \sim v, s = 1$. However, a corresponding particle cannot reach the horizon because, as follows from Table I, s has to satisfy $0 < s \leq \frac{p}{2}$, which, in our case, gives $0 < s \leq \frac{1}{2}$.

If $\tilde{E} \neq \omega_H\tilde{L} + e(A_t)_H(1 - a\omega_H)$, $\mathcal{X}$ is not zero on the horizon; that gives us $s = 0$ (a usual trajectory). Using (111), we deduce that $s_1 = 1$. According to (90), that gives us $n_1 = \frac{q}{2} - p = -\frac{1}{2}$. We see that in this case, acceleration in OZAMO frame diverges.

Thus, we conclude that for nonextremal horizons electrogeodesics in the Kerr-Newman-(anti-)de Sitter space-time, the existence of critical particles is forbidden, while usual ones experience infinite acceleration in the OZAMO frame.

2. Extremal horizon

In the case of extremal horizon, $q = p = 2$. Expansion for $\mathcal{X}$ has the similar structure:

$$\mathcal{X} = \tilde{\mathcal{X}}_H - \frac{e}{m}(\mathcal{A}_t)_H(1 - a\omega_H) + O(v). \quad (112)$$

As in the previous case, if $\tilde{E} = \omega_H\tilde{L} + e(A_t)_H(1 - a\omega_H)$, then $\mathcal{X} \sim v, s = 1$. Such the near-horizon behavior of $\mathcal{X}$ is allowed, because now $s = \frac{p}{2}$, which gives us the critical particle. Also note that $k = 1$. As in this case, $k = p/2$, we are in region IV. According to Table V, this gives us the critical trajectory (second line). As follows from (76), in this case, $n_1 = \frac{q-2}{2}$. As now $q = 2$, we obtain $n_1 = 0$; thus, such a particle experiences an action of a finite force.

If $\tilde{E} \neq \omega_H\tilde{L} + e(A_t)_H(1 - a\omega_H)$, $\mathcal{X}_H \neq 0$, which gives us $s = 0$. From (111), we derive that $s_1 = 1$. In this case, using (90), we obtain $n_1 = -1$. Thus, acceleration is divergent. Thus, for extremal horizons, only critical electro-geodesics can reach the horizon with a finite force in the OZAMO frame.

3. Ultraextremal horizon

In this case, the horizon is triple, $p = q = 3$. We can write the radial function Δ_r in the form

$$\Delta_r = -\frac{\Lambda}{3}(r-b)^3(r+r_0). \tag{113}$$

Comparing this with (102), we see that such factorization is possible only if

$$b = \frac{1}{\sqrt{2\Lambda}}\sqrt{1-\frac{\Lambda a^2}{3}}, \qquad r_0 = \frac{3}{\sqrt{2\Lambda}}\sqrt{1-\frac{\Lambda a^2}{3}}, \tag{114}$$

with additional restriction on parameters:

$$Q^2 = \frac{\Lambda}{3}b^3 r_0, \qquad M = \frac{\Lambda}{2}\left(b^2 r_0 - \frac{b^3}{3}\right). \tag{115}$$

As Q and M have to be positive, we have the condition $\Lambda > 0$. This restricts us by the Kerr-Newman-(anti-)de Sitter space-time. Also note that the triple horizon is a cosmological one. However, this does not influence the behavior of accelerations, which we discuss in this section.

In this case, expansion for $\mathcal{X}$ is the same:

$$\mathcal{X} = \frac{\tilde{E}-\omega_H\tilde{L}}{m} - \frac{e}{m}(\mathcal{A}_t)_H(1-a\omega_H) + O(v). \tag{116}$$

If $\tilde{E} = \omega_H\tilde{L} + e(A_t)_H(1-a\omega_H)$, then $\mathcal{X} \sim v, s = 1$. This value of s gives us subcritical particles because the value $s = 1$ is lower than $p/2$, typical of critical and ultracritical particles. According to our classification, k lies in region II since (78) gives us $\frac{p-q}{2} + 1 = 1 \le k = 1 < \frac{p+1-q/2}{2} = \frac{5}{4}$. According to Table IV, this gives us the second line (subcritical particle) with $s = c = 1$ and $n_1 = -\frac{1}{2}$. Thus, we see that the force diverges.

If $\tilde{E} \neq \omega_H\tilde{L} + e(A_t)_H(1-a\omega_H)$, $\mathcal{X}_H \neq 0$, which gives us $s = 0$. From (111) we, as before, deduce that $s_1 = 1$. In this case, using (90), we obtain $n_1 = -\frac{3}{2}$, so acceleration is divergent. Thus, for ultraextremal horizons, both subcritical and usual particles experience infinite forces in the OZAMO frame.

B. Verifying results

Now let us verify our predictions explicitly. In [20,21], a reader can find equations of trajectory of particle in Kerr-Newman-(anti-)de Sitter space-time:

$$\frac{dr}{d\tau} = -\frac{\sqrt{R(r)}}{r^2}, \tag{117}$$

where for equatorial motion

$$R(r) = \left(\left(1+\frac{\Lambda}{3}a^2\right)(E(r^2+a^2)-aL)+eQ\right)^2 - \Delta_r\left(\left(1+\frac{\Lambda}{3}a^2\right)(aE-L)^2+m^2r^2\right). \tag{118}$$

In the nonextremal case $p = q = 1$, we have $\Delta_r \approx \alpha v$, where α is a constant. Calculating acceleration [using (46) and (44)], we find

$$a^r = \frac{3eQ}{r_h^4(3a^2+\Lambda)^2}((3a^2+\Lambda)(E(a^2+r_h^2)-aL) - 3eQr_h) + O(v). \tag{119}$$

We see that a^r is finite. However, in the OZAMO frame, $a_o^{(r)} = \frac{a^r}{\sqrt{A}} \sim \frac{1}{\sqrt{v}}$, so it diverges as we concluded in Sec. XI A 1.

For the extremal case $p = q = 2$, we have $\Delta \approx \alpha v^2$, and

$$a^r = \frac{3eQ}{r_h^4(3a^2+\Lambda)^2}((3a^2+\Lambda)(E(a^2+r_h^2)-aL)-3eQr_h)+ \tag{120}$$

$$+\frac{3eQ}{r_h^5(3a^2+\Lambda)^2}(9eQr_h - 2(3a^2+\Lambda)(E(2a^2+r_h^2) - 2aL))v + O(v^2). \tag{121}$$

If $E = \frac{aL}{r^2+a^2} + \frac{eQr}{r^2+a^2}\frac{1}{1+\frac{\Lambda}{3}a^2}$, this is equivalent to the condition $E = \omega_H L - eA_H(1-a\omega_H)$, which gives the critical trajectory, and only the second term survives, so $a^r \sim v$. Thus, $a_o^{(r)} = \frac{a^r}{\sqrt{A}} = O(1)$. If $E \neq \frac{aL}{r^2+a^2} + \frac{eQr}{r^2+a^2}\frac{1}{1+\frac{\Lambda}{3}a^2}$, that gives rise to a usual trajectory, $a^r = O(1)$. Thus, we have $a_o^{(r)} = \frac{a^r}{\sqrt{A}} \sim \frac{1}{v}$. So, as we concluded in Sec. XI A 2, for critical particles, acceleration in the OZAMO frame is finite, while for usual one, it diverges.

In the ultraextremal case $p = q = 3$, we have $\Delta_r \approx \alpha v^3$, and we have

$$a^r = \frac{3eQ}{r_h^4(3a^2+\Lambda)^2}((3a^2+\Lambda)(E(a^2+r_h^2)-aL)-3eQr_h)+ \tag{122}$$

$$+\frac{3eQ}{r_h^5(3a^2+\Lambda)^2}(9eQr_h - 2(3a^2+\Lambda)(E(2a^2+r_h^2) - 2aL))v + O(v^2). \tag{123}$$

We see that if $E = \frac{aL}{r^2+a^2} + \frac{eQr}{r^2+a^2}\frac{1}{1+\frac{\Lambda}{3}a^2}$, that is equivalent to the condition $E = \omega_H L - eA_H(1-a\omega_H)$. As now $s = c = 1 < \frac{p}{2}$, this corresponds to the subcritical trajectory (line 2 in Table I). In the above formula, for a^r, only the second term survives, so $a^r \sim v$. Thus, $a_o^{(r)} = \frac{a^r}{\sqrt{A}} \sim \frac{1}{\sqrt{v}}$. If $E \neq \frac{aL}{r^2+a^2} + \frac{eQr}{r^2+a^2}\frac{1}{1+\frac{\Lambda}{3}a^2}$, $a^r = O(1)$ and $a_o^{(r)} = \frac{a^r}{\sqrt{A}} \sim \frac{1}{v^{3/2}}$. So, for subcritical and usual particles, the acceleration in the OZAMO frame diverges.

We see that all the results for the metric under discussion agree completely with our general scheme.

XII. RESTRICTIONS AND RESERVATIONS

In the present work, we considered collisions in the test particles approximation. This means that we neglect backreaction of particles on the metric. This is just the same approximation that was made in pioneering works [1–4]. Clearly, account of self-gravitation can change the results qualitatively. For example, in the paper [22], collisions of massive spherically symmetric charged shells were considered. It was shown that the self-gravitation bounds the energy in the center of mass frame that otherwise would be as large as one likes [23]. In doing so, these authors found that according to their Eq. (42), the factor $\eta = (\frac{M_1}{\mu})^{1/4}$ appears, which restricts $E_{\mathrm{c.m.}}$, where M_1 is the mass of a central black hole, μ being the proper mass of the shell, $M_1 \gg \mu$. Although the restriction does indeed take place, this factor, being finite, is nonetheless very large, $\eta \gg 1$, so qualitatively the effect remains.

In [24], another approach was used for collisions of particles (not shells) that was based on the hoop conjecture [25]. Remarkably, the same factor η was obtained there. Quite recently, Ref. [26] appeared in which the hoop conjecture was applied to scenarios in the background of rotating black holes, with the conclusion that this conjecture regularizes $E_{\mathrm{c.m.}}$, making it finite, but this quantity remains quite high. For example, these authors found that $E_{\mathrm{c.m.}}$ can be only 3 orders of magnitude less than the Planck energy.

In our view, all this is quite natural. If there exists an effect that leads to indefinitely large values of energy, something should exist that bounds it from above, leaving it finite but large. In this respect, the test particle approximation used in our work (as well as almost all works on the BSW effect) is the only first step. The scenarios considered in our paper are much more involved than those in [1,23], since they take into account forces of different nature and more types of trajectories. Therefore, in our context, the effect of self-gravity is less obvious in advance. It looked reasonable to study collisions firstly in the text particle approximation and only afterward to include into consideration self-gravitation in addition to aforementioned factors. We hope to return to this issue in our future work.

XIII. SUMMARY AND CONCLUSIONS

Thus, we constructed classification of near-horizon trajectories according to their four-velocities and four-accelerations. We singled out so-called usual particles (without fine-tuned parameters) and fine-tuned ones. As the necessary condition of the BSW effect implies the process with participation of a fine-tuned particle, the main emphasis was made on investigation of properties of such particles. In turn, the set of fine-tuned particles is split to subcritical, critical, and ultracritical ones depending on their near-horizon behavior. We found the conditions when the components of acceleration remains finite for each type of a trajectory. For fine-tuned particles, the relevant frame for measuring these components is the OZAMO frame since a corresponding observer does not cross the horizon, similarly to a fine-tuned particle.

The properties of the metric are characterized by the set of three numbers p, q, k responsible for the near-horizon behavior. We also introduced the numbers n_0, n_1, n_2, which show the rate with which the tetrad components of the acceleration in the OZAMO frame change near the horizon. Then, the requirement of finiteness of acceleration for fine-tuned particles reduces to the conditions $n_i \geq 0$ for $i = 0, 1, 2$. These conditions lead to constraints in the space of relevant parameters describing the metric. The results of our work are presented in the Tables I–VIII.

A separate interesting issue that revealed itself in the course of our investigation is the principle of kinematic censorship. By itself, it looks very simple or even trivial since it is obvious that in any act of collision, the energy $E_{\mathrm{c.m.}}$ cannot be infinite. Meanwhile, as we saw it, the proof of the fact that this principe is indeed realized in all scenarios under study turned out quite nontrivial in our context. We showed that kinematic censorship is indeed preserved. Namely, either (i) the proper time required to reach the horizon for a fine-tuned particle participating in the BSW process is infinite, (ii) the force diverges, or (iii) the horizon fails to be regular. Actually, this principle is a power tool that enables one to select between possible and forbidden scenarios, even without having explicit solutions of particle motion.

To verify the obtained results, we checked them using the Kerr-Newman-(anti-)de Sitter metric as an example.

Although our main motivation was connected with the study of the BSW effect, the obtained results for the relationship between the type of trajectory and acceleration can be of some use in more general contexts.

It is of interest to extend the present results to non-equatorial motion, the BSW processes with circle orbits, and near-critical particles. Also, it would be interesting to take into account the effects of self-gravitation briefly mentioned in the preceding section.

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