Physical nonviability of a wide class of $f(R)$ models and their constant-curvature solutions

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Constant-curvature solutions lie at the very core of gravitational physics, with Schwarzschild and (anti-) de Sitter being two of the most paradigmatic examples. Although such a kind of solutions are very well-known in general relativity, that is not the case for theories of gravity beyond the Einsteinian paradigm. In this article, we provide a systematic overview on $f(R)$ models allowing for constant-curvature solutions, as well as of the constant-curvature solutions themselves. We conclude that the vast majority of these $f(R)$ models suffer, in general, from several shortcomings rendering their viability extremely limited, when not ruled out by physical evidence. Among these deficiencies are instabilities (including previously unforeseen strong-coupling problems) and issues limiting the predictive power of the models. Furthermore, we will also show that most $f(R)$-exclusive constant-curvature solutions also exhibit a variety of unphysical properties.

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I. INTRODUCTION

When modeling physical systems, it is often necessary to resort to simplifying assumptions, either to make the equations describing the problem more tractable or to gain further insight on the relevant physics. It seems obvious that such assumptions should be physically well-motivated and consistent with both experiments and the theoretical framework being employed. For example, within the general relativity (GR) framework, the explanation of a variety of cosmological observations relies on the crucial assumption that the Universe is *approximately* homogeneous and isotropic at sufficiently large scales.

However, despite its success in describing most gravitational phenomena, GR still suffers from several shortcomings, such as its inability to describe dark energy without introducing a new, *ad hoc* fluid in the theory. This and other weaknesses can be solved by generalizing GR, for instance by postulating that the gravitational Lagrangian is given by a function $f(R)$ of the Ricci scalar R, instead of just R (as in Einsteinian gravity). Though simple, the $f(R)$ ansatz turns out to comply with the basic consistency requirements outlined above in most circumstances. Except in some pathological examples, $f(R)$ models are mathematically consistent, can be reduced to standard GR in the appropriate limit, and yield predictions which are in accordance with observations. Indeed, appropriate choices of function f lead to a correct description of both early- and late-universe physics, such as inflation and the aforementioned dark-energy-dominated epoch. $f(R)$ models have also found applications in stellar physics, with some of them being compatible with neutron-star and gravitational-wave observations [1–7].

Nonetheless, outside of the highly symmetric cosmological scenarios, it is often very complicated to solve the fourth-order equations of metric $f(R)$ gravity without making further simplifying assumptions. Currently, one of the most popular choices is to find solutions with constant scalar curvature R. Indeed, some of the most well-known solutions of GR have constant curvature, such as the Schwarzschild or Vaidya spacetimes, their generalizations including a cosmological constant, or the Friedmann-Lemaître-Robertson-Walker (FLRW) spacetime sourced by radiation. It has been known for a long time that many $f(R)$ gravity models host only the same vacuum constant-curvature solutions as GR [8]. However, as will be further detailed in Sec. II, there is a particular set of $f(R)$ models satisfying some additional assumptions [9,10] for which *any* metric with a given constant Ricci scalar is a solution of said $f(R)$ model. Because of this, in what follows we shall refer to these special $f(R)$ models admitting all spacetimes with $R = R_0 = \text{const}$ as R_0-

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degenerate $f(R)$ *models*. In order to find new constant-curvature solutions of R_0-degenerate $f(R)$ models which are not present in GR, one merely needs to solve equation $R = R_0 = \text{const}$ for some particular metric ansatz and initial conditions.

For these reasons, it is almost immediate to obtain novel, $f(R)$-exclusive constant-curvature solutions that provide an answer to virtually *every* open problem in gravitational physics. For example, $f(R)$-exclusive constant-curvature solutions describing wormholes made out of pure vacuum (and thus complying with the standard energy conditions), exotic black holes, or even spacetimes giving rise to the observed rotation curves of galaxies without introducing dark matter have been reported in the literature [10–13]. The approach has furthermore been generalized to other modified gravity theories, such as $f(Q)$, where a similar situation occurs [14].

In the present work, we shall show that the vast majority of R_0-degenerate $f(R)$ models, as well as their constant-curvature solutions themselves, are most often pathological in nature, and thus not physically viable. In particular, we will prove that this special class of $f(R)$ models only propagates one scalar degree of freedom at linear level [in contrast with generic $f(R)$ models and GR], thus being incompatible with gravitational-wave observations. Furthermore, as we shall discuss, R_0-degenerate $f(R)$ models apparently lack predictability, because of the aforementioned infinite degeneracy of their constant-curvature solutions. Finally, to make matters worse, we will also show that most of the novel constant-curvature solutions are unstable and host a number of unphysical properties, such as regions in which the metric signature changes abruptly, naked curvature singularities, and more.

We remark that the instabilities we present in this communication are different in nature from the well-known matter instability displayed by some $f(R)$ models in the presence of matter [15]. In particular, we have discovered that $f(R)$-exclusive constant-curvature vacuum solutions can be unstable, and that the all important Minkowski background (being a constant-curvature vacuum solution itself) is strongly coupled in the $f(R)$ models we denote as $(R_0 = 0)$-degenerate.

The contents of this work will be organized as follows. First, in Sec. II, we shall briefly review the conditions $f(R)$ models must satisfy so as to harbor any metric with a given constant Ricci scalar. Second, the pathological character of the $f(R)$ models within this special class will be discussed in Sec. III. More precisely, the fact that the linearized spectrum of said models contains at most one massless scalar field only will be proven therein. Next, in Sec. IV, we shall assess the stability of $f(R)$-exclusive constant-curvature solutions under small perturbations of their Ricci scalar. Finally, Sec. V will be devoted to a characterization of several classes of novel constant-curvature solutions, some of which have not yet been reported in the literature, as far as we are concerned. We shall then conclude that most of the solutions analyzed in the latter section display a variety of unphysical properties. Supplementary discussions and examples are provided in the appendixes. Appendix A shall be devoted to clarifying the subtleties appearing when one analyzes R_0-degenerate constant-curvature solutions using the Einstein frame representation of the theory (as done in Sec. IV). Next, in Appendix B, we show that the representation of the novel constant-curvature solutions on the Brans-Dicke (scalar-tensor) representation of R_0-degenerate $f(R)$ models is trivial. Subsequently, so as to better illustrate the conditions under which an $f(R)$ model is R_0-degenerate, in Appendix C, we consider a particularly simple $f(R)$ model which is R_0-degenerate for one particular value of the Ricci scalar, but which also hosts nondegenerate constant-curvature solutions for a different value of R. Finally, in Appendix D, we collect all the mathematical definitions and conventions which are employed in the analysis performed in Sec. II.

The busy reader is encouraged to focus on Sec. II, containing our precise definitions of *constant-curvature solutions* and R_0-*degenerate* $f(R)$ *models*; Results 1 and 2, containing our most relevant findings regarding the existence of strong-coupling instabilities in R_0-degenerate $f(R)$ models; Results 3 and 4, concerning the stability of constant-curvature solutions within R_0-degenerate $f(R)$ models; Table I, summarizing all the pathological traits displayed by the $f(R)$-exclusive solutions discussed herein; and, finally, the conclusions and final discussions collected in Sec. VI.

Before proceeding with the results of our investigations, let us enumerate, for the sake of clarity, the various notational conventions to be followed hereafter. Our sign choice shall be the one denoted as $(-,-,-)$ by Misner *et al.* [17]: the metric signature will be $(+,-,-,-)$, the Riemann and Ricci tensors are defined as $R^{\rho}{}_{\sigma\mu\nu} \equiv -2(\partial_{[\mu|}\Gamma^{\rho}{}_{\sigma|\nu]} + \Gamma^{\rho}{}_{\lambda[\mu|}\Gamma^{\lambda}{}_{\sigma|\nu]})$ and $R_{\mu\nu} \equiv R^{\rho}{}_{\mu\rho\nu}$, respectively, and the Einstein field equations read $G_{\mu\nu} = -\kappa T_{\mu\nu}$, with $\kappa \equiv 8\pi G \quad (c = 1)$ and $T_{\mu\nu} \equiv +(2/\sqrt{-g})\delta S_{\text{matter}}/\delta g^{\mu\nu}$, where S_{matter} is the matter action sourcing the gravitational sector. Moreover, as widely known, the total action of metric $f(R)$ gravity coupled to matter reads

$$S = \frac{1}{2\kappa}\int \mathrm{d}^4x\sqrt{-g}f(R) + S_{\text{matter}}, \tag{1}$$

whose associated equations of motion are

$$f'(R)R_{\mu\nu} - \frac{f(R)}{2}g_{\mu\nu} + \mathcal{D}_{\mu\nu}f'(R) = -\kappa T_{\mu\nu}, \tag{2}$$

where $\mathcal{D}_{\mu\nu} \equiv \nabla_\mu\nabla_\nu - g_{\mu\nu}\Box$ and n primes right after any function denote the nth derivative of said function with respect to its argument. For instance, $f'(R) \equiv \mathrm{d}f(R)/\mathrm{d}R$ and $A''(r) \equiv \mathrm{d}^2A(r)/\mathrm{d}r^2$.

TABLE I. Overview of the constant-curvature solutions considered in this work and their pathological characteristics.

Type	Parameters (dimensions)	Subclass	Line element	Issues and oddities
Class 1	$C \neq 0$ (L)	$C > 0$	(46)	(i) Protected Królak-strong and Tipler-strong curvature singularity at the origin $r = 0$ (i.e. it cannot be reached by causal observers in finite proper time). (ii) Can be smoothly matched with an interior Minkowski spacetime at any given radius for every C [16].
		$C < 0$	(52)	(i) Protected Królak-strong and Tipler-strong curvature singularity at $r = 0$. (ii) Accessible Królak-strong and Tipler-strong curvature singularity at $r = \|C\|$ (i.e. causal observers can reach it in finite proper time). (iii) Surface $r = \|C\|$ also appears to be a Killing horizon, but it is not null. (iv) As their counterparts with $C < 0$, they can be smoothly matched with an interior Minkowski spacetime at any given radius for every C [16].
Class 2	R_0 (L^{-2})[a] D (L^0) $C \neq 0$ (L)[b]	$R_0 > 0$	(59)	(i) Infinite number of accessible curvature singularities. (ii) Infinite number of would-be (non-null) Killing horizons, all coincident with curvature singularities. (iii) Generically unstable, as per Result 3.
		$R_0 < 0$	(64)	(i) Protected Królak-strong and Tipler-strong curvature singularity at the origin. (ii) Generically unstable, as per Result 3.
		$R_0 = 0$[c]	(48)	(i) Protected Królak-strong and Tipler-strong curvature singularity at the origin. (ii) Describes a pair of disconnected parallel universes, one at each side of the central singularity.
Class 3	R_0 (L^{-2})[d] M (L^{-1})[e]	$M = 0$, $R_0 > 0$	(73)	(i) The metric signature becomes unphysical, i.e. $(+,+,-,-)$, for radii r larger than the critical value $r = \sqrt{6/R_0}$. (ii) The region with an unphysical metric signature is accessible in finite proper time. (iii) $r = \sqrt{6/R_0}$ is also an apparent horizon (a null surface in which $g^{rr} = 0$). (iv) Generically unstable, as per Result 3.
		$M = 0$, $R_0 < 0$	(76)	(i) Describes a traversable wormhole made out of pure vacuum. (ii) However, it is generically unstable, as per Result 3.
		$M > 0$, $R_0 = 0$	(77)	(i) The metric signature becomes unphysical for $r < 2GM$. (ii) The region with an unphysical metric signature is accessible in finite proper time. (iii) $r = 2GM$ is also the location of an apparent horizon. (iv) Curvature singularity at $r = 0$ (i.e. within the unphysical region).
		$M < 0$, $R_0 = 0$	(81)	(i) Accessible Królak-strong and Tipler-strong curvature singularity at $r = 0$.
		$M < 0$, $R_0 < 0$	(84)	(i) Accessible Królak-strong and Tipler-strong curvature singularity at $r = 0$. (ii) Generically unstable, as per Result 3.

(Table continued)

TABLE I. *(Continued)*

Type	Parameters (dimensions)	Subclass	Line element	Issues and oddities
		$M > 0$, $R_0 < 0$	(87)	(i) The metric signature becomes unphysical for radii r smaller than some critical value $r_{\rm ah}$ given by expression (88). (ii) Curvature singularity at $r = 0$ (i.e. within the unphysical region). (iii) Generically unstable, as per Result 3.
		$M < 0$, $R_0 > 0$	(89)	(i) The metric signature becomes unphysical for radii r larger than some critical value $r_{\rm ah}$ given by expression (90). (ii) Królak-strong and Tipler-strong curvature singularity at $r = 0$. (iii) Generically unstable, as per Result 3.
		$M > 0$, $R_0 > 0$	(91)	(i) If $3GM\sqrt{R_0/2} \geq 1$, the metric signature is unphysical for all r. If $3GM\sqrt{R_0/2} = 1$, then $r = 3GM = \sqrt{2/R_0}$ would be a wormhole throat (were the metric signature physical). (ii) If $0 < 3GM\sqrt{R_0/2} < 1$, the metric signature is unphysical for $r < r_0$ and for $r > r_1$, where r_0 and r_1 are two critical values given by expressions (92) and (93), respectively. $r = r_0$ and $r = r_1$ correspond to apparent horizons. (iii) Curvature singularity at $r = 0$ (i.e. within the unphysical region). (iv) Generically unstable, as per Result 3.
Class 4	$z \neq 0$ (L^0)[f] $c > 0$ (L^b)[g] $R_0 > 0$ (L)[h]	$b = b_\pm$ (both)	(95)	(i) They are not asymptotically flat (they do not describe fully isolated objects), but they comply with Solar System tests for $b = b_-$ and $\lvert z\rvert \ll 1$. (ii) Curvature singularity at $r = 0$, hidden within an event horizon. (iii) The central singularity is unprotected if $z > -10 + 4\sqrt{6} = -0.202\ldots$, and protected if $z < -10 - 4\sqrt{6} = -19.798\ldots$

[a]Notice that, by changing the value of R_0, one is actually changing the set of $f(R)$ models in which this spacetime is a vacuum solution; cf. conditions (6) and (7).
[b]As explained in Sec. V B in the bulk of the text, this constant can always be absorbed by a redefinition of the time coordinate. We have deliberately kept it in the metric for purely dimensional purposes (i.e. so as to have a time coordinate with dimensions of length), and also to explicitly demonstrate that it represents the same physical quantity as similarly named parameter C in Class 1 solutions.
[c]In this case, dimensionless constant D can always be absorbed by a coordinate redefinition.
[d]See footnote a.
[e]R_0 and M cannot vanish at the same time because, in that case, Class 3 solutions would trivially reduce to Minkowski spacetime.
[f]For consistency, conditions $z < -10 - 4\sqrt{6}$ or $-10 + 4\sqrt{6} < z$ (i.e. $z < -19.798\ldots$ and $z > -0.202\ldots$) are also necessary; see text.
[g]Exponent b depends on parameter z. Its two allowed values, $b_\pm$, are given in Eq. (98).
[h]Again, this constant is left in the metric solely on dimensional grounds and might be removed with a coordinate redefinition.

II. CONSTANT-CURVATURE VACUUM SOLUTIONS OF $f(R)$ GRAVITY

Throughout this article, we shall define a *constant-curvature spacetime* as the one represented by a metric whose Ricci scalar is constant, i.e.

$$R = \mathrm{const} \equiv R_0. \tag{3}$$

When the equations of motion of $f(R)$ gravity (2) are evaluated in vacuum—i.e. $T_{\mu\nu} = 0$—and constant scalar curvature R_0 solutions are sought, the last term on the left-hand side of (2) vanishes. Thus, the equations of motion reduce to

$$f'(R_0)R_{\mu\nu} = \frac{f(R_0)}{2} g_{\mu\nu}. \tag{4}$$

Taking the trace of (4), one finds that, in vacuum, such a constant-curvature solution satisfies

$$f'(R_0)R_0 = 2f(R_0). \tag{5}$$

Thus, in the event that vacuum solutions with constant-curvature R_0 are present in a given $f(R)$ model, Eqs. (2), (4), and (5) hold simultaneously, giving rise to the following scenarios:

(i) If $R_0 = 0$, Eq. (5) necessarily implies that $f(0) = 0$. As a result, Eq. (4) then entails that either $f'(0) = 0$ or $R_{\mu\nu} = 0$. This means that an $f(R)$ model satisfying $f(0) = 0$ always admits the same $R_0 = 0$ solutions as GR (for which $R_{\mu\nu} = 0$). If, in addition, $f'(0) = 0$, then the full equations of motion (4)—or, equivalently, (2)—are satisfied automatically, and the theory admits *any* metric having $R_0 = 0$ as a solution, even if said vanishing-curvature metrics are not solutions of GR. Notice that these novel, $f(R)$-exclusive solutions would coexist with those of GR in $f(R)$ models satisfying $f'(0) = 0$.

(ii) If $R_0 \neq 0$, there are two possibilities within this scenario.

On the one hand, for $f(R)$ models satisfying $f(R_0) \neq 0$, Eq. (4) necessarily implies that $f'(R_0) \neq 0$ (since $R_0 \neq 0$). Thus, Eqs. (2) and (4) turn into $R_{\mu\nu} = (R_0/4)g_{\mu\nu}$, and only the constant-curvature solutions of $\mathrm{GR} + \Lambda$—with $\Lambda = R_0/4 = f(R_0)/2f'(R_0)$—solve the equations of motion of the $f(R)$ model under consideration.

On the other hand, for $f(R)$ models satisfying $f(R_0) = 0$, Eq. (4) forces $f'(R_0) = 0$. As such, Eqs. (2) and (4) are trivially satisfied, and *any* metric with constant scalar curvature R_0 is a solution of the $f(R)$ model. In particular, the constant-curvature solutions of $\mathrm{GR} + \Lambda$ (with $\Lambda = R_0/4$) would also be solutions of this particular set of $f(R)$ models. Thus, in these models, the novel, $f(R)$-exclusive constant-curvature solutions (which do not satisfy the usual Einstein equations in the presence of a cosmological constant) would coexist with the constant-curvature solutions of Einsteinian gravity (with an appropriate cosmological constant).

In summary, *any* metric with a vanishing Ricci scalar trivially solves the vacuum equations of motion of *all* $f(R)$ models such that

$$f(0) = 0, \qquad f'(0) = 0, \tag{6}$$

with the first condition being necessary for the model to harbor the vanishing-curvature solutions of GR. Analogously, *every* metric with constant Ricci scalar R_0 is a vacuum solution of *any* $f(R)$ model satisfying

$$f(R_0) = 0, \qquad f'(R_0) = 0. \tag{7}$$

$f(R)$ models fulfilling conditions (6) or (7) shall thus be the object of study of the present work. As explained in the Introduction, we will generically refer to these special choices of function f as R_0-*degenerate models*. In addition, we shall further distinguish between $(R_0 = 0)$-*degenerate models*, which satisfy conditions (6), and $(R_0 \neq 0)$-*degenerate models*, which comply with (7) instead. Finally, constant-curvature solutions exclusive to R_0-degenerate $f(R)$ models will hereafter be referred to as R_0-*degenerate* (in analogy with the models themselves), or as $f(R)$-*exclusive*, or even as *novel* solutions.

The reader should note that it is straightforward to find nontrivial R_0-degenerate $f(R)$ models which appear to be, at least *a priori*, physically well-motivated. The most paradigmatic examples would be the so-called "power-of-GR" models, $f(R) \propto R^{1+\delta}$, which fulfill conditions (6) provided that $\delta > 0$. These models have interesting applications in cosmology and might be compatible with Solar System experiments, depending on the value of δ [18,19]. Another simple instance of a $(R_0 \neq 0)$-degenerate model would be $f(R) = R - R_0/2 - R^2/(2R_0)$, which satisfies (7) while behaving as $R - 2\Lambda + \mathcal{O}(R^2)$ for $R \ll R_0$, with R_0 being the only dimensional parameter. The fact that there exist R_0-degenerate models which reduce to $\mathrm{GR} + \Lambda$ in the appropriate limit is a remarkable result, given that GR with (or without) a cosmological constant is not R_0-degenerate by itself (for any R_0).

At this point, it is important to remark that R_0-degenerate $f(R)$ models may admit solutions which are not degenerate. For example, in Appendix C, we present a simple $f(R)$ model harboring constant-curvature solutions for two different values of R_0. However, the model is R_0-degenerate for only one of these two R_0's; for the other value, the $f(R)$ model only hosts the (nondegenerate) constant-curvature solutions of $\mathrm{GR} + \Lambda$.

Another important observation is that $f(R)$ models do not need to be fine-tuned in order to become R_0-degenerate. For example, the power-of-GR models $f(R) \propto R^{1+\delta}$ with $\delta > 0$ discussed above are always $(R_0 = 0)$-degenerate. These models have been well-studied in the literature in the context of cosmology and, certainly, were not purposefully designed to be $(R_0 = 0)$-degenerate. Thus, as a matter of fact, whenever conditions (6) or (7) are satisfied—either accidentally or on purpose—the $f(R)$ model in question is constant-curvature degenerate for the appropriate values of R_0. It is true, however, that nothing prevents one from constructing a *designer* R_0-degenerate $f(R)$ model hosting as many solutions as one wishes with a particular constant-curvature R_0.

Before closing this section, a brief comment on the predictability of R_0-degenerate $f(R)$ models is pertinent. As their name suggests, it is not clear whether $f(R)$ models complying with either (6) or (7) have full predictive power. In any such R_0-degenerate model, there is a set of initial conditions (namely, those requiring the Ricci scalar to be R_0) whose evolution is not dictated by the vacuum equations of motion; recall Eq. (2) hold trivially for (the infinite number of) metrics with $R = R_0$. Moreover, there are indications that some metrics with the privileged Ricci scalar R_0 can almost always be smoothly glued to each other [16], thus suggesting that R_0-degenerate models might be unable to discern between its (infinitely many) constant-curvature solutions.

III. STRONG-COUPLING PATHOLOGIES IN R_0-DEGENERATE $f(R)$ MODELS

Innocent as they might seem at first sight, the special class of R_0-degenerate $f(R)$ models—i.e. those fulfilling either conditions (6) or (7)—can be shown to be inherently pathological, as stated in the Introduction. In the following, we shall concentrate in a physically relevant shortcoming of said models, namely, an apparent strong-coupling instability—i.e. the nonpropagation of all expected degrees of freedom at linear level around a flat background.

The linearized spectrum of a given gravity theory comprises all the independent fields which propagate on top of a suitable background when the equations of motion are expanded up to linear order in perturbations. The linearized spectrum around flat Minkowski spacetime thus coincides with the possible gravitational-wave polarization modes which can be observationally detected, since the weak-field approximation is appropriate near current experimental settings.

It is well-known that, generically, the gravitational wave spectrum of metric $f(R)$ models consists of a massless and traceless graviton akin to that of GR (with two polarization modes, the so-called "+" and "×" polarizations) plus an additional longitudinal (i.e. massive) scalar degree of freedom [6,7], in consonance with the fact that $f(R)$ theories of gravity are dynamically equivalent to a scalar-tensor theory [20,21].[1]

The fact that *most* $f(R)$ models propagate a massless and traceless graviton renders them compatible with gravitational wave observations [27] (notice that no current gravitational-wave detectors are sensible to nontensorial modes, including scalar modes, which remain unobserved). However, it is important to remark that, in general, previous analyses of gravitational waves in $f(R)$ gravity made no assumptions on function f itself (apart from analyticity at $R = 0$, which is necessary to perform the linearization of the field equations, as we shall see later). For these reasons, these investigations failed to recognize that *not all* $f(R)$ models propagate the expected linearized degrees of freedom (graviton + scalar) around a Minkowski background. Indeed, we have found that $(R_0 = 0)$-degenerate models feature such evanescence of the expected degrees of freedom, signaling the presence of a previously undiscovered strong-coupling instability in these models.[2] In particular, we have been able to establish the following two results:

Result 1. At linear level in perturbations, $(R_0 = 0)$-degenerate $f(R)$ models—i.e. those complying with conditions (6)—propagate, at most, one single massless scalar mode atop a flat background. In other words, these models do not contain the expected spin-2 graviton in their linearized spectrum, and thus Minkowski spacetime is strongly coupled in $(R_0 = 0)$-degenerate $f(R)$ models.

Result 2. Around a Minkowski background, the linearized spectrum of $(R_0 = 0)$-degenerate $f(R)$ models satisfying $f''(0) = 0$ does not contain any dynamical degrees of freedom whatsoever.

As mentioned before, Result 1 puts into question the physical viability of $(R_0 = 0)$-degenerate $f(R)$ models, since the two polarization modes corresponding to a massless and traceless spin-2 graviton have been detected in all gravitational-wave experiments carried out by the LIGO and VIRGO Collaborations since 2015 [29,30]. We must also stress at this point that it is not very difficult to find $(R_0 = 0)$-degenerate $f(R)$ models that comply with the hypotheses of Result 2; for instance, all power-of-GR models $f(R) \propto R^{1+\delta}$ with $\delta > 1$ (we shall also assume that δ is a natural number for the series expansion around $R = 0$ to exist).

In order to prove the assertions in Results 1 and 2, we will proceed as follows. First, we will review the linearization of the $f(R)$ field equations (2) for any choice of function f. After that, we will particularize the results to the special case of $(R_0 = 0)$-degenerate models—i.e. we will make use of conditions (6)—to demonstrate the existence of aforementioned apparent strong-coupling instabilities.

To perform the linear expansion of the $f(R)$ equations of motion (2) around Minkowski spacetime, one starts by choosing a suitable coordinate system in which the metric $g_{\mu\nu}$ can be decomposed as

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \tag{8}$$

where $\eta_{\mu\nu}$ is the Minkowski background and $h_{\mu\nu}$ is the metric perturbation, i.e. $|h_{\mu\nu}| \ll 1$ in this special coordinate

[1]Some studies [22–24] claimed that the linearized spectrum of $f(R)$ contained a second scalar polarization mode, dubbed *breathing mode*, in disagreement with previous results. The controversy was finally settled against the existence of such a breathing mode resorting to the Hamiltonian formalism [25] and gauge-invariant methods [26].

[2]A background is said to be *strongly coupled* whenever at least one of the expected perturbative degrees of freedom fails to propagate atop said background. In other words, the kinetic term(s) of the evanescent field(s) vanish when evaluated in the strongly coupled background; equivalently, interaction terms blow up upon canonicalization of the equations of motion, hence the name *strong-coupling*. Comprehensive accounts of the generalities of strong-coupling phenomena may be found in works investigating the appearance of such instabilities in physical theories. We refer the interested reader to references such as [28], for instance.

system. As is widely known, the following expressions are true at first order in $h_{\mu\nu}$:

$$g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu} + \mathcal{O}(h^2), \tag{9}$$

$$R_{\mu\nu} = R^{(1)}_{\mu\nu} + \mathcal{O}(h^2), \tag{10}$$

$$R = R^{(1)} + \mathcal{O}(h^2), \tag{11}$$

where

$$R^{(1)}_{\mu\nu} \equiv \frac{1}{2}\left[\Box h_{\mu\nu} + \partial_\mu\partial_\nu h - 2\partial_\lambda\partial_{(\mu}h^\lambda{}_{\nu)}\right], \tag{12}$$

$$R^{(1)} \equiv \eta^{\mu\nu}R^{(1)}_{\mu\nu} = \Box h - \partial_\mu\partial_\nu h^{\mu\nu}, \tag{13}$$

$$h^{\mu\nu} \equiv \eta^{\mu\rho}\eta^{\nu\sigma}h_{\rho\sigma}, \qquad h^\mu{}_\nu \equiv \eta^{\mu\lambda}h_{\lambda\nu}, \qquad h \equiv \eta^{\mu\nu}h_{\mu\nu}. \tag{14}$$

In expressions (12)–(14) and hereafter, $\Box$ shall denote the Minkowski-space d'Alembertian, i.e. $\Box = \eta^{\mu\nu}\partial_\mu\partial_\nu$.

The existence of a linearized regime of $f(R)$ theories requires one additional assumption to be made, namely that f must be analytic at $R = 0$, and therefore series-expandable up to linear order in the Ricci-scalar perturbation $R^{(1)}$, i.e. up to linear order in metric perturbations. This is to guarantee that the resulting linearized equations of motion remain first-order in $h_{\mu\nu}$, something which is impossible whenever f and its derivatives cannot be linearized in the first place.

For a generic $f(R)$ theory of gravity which is analytic around $R = 0$, taking into account all the previous considerations results in the following set of linearized vacuum equations of motion:

$$f'(0)G^{(1)}_{\mu\nu} + f''(0)(\partial_\mu\partial_\nu - \eta_{\mu\nu}\Box)R^{(1)} + \mathcal{O}(h^2) = 0, \tag{15}$$

where we have defined the Einstein-like tensor

$$G^{(1)}_{\mu\nu} \equiv R^{(1)}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}R^{(1)}. \tag{16}$$

Taking the trace of (15), one finds

$$3f''(0)\Box R^{(1)} + f'(0)R^{(1)} + \mathcal{O}(h^2) = 0, \tag{17}$$

which is a noncanonical Klein-Gordon equation for $R^{(1)}$ provided that $f''(0) \neq 0$. Direct inspection of this equation clearly reveals that the kinetic term for $R^{(1)}$ vanishes if $f''(0) = 0$. Therefore, expression (17) alone suffices to conclude that, only in cases where $f''(0) \neq 0$, the Ricci-scalar perturbation $R^{(1)}$ behaves as an independent, propagating scalar degree of freedom at the linearized level. One might then divide both sides of Eq. (17) by $f''(0)$ so as to canonicalize the kinetic term, yielding

$$\Box R^{(1)} + \frac{f'(0)}{3f''(0)}R^{(1)} + \mathcal{O}(h^2) = 0. \tag{18}$$

Thus, for $f''(0) \neq 0$, the propagating scalar degree of freedom $R^{(1)}$ has an effective mass m_{eff} given by

$$m^2_{\mathrm{eff}} = \frac{f'(0)}{3f''(0)}. \tag{19}$$

Turning back to the full linearized equations of motion (15), we are now in a position that will allow us to understand intuitively why the spin-2 sector of the theory does not propagate atop the Minkowski background in $(R_0 = 0)$-degenerate $f(R)$ models. Aside from the higher-order terms, Eq. (15) contains (i) the term proportional to $f'(0)$ and the Einstein-like tensor $G^{(1)}_{\mu\nu}$ given by (16), which encapsulates all terms depending on $h_{\mu\nu}$ and its derivatives; and (ii) the term containing derivatives of the scalar mode $R^{(1)}$, which is proportional to $f''(0)$ and does *not* depend on $h_{\mu\nu}$ or its derivatives. From this, it is clear that:

(i) The first term (i) vanishes whenever function f is such $f'(0) = 0$, which is precisely one of the defining conditions of $(R_0 = 0)$-generate $f(R)$ models; cf. (6). Given that $G^{(1)}_{\mu\nu}$ contains all the derivatives of $h_{\mu\nu}$ appearing in the equations of motion, the absence of this term entails that the spin-2 mode does not propagate, as stated in Result 1. The strong-coupling problem becomes evident once one notices that the interaction terms—i.e. the $\mathcal{O}(h^2)$ terms—blow up when one divides Eq. (15) by $f'(0)$ (in order to canonicalize the graviton kinetic terms) and then takes the limit $f'(0) \to 0$.

(ii) The second term (ii) will not be present either whenever $f''(0) = 0$, as discussed above. Thus, for $(R_0 = 0)$-generate $f(R)$ models such that $f''(0) = 0$, Eqs. (15) and (17) contain no kinetic terms at all, only the $\mathcal{O}(h^2)$ interaction terms survive, and thus those theories do not possess a linearized spectrum, as asserted in Result 2.

However, there is a more insightful (and more mathematically explicit) way of understanding why the spin-2 degree of freedom fully decouples when $f'(0) = 0$, i.e. for $(R_0 = 0)$-degenerate $f(R)$ models. The argument goes as follows. As in GR, in $f(R)$ gravity it is possible [6] to define a new symmetric rank-two tensor field, $\bar{h}_{\mu\nu}$, such that Eq. (15) reduces to the wave equation

$$\Box\bar{h}_{\mu\nu} + \mathcal{O}(h^2) = 0 \tag{20}$$

in the de Donder gauge, i.e. after setting

$$\partial_\mu\bar{h}^{\mu\nu} = 0 \tag{21}$$

using some of the available gauge freedom in the theory. The remaining gauge freedom is then employed to impose the transverse-traceless (TT) condition. In a generic $f(R)$ theory of gravity, after expanding (15) in the de Donder gauge and comparing the result with (20), one finds that $\bar{h}_{\mu\nu}$ is given by[3]

$$\bar{h}_{\mu\nu} = f'(0)\bar{h}^{\mathrm{GR}}_{\mu\nu} - f''(0)R^{(1)}\eta_{\mu\nu}, \tag{22}$$

where

$$\bar{h}^{\mathrm{GR}}_{\mu\nu} \equiv h_{\mu\nu} - \frac{h}{2}\eta_{\mu\nu} \tag{23}$$

is the usual spin-2 degree of freedom of GR. In consequence, Eqs. (20) and (22) evince that what propagates at the speed of light (in vacuum) in a generic $f(R)$ gravity model is a mixture of the GR spin-2 graviton and the extra scalar mode. As per Eq. (22), such a propagating mixture reduces to its scalar component provided that $f'(0) = 0$. In such a situation, $R^{(1)}$ becomes effectively massless, due to (19), and the Klein-Gordon equation (17) becomes equivalent to the wave equation (20). As a result, only the massless scalar degree of freedom propagates in $(R_0 = 0)$-degenerate $f(R)$ models such that $f''(0) \neq 0$, as previously stated in Result 1. Again, one clearly sees that no propagating degree of freedom survives the limit $f''(0) \to 0$, in agreement with the strong-coupling instability described in Result 2.

IV. STABILITY OF THE NOVEL CONSTANT-CURVATURE SOLUTIONS

Even though R_0-degenerate $f(R)$ models possess an infinite number of solutions having constant scalar curvature $R = R_0$, it is actually possible to study the stability of all such solutions at once within a given model, without needing to perform a case-by-case analysis. In order to do so, we will resort to the Einstein-frame (i.e. scalar-tensor) representation of $f(R)$ gravities, which is ideally suited to study stability against small perturbations about a given constant value of R.

As previously stated, it is well-known that metric $f(R)$ theories can be regarded as equivalent to a scalar-tensor gravitational theory. More precisely, in the so-called Einstein frame [20,21],

$$\bar{g}_{\mu\nu} = f'(R)g_{\mu\nu}, \tag{24}$$

[3]Our expression (22) for $\bar{h}_{\mu\nu}$ is slightly different from the one commonly found in the literature [6,24],

$$\bar{h}_{\mu\nu} = \bar{h}^{\mathrm{GR}}_{\mu\nu} - \frac{f''(0)}{f'(0)}R^{(1)}\eta_{\mu\nu},$$

which, as the reader may immediately notice, is not valid if $f'(0) = 0$, i.e. precisely in the case we are interested in.

the action (1) of metric $f(R)$ gravity transforms into that of GR plus a dynamical gravitational scalar field ϕ, with the latter being given by

$$\phi(R) = \sqrt{\frac{3}{2\kappa}}\ln f'(R). \tag{25}$$

This scalar field, also known as the *scalaron*, is subject to the $f(R)$-model-dependent potential

$$V(R) = \frac{f'(R)R - f(R)}{2\kappa f'^2(R)}. \tag{26}$$

Notice that the Ricci scalar R appearing in the previous expressions is that of the so-called Jordan-frame metric, i.e. the original, physical metric $g_{\mu\nu}$. As mentioned before, the scalaron (25) is related to the scalar polarization mode found in the gravitational-wave spectrum of the theory.

When working in the Einstein frame, the stability of a Jordan-frame constant-curvature solution will depend on whether $R = R_0$ is a minimum of the scalaron potential, and on whether such a minimum is either global or local (in the latter case, the solution will only be metastable). Nonetheless, we must stress that some subtleties arise when using the Einstein frame in R_0-degenerate models. The ones relevant to our work will be comprehensively discussed in Appendix A. Also, it is straightforward to notice in (26) that, if the $f(R)$ model is R_0-degenerate, a naive evaluation of $V(R)$ at $R = R_0$ leads to a $0/0$ indetermination. The reason is that, in such a case, both the numerator and the denominator in Eq. (26) become zero when $R \to R_0$, owing to conditions (7). Therefore, the limit must be evaluated carefully.

There are two possible ways of computing limits which naively evaluate to indeterminations of the $0/0$ kind: (i) performing series expansions or (ii) applying L'Hôpital's rule. The only difference between the aforementioned methods is their range of applicability: Taylor series require analyticity around the expansion point, while L'Hôpital's rule only requires differentiability of the numerator and denominator. In our assessment of the stability of constant-curvature solutions in R_0-degenerate models, we have made use of both methods, obtaining exactly the same outcomes, as shown in Results 3 and 4 (recall that analyticity requires differentiability, and thus the results obtained using L'Hôpital's rule imply those obtained using series expansions). However, for the sake of clarity, and to avoid cluttering up this communication with long formulas, we shall only present the derivation using Taylor series, which produces shorter expressions at the expense of a more limited scope. However, we insist that the final results apply to nonanalytic but differentiable f's as well.

It is worth noting that, regardless of the method employed in the stability analysis, one is forced to assume that $f''(R_0) \neq 0$ to prevent some denominators from blowing up at $R = R_0$.[4] In addition, it is well-known that $f(R)$ models having $f''(R) < 0$ develop the so-called Dolgov-Kawasaki matter instability [31–33]. As such, the additional constraint $f''(R_0) > 0$ is unavoidable on physical grounds.

Having explained why we have chosen to present just the computations using the series-expansion method, let us proceed with the stability analysis. As stated above, apart from demanding conditions (7) to hold (so that the model is R_0-degenerate and harbors any solution with constant scalar curvature R_0), we shall only make two additional assumptions, in particular, that function f is analytic around $R = R_0$ (so that it can be Taylor-expanded around R_0, as explained before), and also that $f''(R_0) > 0$. As a result, it is possible to expand both the numerator and the denominator of the scalaron potential (26) around $R = R_0$. Indeed, close to R_0, the denominator of $V(R)$ behaves as

$$f'^{-2}(R) \underset{R=R_0}{\sim} f''^{-2}(R_0)(R - R_0)^{-2} + \frac{f'''(R_0)}{f''^{3}(R_0)}(R - R_0)^{-1} + \mathcal{O}[(R - R_0)^0], \tag{27}$$

whereas the numerator can be expanded as

$$f'(R)R - f(R) \underset{R=R_0}{\sim} R_0 f''(R_0)(R - R_0) + \frac{f''(R_0) + R_0 f'''(R_0)}{2}(R - R_0)^2 + \mathcal{O}[(R - R_0)^3]. \tag{28}$$

As a result, we have that

$$2\kappa V(R) \underset{R=R_0}{\sim} \frac{R_0}{f''(R_0)}(R - R_0)^{-1} + \frac{f''(R_0) - R_0 f'''(R_0)}{2f''^{2}(R_0)} + \mathcal{O}(R - R_0). \tag{29}$$

We can now clearly infer from this last expression that the limit of $V(R)$ as $R \to R_0$ does not exist unless $R_0 = 0$. Certainly, should R_0 be different from zero, series expansion (29) would be dominated by the order $(R - R_0)^{-1}$ term, which is a hyperbola tending to either positive or negative infinity depending whether R_0 is approached from the left or the right. More precisely,

$$\lim_{\substack{R\to R_0^{\pm} \\ R_0\neq 0}} V(R) = \mathrm{sign}\left[\frac{R_0}{f''(R_0)}\right] \times (\pm\infty). \tag{30}$$

[4] Condition $f''(R) \neq 0$ ensures that the correspondence between the Jordan and Einstein frames is well-posed [20]. Moreover, the extrema of $V(R)$ and $V(\phi)$ coincide if and only if $f''(R) \neq 0$, as discussed in Appendix A.

Requiring the Dolgov-Kawasaki stability condition $f''(R) > 0$ to hold on physical grounds, this expression further simplifies to

$$\lim_{\substack{R\to R_0^{\pm} \\ R_0\neq 0}} V(R) = \mathrm{sign}(R_0) \times (\pm\infty), \tag{31}$$

but, as we can see, the limit still does not exist under the new condition.

As mentioned earlier in this section, even though the previous findings have been obtained using Taylor expansions (and thus under the assumption that f is analytic at $R = R_0$), the equivalent computation using L'Hôpital's rule yields exactly the same results (30) and (31) for choices of f which might not be analytic. We can therefore establish the following general result:

Result 3. Consider an $(R_0 \neq 0)$-degenerate $f(R)$ model —i.e. one fulfilling conditions (7)—such that $f''(R_0) \neq 0$. Then its infinitely many solutions with constant-curvature $R = R_0$ are generically unstable.

On the contrary, if $R_0 = 0$, series expansion (29) yields

$$\lim_{R\to 0} V(R) = \frac{1}{4\kappa f''(0)}; \tag{32}$$

i.e. the potential (26) is analytic and perfectly well-defined at $R = 0$. Consequently, whenever $R = 0$ is a global minimum of the potential, zero-scalar-curvature solutions will be stable. If $R = 0$ is a local but nonglobal minimum, then the solutions will be just metastable (i.e. stable only under small-enough perturbations). In either case, the additional constraints one must impose on $V(R)$ are the usual minimum conditions:

$$V'(0) = \lim_{R\to 0} V'(R) = 0, \tag{33}$$

$$V''(0) = \lim_{R\to 0} V''(R) > 0 \quad \text{but finite.} \tag{34}$$

Again, should the limits of $V'(R)$ and $V''(R)$ as R tends to zero be taken directly, indeterminations of the 0/0-type would emerge in each case. Consequently, one has to proceed exactly as we have done above when evaluating the limit of the potential itself. As before, we shall only assume that f is analytic around $R = R_0 = 0$, and that $f''(0) \neq 0$ (and, eventually, that $f''(0) > 0$).

Taking this into account, the limit of $V'(R)$ as $R \to 0$ may again be computed by expanding its numerator and denominator around $R = 0$, yielding

$$\lim_{R\to 0} V'(R) = -\frac{f'''(0)}{12\kappa f''^{2}(0)}. \tag{35}$$

As per local minimum condition (33), metastability of the solutions with $R = 0$ then requires

$$f'''(0) = 0. \tag{36}$$

Similarly, one finds that the limit of $V''(R)$ as $R \to 0$ is also well-defined, being

$$\lim_{R\to 0} V''(R) = -\frac{f''''(0)}{24\kappa f''^2(0)}. \tag{37}$$

As a result, local minimum condition (34) implies that an $(R_0 = 0)$-degenerate $f(R)$ model must be such that

$$f''''(0) < 0 \tag{38}$$

for its infinitely many vanishing-scalar-curvature solutions to be at least metastable. Once more, as in the $R_0 \neq 0$ case, results (30), (31), and (35)–(38) are also found using L'Hôpital's rule, and therefore hold for functions f which need not be analytic. In consequence, we can formulate the following generic result:

Result 4. All vanishing-scalar-curvature spacetimes are at least metastable vacuum solutions of $(R_0 = 0)$-degenerate $f(R)$ models—i.e. those satisfying conditions (6)—provided that function f is such that (i) $f''(0) \neq 0$, (ii) $f'''(0) = 0$, and (iii) $f''''(0) < 0$. It should also be taken into account that compliance with the Dolgov-Kawasaki stability criterion forces $f''(R) > 0$ for all R, and, *a fortiori*, $f''(R_0) > 0$, instead of condition (i) above.

Analytic $f(R)$ models satisfying all the assumptions of Result 4 [including $f''(0) > 0$] would admit the following Taylor expansion around $R = 0$:

$$f(R) = \alpha R^2 - \beta R^4 + \mathcal{O}(R^5), \tag{39}$$

where $\alpha > 0$ and $\beta > 0$ are real constants. The simplest such models are, evidently, the polynomic ones with

$$f(R) = \alpha R^2 - \beta R^4. \tag{40}$$

It is not difficult to show that, for $\alpha > 0$, $R = 0$ is just a local minimum of the potential associated with model (40). The only other extrema of $V(R)$ are two maxima at $R = \pm\sqrt{\alpha/6\beta}$. Thus, for $|R| > \sqrt{\alpha/6\beta}$ the potential is monotonically decreasing, i.e. $V(R) \to -\infty$ as $R \to \pm\infty$. In other words, the model has a potential not bounded below and lacks a ground state. As a result, the metastable solutions with $R = 0$ could be driven to infinite scalar curvature, should the perturbations applied on them be large enough. On the other hand, if α were negative, the extremum at $R_0 = 0$ would be the true, global minimum of the potential; in such a case, however, the Dolgov-Kawasaki stability condition $f''(R) > 0$ would be violated, and the model would be unstable.

V. SOME PARADIGMATIC $f(R)$-EXCLUSIVE CONSTANT-CURVATURE SOLUTIONS

As we saw before in Sec. II, *any* spacetime with constant-curvature R_0 is a solution of *any* R_0-degenerate $f(R)$ model. Therefore, the problem of obtaining new constant-curvature solutions for such R_0-degenerate models reduces to solving the differential equation $R = R_0$. This can be accomplished by postulating several simple ansätze for the metric.

As a first approximation to the problem, one may require the constant-curvature solution to be static and spherically symmetric. In such a case, it is always possible to choose "areal-radius" coordinates (t, r, θ, φ) such that the most generic static and spherically symmetric line element can be written as

$$\mathrm{d}s^2 = A(r)\mathrm{d}t^2 - B(r)\mathrm{d}r^2 - r^2\mathrm{d}\Omega^2, \tag{41}$$

where A and B are the only independent metric functions. Line element (41) has Ricci scalar

$$R = -\frac{A''}{AB} + \frac{A'}{2AB}\left(\frac{A'}{A} + \frac{B'}{B}\right) - \frac{2}{r}\left(\frac{A'}{AB} - \frac{B'}{B^2}\right) + \frac{2}{r^2}\left(1 - \frac{1}{B}\right). \tag{42}$$

Notice that (41) is directly expressed in the so-called Abreu-Nielsen-Visser gauge,

$$\mathrm{d}s^2 = \mathrm{e}^{2\Phi}\left(1 - \frac{2GM_{\mathrm{MSH}}}{r}\right)\mathrm{d}t^2 - \left(1 - \frac{2GM_{\mathrm{MSH}}}{r}\right)^{-1}\mathrm{d}r^2 - r^2\mathrm{d}\Omega^2, \tag{43}$$

with Misner-Sharp-Hernández (MSH) mass [34,35]

$$M_{\mathrm{MSH}}(r) = \frac{r}{2G}\left(1 - \frac{1}{B(r)}\right) \tag{44}$$

and anomalous redshift function

$$\Phi(r) = \frac{1}{2}\ln\left[A(r)B(r)\right]. \tag{45}$$

These two quantities will help us interpret the various solutions to be discussed in what follows.

Equation $R = R_0 = \mathrm{const}$—with R given by expression (42)—has infinitely many solutions; i.e. there is an infinite number of static, spherically symmetric spacetimes having a constant Ricci scalar.[5] For this reason, we will focus on spherically symmetric spacetimes which have been previously reported in the literature [10,16], or generalizations thereof. Even though some of these spacetimes might look like mere deviations from either Minkowski or Schwarzschild spacetimes, it is evident that hardly one family (namely, Class 4, to be discussed in Sec. V D) passes standard weak-field or Solar System experiments. Thus, if they existed in nature, the solutions we are considering in this work would describe other, more exotic compact objects, such as black holes, wormholes, etc. Moreover, it is important to remark that none of the infinitely many R_0-degenerate solutions of R_0-degenerate $f(R)$ models is *a priori* privileged (on physical grounds) over the others—recall, for instance, that Birkhoff's theorem does not hold in $f(R)$ theories of gravity in general, least of all in R_0-degenerate models. As such, even the simplest such R_0-degenerate solutions must be taken into full consideration within R_0-degenerate $f(R)$ models.[6] Proving that these degenerate solutions display unphysical traits provides further evidence against the viability of their host R_0-degenerate $f(R)$ models (adding on to the one already presented in Results 1–4 above). For these reasons, we will devote the sections below to study the simple (yet archetypal) solutions we have chosen in detail.

Once we have presented the paradigmatic solutions in Secs. V A–V D, we will proceed to characterize them. That is to say, we would like to know which kind of objects (wormholes, black holes, etc.) such solutions represent, and whether they have physically meaningful properties. With that purpose, we will analyze the following five aspects:

(i) Apparent and Killing horizons.
(ii) Coordinate singularities.
(iii) Curvature singularities. In cases where it is feasible to perform the relevant computations, we will also determine whether said singularities are *strong* or *weak*, according to the criteria laid out by Tipler and Królak [37–39].
(iv) Geodesic completeness, i.e. whether the curvature singularities can be reached in finite time by freely falling observers.
(v) Regions in which the metric has an unphysical signature (e.g. regions in which there are two time coordinates or the metric becomes Euclidean), and whether these pathological regions can be reached in finite time by freely falling observers.

The precise definitions of the properties listed above may be found in Appendix D, along with useful formulas that will be employed during our characterization of the solutions. For instance, Eq. (D8) will allow us to determine whether a singularity or a region with an unphysical metric signature is out of reach for causal observers.

The R_0-degenerate solutions we are about to consider have been chosen because they allow for a fully analytic examination. Our findings concerning the points above are summarized in Table I. As may be deduced from these results, the paradigmatic solutions to be described in what follows exhibit a number of unusual characteristics which put their physical viability into question. As said before, given that the number of constant-curvature solutions a certain R_0-degenerate model hosts is infinite, one expects the majority of those constant-curvature spacetimes to be pathological. This section is thus intended to provide a limited but illustrative picture of the kind of issues one should expect to find when dealing with novel, $f(R)$-exclusive constant-curvature solutions.

A. Novel solutions of Class 1

The first set of $f(R)$-exclusive constant-curvature solutions we would like to discuss (hereafter to be known as Class 1 solutions) shall be those metrics whose line element can be expressed as

$$\mathrm{d}s^2 = \left(1+\frac{C}{r}\right)^2 \mathrm{d}t^2 - \mathrm{d}r^2 - r^2\mathrm{d}\Omega^2, \tag{46}$$

where C is a free parameter with dimensions of length, which might be either positive or negative. These spacetimes have a vanishing Ricci scalar for any value of C, and thus solve the equations of motion of all $f(R)$ models complying with conditions (6). An interesting property of Class 1 spacetimes is that they can be smoothly matched to a Minkowski interior at any given spherical surface $r = r_* = \text{const}$, as shown in [16]. Metric (46) thus represents the spacetime outside of such a static vacuole solution.

The most straightforward way of obtaining line element (46) consists in setting $R = 0$ in Eq. (42) and solving for

[5] A remarkable observation is that, given any function $A(r)$, there is a closed-form expression for the precise function $B(r)$ which solves equation $R = R_0$, namely

$$B(r) = \frac{I(r)}{B_0 + B_1(r)},$$

where B_0 is an integration constant,

$$B_1(r) \equiv \int_1^r \frac{\mathrm{d}x}{x}\frac{4I(x)A(x)}{4A(x)+xA'(x)}\left(1-\frac{R_0}{2}x^2\right),$$

and function $I(r)$ is defined as

$$I(r) \equiv \exp\left[\int_1^r \frac{\mathrm{d}x}{x}\frac{4A(x)+4xA'(x)-\frac{x^2A'^2(x)}{A(x)}+2x^2A''(x)}{4A(x)+xA'(x)}\right].$$

[6] We would like to stress at this point that, in $f(R)$ gravity, Birkhoff's theorem is absent and, in addition, Schwarzschild cannot describe the spacetime outside *any* matter source, and it can only exist as a black hole [36]. Therefore, novel exterior solutions must exist. In particular, nothing prevents, in principle, the exterior to a certain physical body to be described by any of the R_0-degenerate solutions of degenerate $f(R)$ models.

$A(r)$ using the simple ansatz $B(r) = 1$. By doing so, one finds that the general solution is given by the (*a priori*) two-parameter family of metrics

$$\mathrm{d}s^2 = \left(D + \frac{C}{r}\right)^2 \mathrm{d}t^2 - \mathrm{d}r^2 - r^2\mathrm{d}\Omega^2, \tag{47}$$

where C and D are, again, real constants, with D being dimensionless. However, it is not difficult to realize that parameter D can always be removed from the metric. If $D \neq 0$, it is always possible to transform line element (47) into the Class 1 form (46) presented above by redefining $C/D \to C$ and $D^2\mathrm{d}t^2 \to \mathrm{d}t^2$. Nonetheless, if $D = 0$, metric (47) becomes

$$\mathrm{d}s^2 = \left(\frac{C}{r}\right)^2 \mathrm{d}t^2 - \mathrm{d}r^2 - r^2\mathrm{d}\Omega^2, \tag{48}$$

Spacetimes of this form (48) are a particular instance of one of the families of $f(R)$-exclusive constant-curvature solutions originally discovered by Calzà, Rinaldi, and Sebastiani in Ref. [10].[7] We shall consider metrics of the form (48) to be solutions of Class 2, given that they cannot be recovered from the standard Class 1 form (46), and for further reasons that will become apparent in Sec. V B. Consequently, in the present subsection we shall focus on analyzing only the properties of metrics of the form (46).

1. General properties of Class 1 solutions

It is clear from line element (46) that Class 1 solutions are asymptotically flat, and that they also reduce to Minkowski spacetime in the limit $C \to 0$. Using (44) and (45), one might immediately deduce that metrics of the form (46) have vanishing MSH mass and anomalous redshift function

$$\Phi(r) = \ln\left|1 + \frac{C}{r}\right|. \tag{49}$$

In consequence, Class 1 solutions may be regarded as metrics deviating from Minkowski spacetime only in a gravitationally induced anomalous redshift. The interpretation of length scale C appearing in (46) is far less clear. As shown in [16], the value of this parameter cannot be fixed by glueing (46) to a Minkowski spacetime at a given spherical surface $r = r_* = \text{const}$. Class 1 solutions are also strikingly similar to the extremal Reissner-Nordström spacetime, sharing a common Newtonian limit $g_{tt} \simeq 1 + 2\phi_\mathrm{N}$ (where ϕ_N is the Newtonian potential) upon identification of C with (Newton's constant times) the mass or the charge of the black hole. In spite of this, Class 1 metrics are not sourced by any electromagnetic field. Instead, as mentioned before, the correction to the point-like-mass potential in the weak-field limit of (46) is entirely due to anomalous gravitational effects (recall this is a vacuum solution). Therefore, the interpretation of C as an "extremal charge" seems not to be appropriate.

Expressions (46) and (49) reveal that the remaining properties of Class 1 solutions depend crucially on the sign of C. As such, we will study separately the subclass of solutions with $C > 0$ and the subclass with $C < 0$.

2. Class 1 solutions with $C > 0$

Direct inspection of line element (46) reveals that Class 1 solutions with $C > 0$ harbor only one coordinate singularity at $r = 0$, with the metric remaining regular for any other $r > 0$. Moreover, when $C > 0$ there are neither apparent nor Killing horizons, nor any regions in which the signature of line element (46) becomes unphysical.

Given that the Kretschmann scalar corresponding to (46) is

$$\mathcal{K} = \frac{24C^2}{r^6}\left(1 + \frac{C}{r}\right)^{-2}, \tag{50}$$

one immediately realizes that the coordinate singularity at the origin is an actual curvature singularity. Despite this, the singularity is inaccessible for any causal observer. Integral (D8) may be computed for Class 1 solutions with $C > 0$ directly, yielding

$$\Delta\lambda(r_\mathrm{ini} \to r_\mathrm{fin}) = \left| r_\mathrm{fin} - r_\mathrm{ini} + C\ln\left(\frac{r_\mathrm{fin}}{r_\mathrm{ini}}\right)\right|. \tag{51}$$

It is thus apparent that, along a radial null geodesic, the affine parameter separating any $r > 0$ and the central singularity is infinite. Because photons take an infinite amount of affine parameter to reach the central singularity, no other particle can reach $r = 0$ in finite proper time either. In consequence, for all practical purposes, the singularity at $r = 0$ is inaccessible, and spacetime (46) is geodesically complete for $C > 0$.

The previous result can easily be understood in view of the form of the anomalous redshift function (49) for Class 1 spacetimes with $C > 0$. As we can clearly deduce from this expression, the anomalous redshift becomes infinite at $r = 0$. In other words, modified-gravity effects induce an ever-increasing redshift function which *protects* observers from the curvature singularity.

Finally, it is worth noting that the central singularity in Class 1 spacetimes with $C > 0$ turns out to be both Królak-strong and Tipler-strong, as one might infer using (D13) and (D14).

[7] More precisely, line element (48) can be obtained by setting $b = 2$, $z = -2$, and $c_0 = 0$ in Eqs. (33)–(35) of [10].

3. Class 1 solutions with $C < 0$

The metric of Class 1 spacetimes with $C < 0$, viz.

$$\mathrm{d}s^2 = \left(1 - \frac{|C|}{r}\right)^2 \mathrm{d}t^2 - \mathrm{d}r^2 - r^2 \mathrm{d}\Omega^2, \qquad (52)$$

exhibits two coordinate singularities: the one at $r = 0$ (which was already present in the $C > 0$ subclass), and a second one at $r = |C|$. Both prove to be curvature singularities, since the Kretschmann scalar becomes

$$\mathcal{K} = \frac{24C^2}{r^6}\left(1 - \frac{|C|}{r}\right)^{-2} \qquad (53)$$

for $C < 0$. As we shall see now, the singularity at $r = 0$ remains inaccessible for radially infalling photons. However, the singularity at $r = |C|$ is causally connected to the rest of the spacetime.

On the one hand, for $r_{\text{ini}}, r_{\text{fin}} < |C|$, integral (D8) yields

$$\Delta\lambda(r_{\text{ini}} < |C| \to r_{\text{fin}} < |C|) = \left| r_{\text{fin}} - r_{\text{ini}} - |C| \ln\left(\frac{r_{\text{fin}}}{r_{\text{ini}}}\right)\right|. \qquad (54)$$

One may readily substitute $r_{\text{fin}} = 0$ in this expression to find that the curvature singularity at $r = 0$ cannot be reached in finite time by causal observers. In parallel with the $C > 0$ case, the curvature singularity is protected by an $f(R)$-induced infinite redshift, and is Królak- and Tipplerstrong, as may be inferred using (D13) and (D14).

On the other hand, if $r_{\text{ini}} \neq |C|$, but $r_{\text{fin}} = |C|$, then integral (D8) becomes

$$\Delta\lambda(r_{\text{ini}} \to |C|) = \left| |C| - r_{\text{ini}} - |C| \ln\left(\frac{|C|}{r_{\text{ini}}}\right)\right|, \qquad (55)$$

which is finite for every $r_{\text{ini}} \neq 0$. Therefore, a photon traveling along a null radial geodesic will reach the singularity at $r = |C|$ in finite time, should it not be "emitted" at the other singularity at the origin. Furthermore, Eqs. (D13) and (D14) reveal that the naked singularity at $r = |C|$ is also both Królak- and Tippler-strong.

For completeness, we shall also compute $\Delta\lambda(r_{\text{ini}} \to r_{\text{fin}})$ for $r_{\text{ini}} > |C|$ and $r_{\text{fin}} \leq |C|$ (without loss of generality[8]). To do so, we must first remark that the integral in (D8) breaks into two pieces in this case:

$$\Delta\lambda(r_{\text{ini}} \to r_{\text{fin}}) = \left| \int_{r_{\text{ini}}}^{|C|} \mathrm{d}r \left(1 - \frac{|C|}{r}\right) - \int_{|C|}^{r_{\text{fin}}} \mathrm{d}r \left(1 - \frac{|C|}{r}\right)\right|. \qquad (56)$$

As a result,

$$\Delta\lambda(r_{\text{ini}} \to r_{\text{fin}}) = \left| r_{\text{ini}} + r_{\text{fin}} - 2|C|\left[1 + \ln\left(\frac{\sqrt{r_{\text{ini}} r_{\text{fin}}}}{|C|}\right)\right]\right|. \qquad (57)$$

This expression attests once more that, whatsoever value r_{ini} takes, $\Delta\lambda(r_{\text{ini}} \to r_{\text{fin}})$ remains finite unless $r_{\text{fin}} = 0$. Thus, the central singularity is protected, with the one at $r = |C|$ being truly naked.

Another surprising feature of surface $r = |C|$ is that it appears to be a Killing horizon for $\xi = \partial/\partial t$, since

$$g_{\mu\nu}\xi^\mu \xi^\nu = \left(1 - \frac{|C|}{r}\right)^2 \qquad (58)$$

vanishes at $r = |C|$. However, $r = |C|$ is not a null surface because its normal vector $n_\mu = \partial_\mu r = \delta^r{}_\mu$ is everywhere timelike. Therefore, strictly speaking, it cannot be a Killing horizon.[9] Given the fact that $g_{\mu\nu}\xi^\mu\xi^\nu > 0$ for any $r \neq |C|$, the would-be Killing horizon at $r = |C|$ would be degenerate, in analogy with the true Killing horizon of extremal Reissner-Nordström black holes. A remarkable difference between Class 1 solutions with $C < 0$ and extremal Reissner-Nordström black holes is that the former do not possess any apparent horizons, while the Killing horizon of extremal Reissner-Nordström black holes is also an apparent horizon.[10]

B. Novel solutions of Class 2

Class 2 shall encompass three different kinds of constant-curvature solutions, all of which are characterized by their compliance with the simple ansatz $B(r) = 1$, as those of Class 1. As a result, all Class 2 spacetimes will have vanishing MSH mass, and thus only differ from Minkowski spacetime in an $f(R)$-induced anomalous redshift factor.

The first two subclasses within Class 2 will contain, respectively, the solutions of equation $R = R_0$ with $B(r) = 1$ and either positive (first subclass) or negative (second subclass) constant-curvature R_0—recall that the Ricci scalar of static spacetimes is given by expression (42). The third subclass will be conformed by those spacetimes resulting from setting $R_0 \to 0$ in the aforementioned Class 2 solutions with $R_0 \neq 0$.

[8]As mentioned in Appendix D, the absolute value in (D8) entails that $\Delta\lambda(r_{\text{ini}} \to r_{\text{fin}}) = \Delta\lambda(r_{\text{fin}} \to r_{\text{ini}})$, so it is always possible to exchange $r_{\text{ini}} \leftrightarrow r_{\text{fin}}$.

[9]Given that $r = |C|$ is also a curvature singularity, as seen before, it is technically not part of the spacetime.

[10]Furthermore, the horizon of extremal Reissner-Nordström black holes is regular, i.e. free of curvature singularities.

As done in the previous section, we will analyze each subclass within Class 2 separately.

1. Class 2 solutions with $R_0 > 0$

Substituting the simple ansatz $B(r) = 1$ in (42) and solving the equation $R = R_0 > 0$ for $A(r)$, one finds the following family of constant-curvature solutions of $(R_0 \neq 0)$-degenerate $f(R)$ gravity models:

$$\mathrm{d}s^2 = \left(\frac{C}{r}\right)^2 \cos^2\left(\sqrt{\frac{R_0}{2}}r + D\right)\mathrm{d}t^2 - \mathrm{d}r^2 - r^2\mathrm{d}\Omega^2, \quad (59)$$

where $C \neq 0$ and D are integration constants, C having dimensions of length, and D being dimensionless. Notice that, even though C can always be removed from the line element through the coordinate redefinition $t \to t/C$, we have kept it in (59) for dimensional reasons (i.e. the new, rescaled time coordinate would be dimensionless, while the new g_{tt} would have dimensions of length squared).

Class 2 spacetimes with $R_0 > 0$ are highly pathological; despite their simple appearance, they are, perhaps, the least physically well-founded solutions we will consider in this work. For instance, it is immediate to infer from expression (59) that these solutions have an infinite number of coordinate singularities. One of them is located at $r = 0$, where $g_{tt} \to \infty$, while the remaining ones are located at the points in which the cosine vanishes, that is to say, at

$$r_n = \sqrt{\frac{2}{R_0}}\left[\frac{\pi}{2}(2n+1) - D\right], \quad (60)$$

with n an integer.[11] Direct computation of the Kretschmann scalar provides a long expression (which we shall not include here) which diverges at $r = 0$ and all the r_n above. As a result, the infinite coordinate singularities represent actual curvature singularities.

What is worse, only the curvature singularity at $r = 0$ can be out of reach for any causal observer, and only in very particular circumstances, as we shall see. To obtain this result, we first recall that $\Delta\lambda(r_{\text{ini}} \to r_{\text{fin}})$ can always be expressed in terms of the primitive (D10), evaluated at some particular values of r.[12] For solutions of the form (59), it turns out that

$$\lambda(r) = |C|\left[\sin D\,\mathrm{Si}\left(\sqrt{\frac{R_0}{2}}r\right) + \cos D\,\mathrm{Ci}\left(\sqrt{\frac{R_0}{2}}r\right)\right], \quad (61)$$

where Si and Ci are the so-called sine integral and cosine integral functions, respectively:

$$\mathrm{Si}(z) = \int_0^z \frac{\mathrm{d}x}{x}\sin x, \quad (62)$$

$$\mathrm{Ci}(z) = \gamma + \ln z + \int_0^z \frac{\mathrm{d}x}{x}(\cos x - 1), \quad (63)$$

where γ is the Euler-Mascheroni constant. The sine integral function is regular for all r, while the cosine integral only blows up (to negative infinity) when $r = 0$. As a result, $\lambda(r)$ solely becomes infinite at $r = 0$ provided that $\cos D \neq 0$. There are, thus, two different scenarios:

(i) If $\cos D \neq 0$, then $\Delta\lambda(r_{\text{ini}} \to r_{\text{fin}})$ blows up if and only if either r_{ini} or r_{fin} is equal to zero, and thus “only” the infinite number of curvature singularities located at r_n—with r_n given by (60)—are accessible for radially falling causal observers.

(ii) If $\cos D = 0$, then $\Delta\lambda(r_{\text{ini}} \to r_{\text{fin}})$ remains finite regardless of the values of r_{ini} and r_{fin}. As a result, all curvature singularities can be reached in finite time by causal observers.

In either case, causally propagating particles in Class 2 spacetimes with $R_0 > 0$ may encounter an infinite number of curvature singularities in finite time, and thus we can immediately assert that these solutions are physically unjustifiable just for this reason. Other pathological properties of Class 2 solutions with $R_0 > 0$ include the existence of infinite would-be Killing horizons located at each of the r_n given by (60)—which cannot be true Killing horizons since their normal remains timelike for all r, as in Class 1 solutions with $C < 0$—as well as the inherent instability of $f(R)$-exclusive constant-curvature solutions with $R_0 \neq 0$ demonstrated in Result 3.

2. Class 2 solutions with $R_0 < 0$

Setting $B(r) = 1$ and solving equation $R = R_0 < 0$ for $A(r)$, one obtains instead another set of novel solutions, characterized by

$$\mathrm{d}s^2 = \left(\frac{C}{r}\right)^2 \cosh^2\left(\sqrt{\frac{|R_0|}{2}}r + D\right)\mathrm{d}t^2 - \mathrm{d}r^2 - r^2\mathrm{d}\Omega^2, \quad (64)$$

where $C \neq 0$ and D are once again integration constants. Since the hyperbolic cosine never vanishes, Eq. (64) possesses only one coordinate singularity, which is located at the origin $r = 0$ of the spherical coordinate system. As in the $R_0 > 0$ case, this coordinate singularity turns out to be a curvature singularity. The Kretschmann scalar for Class 2 solutions with $R_0 < 0$ is given again by a convoluted expression (which we shall not include here); careful examination of such an expression reveals that there are no more curvature singularities apart from the one at $r = 0$.

[11] We remark that the allowed values of n depend on the value of D. In particular, the minimum possible n should be such that $r_n > 0$ for all permitted ns.

[12] Note, however, that expression (D9) does not hold for arbitrary r_{ini}, r_{fin} in this case. The reason is that, for Class 2 solutions with $R_0 > 0$, the integrand in (D10) is a cosine, which changes sign periodically.

Once more, we turn to determine whether the central curvature singularity is accessible to causal observers in spacetimes of the form (64). In this particular case, primitive (D10) might be readily expressed in terms of special functions, namely the hyperbolic sine integral (Shi) and the hyperbolic cosine integral (Chi), as[13]

$$\lambda(r) = |C|\left[\sinh D\,\mathrm{Shi}\left(\sqrt{\frac{|R_0|}{2}}r\right) + \cosh D\,\mathrm{Chi}\left(\sqrt{\frac{|R_0|}{2}}r\right)\right]. \tag{65}$$

Said special functions are defined as

$$\mathrm{Shi}(z) = \int_0^z \frac{\mathrm{d}x}{x}\sinh x, \tag{66}$$

$$\mathrm{Chi}(z) = \gamma + \ln z - \int_0^z \frac{\mathrm{d}x}{x}(\cosh x - 1), \tag{67}$$

in analogy with their trigonometric counterparts (62) and (63). Moreover, Shi, as Si, is regular for all r, while Chi, as Ci, diverges at $r = 0$. Hence, $\lambda(r)$—and thus $\Delta\lambda(r_{\mathrm{ini}} \to r_{\mathrm{fin}})$—diverges only if either r_{ini} or r_{fin} is zero [notice that, this time, the prefactor accompanying Chi in (65), which is $\cosh D$, does not vanish for any D]. As a result, the only singularity in (64) is out of reach of radially infalling photons, and for this reason causal observers cannot encounter it in finite time. Moreover, using Eqs. (D13) and (D14), we find that the central singularity hosted by Class 2 line elements with $R_0 < 0$ is both Królak-strong and Tipler-strong.

The metric components of Class 2 solutions with $R_0 > 0$ are always positive, so these metrics contain no apparent or Killing horizons. As a result, they describe the spacetime outside the singularity at $r = 0$. As this singularity cannot interfere in finite time with causal observers, spacetimes of the form (64) are much more physically well-founded than their counterparts (59) with $R_0 < 0$. Nonetheless, we insist that Class 2 spacetimes with $R_0 \neq 0$ are automatically unstable, as per Result 3.

3. *Class 2 solutions with* $R_0 = 0$

The only Class 2 solutions which can be stable (in accordance with Result 4) are the ones which result from taking the limit of (59) or (64) as $R_0 \to 0$. It turns out that those solutions have a line element given by expression (48). This is the reason why we have grouped these solutions within Class 2 and not Class 1.

Line element (48) appears to describe a wormhole with its throat at $r = 0$ upon extension of coordinate r to negative values [10]. This would be a remarkable result, since such a wormhole would not require negative energies to form; in fact, it would be a *vacuum* solution of the equations of motion.

However, Class 2 solutions with $R_0 = 0$ cannot describe wormholes since (i) they contain a naked singularity at $r = 0$, and (ii) light rays cannot arrive at the locus of such a singularity in a finite affine parameter by following radial null geodesics. On the one hand, feature (ii) implies that the singularity is *protected*; however, it also entails that, even if there were no curvature singularity, light rays would never be able to cross the wrongly identified wormhole throat in finite time.

The previous assertions (i) and (ii) are easy to verify. Indeed, the Kretschmann scalar corresponding to line element (48) evaluates to

$$\mathcal{K} = \frac{24}{r^4}, \tag{68}$$

which diverges at $r = 0$ independently of the value of C, as expected.[14] Using (D8), we find that

$$\Delta\lambda(r_{\mathrm{ini}} \to r_{\mathrm{fin}}) = |C|\ln\left|\frac{r_{\mathrm{fin}}}{r_{\mathrm{ini}}}\right|, \tag{69}$$

which only diverges when either r_{ini} or r_{fin} equals zero. Therefore, we conclude that these solutions cannot describe a traversable wormhole. In reality, what they model is two parallel and causally disconnected universes which are separated by an unreachable curvature singularity at the origin. The central curvature singularity is found to be both Królak-strong and Tipler-strong using (D13) and (D14), respectively.

Apart from the central singularity, Class 2 solutions with $R_0 = 0$ do not have any other remarkable features; in particular, they have no apparent or Killing horizons.

C. Solutions of Class 3

Class 3 solutions, discovered in Ref. [10], can be obtained by solving equation $R = R_0$—with R_0 given, as always, by expression (42)—for $B(r)$, under the simple assumption that $A(r) = 1$. Class 3 comprises a family of Kottler[15] lookalikes:

$$\mathrm{d}s^2 = \mathrm{d}t^2 - \left(1 - \frac{2GM}{r} - \frac{R_0}{6}r^2\right)^{-1}\mathrm{d}r^2 - r^2\mathrm{d}\Omega^2, \tag{70}$$

[13]Notice that the integrand leading to (65) is a hyperbolic cosine, which is always positive. Thus, for Class 2 solutions with $R_0 < 0$, identity (D9) holds for whatsoever r_{ini}, r_{fin}.

[14]Notice that constant C in line element (48) can always be removed by redefining $C^2\mathrm{d}t^2 \to \mathrm{d}t^2$; however, as we have done before for the other Class 2 solutions, we have explicitly included it in the metric for dimensional reasons.

[15]The Kottler spacetime is also known as Schwarzschild–de Sitter or Schwarzschild–anti–de Sitter, depending on whether R_0 is positive or negative, respectively.

where M is a free parameter with units of mass, which can be (in principle) either positive or negative. Class 3 solutions with constant curvature $R_0 \neq 0$ are generically unstable as per Result 3; Class 3 solutions with $R_0 = 0$ could be at least metastable should the hypotheses of Result 3 hold.

Even thought line element (70) is reminiscent of the Kottler metric, there are two notable differences between them. First, unlike the latter, the former has an anomalous redshift function distinct from unity; specifically,

$$\Phi(r) = -\frac{1}{2}\ln\left(1 - \frac{2GM}{r} - \frac{R_0}{6}r^2\right). \tag{71}$$

This modified-gravity-induced redshift function (71) allows (70) to have $g_{tt} = 1$ instead of $g_{tt}g_{rr} = -1$. Second, the "cosmological-constant term" of Class 3 solutions, $R_0 r^2/6$, is exactly half of the one present in Kottler spacetime, namely $R_0 r^2/12$. As a result, Class 3 solutions have MSH mass

$$M_{\rm MSH}(r) = M + \frac{R_0}{12G}r^3. \tag{72}$$

Notice that, while Class 3 solutions might harbor apparent horizons, they do not exhibit any Killing horizon corresponding to the generator of time translations ∂_t, since its norm remains equal to unity throughout all spacetime, as per (D2) and (70).

While all members of this family were originally thought to describe traversable wormholes [10], we shall see that they might exhibit several unphysical properties depending on the values of M and R_0. We shall sort Class 3 solutions into subclasses according to the signs of their free parameters M and R_0. We shall also consider the simple cases $M = 0$, $R_0 \neq 0$ and $M \neq 0$ and $R_0 = 0$ separately.

1. Class 3 solutions with $M = 0$ and $R_0 > 0$

In the simple case in which parameter M vanishes and R_0 is positive, the metric reduces to a de Sitter lookalike:

$$\mathrm{d}s^2 = \mathrm{d}t^2 - \left(1 - \frac{R_0}{6}r^2\right)^{-1}\mathrm{d}r^2 - r^2\mathrm{d}\Omega^2. \tag{73}$$

Class 3 solutions with $M = 0$ have no curvature singularities, since their Kretschmann scalar, $\mathcal{K} = R_0^2/3$, remains constant for all r. Nonetheless, direct inspection of line element (73) reveals that these spacetimes suffer from an evident physical pathology: if $r > r_{\rm ah}$, where

$$r_{\rm ah} = \sqrt{\frac{6}{R_0}}, \tag{74}$$

then the metric has two time coordinates, t and r, since g_{rr} changes sign. What is more, the region with an unphysical metric signature can be reached by causal observers in finite time. When evaluated for spacetime (73) and $r_{\rm ini}$, $r_{\rm fin} \leq r_{\rm ah}$, integral (D8) can be computed using (D9), with primitive (D10) being given by

$$\lambda(r) = \arcsin\left(\sqrt{\frac{R_0}{6}}r\right). \tag{75}$$

It is evident from this expression that $\Delta\lambda(r_{\rm ini} \to r_{\rm fin})$ remains finite when either $r_{\rm ini}$ or $r_{\rm fin}$ is equal to $r_{\rm ah}$.

Before closing this section, it is interesting to emphasize that $r_{\rm ah}$ corresponds to an apparent horizon of (73), being a zero of $g^{rr} = 1/B(r)$.

2. Class 3 solutions with $M = 0$ and $R_0 < 0$

One way to cure the pathology in the metric signature exhibited by (70) is to change the sign of the constant-curvature R_0. In such a case, one obtains a line element given by

$$\mathrm{d}s^2 = \mathrm{d}t^2 - \left(1 + \frac{R_0}{6}r^2\right)^{-1}\mathrm{d}r^2 - r^2\mathrm{d}\Omega^2, \tag{76}$$

whose components remain non-negative for all r, and has neither horizons nor curvature singularities.[16] In fact, Eq. (76) is the only Class 3 solution which truly describes a traversable wormhole centered at the origin, since the metric can be effortlessly extended to negative values of r. The remarkable fact that a traversable wormhole can exist as a vacuum solution of an infinite number of $f(R)$ models is downplayed by the fact that such a wormhole is generically unstable, as per Result 3.

3. Class 3 solutions with $M > 0$ and $R_0 = 0$

In this case, the metric is a Schwarzschild lookalike:

$$\mathrm{d}s^2 = \mathrm{d}t^2 - \left(1 - \frac{2GM}{r}\right)^{-1}\mathrm{d}r^2 - r^2\mathrm{d}\Omega^2. \tag{77}$$

The metric signature changes to $(+,+,-,-)$ if $r < 2GM$. Within this region also lies a curvature singularity, located at the origin $r = 0$, as revealed by the form of the Kretschmann scalar, which is

$$\mathcal{K} = \frac{24G^2M^2}{r^6}. \tag{78}$$

Exactly as in the $M = 0$, $R_0 > 0$ case, the unphysical region can be reached by radially infalling photons for a

[16]Notice that, as in the previous case, the Kretschmann scalar is constant and equal to $\mathcal{K} = R_0^2/3$. As a result, the usual coordinate singularity at $r = 0$ is not an actual curvature singularity, but an artifact caused by the choice of "areal-radius" coordinates.

finite value of the affine parameter. This is because, for spacetime (77), integral (D8) is of the form (D9), where primitive (D10) is now given by

$$\lambda(r) = \sqrt{r(r-2GM)} + GM\ln\left[r - GM + \sqrt{r(r-2GM)}\right]. \quad (79)$$

As one may readily infer from this expression, $\lambda(r)$ remains finite and positive for all r_{ini} and r_{fin} equal to or larger than $2GM$.[17] In particular,

$$\lambda(2GM) = GM\ln(GM). \quad (80)$$

Hence, we conclude that line element (77) is physically unsatisfactory, as causal observers can access the region with an unphysical metric signature in finite time.

4. Class 3 solutions with $M < 0$ and $R_0 = 0$

A change in the sign of the mass parameter turns line element (73) into

$$\mathrm{d}s^2 = \mathrm{d}t^2 - \left(1 + \frac{2G|M|}{r}\right)^{-1}\mathrm{d}r^2 - r^2\mathrm{d}\Omega^2. \quad (81)$$

As a result, g_{rr} remains positive for all $r > 0$, and thus the metric signature remains physical throughout all spacetime. Nonetheless, metric (81) remains pathological, since the singularity at the origin is naked and reachable in finite proper time by causal observers. In this case, the Kretschmann scalar is still given by (78), and thus $r = 0$ is a true curvature singularity of (81). Furthermore, the affine-parameter interval between any two radii is given again by (D9), but with primitive $\lambda(r)$ now being

$$\lambda(r) = \sqrt{r(r+2G|M|)} - G|M|\ln\left[r + G|M| + \sqrt{r(r+2G|M|)}\right]. \quad (82)$$

This quantity remains finite for all r, and in particular

$$\lambda(0) = -G|M|\ln(G|M|); \quad (83)$$

hence, the central singularity is causally connected to the rest of the spacetime. The central singularity is also Królak-strong and Tipler-strong, as may be deduced using (D13) and (D14). From this it is clear that, just as its positive-mass counterpart (77), metric (81) is not well-founded from a physical point of view.

[17]Notice that $\lambda(r)$ becomes complex for $r < 2GM$, in yet another example of the pathological character of this subclass of spacetimes.

5. Class 3 solutions with $M < 0$ and $R_0 < 0$

When both the mass scale M and the constant-curvature R_0 in (70) are negative, the line element becomes

$$\mathrm{d}s^2 = \mathrm{d}t^2 - \left(1 + \frac{2G|M|}{r} + \frac{|R_0|}{6}r^2\right)^{-1}\mathrm{d}r^2 - r^2\mathrm{d}\Omega^2. \quad (84)$$

Thus, in this case, $B(r)$ remains positive for all positive radii, and the metric signature remains physical through all spacetime. However, the metric harbors a curvature singularity at $r = 0$, where the Kretschmann scalar,

$$\mathcal{K} = \frac{24G^2M^2}{r^6} + \frac{R_0^2}{3}, \quad (85)$$

diverges. Radially infalling photons can have access to this singularity in finite time. This is because

$$\Delta\lambda(r_{\mathrm{ini}} \to 0) = \left|\int_{r_{\mathrm{ini}}}^{0}\mathrm{d}r\left(1 + \frac{2G|M|}{r} + \frac{|R_0|}{6}r^2\right)^{-1/2}\right| < \left|\int_{r_{\mathrm{ini}}}^{0}\mathrm{d}r\right| = |r_{\mathrm{ini}}| < \infty \quad (86)$$

for any $r_{\mathrm{ini}} < \infty$. Once again, the reachable singularity turns out to be Królak-strong and Tipler-strong. Therefore, this subclass is also pathological.

6. Class 3 spacetimes with $M > 0$ and $R_0 < 0$

In this case, the metric is

$$\mathrm{d}s^2 = \mathrm{d}t^2 - \left(1 - \frac{2GM}{r} + \frac{|R_0|}{6}r^2\right)^{-1}\mathrm{d}r^2 - r^2\mathrm{d}\Omega^2. \quad (87)$$

Apart from the curvature singularity at the origin, Eq. (87) also has a coordinate singularity whenever there is an apparent horizon, i.e. when g^{rr} vanishes. This occurs only at

$$r_{\mathrm{ah}} = \sqrt{\frac{8}{|R_0|}\sinh\left[\frac{1}{3}\operatorname{arcsinh}\left(3GM\sqrt{\frac{|R_0|}{2}}\right)\right]}, \quad (88)$$

What is more, g^{rr} is a strictly increasing function of r for $r > 0$. The monotonicity of g^{rr}, together with the fact that (88) is its only root, guarantees that $g^{rr} < 0$ for $r < r_{\mathrm{ah}}$, and thus the metric signature changes inside the apparent horizon. As such, the central curvature singularity at $r = 0$ lies within the region with an unphysical metric signature.

7. Class 3 spacetimes with $M < 0$ and $R_0 > 0$

This case is analogous to the previous one. The metric now reads

$$ds^2 = dt^2 - \left(1 + \frac{2G|M|}{r} - \frac{R_0}{6}r^2\right)^{-1} dr^2 - r^2 d\Omega^2, \quad (89)$$

and has a curvature singularity at $r = 0$ and a coordinate singularity at

$$r_{\rm ah} = \sqrt{\frac{8}{R_0}} \cosh\left[\frac{1}{3}\operatorname{arccosh}\left(3GM\sqrt{\frac{R_0}{2}}\right)\right], \quad (90)$$

which is also the only zero of g^{rr}, i.e. an apparent horizon. Because g^{rr} is monotonically decreasing for all r, we find that the metric signature changes for $r > r_{\rm ah}$. Thus, in contrast with the previous case, the central curvature singularity at $r = 0$—which can be shown to be Królak-strong and Tipler-strong using (D13) and (D14)—lies within the region where the metric signature is physical.

8. Class 3 spacetimes with $M > 0$ and $R_0 > 0$

To conclude the investigation of Class 3 spacetimes, we study the case in which the mass and the scalar curvature of the Kottler lookalike are both positive. The line element becomes

$$ds^2 = dt^2 - \left(1 - \frac{2GM}{r} - \frac{R_0}{6}r^2\right)^{-1} dr^2 - r^2 d\Omega^2. \quad (91)$$

After some computations, one might realize that:

(i) If $3GM\sqrt{R_0/2} > 1$, then g^{rr} does not vanish for any r.

(ii) If $3GM\sqrt{R_0/2} = 1$, then g^{rr} has a double zero (i.e. a wormhole throat) at $r = 3GM = \sqrt{2/R_0}$, which is also a coordinate singularity.

(iii) If $0 < 3GM\sqrt{R_0/2} < 1$, then g^{rr} vanishes at

$$r_0 = \sqrt{\frac{8}{R_0}} \sin\psi_0, \quad (92)$$

$$r_1 = \sqrt{\frac{2}{R_0}}(\sqrt{3}\cos\psi_0 - \sin\psi_0), \quad (93)$$

where

$$\psi_0 \equiv \frac{1}{3}\operatorname{arcsinh}\left(3GM\sqrt{\frac{R_0}{2}}\right). \quad (94)$$

Both r_0 and r_1 are apparent horizons and coordinate singularities.

Furthermore, it is straightforward to show that g^{rr} always has a maximum at $r_* = (6GM/R_0)^{1/3}$, with $g^{rr}(r_*) = 1\text{–}3GM\sqrt{R_0/2}$. In consequence:

(i) If $3GM\sqrt{R_0/2} > 1$, then $g^{rr}(r_*) < 0$, and thus $g^{rr} < 0$ (i.e. the metric signature is unphysical) for all r, since r_* is a global maximum. Thus, this subclass of solutions is not acceptable from a physical point of view.

(ii) If $3GM\sqrt{R_0/2} = 1$, then $g^{rr}(r_*) = 0$, and again the metric signature is unphysical for all r, due to the fact that r_* is a maximum. For this reason, this subclass of solutions is also not acceptable from a physical point of view.

(iii) If $0 < 3GM\sqrt{R_0/2} < 1$, then $g^{rr}(r_*) > 0$. Thus, because r_0 and r_1—as given by (92) and (93), respectively—are zeros of g^{rr} (with $r_0 < r_1$), we have that g^{rr} must become negative for $r < r_0$ and $r > r_1$. Therefore, the metric signature is physical only for $r_0 < r < r_1$.

Notice that all Class 3 solutions with $M > 0$ and $R_0 > 0$ also harbor a curvature singularity at the origin, as their Kretschmann scalar is given by (85). However, this curvature singularity lies within the region with an unphysical metric signature.

D. Solutions of Class 4

Last but not least, we proceed to analyze the $f(R)$-exclusive, vanishing-curvature solutions

$$ds^2 = \left(\frac{r}{r_0}\right)^z a(r)dt^2 - \frac{dr^2}{a(r)} - r^2 d\Omega^2, \quad (95)$$

where

$$a(r) = k - \frac{c}{r^b}, \quad (96)$$

with $z \neq 0$ a dimensionless real parameter, k given by

$$k = \frac{4}{z^2 + 2z + 4}, \quad (97)$$

exponent b being allowed to take the z-dependent values

$$b_\pm = \frac{1}{4}(3z + 6 \pm \sqrt{z^2 + 20z + 4}), \quad (98)$$

$c > 0$ being an integration constant (with units of length to the power of b) and r_0 a free parameter which is left in line element (95) for dimensional purposes only. Requiring exponents $b = b_\pm$ to be real imposes two additional constraints on z: $z < -10 - 4\sqrt{6}$ and $z > -10 + 4\sqrt{6}$ (numerically, $z < -19.798\ldots$ and $z > -0.202\ldots$).

Metrics of the form (95) are a particular instance of some novel black hole solutions with spherical topology discovered in [10]. One could in principle distinguish between two subclasses within Class 4, depending on whether $b = b_+$ or $b = b_-$. In spite of this, this distinction will be of little help in practice, as both subtypes exhibit the same key features. The main advantage of Class 4 solutions as compared with those of Classes 1–3 is that they are able to pass weak-field and Solar System tests for $b = b_-$ and $|z| \ll 1$. Moreover, their vanishing Ricci scalar allows them to be at least metastable solutions of $(R_0 = 0)$-degenerate $f(R)$ models. However, no Class 4 spacetime is asymptotically flat, as may be immediately deduced from line element (95). This entails that they do not truly describe fully isolated black holes (except in the GR limit $z = 0$). In any case, Class 4 solutions are certainly among the less pathological spacetimes we are considering in this work.

As mentioned before, Class 4 spacetimes describe spherically symmetric black holes. This is because

$$r_{\mathrm{hor}} = \left(\frac{c}{k}\right)^{1/b}, \tag{99}$$

the only positive root of equation $a(r) = 0$, turns out to be both an apparent and Killing horizon for Killing vector ∂_t.[18] This horizon is regular; the Kretschmann scalar associated with (95) can be shown to be of the form

$$\mathcal{K}(r) = \frac{1}{r^4}\left[\mathcal{K}_0(z) + \frac{c}{r^b}\mathcal{K}_1(z) + \left(\frac{c}{r^b}\right)^2\mathcal{K}_2(z)\right], \tag{100}$$

where functions $\mathcal{K}_i = \mathcal{K}_i(z)$ do not depend on r. Thus, we clearly see that the only curvature singularity is the one at the origin $r = 0$, and is hidden inside the horizon. As per (D8), the difference in the affine parameter between any $r_{\mathrm{ini}} > 0$ and $r_{\mathrm{fin}} = 0$ is given by (D9), with primitive (D10) being[19]

$$\lambda(r) = \frac{2r_0}{z+2}\left(\frac{r}{r_0}\right)^{(z+2)/2}, \tag{101}$$

which is finite for every $r > 0$, regardless of z. Considering only the allowed values of z, we deduce that $\lambda(0)$—and thus $\Delta\lambda(r_{\mathrm{ini}} \to 0)$—remains finite for $z > -10 + 4\sqrt{6}$, but diverges for $z < -10 - 4\sqrt{6}$, in which case a photon falling radially into the black hole would never encounter the central singularity (i.e. it would be protected).

VI. CONCLUSIONS

In this work we have performed an assessment of the physical viability of R_0-degenerate $f(R)$ models, as well as of their (infinitely many) constant-curvature solutions. As our results demonstrate, there are reasons to believe that these models cannot be successful in describing nature, even when experimental uncertainties do not rule them out directly. It is important to emphasize that $f(R)$ models which are R_0-degenerate for some values of R_0 are not difficult to find even without fine-tuning or *model-engineering*. In fact, some of these $f(R)$ models are compatible with Solar System experiments and cosmological observations, and may host nondegenerate and pathology-exempt solutions for Ricci scalars different from the degenerate value R_0.

One of the first anomalies we have detected is that $(R_0 = 0)$-degenerate $f(R)$ models feature a strongly coupled Minkowski background (Result 1), meaning that the expected massless and traceless graviton does not propagate on top of such a background. This result renders $(R_0 = 0)$-dependent $f(R)$ models incompatible with gravitational-wave observations. For instance, this result applies to the so-called power-of-GR models $f(R) \propto R^{1+\delta}$ with integer $\delta > 0$. We have also found that some $(R_0 = 0)$-degenerate $f(R)$ models do not even propagate the scalar degree of freedom atop a Minkowski background (Result 2), and thus their linearized spectrum does not contain any polarization modes whatsoever.

Another important weakness of R_0-degenerate $f(R)$ models (either with $R_0 = 0$ or $R_0 \neq 0$) is their apparent lack of predictive power, given the infinite degeneracy of their constant-curvature solutions and the triviality of the equations of motion (2) when evaluated for R_0-degenerate-exclusive constant-curvature solutions. More precisely, there is a subset of all possible initial conditions for the metric (namely, requiring $R = R_0$) whose evolution is not determined by the equations of motion of R_0-degenerate models, which hold automatically for those initial conditions.

In relation to the previous point, there are reasons to believe that novel $f(R)$-exclusive constant-curvature solutions can easily be matched to each other in R_0-degenerate models [16]. This result is actually more disturbing than it seems, given that one expects most of the (infinitely many) constant-curvature solutions to exhibit all sorts of physically undesirable properties. In particular, the Class 1 models presented in Sec. VA, which are not exempt from pathologies, are known to smoothly match with Minkowski spacetime, forming a vacuolelike solution. The fact that one may freely choose the radius where solutions (46) match the interior Minkowski spacetime raises the question of whether a degenerate $f(R)$ model can actually predict the boundaries at which spacetime ceases to be described by one of its constant-curvature solution and makes way for another constant-curvature solution.

Regarding the R_0-degenerate constant-curvature solutions themselves, we have performed a stability analysis of these solutions, finding that, in general terms, the constant-curvature solutions of $(R_0 \neq 0)$-degenerate models are all unstable (Result 3). The constant-curvature solutions of $(R_0 = 0)$-degenerate $f(R)$ models can be metastable provided that function f satisfies some constraints (Result 4). This can be a problem, however, since the conditions

[18]Note that $a(r)$ can be shown to be a monotonic function of r, and one can thus guarantee that r_{hor} is the only horizon of (95). This is a null surface; as in Schwarzschild spacetime, coordinates t and r exchange roles at $r = r_{\mathrm{hor}}$, so there is no region with an unphysical metric signature.

[19]We note that $z = -2$, for which $\lambda(r) = r_0 \ln(r/r_0)$, lies within the forbidden parameter band $-10 - 4\sqrt{6} < z < -10 + 4\sqrt{6}$.

required for the stability of constant-curvature solutions may be incompatible with those guaranteeing the absence of other instabilities (such as the Dolgov-Kawasaki instability). Moreover, the latter result also implies that constant-curvature solutions exhibiting pathological features can be stable.

In order to exemplify the kind of pathologies one may find in $f(R)$-exclusive constant-curvature solutions, we have chosen four representative classes of such spacetimes, so as to thoroughly analyze various key aspects determining their physical viability: coordinate and curvature singularities, singularity strength (according to the Królak and Tipler criteria), regions in which the metric acquires an unphysical signature (i.e. the metric determinant changes sign due to one spatial coordinate abruptly becoming timelike), and geodesic completeness (i.e. whether causal observers can or cannot encounter any of the previous pathologies in finite proper time). As the results in Table I reveal, all the solutions considered in this work exhibit unphysical properties. For example, we have found that in all those cases where the analysis was computationally viable, such curvature singularities turned to be Królak-strong and Tipler-strong, meaning that extended objects are compressed to zero volume when falling into them. This was true even for some of the naked singularities we have found, which are accessible to causal observers in finite proper time. These and all the remaining unsubstantiated traits found in Table I suffice to conclude that the solutions considered herein are unlikely to exist in nature.

Finally, we must stress that there are some issues we would like to study in more detail in future works. For example, we have not investigated the linearized spectrum of $(R_0 \neq 0)$-degenerate $f(R)$ theories, in which the natural background is no longer Minkowski spacetime, but (anti–) de Sitter (depending on the sign of R_0). Similarly, we have not performed a perturbative expansion around any of the novel, $f(R)$-exclusive constant-curvature solutions, since we have only been concerned with obtaining the linearized spectrum of the theory far from any gravitational-wave source. To shed more light on these issues, a general analysis of strong-coupling instabilities in $f(R)$ gravity theories is currently in preparation.

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APPENDIX A: SOME CAVEATS CONCERNING THE USE OF THE EINSEIN FRAME IN R_0-DEPENDENT $f(R)$ MODELS

In this appendix, we shall present some of the subtleties arising when using the Einstein-frame representation (24)–(26) of R_0-degenerate $f(R)$ models, and in particular when dealing with their constant-curvature solutions and their stability.

The first caveat is that conformal transformation (24) leading to the Einstein frame becomes singular when evaluated on $f(R)$-exclusive constant-curvature solutions, since the existence of those solutions requires $f'(R_0) = 0$. In consequence, the dynamical equivalence between the Einstein and Jordan frames appears to break down in this scenario. However, given that the transformation between frames becomes singular only if $R = R_0$ (i.e. in a null-measure set of values of R), one may argue that the transformation does not indeed fail provided that all the relevant physical quantities remain well-defined after taking the limit $R \to R_0$.[20] In particular, the scalaron potential naively evaluates to a $0/0$ indetermination for R_0-degenerate constant-curvature solutions (as mentioned in Sec. IV). Only in cases where said indetermination can be resolved shall we consider the Einstein frame to be a valid representation of the $f(R)$ dynamics.

A second crucial observation is that, in our stability analysis, we have treated the scalaron potential V as a function of R instead of as a function of the scalaron ϕ, the reason being that it is more enlightening to perturb the scalar curvature instead of the abstract scalaron. However, in doing so, one must check that the extrema of V as a function of R coincide with those of V seen as a function of ϕ. This is indeed the case provided that $f''(R) \neq 0$. More precisely, given that

$$\frac{\mathrm{d}V}{\mathrm{d}R} = \frac{\mathrm{d}\phi}{\mathrm{d}R}\frac{\mathrm{d}V}{\mathrm{d}\phi} = \sqrt{\frac{3}{2\kappa}}\frac{f''(R)}{f'(R)}\frac{\mathrm{d}V}{\mathrm{d}\phi}, \tag{A1}$$

the zeros of $\mathrm{d}V/\mathrm{d}R$ coincide with those of $\mathrm{d}V/\mathrm{d}\phi$ if and only if $f''(R) \neq 0$. Moreover, one also has that

[20] Under this scope, the fact that the conformal transformation (24) becomes singular at $R = R_0$ is simply a reflection of the fact that the scalaron (25) tends to negative infinity as $R \to R_0$.

$$\frac{\mathrm{d}^2 V}{\mathrm{d}R^2} = \frac{\mathrm{d}^2\phi}{\mathrm{d}R^2}\frac{\mathrm{d}V}{\mathrm{d}\phi} + \left(\frac{\mathrm{d}\phi}{\mathrm{d}R}\right)^2 \frac{\mathrm{d}^2 V}{\mathrm{d}\phi^2}, \tag{A2}$$

by virtue of which

$$\mathrm{sign}\left[\frac{\mathrm{d}^2 V}{\mathrm{d}R^2}\bigg|_{\frac{\mathrm{d}V}{\mathrm{d}\phi}=0}\right] = \mathrm{sign}\left[\frac{\mathrm{d}^2 V}{\mathrm{d}\phi^2}\right], \tag{A3}$$

i.e. the character of the extremum (maximum, minimum, or saddle point) does not change when one regards the scalaron potential to be a function of R instead of ϕ.

Finally, it is worth mentioning that

$$\frac{\mathrm{d}V}{\mathrm{d}R} = \frac{f''(R)}{2\kappa}\frac{2f(R) - f'(R)R}{f'^3(R)}. \tag{A4}$$

Thus, a constant-curvature solution with $R = R_0$ will extremize the scalaron potential—with respect to both R and ϕ—provided that (i) $f''(R_0) \neq 0$, (ii) $f'(R_0) \neq 0$, and (iii) the trace (5) of the equations of motion holds. If $f'(R_0) = 0$, i.e. in R_0-degenerate models, the stability analysis is more complicated and must be performed as done in Sec. IV, as the potential and its derivatives might not be well-defined in the limit $R \to R_0$. Moreover, when $f'(R_0) = 0$, mere compliance with Eq. (5)—which now holds automatically—does not guarantee that R_0-degenerate solutions extremize the potential. Results 3 and 4 provide the conditions under which R_0-degenerate constant-curvature solutions are not stable.

In order to shed more light on these issues, in Appendix C we shall consider a particular instance of $f(R)$ model hosting both unstable degenerate constant-curvature solutions and stable non-degenerate constant-curvature solutions.

APPENDIX B: R_0-DEGENERATE SOLUTIONS IN THE BRANS-DICKE REPRESENTATION OF $f(R)$ GRAVITY

As widely known [20,21], apart from the Einstein-frame representation (24)–(26), the $f(R)$ action (1) can also be cast in the form of a Brans-Dicke scalar-tensor theory [40] with parameter $\omega_0 = 0$, provided that condition $f''(R) \neq 0$ is fulfilled for all R. To accomplish this, one first has to introduce a Lagrange multiplier field χ in (1) as follows:

$$S = \frac{1}{2\kappa}\int \mathrm{d}^4x\sqrt{-g}[f(\chi) + f'(\chi)(R-\chi)] + S_{\mathrm{matter}}. \tag{B1}$$

This entails that $\chi = R$ if and only if $f''(R) \neq 0$, in which case actions (1) and (B1) are dynamically equivalent to each other. If one redefines χ to ψ through the implicit relation $\psi = f'(\chi)$, Eq. (B1) becomes the $\omega_0 = 0$ Brans-Dicke action

$$S = \frac{1}{2\kappa}\int \mathrm{d}^4x\sqrt{-g}[\psi R - U(\psi)] + S_{\mathrm{matter}}, \tag{B2}$$

where we have introduced the Brans-Dicke potential

$$U(\psi) = \chi(\psi)\psi - f(\chi(\psi)). \tag{B3}$$

Then the equations of motion corresponding to (B2) are

$$G_{\mu\nu}\psi = \mathcal{D}_{\mu\nu}\psi - \frac{U(\psi)}{2}g_{\mu\nu} - \kappa T_{\mu\nu}, \tag{B4}$$

$$R = U'(\psi), \tag{B5}$$

with the interactions between ψ and matter being given by the trace of (B4), which is

$$3\Box\psi + 2U(\psi) - \psi U'(\psi) = -\kappa T. \tag{B6}$$

Notice that the transformations leading to the Brans-Dicke representation (B2)–(B6) of $f(R)$ gravities do not affect the metric, which is still $g_{\mu\nu}$ in all the aforementioned equations. This is in stark contrast with the Einstein frame, where the corresponding metric (24) is conformally related, but (in general) is distinct from $g_{\mu\nu}$.

Provided that $f''(R) = 0$, dynamical equivalence of (1), (B1), and (B2) entails that $\psi = f'(R)$ and

$$U(\psi) = \psi R(\psi) - f(R(\psi)). \tag{B7}$$

Hence, in R_0-degenerate $f(R)$ models, R_0-degenerate constant-curvature solutions are trivial solutions of (B4)–(B6) in vacuum, having Brans-Dicke scalar $\psi = 0$ and potential $U(\psi) = 0$, as per conditions (7). Therefore, in R_0-degenerate $f(R)$ models, the Jordan-frame and Brans-Dicke representations of R_0-degenerate constant-curvature solutions coincide. It should be noted, however, that other solutions of R_0-degenerate models can feature a nontrivial scalar field ψ in the Brans-Dicke representation.

APPENDIX C: A SIMPLE ILLUSTRATIVE MODEL

For illustrative purposes, let us consider the simple, one-parameter model

$$f(R) = R_* - R + R\ln\left(\frac{R}{R_*}\right), \tag{C1}$$

where $R_* > 0$ is a constant with units of inverse length squared. This model harbors constant-curvature solutions for two different values of R, namely:

(i) $R_0 = R_*$. Since $f(R_*) = 0$ and $f'(R_*) = 0$, all spacetimes with $R_0 = R_*$ trivially solve the equations of motion associated with (C1); i.e. the model is R_*-degenerate.

(ii) $R_0 = \eta R_*$, with $\eta = 4.92155\ldots$ being the nontrivial solution of $2 - 2\eta + \eta \ln \eta = 0$ ($\eta = 1$ is the trivial solution of this transcendental equation). In this case, $f(\eta R_*) \neq 0$ and $f'(\eta R_*) \neq 0$, but the trace of the equations of motion (5) holds. As a result, the $f(R)$ equations of motion (2) reduce to the Einstein equations with cosmological constant (4), and model (C1) admits the same constant-curvature solutions as GR + Λ, provided $\Lambda = \eta R_*/4$, without being (ηR_*)-degenerate.

Due to Result 3, the constant-curvature solutions of (C1) having $R_0 = R_*$ are all unstable. However, the nondegenerate constant-curvature solutions with $R_0 = \eta R_*$ are all stable. Indeed, it is straightforward to check that (i) $f''(\eta R_*) \neq 0$, (ii) $f'(\eta R_*) \neq 0$, and (iii) Eq, (5) is satisfied for constant-curvature solutions with $R_0 = \eta R_*$. Thus, as per the discussion in Appendix A, nondegenerate solutions having $R_0 = \eta R_*$ extremize the scalaron potential, regardless of whether it is understood to be a function of R or of ϕ. Moreover,

$$\left.\frac{\mathrm{d}V}{\mathrm{d}R}\right|_{R=\eta R_*} = \frac{\eta}{8R_*}\frac{\eta - 2}{(\eta - 1)^3} > 0, \tag{C2}$$

given that $R_* > 0$ and $\eta > 2$. Thus, $R_0 = \eta R_*$ is a minimum of the scalaron potential. It can be shown that there are no other extrema in the region $R > R_0$, and given that $V(R) \to +\infty$ as $R \to R_0^+$, we conclude that nondegenerate constant-curvature solutions of model (C1) having $R_0 = \eta R_*$ are stable.

APPENDIX D: QUANTITIES AND FORMULAS OF INTEREST FOR THE PHYSICAL CHARACTERIZATION OF SOLUTIONS

Given the extraordinary amount of symmetry exhibited by static, spherically symmetric spacetimes of the form (41), the study of all the points listed at the beginning of Sec. II simplifies considerably. For any given solution, one essentially needs to determine the values of r where functions $A(r)$ and $B(r)$ either vanish or become infinite, and which of these points or regions are pathological, as explained in what follows. As stated in the bulk of the text, this appendix is also meant to further clarify our nomenclature and symbol conventions.

1. Apparent and Killing horizons

It can be shown that, when a static spherically symmetric spacetime is expressed in the Abreu-Nielsen-Visser gauge (43)—equivalently, in areal-radius coordinates (41)—its apparent horizons are located at the *simple* roots r_{ah} of the algebraic equation $g^{rr}(r_{\mathrm{ah}}) = 0$.[21] In other words, Eq. (41) has an apparent horizon at $r = r_{\mathrm{ah}}$ when $B(r_{\mathrm{ah}}) \to \infty$ or, equivalently, if

$$r_{\mathrm{ah}} = 2GM_{\mathrm{MSH}}(r_{\mathrm{ah}}). \tag{D1}$$

On the other hand, $\xi = \partial/\partial t$ is a Killing vector of every line element of the form (41), since they are all static. As a result, these spacetimes will harbor a Killing horizon provided that Killing vector ξ becomes null in some region of spacetime. In areal-radius coordinates (t, r, θ, φ), $\xi^\mu = \delta^\mu{}_t$, so the norm of ξ is given by

$$g_{\mu\nu}\xi^\mu\xi^\nu = g_{tt} = A(r). \tag{D2}$$

We then conclude that spacetimes of the form (41) will host a Killing horizon at $r = r_{\mathrm{Kh}}$ provided that r_{Kh} is a root of

$$A(r_{\mathrm{Kh}}) = 0. \tag{D3}$$

Expressions (D1) and (D3) imply that line element (41) exhibits coordinate singularities at the locations of the apparent and Killing horizons, respectively. However, we must stress that the aforementioned conditions (D1) and (D3) are obtained by computing scalar quantities, which should remain invariant even when they are expressed in singular coordinates, such as the areal radius coordinates. For the purpose of our analysis, this level of rigor shall be sufficient; we are nonetheless aware that a more detailed and mathematically precise computation can be performed.

2. Singularities and geodesic completeness

The existence of coordinate singularities [points in which $A(r)$ and/or $B(r)$ become zero or infinite] may also point out the presence of curvature singularities. The existence of such curvature singularities, however, must be determined in a coordinate-invariant way; for example, we will compute the Kretschmann scalar

$$\mathcal{K} = R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} \tag{D4}$$

for each solution, and then determine the values of r where this quantity diverges.

The mere existence of curvature singularities is not a sign of unphysical dynamics *per se*, even though they represent points or regions in which tidal forces become infinite. Should these singularities be unreachable in a finite affine parameter for causal observers, then none of such observers would experience infinite tidal forces at any point along their world lines.

In order to compute whether a photon can reach a singularity for a finite value of the affine parameter, we will analyze the corresponding geodesic equation. Given that all the spacetimes we are considering are spherically symmetric, we may always choose, without loss of generality, to perform all computations on the equatorial plane

[21] In addition, spacetimes (43) possess a wormhole throat wherever g^{rr} has a *double* root.

($\theta = \pi/2$). As a result, the equation for the null geodesics of (41) reduces to

$$\left(\frac{\mathrm{d}r}{\mathrm{d}\lambda}\right)^2 = \frac{E^2}{A(r)B(r)} - \frac{L^2}{r^2 B(r)}, \tag{D5}$$

where λ is the affine parameter of the trajectory and E and L are the observer's conserved quantities, namely

$$E = A(r)\frac{\mathrm{d}t}{\mathrm{d}\lambda} = \mathrm{const}, \tag{D6}$$

$$L = r^2 \frac{\mathrm{d}\varphi}{\mathrm{d}\lambda} = \mathrm{const}. \tag{D7}$$

From expression (D6), it is evident that one can always rescale λ in such a way that $E = 1$. Accordingly, a photon with unit energy traveling from a given r_{ini} following a radial geodesic ($L = 0$) takes the following variation of affine parameter λ to reach any other value of the areal radius r_{fin}:

$$\Delta\lambda(r_{\mathrm{ini}} \to r_{\mathrm{fin}}) = \left|\int_{r_{\mathrm{ini}}}^{r_{\mathrm{fin}}} \mathrm{d}r\sqrt{A(r)B(r)}\right|. \tag{D8}$$

[Notice that an absolute value has been intentionally included in the previous expression in order to produce the same value of $\Delta\lambda(r_{\mathrm{ini}} \to r_{\mathrm{fin}})$ regardless of whether $r_{\mathrm{ini}} < r_{\mathrm{fin}}$ or $r_{\mathrm{ini}} > r_{\mathrm{fin}}$.] In most practical cases, the evaluation of (D8) will reduce to

$$\Delta\lambda(r_{\mathrm{ini}} \to r_{\mathrm{fin}}) = |\lambda(r_{\mathrm{fin}}) - \lambda(r_{\mathrm{ini}})|, \tag{D9}$$

where we have defined the primitive

$$\lambda(r) \equiv \int \mathrm{d}r\sqrt{A(r)B(r)}. \tag{D10}$$

However, it is evident that (D9) does not hold when the integrand $\sqrt{A(r)B(r)}$ changes sign within the integration interval, as is the case with some of the spacetimes and intervals considered in this work.

Finally, a singularity can be considered to be *strong* if any object falling into it is inevitably compressed to zero volume [41]. There exist criteria, due to Tipler [37] and Królak [38], that allow one to discern whether a singularity is strong or not. These criteria are as follows [39]. Consider an affinely parametrized timelike or null geodesic passing through two given spacetime points at $\lambda = \lambda_{\mathrm{ini}}$ and $\lambda = \lambda_{\mathrm{fin}}$, respectively. A singularity is said to be *Królak-strong* if the integral

$$\Delta K(\lambda_{\mathrm{ini}} \to \lambda_{\mathrm{fin}}) \equiv \left|\int_{\lambda_{\mathrm{ini}}}^{\lambda_{\mathrm{fin}}} \mathrm{d}\lambda R_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}\frac{\mathrm{d}x^\nu}{\mathrm{d}\lambda}\right| \tag{D11}$$

diverges when the final point approaches the singularity, for any starting point. Similarly, a singularity is considered to be *Tipler-strong* if

$$\Delta T(\lambda_{\mathrm{ini}} \to \lambda_{\mathrm{fin}}) \equiv \left|\int_{\lambda_{\mathrm{ini}}}^{\lambda_{\mathrm{fin}}} \mathrm{d}\lambda \int_{\lambda_{\mathrm{ini}}}^{\lambda} \mathrm{d}\chi R_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}\chi}\frac{\mathrm{d}x^\nu}{\mathrm{d}\chi}\right| \tag{D12}$$

diverges when the final point approaches the singularity, for any starting point. Notice that the curvature singularity in the Schwarzschild spacetime (having $R_{\mu\nu} = 0$) is automatically both Królak-weak and Tipler-weak.

For a radial null geodesic of a static spherically symmetric spacetime of the form (41), the integrals in Królak and Tipler conditions (D11) and (D12) can be expressed in terms of radial coordinate r using Eqs. (D5)–(D7):

$$\Delta K(r_{\mathrm{ini}} \to r_{\mathrm{fin}}) = \left|\int_{r_{\mathrm{ini}}}^{r_{\mathrm{fin}}} \mathrm{d}r\sqrt{A(r)B(r)} \times \left[\frac{R_{tt}(r)}{A^2(r)} + \frac{R_{rr}(r)}{A(r)B(r)}\right]\right|, \tag{D13}$$

$$\Delta T(r_{\mathrm{ini}} \to r_{\mathrm{fin}}) = \left|\int_{r_{\mathrm{ini}}}^{r_{\mathrm{fin}}} \mathrm{d}r\sqrt{A(r)B(r)} \times \int_{r_{\mathrm{ini}}}^{r} \mathrm{d}x\sqrt{A(x)B(x)} \times \left[\frac{R_{tt}(x)}{A^2(x)} + \frac{R_{rr}(x)}{A(x)B(x)}\right]\right|, \tag{D14}$$

where the Ricci-tensor components R_{tt} and R_{rr} may be readily computed from (41). Equations (D13) and (D14) are ideally suited to determine the strength of the curvature singularities present in some of the R_0-degenerate spacetimes encapsulated in Table I.

3. Regions in which the metric signature becomes unphysical

Last but not least, the existence of zeros of functions A and B can also lead to changes in the metric signature. For example, if A becomes negative for some values of r, then the metric becomes Euclidean (i.e. all coordinates become spacelike). If, on the contrary, B becomes negative, then there are two time coordinates. Both situations are clearly unphysical and should be avoided. In particular, we can use again Eq. (D8) to know whether such regions with unphysical metric signatures can be reached by causal observers in a finite affine parameter.

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