Uniqueness of dark matter magnetized static black hole spacetime

Marek Rogatko [*]

Institute of Physics Maria Curie-Sklodowska University 20-031 Lublin, pl. Marii Curie-Sklodowskiej 1, Poland

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The uniqueness problem of static axially symmetric black hole in a magnetic universe filled with dark matter component is considered in this paper. The dark matter model comprises of the additional $U(1)$ gauge field (dark photon) interacting with the Maxwell one through the kinetic mixing term. We show that all the solutions of Einstein-Maxwell dark photon gravity subject to the same boundary and regularity conditions authorize the only static axially symmetric black hole solutions with nonvanishing time and azimuthal components of Maxwell and hidden-sector gauge fields, e.g., Schwarzschild-like black hole immersed in a dark matter Melvin universe.

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I. INTRODUCTION

Elucidating the dark matter sector which comprises of over 23% of the mass of the observable Universe, and interacts principally with ordinary visible sector through gravity, is one of the predominant pursuit in observational astrophysics; experimental high-energy physics and theoretical attempts of explaining its origin. Astrophysical observations reveal that nonbaryonic cold dark matter comprises the dominant factor for the formation of large-scale structures in the Universe, motion of galaxies, and clusters of galaxies, as well as, playing a crucial role in light bending coming from outer space [1–4].

On the other hand, our times are known for black hole physics, from the first LIGO gravitational wave detection to the Event Horizon Telescope (EHT) images of a black hole shadow.

Studies of black holes in magnetic field introduce an interesting problem on its own; namely, the effect of cosmological magnetic fields might lead to interesting astrophysical behaviors in the nearby of them. Secondly, because of the fact that black hole magnetic solutions are not asymptotically flat ones, they also constitute an interesting mathematical problem. Moreover, the problem of a magnetic field in close to a black hole is interesting from the point of view of the recent measurements done by the EHT team.

For the first time, a regular static cylindrically symmetric solution describing a uniform magnetic field in general relativity were presented by Melvin in [5,6]. Next, the problem of a rotating time-dependent magnetic universe was contemplated in [7], while the case of gravitational waves and charged matter ones, traveling through a magnetic universe was studied in Refs. [8,9]. The influence of the cosmological constant on the properties of Melvin universe has been analyzed in [10,11].

Black holes immersed in a magnetic universe have also attracted much attention too. Namely, the magnetic Kerr and Kerr-Newman solutions were revealed in [12], while the magnetized Kerr-Taub-NUT and Kerr-Newman-Taub-NUT solutions were elaborated in Refs. [13,14]. Additionally, the ultrarelativistic boosts of black holes in an external electromagnetic field was studied in [15,16] and studies of the ergoregions and thermodynamics of a magnetized black hole were presented in [17] [see also the earlier works connected with black holes in a magnetized universe, e.g., [18])].

Magnetized black hole solutions were also scrutinized in generalizations of Einstein theory of gravity (i.e., a Melvin universe with nontrivial dilaton and axion fields was founded [19,20]) where the dilaton C-metric in a dilaton magnetic universe was presented [21], and the case of a pair creation of the extremal black hole and Kaluza-Klein monopoles was examined in [22]. A Melvin-like solution with a Liouville-type potential was given in Ref. [23], an electrically charged dilaton black hole in magnetic field was analyzed in [24], while the generalization of the aforementioned problems to a higher-dimensional gravity was the subject of examination in Refs. [25,26].

This interesting class of subjects was developed using the Ernst's solution generation technique which enabled us to scrutinize black hole solutions in a magnetic universe in Einstein-Maxwell theory coupled conformally to a scalar field [27], axisymmetric stationary black holes with cosmological constant [28], the C-metric with conformally coupled scalar field in a magnetic universe [29], as well as the regular solution describing a couple of charged spinning black holes in an external electromagnetic field [30].

[*] rogat@kft.umcs.lublin.pl

Moreover, the solution unifying both the magnetic Bertotti-Robinson and the Melvin solution as a single axisymmetric line element was revealed in [31].

Recent studies revealed that Melvin-type solution could be found in gravity theories minimally coupled to any nonlinear electromagnetic theory, including Born-Infeld electrodynamics [32].

Furthermore, gravitational collapse, physics of black holes, as well as the uniqueness theorem for black holes (the mathematical formulation of Wheeler's black hole no-hair conjecture) attract much attention. The problem of the classification of the domains of the outer communication of suitably regular black hole spacetimes in Einstein gravity has been widely elaborated on in [33].

Higher-dimensional generalization of gravity theory motivated by contemporary unifications schemes such as M/string theories the classification of higher-dimensional charged black holes both with nondegenerate and degenerate component of the event horizon has been exploited in [34], while the nontrivial case of n-dimensional rotating black objects (black holes, black rings, or black lenses) uniqueness theorem were revealed in [35].

The quest for a consistent quantum gravity theory triggered interest in the mathematical aspects of black holes in the low-energy limit of the string theories and supergravity [36], and various modifications of the Einstein gravitylike Gauss-Bonnet extension [37,38] such as Chern-Simons modified gravity [39,40], while the classification of static black holes in the Einstein phantom-dilaton Maxwell/anti-Maxwell gravity systems has been given in [41].

On the other hand, the uniqueness theorem for a black hole in a magnetic universe in Ernst-Maxwell theory was elaborated on in [42], whereas the magnetic Einstein-Maxwell dilaton gravity case was treated in [43].

Motivated by the above key problems (i.e., dark matter, black hole classifications, and the influence of the magnetic field on the no-hair theorem) the main aim of our paper is to study the uniqueness of static black hole solutions in a magnetic universe. An additional point in our research will be boundedness with the influence of the dark sector on these objects. We shall pay attention to the so-called dark photon model, where the ordinary Maxwell gauge field is supplemented by an auxiliary $U(1)$ gauge field, which interacts with the Maxwell one by the kinetic mixing term.

The organization of our paper is as follows. In Sec. I we describe the main assumptions leading to dark photon theory and derive equations governed by dark matter. Then, one rewrites the Einstein-Maxwell dark photon relations in the form of complex equations fulfilled by redefined gauge field strengths. We also pay attention to the derivation of the dark matter magnetic Melvin solution, which will be needed for the uniqueness theorem for static magnetized black holes with a dark matter sector. Section III is devoted to the boundary conditions of the aforementioned equations of motion. In Sec. IV we achieve the uniqueness of the static magnetized Schwarzschild-like black hole solution in the Melvin universe, i.e., the dark Melvin universe Schwarzschild black hole. Section V concludes our investigations.

II. EQUATIONS OF MOTION

The idea that the dark photon can be a candidate for dark matter has been widely exploited on various backgrounds, both theoretically ([44–49]) and experimentally ([50–55]). Additionally, the model in question possesses some possible astrophysical confirmations [56–60]. We have cited only some illustrative examples due to the vast amount of work authorizing this blossoming field of researches.

To begin with, let us consider the Einstein-Maxwell dark matter gravity, where the dark sector will be described by the additional $U(1)$ gauge field (dark photon) coupled to the ordinary Maxwell one by the so-called kinetic mixing term, describing the interactions of both gauge fields. The action related to Einstein-Maxwell dark photon gravity is provided by

$$S_{\text{EM}-\text{dark photon}} = \int \sqrt{-g} d^4x (R - F_{\mu\nu}F^{\mu\nu} - B_{\mu\nu}B^{\mu\nu} - \alpha F_{\mu\nu}B^{\mu\nu}), \quad (1)$$

where α is taken as a coupling constant between the Maxwell and dark matter field strength tensors.

Introducing the redefined gauge fields $\tilde{A}_\mu$ and $\tilde{B}_\mu$, in the forms as follows:

$$\tilde{A}_\mu = \frac{\sqrt{2-\alpha}}{2}(A_\mu - B_\mu), \quad (2)$$

$$\tilde{B}_\mu = \frac{\sqrt{2+\alpha}}{2}(A_\mu + B_\mu), \quad (3)$$

one can get rid of the kinetic mixing term. Namely, one arrives at

$$F_{\mu\nu}F^{\mu\nu} + B_{\mu\nu}B^{\mu\nu} + \alpha F_{\mu\nu}B^{\mu\nu} \Rightarrow \tilde{F}_{\mu\nu}\tilde{F}^{\mu\nu} + \tilde{B}_{\mu\nu}\tilde{B}^{\mu\nu}, \quad (4)$$

where we have denoted $\tilde{F}_{\mu\nu} = 2\partial_{[\mu}\tilde{A}_{\nu]}$ and $\tilde{B}_{\mu\nu} = 2\partial_{[\mu}\tilde{B}_{\nu]}$, respectively.

The rewritten action (1) is given by

$$S_{\text{EM}-\text{dark photon}} = \int \sqrt{-g} d^4x (R - \tilde{F}_{\mu\nu}\tilde{F}^{\mu\nu} - \tilde{B}_{\mu\nu}\tilde{B}^{\mu\nu}). \quad (5)$$

Variation of the action (5) with respect to $g_{\mu\nu}, \tilde{A}_\mu$, and $\tilde{B}_\mu$ reveals the following equations of motion for Einstein-Maxwell dark matter gravity:

$$R_{\mu\nu} = 2\tilde{F}_{\mu\rho}\tilde{F}_\nu{}^\rho - \frac{1}{2}g_{\mu\nu}\tilde{F}^2 + 2\tilde{B}_{\mu\rho}\tilde{B}_\nu{}^\rho - \frac{1}{2}g_{\mu\nu}\tilde{B}^2, \quad (6)$$

$$\nabla_\mu \tilde{F}^{\mu\nu} = 0, \qquad \nabla_\mu \tilde{B}^{\mu\nu} = 0. \tag{7}$$

In what follows we shall consider the static axially symmetric background, due to the physical meaning of the Melvin spacetime with a magnetic field. In our case, the magnetic field will originate both from the visible and the hidden sector components.

The static axially symmetric line element under inspection yields

$$ds^2 = -e^{2\psi}dt^2 + e^{-2\psi}[e^{2\gamma}(dr^2 + dz^2) + r^2 d\phi^2], \tag{8}$$

where we assume that the functions ψ and γ depend on the r and z coordinates. On the other hand, the symmetry of the elaborated problem enforces that one supposes the existence of time and azimuthal components of the $U(1)$ gauge fields. Consequently they yield

$$\tilde{A}_\mu dx^\mu = \tilde{A}_t dt + \tilde{A}_\phi d\phi, \qquad \tilde{B}_\mu dx^\mu = \tilde{B}_t dt + \tilde{B}_\phi d\phi. \tag{9}$$

As metric components, the gauge fields depend only on the r and z coordinates.

The equations of motion for the considered dark matter Melvin axisymmetric spacetime are provided by

$$\nabla^2\psi - e^{-2\psi}(\tilde{A}_{t,r}^2 + \tilde{A}_{t,z}^2 + \tilde{B}_{t,r}^2 + \tilde{B}_{t,z}^2) - \frac{e^{2\psi}}{r^2}(\tilde{A}_{\phi,r}^2 + \tilde{A}_{\phi,z}^2 + \tilde{B}_{\phi,r}^2 + \tilde{B}_{\phi,z}^2) = 0, \tag{10}$$

$$\nabla_r(re^{-2\psi}\tilde{A}_{t,r}) + \nabla_z(re^{-2\psi}\tilde{A}_{t,z}) = 0, \tag{11}$$

$$\nabla_r(re^{-2\psi}\tilde{B}_{t,r}) + \nabla_z(re^{-2\psi}\tilde{B}_{t,z}) = 0, \tag{12}$$

$$\nabla_r\left(\frac{e^{2\psi}}{r}\tilde{A}_{\phi,r}\right) + \nabla_z\left(\frac{e^{2\psi}}{r}\tilde{A}_{\phi,z}\right) = 0, \tag{13}$$

$$\nabla_r\left(\frac{e^{2\psi}}{r}\tilde{B}_{\phi,r}\right) + \nabla_z\left(\frac{e^{2\psi}}{r}\tilde{B}_{\phi,z}\right) = 0, \tag{14}$$

$$\frac{\gamma_{,z}}{r} - 2\psi_{,r}\psi_{,z} = -2e^{-2\psi}\tilde{A}_{t,r}\tilde{A}_{t,z} + \frac{2}{r^2}e^{2\psi}\tilde{A}_{\phi,r}\tilde{A}_{\phi,z} - 2e^{-2\psi}\tilde{B}_{t,r}\tilde{B}_{t,z} + \frac{2}{r^2}e^{2\psi}\tilde{B}_{\phi,r}\tilde{B}_{\phi,z}, \tag{15}$$

$$e^{-2\psi}(\tilde{A}_{t,r}^2 - \tilde{A}_{t,z}^2 + \tilde{B}_{t,r}^2 - \tilde{B}_{t,z}^2) + \frac{1}{r}e^{2\psi}(\tilde{A}_{\phi,r}^2 - \tilde{A}_{\phi,z}^2 + \tilde{B}_{\phi,r}^2 - \tilde{B}_{\phi,z}^2) = \psi_{,r}^2 - \psi_{,z}^2 - \frac{\gamma_{,r}}{r}. \tag{16}$$

Equations (13) and (14) can be regarded as integrability conditions for the magnetic scalar potentials $\tilde{A}_3$ and $\tilde{B}_3$. Thus, one obtains

$$\vec{e}_\phi \times \vec{\nabla}\tilde{A}_3 = \frac{e^{2\psi}}{r}\vec{\nabla}\tilde{A}_\phi, \quad \vec{e}_\phi \times \vec{\nabla}\tilde{B}_3 = \frac{e^{2\psi}}{r}\vec{\nabla}\tilde{B}_\phi, \tag{17}$$

where for brevity of the notation we set $\vec{\nabla}g = (\partial_r g, \partial_z g)$, while $\vec{e}_\phi$ is one of the orthonormal triad in the coordinate system (r, z, ϕ). It can be noticed that Eqs. (17) lead to the conditions

$$\partial_r\partial_z\tilde{A}_3 = \partial_z\partial_r\tilde{A}_3, \qquad \partial_r\partial_z\tilde{B}_3 = \partial_z\partial_r\tilde{B}_3. \tag{18}$$

Further, the expressions for $\vec{e}_\phi \times \vec{\nabla}\tilde{A}_\phi$ and $\vec{e}_\phi \times \vec{\nabla}\tilde{B}_\phi$, have been used and the complex potentials are written as

$$\Phi_{(\tilde{F})} = \tilde{A}_t + i\tilde{A}_3, \qquad \Phi_{(\tilde{B})} = \tilde{B}_t + i\tilde{B}_3. \tag{19}$$

In what follows, for brevity of notation, we rewrite the potentials given by Eqs. (19) in the form given by

$$\Phi_{(\tilde{F})} = E_{(\tilde{F})} + iB_{(\tilde{F})}, \tag{20}$$

$$\Phi_{(\tilde{B})} = E_{(\tilde{B})} + iB_{(\tilde{B})}. \tag{21}$$

In view of the definitions (20) and (21) the Maxwell equations yield

$$\partial_r\left(re^{-2\psi}\partial_r\Phi_{(\tilde{F})}\right) + \partial_z\left(re^{-2\psi}\partial_z\Phi_{(\tilde{F})}\right) = 0, \tag{22}$$

$$\partial_r\left(re^{-2\psi}\partial_r\Phi_{(\tilde{B})}\right) + \partial_z\left(re^{-2\psi}\partial_z\Phi_{(\tilde{B})}\right) = 0. \tag{23}$$

In the next step we define complex functions bounded with each of the gauge fields

$$\epsilon_{(i)} = Z - |\Phi_{(i)}|^2 + iY_{(i)}, \tag{24}$$

where $Z = e^{2\psi}$ and $i = \tilde{F}, \tilde{B}$ and one introduces new potentials $k_{(i)}$, provided by

$$\vec{\nabla}Y_{(i)} = -2\,\mathrm{Im}\left(\Phi_{(i)}^*\vec{\nabla}\Phi_{(i)}\right). \tag{25}$$

Consequently the Einstein-Maxwell dark matter system of relations can be rewritten in a couple of complex equations, fulfilled by each of the gauge field $\tilde{F}_{\mu\nu}$ and $\tilde{B}_{\mu\nu}$, which implies the following:

$$\sum_{i=\tilde{F},\tilde{B}}(\mathrm{Re}\epsilon_{(i)} + |\Phi_{(i)}|^2)\nabla^2\epsilon_{(i)} = \sum_{i=\tilde{F},\tilde{B}}\left(\vec{\nabla}\epsilon_{(i)} + 2\Phi_{(i)}^*\vec{\nabla}\Phi_{(i)}\right)\vec{\nabla}\epsilon_{(i)}, \tag{26}$$

$$\sum_{i=\tilde{F},\tilde{B}}(\mathrm{Re}\epsilon_{(i)} + |\Phi_{(i)}|^2)\nabla^2\Phi_{(i)} = \sum_{i=\tilde{F},\tilde{B}}\left(\vec{\nabla}\epsilon_{(i)} + 2\Phi_{(i)}^*\vec{\nabla}\Phi_{(i)}\right)\vec{\nabla}\Phi_{(i)}. \tag{27}$$

The differential operators appearing in (26) and (27) are defined as $\overrightarrow{\nabla} = (\partial_r, \partial_z)$ and $\nabla^2 = (\partial_r^2 + \partial_z^2 + 1/r\partial_r)$ and constitute flat gradient and Laplacian operators in cylindrical coordinates (r, z, ϕ); however, in our considerations we restrict our attention to the functions depending on the (r, z) coordinates.

As in the case of Ernst attitude to the Einstein-Maxwell system of differential equations, the real and imaginary parts of the above relations represent adequate equations of motion for the theory in question. Moreover, it can be noticed that the effective action for the stationary axisymmetric Ernst potentials given by

$$S(\epsilon_{(i)}, \Phi_{(i)}) = \int drdz \sum_{i=\tilde{F},\tilde{B}} \frac{\left(\overrightarrow{\nabla}\epsilon_{(i)} + 2\Phi^*_{(i)}\nabla\Phi_{(i)}\right)\left(\overrightarrow{\nabla}\epsilon^*_{(i)} + 2\Phi_{(i)}\overrightarrow{\nabla}\Phi^*_{(i)}\right)}{\left(\epsilon_{(i)} + \epsilon^*_{(i)} + 2\Phi^*_{(i)}\Phi_{(i)}\right)^2}, \tag{28}$$

leads to the aforementioned system of relations.

A. Melvin dark universe black hole

In the latter section we arrive at the equation of motion for Einstein-Maxwell dark photon system. In order to proceed to the uniqueness proof of static magnetized black hole solution in the theory under considerations, one should specify the boundary conditions at infinity. As in the case of Einstein-Maxwell static black hole solutions, they tend asymptotically to the Melvin magnetic universe solution [42,43]. In the present case the magnetized static black hole solution with dark photon sector ought to tend asymptotically to the dark Melvin universe one. Thus, firstly in this section, we scrutinize the magnetostatic axisymmetric solution of the Einstein-Maxwell dark matter system which we shall call the dark Melvin universe.

The solution will be describe a static magnetic fields stemming from both the visible and dark sectors. As in ordinary Melvin solution in general relativity, the magnetic fields will be given as a bundle of magnetic flux lines being in magnetostatic equilibrium with gravity. The Killing vectors of the underlying spacetime are bounded with time translation symmetry, spatial translation along the axis, rotational symmetry, as well as, boost along the axis. Moreover, the fields under considerations will have zero electric components, i.e., $\tilde{F}_{\alpha\beta}n^\beta = 0$ and $\tilde{B}_{\alpha\beta}n^\beta = 0$.

Having in mind the Maxwell equations of motion for both gauge fields, we define the pseudopotentials

$$\Phi^{(\tilde{F})}_{,z} = -\frac{e^{2\psi}}{r}\tilde{A}_{\phi,r}, \qquad \Phi^{(\tilde{F})}_{,r} = \frac{e^{2\psi}}{r}\tilde{A}_{\phi,z}, \tag{29}$$

$$\Phi^{(\tilde{B})}_{,z} = -\frac{e^{2\psi}}{r}\tilde{B}_{\phi,r}, \qquad \Phi^{(\tilde{B})}_{,r} = \frac{e^{2\psi}}{r}\tilde{B}_{\phi,z}, \tag{30}$$

which enables us to rewrite the equations as follows:

$$\frac{\partial^2\psi}{\partial z^2} + \frac{1}{r}\frac{\partial\psi}{\partial r} + \frac{\partial^2\psi}{\partial r^2} = e^{-2\psi}\left[\left(\frac{\partial\Phi^{(\tilde{F})}}{\partial z}\right)^2 + \left(\frac{\partial\Phi^{(\tilde{F})}}{\partial r}\right)^2 + \left(\frac{\partial\Phi^{(\tilde{B})}}{\partial z}\right)^2 + \left(\frac{\partial\Phi^{(\tilde{F})}}{\partial z}\right)^2\right], \tag{31}$$

$$\frac{1}{r}\frac{\partial\gamma}{\partial r} - 2\frac{\partial\psi}{\partial r}\frac{\partial\psi}{\partial z} = -2e^{-2\psi}\left(\frac{\partial\Phi^{(\tilde{F})}}{\partial z}\frac{\partial\Phi^{(\tilde{F})}}{\partial r} + \frac{\partial\Phi^{(\tilde{B})}}{\partial z}\frac{\partial\Phi^{(\tilde{B})}}{\partial r}\right), \tag{32}$$

$$\frac{1}{r}\frac{\partial\gamma}{\partial r} + \left(\frac{\partial\psi}{\partial z}\right)^2 - \left(\frac{\partial\psi}{\partial r}\right)^2 = e^{-2\psi}\left[\left(\frac{\partial\Phi^{(\tilde{F})}}{\partial z}\right)^2 - \left(\frac{\partial\Phi^{(\tilde{F})}}{\partial r}\right)^2 + \left(\frac{\partial\Phi^{(\tilde{B})}}{\partial z}\right)^2 - \left(\frac{\partial\Phi^{(\tilde{F})}}{\partial z}\right)^2\right], \tag{33}$$

In order to obtain a bundle of magnetic flux lines, one assumes that magnetic fields stemming from both gauge fields are directed along the z-axis and the metric functions depend only on radial coordinates, one obtains

$$\Phi^{(\tilde{F})} = B_0^{(\tilde{F})} z, \qquad \Phi^{(\tilde{B})} = B_0^{(\tilde{B})} z, \tag{34}$$

$$\psi(r) = \ln\left[1 + \frac{1}{4}\left(B_0^{(\tilde{F})2} + B_0^{(\tilde{B})2}\right)r^2\right], \tag{35}$$

$$\gamma(r) = 2\ln\left[1 + \frac{1}{4}\left(B_0^{(\tilde{F})2} + B_0^{(\tilde{B})2}\right)r^2\right], \tag{36}$$

where $B_0^{(\tilde{F})}$ and $B_0^{(\tilde{B})}$ are constant bounded with the strength of the adequate magnetic fields, pertaining to both the visible and dark sectors.

From Eqs. (29) and (30) one finds that

$$\tilde{A}_\phi = \frac{2B_0^{(\tilde{F})}}{\left(B_0^{(\tilde{F})2} + B_0^{(\tilde{B})2}\right)} \frac{1}{\left[1 + \frac{1}{4}\left(B_0^{(\tilde{F})2} + B_0^{(\tilde{B})2}\right) r^2\right]}, \tag{37}$$

$$\tilde{B}_\phi = \frac{2B_0^{(\tilde{B})}}{\left(B_0^{(\tilde{F})2} + B_0^{(\tilde{B})2}\right)} \frac{1}{\left[1 + \frac{1}{4}(B_0^{(\tilde{F})2} + B_0^{(\tilde{B})2}\right) r^2\right]}. \tag{38}$$

It can be seen that the dark photon field influences the obtained potentials. In order to envisage the dark photon impact let us use Eqs. (2) and (3) and rewrite $B_0^{(\tilde{F})}$ and $B_0^{(\tilde{B})}$ as follows:

$$B_0^{(\tilde{F})} = \frac{\sqrt{2-\alpha}}{2}\left(B_0^{(F)} - B_0^{(B)}\right),$$
$$B_0^{(\tilde{B})} = \frac{\sqrt{2+\alpha}}{2}\left(B_0^{(F)} + B_0^{(B)}\right), \tag{39}$$

where $B_0^{(F)}$ denotes constant Maxwell magnetic field and $B_0^{(B)}$ stands for the constant dark photon magnetic component. After some algebra we obtain

$$\tilde{A}_\phi = \frac{\sqrt{2-\alpha}}{2}(P^{(F)} - P^{(B)}), \tag{40}$$

$$\tilde{B}_\phi = \frac{\sqrt{2+\alpha}}{2}(P^{(F)} + P^{(B)}), \tag{41}$$

where we set

$$P^{(F)} = \frac{2B_0^{(F)}}{\left[1 + \frac{1}{4}\left(B_0^{(F)2} + B_0^{(B)2}\right) r^2\right]}, \tag{42}$$

$$P^{(B)} = \frac{2B_0^{(B)}}{\left[1 + \frac{1}{4}\left(B_0^{(F)2} + B_0^{(B)2}\right) r^2\right]}. \tag{43}$$

On the other hand, at large distances of r, the obtained metric reveals that $\psi(r) \simeq 2 \ln r$, as in ordinary Einstein-Maxwell Melvin case [6]. However at infinity, the dark Melvin universe solution approaches to a nonflat solution, which will constitute the crucial point in the boundary conditions and then in the uniqueness theorem.

III. BOUNDARY CONDITIONS

This section will be devoted to the relevant boundary conditions in the case under consideration. In the present case the spacetime is asymptotically cylindrical, i.e., the static magnetized black hole solution will tend asymptotically to the dark Melvin universe, describing the bundle of magnetic flux lines. This fact constitutes the main difference between the studied case and the asymptotically flat one.

To begin with we introduce the two-dimensional manifold, equipped with the spheroidal coordinates provided by the relations

$$r^2 = (\lambda^2 - c^2)(1 - \mu^2), \qquad z = \lambda\mu, \tag{44}$$

where $\mu = \cos\theta$ is chosen in such a way that the black hole event horizon boundary is situated at a constant value of $\lambda = c$. On the other hand, two rotation-axis segments which distinguish the south and the north segments of the event horizon are described by the respective limit $\mu = \pm 1$. We obtain the line element in the form as

$$dr^2 + dz^2 = (\lambda^2 - \mu^2 c^2)\left(\frac{d\lambda^2}{\lambda^2 - c^2} + \frac{d\mu^2}{1 - \mu^2}\right). \tag{45}$$

We choose the domain of outer communication $\langle\langle \mathcal{D} \rangle\rangle$ as a rectangle

$$\partial\mathcal{D}^{(1)} = \{\mu = 1, \lambda = c, \ldots, R\},$$
$$\partial\mathcal{D}^{(2)} = \{\lambda = c, \mu = 1, \ldots, -1\},$$
$$\partial\mathcal{D}^{(3)} = \{\mu = -1, \lambda = c, \ldots, R\},$$
$$\partial\mathcal{D}^{(4)} = \{\lambda = R, \mu = -1, \ldots, 1\}. \tag{46}$$

The relevant boundary conditions may be cast as follows. At infinity, we insist that $Z, \tilde{A}_\phi, \tilde{A}_t$ and $\tilde{B}_\phi, \tilde{B}_t$ are well-behaved functions and the solution under inspection asymptotically tends to the Melvin dark matter universe line element, presented in the preceding section. Namely, they satisfy

$$Z = \left[1 + \frac{1}{4}\left(B_0^{(\tilde{F})2} + B_0^{(\tilde{B})2}\right) r^2\right]^2 (1 + \mathcal{O}(\lambda^{-1})), \tag{47}$$

$$\tilde{A}_\phi = \frac{2B_0^{(\tilde{F})}}{\left(B_0^{(\tilde{F})2} + B_0^{(\tilde{B})2}\right)} \left[1 + \frac{1}{\frac{1}{4}\left(B_0^{(\tilde{F})2} + B_0^{(\tilde{B})2}\right) r^2}\right]^{-1} \times (1 + \mathcal{O}(\lambda^{-1})), \tag{48}$$

$$\tilde{A}_t = \mathcal{O}(\lambda^{-1}), \tag{49}$$

$$\tilde{B}_\phi = \frac{2B_0^{(\tilde{B})}}{(B_0^{(\tilde{F})2} + B_0^{(\tilde{B})2})} \left[1 + \frac{1}{\frac{1}{4}\left(B_0^{(\tilde{F})2} + B_0^{(\tilde{B})2}\right) r^2}\right]^{-1} \times (1 + \mathcal{O}(\lambda^{-1})), \tag{50}$$

$$\tilde{B}_t = \mathcal{O}(\lambda^{-1}), \tag{51}$$

where now r stands for the asymptotical cylindrical coordinate given by $r^2 \to \lambda^2(1-\mu^2)$. The difference in the boundary behaviors at infinity comprises the main distinction between the considered case and the asymptotically flat one. As was previously mentioned the solution should display the asymptotically dark Melvin universe one, in order to reveal the cylindrical nature of the elaborated spacetime.

On the black hole event horizon, where $\lambda \to c$, the quantities in question should behave regularly (see, e.g., [33]), i.e., they yield the following relations:

$$Z = \mathcal{O}(1-\mu^2), \qquad \frac{1}{Z}\partial_\mu Z = -\frac{2\mu}{1-\mu^2} + \mathcal{O}(1), \tag{52}$$

$$\partial_\lambda \tilde{A}_\phi = \mathcal{O}(1-\mu^2), \qquad \partial_\mu \tilde{A}_\phi = \mathcal{O}(1), \tag{53}$$

$$\partial_\lambda \tilde{A}_t = \mathcal{O}(1), \qquad \partial_\mu \tilde{A}_t = \mathcal{O}(1), \tag{54}$$

$$\partial_\lambda \tilde{A}_\phi = \mathcal{O}(1-\mu^2), \qquad \partial_\mu \tilde{A}_\phi = \mathcal{O}(1), \tag{55}$$

$$\partial_\lambda \tilde{A}_t = \mathcal{O}(1), \qquad \partial_\mu \tilde{A}_t = \mathcal{O}(1), \tag{56}$$

$$\partial_\lambda \tilde{B}_\phi = \mathcal{O}(1-\mu^2), \qquad \partial_\mu \tilde{B}_\phi = \mathcal{O}(1), \tag{57}$$

$$\partial_\lambda \tilde{B}_t = \mathcal{O}(1), \qquad \partial_\mu \tilde{B}_t = \mathcal{O}(1), \tag{58}$$

$$\partial_\lambda \tilde{B}_\phi = \mathcal{O}(1-\mu^2), \qquad \partial_\mu \tilde{B}_\phi = \mathcal{O}(1), \tag{59}$$

$$\partial_\lambda \tilde{B}_t = \mathcal{O}(1), \qquad \partial_\mu \tilde{B}_t = \mathcal{O}(1). \tag{60}$$

On the other hand, in the vicinity of the symmetry axis, where $\mu \to 1$ (North Pole segment) and $\mu \to -1$ (South Pole segment), one requires that $\tilde{A}_\phi, \tilde{A}_t, \tilde{B}_\phi, \tilde{B}_t, Z$ should be regular functions of λ and μ, such that

$$Z = \mathcal{O}(1), \qquad \frac{1}{Z} = \mathcal{O}(1), \tag{61}$$

$$\tilde{A}_\phi = \mathcal{O}(1), \qquad \partial_\lambda \tilde{A}_\phi = \mathcal{O}(1), \tag{62}$$

$$\tilde{A}_t = \mathcal{O}(1), \qquad \partial_\lambda \tilde{A}_t = \mathcal{O}(1), \tag{63}$$

$$\tilde{B}_\phi = \mathcal{O}(1), \qquad \partial_\lambda \tilde{B}_\phi = \mathcal{O}(1), \tag{64}$$

$$\tilde{B}_t = \mathcal{O}(1), \qquad \partial_\lambda \tilde{B}_t = \mathcal{O}(1). \tag{65}$$

IV. UNIQUENESS OF SOLUTIONS

To commence with, we recall that the Ernst equations of the type described by relations (26) and (27), can be cast in the matrix type system of equations

$$\partial_r[P^{-1}\partial_r P] + \partial_z[P^{-1}\partial_z P] = 0, \tag{66}$$

where P are 3×3 Hermitian matrices with unit determinants, while including parts bounded with the adequate gauge fields, described by the field strengths $\tilde{F}_{\mu\nu}, \tilde{B}_{\mu\nu}$. For the first time this problem was investigated in [61].

Moreover if one considers any constant invertible matrix A, the matrix built in the form as APA^{-1} constitute the solution of (66). The different forms of these matrices enable us to construct all the transformations referred to the Ernst's system of partial differential equations.

To proceed further let us examine a domain of outer communication $\langle\langle\mathcal{D}\rangle\rangle$ of the two-dimensional manifold $\mathcal{M}$, with boundary $\partial\mathcal{D}$. Suppose next, that the matrix P components are differentiable enough in the domain of outer communication in question. Let us inspect the two different matrix solutions of (66), i.e., P_1 and P_2, subject to the same boundary and differentiability conditions.

The difference between the aforementioned relations fulfils the equation as follows:

$$\nabla(P_1^{-1}(\nabla Q)P_2) = 0, \tag{67}$$

where we have denoted by $Q = P_1 P_2^{-1}$. In the next step one can multiply the equation (67) by $Q^\dagger$ and taking the trace of the result. One arrives at the following:

$$\nabla^2 q = \mathrm{Tr}[(\nabla Q^\dagger)P_1^{-1}(\nabla Q)P_2]. \tag{68}$$

In the above relation we set $q = \mathrm{Tr}Q$. Further the hermicity and positive definiteness of P allow to postulate the form of it given by $P = MM^\dagger$, which leads to the relation

$$\nabla^2 q = \mathrm{Tr}(\mathcal{J}^\dagger \mathcal{J}), \tag{69}$$

where $\mathcal{J} = M_1^{-1}(\nabla Q)M_2$.

Defining homographic change of the variables, for the previously defined quantities connected with both gauge fields, provided by

$$\epsilon_{(i)} = \frac{\xi_{(i)} - 1}{\xi_{(i)} + 1}, \qquad \psi_{(i)} = \frac{\eta_{(i)}}{\xi_{(i)} + 1}, \tag{70}$$

enables us to find that the P matrix implies

$$P_{\alpha\beta} = \eta_{\alpha\beta} - \frac{2\xi_\alpha \bar{\xi}_\beta}{\langle \xi_\delta \bar{\xi}^\delta \rangle}, \tag{71}$$

where we define the scalar product in the form as

$$\langle \xi_\delta \bar{\xi}^\delta \rangle = -1 + \sum_\gamma \xi_\gamma \bar{\xi}^\gamma, \qquad \gamma = 1, \ldots, q. \tag{72}$$

To proceed to the uniqueness of the considered solution, one has to calculate $q_{(i)} = \mathrm{Tr}(P_1 P_2^{-1})$, having in mind the adequate Ernst's potentials $\epsilon_{(i)(1)}$ and $\epsilon_{(i)(2)}$ and their ingredients given by Eqs. (20), (21), and (24), like $E_{(i)1}, B_{(i)1}, Y_{(i)1}, Z_{(i)1}$ and $E_{(i)2}, B_{(i)2}, Y_{(i)2}, Z_{(i)2}$, where $i = \tilde{F}, \tilde{B}$. Consequently, after some algebra, one achieves the relation provided by

$$\begin{aligned} q &= P_{\alpha\beta(1)} P^{\alpha\beta}_{(2)} \\ &= 3 + \frac{1}{Z_1 Z_2} \sum_{i=\tilde{F},\tilde{B}} \bigg[(Z_1 - Z_2)^2 + \frac{1}{4} \Big[(E_{(i)1} - E_{(i)2})^2 + (B_{(i)1} - B_{(i)2})^2 \Big]^2 \\ &\quad + (Z_1 + Z_2) \Big[(E_{(i)1} - E_{(i)2})^2 + (B_{(i)1} - B_{(i)2})^2 \Big] + \Big[\Big(B_{(i)1} E_2^{(i)} - B_{(i)2} E_1^{(i)} \Big) + \frac{1}{2} (Y_{(i)1} - Y_{(i)2}) \Big]^2 \bigg]. \end{aligned} \tag{73}$$

On the other hand, the use of the Stoke's theorem authorizes the integration of the relation given by (69) over the chosen domain of outer communication $\langle\langle \mathcal{D} \rangle\rangle$, described by the relation (46), reveals

$$\begin{aligned} \int_{\partial\langle\langle \mathcal{D} \rangle\rangle} \nabla_m q dS^m &= \int_{\partial\langle\langle \mathcal{D} \rangle\rangle} d\lambda \sqrt{\frac{h_{\lambda\lambda}}{h_{\mu\mu}}} \partial_\mu q |_{\mu=\text{const}} + \int_{\partial\langle\langle \mathcal{D} \rangle\rangle} d\mu \sqrt{\frac{h_{\mu\mu}}{h_{\lambda\lambda}}} \partial_\lambda q |_{\lambda=\text{const}} \\ &= \int_\infty^c d\lambda \sqrt{\frac{h_{\lambda\lambda}}{h_{\mu\mu}}} \partial_\mu q |_{\mu=-1} + \int_c^\infty d\lambda \sqrt{\frac{h_{\lambda\lambda}}{h_{\mu\mu}}} \partial_\mu q |_{\mu=1} + \int_1^{-1} d\mu \sqrt{\frac{h_{\mu\mu}}{h_{\lambda\lambda}}} \partial_\lambda q |_{\lambda=c} + \int_{-1}^1 d\mu \sqrt{\frac{h_{\mu\mu}}{h_{\lambda\lambda}}} \partial_\lambda q |_{\lambda\to\infty} \\ &= \int_{\langle\langle \mathcal{D} \rangle\rangle} \text{Tr}(\mathcal{J}^\dagger \mathcal{J}) dV. \end{aligned} \tag{74}$$

To proceed further, we ought to indicate the behavior of the left-hand side of the equation (74), taking into account the integrals over each part of the domain of outer communication $\langle\langle \mathcal{D}^{(i)} \rangle\rangle$, where $i = 1, \ldots, 4$, taken as the rectangle in the two-dimensional space (μ, λ) and described by the relations (46).

On the black hole event horizon, $\partial\mathcal{D}^{(2)}$, the functions are well-behaved, with the asymptotic given by $\mathcal{O}(1)$. For the r-coordinate given by Eq. (44) we have that $r \simeq \mathcal{O}(\sqrt{\lambda - c})$ as $\lambda \to c$. Moreover, the square root $\sqrt{\frac{h_{\mu\mu}}{h_{\lambda\lambda}}} \simeq \mathcal{O}(\sqrt{\lambda - c})$. It all leads to the conclusion that $\nabla_m q$ vanishes on the black hole event horizon.

On the other hand, on the symmetry axis, $\partial\mathcal{D}^{(1)}$ and $\partial\mathcal{D}^{(3)}$, when $\mu \pm 1$, all the considered quantities are of order $\mathcal{O}(1)$. The r-coordinate tends to $\mathcal{O}(\sqrt{1 - \mu})$, as $\mu \to 1$. When $\mu \to -1$, one obtains that $r \simeq \mathcal{O}(\sqrt{1 + \mu})$. As far as the behavior of the square roots is concerned they are provided by $\sqrt{\frac{h_{\lambda\lambda}}{h_{\mu\mu}}} \simeq \mathcal{O}(\sqrt{1 + \mu})$, when $\mu \to -1$ and $\sqrt{\frac{h_{\lambda\lambda}}{h_{\mu\mu}}} \simeq \mathcal{O}(\sqrt{1 - \mu})$, when $\mu \to 1$. Bearing in mind Eq. (73) enables us to conclude that $\nabla_m q = 0$, for $\mu \pm 1$.

It remains to take into account the contribution for the integration along $\partial\mathcal{D}^{(4)}$, when $\lambda = R \to \infty$. We remark that the main difference between the case under consideration and asymptotically flat one, is in the boundary conditions at infinity, where we insist that all functions in question are well-behaved and have asymptotic behaviors given by Eqs. (47)–(51). The square root in the considered limit tends to $\sqrt{\frac{h_{\mu\mu}}{h_{\lambda\lambda}}} \simeq \mathcal{O}(\lambda)$.

Let us analyze q given by the equation (73), term by term, in the considered limit of λ. They will have the same behavior for each gauge field, described by $\tilde{F}_{\mu\nu}$ and $\tilde{B}_{\mu\nu}$. The first one is equal to the constant value, while the second one is given by

$$\frac{(Z_1 - Z_2)^2}{Z_1 Z_2} \simeq \frac{\Delta B^4_{(1)} - \Delta B^4_{(2)}}{\Delta B^4_{(1)} \Delta B^4_{(2)}} \left(1 + \mathcal{O}\left(\frac{1}{\lambda}\right) \right), \tag{75}$$

where we have denoted

$$\Delta B^2_{(i)} = B^{(\tilde{F},\tilde{B})2}_{0(i)} + B^{(\tilde{B},\tilde{B})2}_{0(i)}, \tag{76}$$

$i = 1, 2$. For the third term in Eq. (73) one obtains the relation

$$\begin{aligned} &\frac{\frac{1}{4} \Big[(E_{(\tilde{F},\tilde{B})1} - E_{(\tilde{F},\tilde{B})2})^2 + (B_{(\tilde{F},\tilde{B})1} - B_{(\tilde{F},\tilde{B})2})^2 \Big]^2}{Z_1 Z_2} \\ &\quad \simeq \frac{\mu^4 \Big(B^{(\tilde{F},\tilde{B})}_{0(1)} - B^{(\tilde{F},\tilde{B})}_{0(1)} \Big)^2}{\Delta B^4_{(1)} \Delta B^4_{(1)} (1 - \mu^2)^2} \mathcal{O}\left(\frac{1}{\lambda^6}\right), \end{aligned} \tag{77}$$

while the fourth one implies

$$\frac{(Z_1+Z_2)}{Z_1Z_2}\left[(E_{(\tilde{F},\tilde{B})1}-E_{(\tilde{F},\tilde{B})2})^2+(B_{(\tilde{F},\tilde{B})1}-B_{(\tilde{F},\tilde{B})2})^2\right] \simeq \frac{\mu^4\left(\Delta B^4_{(1)}+\Delta B^4_{(2)}\right)\left(B^{(\tilde{F},\tilde{B})}_{0(1)}-B^{(\tilde{F},\tilde{B})}_{0(2)}\right)^2}{\Delta B^4_{(1)}\Delta B^4_{(2)}}\mathcal{O}\left(\frac{1}{\lambda^2}\right). \tag{78}$$

For the last one, the fifth term, it can be revealed that

$$\frac{\left[\left(B_{(\tilde{F},\tilde{B})1}E_2^{(\tilde{F},\tilde{B})}-B_{(\tilde{F},\tilde{B})2}E_1^{(\tilde{F},\tilde{B})}\right)+\frac{1}{2}\left(Y_{(\tilde{F},\tilde{B})1}-Y_{(\tilde{F},\tilde{B})2}\right)\right]^2}{Z_1Z_2} \simeq \frac{\left(B^{(\tilde{F},\tilde{B})}_{0(1)}-B^{(\tilde{F},\tilde{B})}_{0(2)}\right)^2(1-2\mu^2)}{\Delta B^4_{(1)}\Delta B^4_{(2)}(1-\mu^2+\mu)^2(1-\mu^2)^2}\mathcal{O}\left(\frac{1}{\lambda^6}\right). \tag{79}$$

Consequently the function q displays the following way of acting:

$$q|_{\lambda\to\infty}\simeq\mathcal{O}(1)+\mathcal{O}\left(\frac{1}{\lambda^6}\right)+\mathcal{O}\left(\frac{1}{\lambda^2}\right)+\mathcal{O}\left(\frac{1}{\lambda^6}\right)\simeq\mathcal{O}(1). \tag{80}$$

In view of the above relations, one has that q tends to a constant value, as $\lambda\to\infty$. All the aforementioned arguments lead to the conclusion that

$$\int_{\langle\langle\mathcal{D}\rangle\rangle}\mathrm{Tr}(\mathcal{J}^\dagger_{(i)}\mathcal{J}_{(i)})=0. \tag{81}$$

This relation implies that $P_{(i)1}=P_{(i)2}$ at all points belonging to the domain of outer communication, being a two-dimensional manifold $\mathcal{M}$ with coordinates (r,z).

Thus, if we consider two black hole solutions of the Einstein-Maxwell dark photon gravity characterized by $(Z_{(1)},\tilde{A}_{t(1)},\tilde{A}_{\phi(1)},\tilde{B}_{t(1)},\tilde{B}_{\phi(1)})$ and $(Z_{(2)},\tilde{A}_{t(2)},\tilde{A}_{\phi(2)},\tilde{B}_{t(2)},\tilde{B}_{\phi(2)})$, respectively, being subject to the same boundary and regularity conditions are identical.

In summary, the consequences of our research can be summarized as follows:

Theorem: Let $\langle\langle\mathcal{D}\rangle\rangle$ be a domain of outer communication constituting a region of a two-dimensional manifold with coordinates (r,z) defined by (44), having a boundary $\langle\langle\partial\mathcal{D}\rangle\rangle$. Suppose, that $P_{(i)}$ are Hermitian positive, three-dimensional matrices, with unit determinants. On the boundary of the domain $\langle\langle\partial\mathcal{D}\rangle\rangle$, matrices $P_{(1)}$ and $P_{(2)}$ being the solution of the equation

$$\partial_r[P^{-1}\partial_rP]+\partial_z[P^{-1}\partial_zP]=0,$$

satisfy the relation $\nabla_mq=0$. Then, $P_{(1)}=P_{(2)}$ in all domain of outer communication $\langle\langle\mathcal{D}\rangle\rangle$, implying that for at least one point $d\in\langle\langle\mathcal{D}\rangle\rangle$, one has that $P_{(1)}(d)=P_{(2)}(d)$.

In other words, all the solutions of Einstein-Maxwell dark photon gravity subject to the same boundary and regularity conditions, say a dark Melvin universe Schwarzschild-type black hole, comprise the only static, axisymmetric symmetric black hole solution, possessing a regular event horizon with nonvanishing $\tilde{A}_t,\tilde{A}_\phi,\tilde{B}_t$, and $\tilde{B}_\phi$ components of the Maxwell visible and hidden-sector gauge fields.

V. CONCLUSIONS

Our paper is devoted to the uniqueness problem of static axially symmetric black hole spacetime in Einstein-Maxwell dark photon gravity.

The dark photon model comprises a new Abelian gauge field coupled to the ordinary Maxwell one, by the kinetic mixing term. The model in question is subject of extensive theoretical and experimental studies.

The equations of motion for the dark photon Einstein-Maxwell gravity can be rewritten in the form of Ernst-like system of complex relations, which can be rearranged in the form of matrix equations. In the studies, the domain of outer communication $\langle\langle\mathcal{D}\rangle\rangle$ was chosen as a two-dimensional manifold with coordinates (r,z). It has been revealed that the two matrix solutions of the equations of motion, subject to the same boundary and regularity conditions, are equal in the considered domain of outer communication.

One may conclude that a Schwarzschild Melvin-like solution with dark photon (representing model of dark matter sector), is the only axisymmetric static black hole in Einstein-Maxwell dark photon gravity with nonzero components of visible Maxwell and hidden-sector $U(1)$ gauge components provided by $\tilde{A}_t,\tilde{A}_\phi,\tilde{B}_t$, and $\tilde{B}_\phi$, being Schwarzschild-type black hole immersed in a magnetic Melvin universe, filled with dark matter.

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