# Compact stars with ultrastrong magnetic fields in the approximation of exact spherical symmetry: Substantiation and application

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A few systematizing remarks concerning a spherically symmetric approximation in the description of compact stars with gravitationally strong magnetic fields are given, including the problem of the definition of mass accounting for the electromagnetic mass contribution. The problem of effective magnetic charges associated with the approximation method is addressed. Two simple models of compact stars with ultrastrong magnetic fields of $10^{17}$–$10^{18}$ Gs are studied in the limiting approximation of exact spherical symmetry, which is the limit of considered spherical approximations. Under this condition the characteristics of the magnetized MIT-bag strange quark star show scaling with the bag constant, which can be shown to be a result of a scaling symmetry of the equations of stellar structure. The model is compared with a neutron star model described by the UV14 + TNI equation of state. At the strongest fields, the masses of the modeled stars exceed values normally expected for compact stars and ascribed rather to black holes or models with exotic equations of state. Based on the considered neutron star example, it seems that too strong magnetic fields may render models of compact stars radially instable.

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## I. INTRODUCTION

This work discusses some of the issues involved in modeling the gravitational effect of strong magnetic fields onto the structure of compact stars in the approximation of spherical symmetry. Within the presented approach one is led to a family of spherical models considered in the above context. The related problem of mass definition in the presence of magnetic fields will also be addressed. Associated with this modeling approach is the issue of effective magnetic charges that deserves a few words of comment. A limiting exactly spherically symmetric model will be applied to two models of compact stars with ultrastrong magnetic fields, namely, a strange quark star described by the simplest equation of state in the framework of MIT bag model, and a neutron star described by the UV14 + TNI equation of state. The obtained results concerning the mass-radius diagram will not wander astray from what is known for models of nearly spherical compact stars with strong electric or magnetic fields. The equations of gravitational equilibrium of the quark star in the considered magnetic field exhibit a scaling symmetry which explains the scaling of the stars' global characteristics with the bag constant, observed for numerical solutions.

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### A. Characteristic scales of magnetic fields in compact stars

Magnetic fields in neutron stars are typically on the order of $10^{12}$–$10^{13}$ Gs [1] or even as large as $10^{14}$–$10^{16}$ Gs ($10^{14}$–$10^{15}$ Gs [2], $8 \times 10^{14}$ Gs [3], $3 \times 10^{15}$ Gs [4]). The surface magnetic fields determined observationally are of $10^{14}$–$10^{15}$ Gs [5]. Originally millisecond rotational period pulsars might generate ultra strong dipole fields, much stronger than $10^{13}$ Gs, even as large as $3 \times 10^{17}$ Gs [2]. With the virial argument at hand, stating that for a uniform star the magnetic energy should not exceed the gravitational energy, one arrives at fields as strong as $10^{18}$ Gs inside neutron stars [6]. Considering ultrastrong fields of $10^{17}$–$10^{18}$ Gs appears natural in the context of compact stars, as suggested also by a simple dimensional analysis discussed below, in Sec. I B. Fields as high as $10^{19}$ Gs can be present at the cores of compact stars [7,8]. An upper limit on the magnetic field sustained by a quark star is roughly $10^{20}$ Gs [9]. A maximum field strength of $1.5 \times 10^{20}$ Gs follows from the requirement that the magnetic energy density should not exceed the energy density of self-bound quark matter [10].

Electromagnetic fields contribute to the stress-energy tensor in the gravitational field equations. The resulting anisotropic pressure and tension additional to the material pressure, when comparable with that of matter, may alter the structure of a gravitationally bound system. For example, the maximum stable mass of magnetized white dwarfs was observed to exceed the Chandrasekhar limit by almost $2M_\odot$ at core magnetic fields of $5\times10^{14}$ Gs [11]. A substantial modification of mass was observed also for neutron stars, quark stars [12] and other models of stars [13] with ultra-strong core magnetic fields of $10^{17}$–$10^{18}$ Gs. It would be interesting to see the effect of fields that strong on the characteristics of compact stars under various modeling assumptions.

### B. The gravitational scale of magnetic fields

The highest field magnitudes as summarized in Sec. I A, agree with a gravitational scale of magnetic fields inferable from a simple dimensional analysis.

In the strong field regime, the gravitational effect of a magnetized star of radius $R$ can be expected for $Gb^2R^2/c^4\sim 1$, which implies the magnetic field strength scale of

$$b\sim\frac{c^2}{R\sqrt{G}}=\left[\frac{10\ \mathrm{km}}{R}\right]\cdot 3.48\times10^{18}\ \mathrm{Gs}.$$

The number $n_b=Gb^2R^2/c^4$ is linear in the gravitational constant $G$ and can be compared with other numbers characterizing the stellar structure and linear in $G$. When compared with $n_M=2GM/Rc^2$ involving a stellar mass $M$, one obtains another scale of magnetic field

$$b\sim\sqrt{\frac{2Mc^2}{R^3}}=\left[\frac{M}{M_\odot}\right]^{1/2}\left[\frac{10\ \mathrm{km}}{R}\right]^{3/2}\cdot 1.89\times10^{18}\ \mathrm{Gs}.$$

At this scale, the magnetic and gravitational effects will be comparable, not necessarily in the strong field regime only (the magnetic energy density and that of matter are then comparable). The numbers $n_b$ and $n_M$ scale differently with the stellar size, however they are comparable for compact stars with radii of several kilometers. In this case the gravitational magnetic field scale is extremely high of $10^{17}$–$10^{18}$ Gs. For quark stars one can make an independent magnetic field scale estimation with the same order of magnitude. At the quark star boundary surface, the energy density of matter is determined by the bag constant: $\epsilon=4B$. On comparing this with the magnetic energy density $b^2/8\pi$, one is led to the following magnetic field scale

$$b\sim\left[\frac{B}{100\ \mathrm{MeV}\cdot\mathrm{fm}^{-3}}\right]^{1/2}\cdot 4.01\times10^{18}\ \mathrm{Gs}.$$

Magnetic fields as strong as $10^{18}$ Gs are not physically implausible. A substitute effective charge $q_m=bR^2$ per stellar mass $M$ can be ascribed to each of the above three estimates, respectively,

$$\left[\frac{R}{10\ \mathrm{km}}\frac{M_\odot}{M}\right]\cdot 6.77\sqrt{G},\qquad \left[\frac{R}{10\ \mathrm{km}}\frac{M_\odot}{M}\right]^{1/2}\cdot 3.68\sqrt{G},$$
$$\text{and}\quad \left[\frac{10\ \mathrm{km}}{R}\right]\cdot\left[\frac{100\ \mathrm{MeV}\cdot\mathrm{fm}^{-3}}{B}\right]^{1/2}\cdot 20.8\sqrt{G}.$$

These ratios are still negligible compared with the genuine electric charge-to-mass ratio of the proton and the magnetic charge-to-mass ratio of the hypothetical magnetic monopole, respectively

$$\frac{q_e}{m_p}\sim 1.11\times10^{18}\sqrt{G},\quad\text{and}\quad \frac{q_m}{m}\sim 9.84\times10^{15}\sqrt{G}$$

(the latter was determined for the Dirac [14] elementary magnetic charge $q_m=e/2\alpha$ and a mass $\alpha^{-1}M_W$, with $M_W\sim 53$ GeV being the upper bound for a typical vector boson mass, as considered within t'Hooft field-theoretical model of magnetic monopole [15]).

## II. SPHERICAL SYMMETRY APPROXIMATION AND EFFECTIVE MAGNETIC CHARGES

The influence of magnetic field on the structure of compact stars, when these fields can be assumed strong enough to have a substantial gravitational effect, can be studied in the simplifying approximation of spherical symmetry. The standard methods developed for spherical stars can be then used, while the anisotropy of stresses characteristic of magnetic fields can be still incorporated.

It was observed for magnetized white dwarfs models [16] that for a toroidally dominated magnetic field the stellar surface indeed deviates very little from a spherical shape, while the star mass and the radius change substantially with the field strength (isodensity contours become prolate as one approaches the center). This quasisphericity substantiated the assumption of spherical symmetry in models of strongly magnetized white dwarfs [16]. But quasisphericity is realistic only to some degree. For example, the effect of poloidal fields is similar in effect to a rotation—the star no longer remains spherical, oblateness of the star surface is observed to increase with the magnetic field strength [17]. Nonspherical corrections are not included for fields below $2\times10^{18}$ Gs as insignificant by other arguments. However, for fields an order of magnitude larger, magnetized strange stars may become more susceptible to radial instabilities [9].

Spherical symmetry is incompatible with sourceless magnetic fields. However, this symmetry can be assumed as an approximation at the cost of introducing effective magnetic charges. As it will become clear later, the charges appear as the result of the averaging scheme adopted to erase non-spherical components and should be regarded as effective. One could rightfully object against considering magnetic sources as realistic. Genuine magnetic monopoles have not been detected so far, despite many dedicated experiments (for the recent advances in the theoretical and experimental physics of magnetic monopoles and the current state of magnetic monopole searches, see [18–20]). Moreover, one can formulate theoretical arguments against the existence of magnetic monopoles, in particular as incompatible with the requirement of the positivity of norm defined for quantum states of the infrared part of the Maxwell electromagnetic field carrying the information about the total electric and magnetic charges of an isolated system [21–23]. On the other hand, there are also compelling theoretical arguments for considering magnetic monopoles as solitonic solutions in the theories of interactions with an extended gauge symmetry group [15,24].

The spherical symmetry of the metric tensor assumed for nearly spherical stars is broken by the gravitationally strong magnetic fields. Nevertheless, this symmetry can be retained approximately and some insight into the gravitating effect of magnetic stresses on the structure of compact stars can be still gained. This is possible at the cost of neglecting nonspherical corrections introduced by realistic sourceless field both at the level of the equation of state and at the level of the equations of the gravitational equilibrium. There are two sources of anisotropy in the magnetic stress-energy tensor, local and global. Local fields are microscopically stochastic fields of which directions can be averaged out to produce a large-scale and locally ordered field. At the microscopic scale, the fields alter the equation of state of the material constituting compact stars [9,10]. In particular, the density of available states gets modified via the Landau quantization mechanism, significant for fields higher than $10^{19}$ Gs [5]. Global fields are considered on the macroscopic scale. When strong enough they will affect the spherical shape of the star, and therefore cannot be described under spherical symmetry unless a coarse grained order-of-magnitude description is sufficient.

In what follows, complete construction from the first principles of a spherical model with magnetic fields is discussed. A signature $(-,+,+,+)$ for the metric tensor will be assumed from now on.

### A. Magnetic stress-energy tensor

Having neglected electric forces, the stress-energy tensor $T^b_{\mu\nu}$ of Maxwell field can be rewritten as one for pure magnetic field orthogonal to the 4-velocity vector $u^\mu$ of the frame in which the magnetic vector $b_\alpha$ is defined as a linear form $\frac{1}{2}\epsilon_{\alpha\beta\mu\nu}u^\beta F^{\mu\nu}$ (up to a sign) of the Faraday tensor $F_{\mu\nu}$. The $T^b_{\mu\nu}$ must be symmetric, traceless and homogeneous of degree 2 in $b_\mu$. Additionally, the plane spanned by null directions $u_\mu\sqrt{b_\alpha b^\alpha} \pm c b_\mu$ should be an invariant plane of this tensor, with any vector from this plane being an eigenvector to the eigenvalue $-\frac{1}{8\pi}b_\alpha b^\alpha$, the latter identified as the energy density of the field $b_\mu$. $T^b_{\mu\nu}$ has to be found among linear combinations of symmetric tensors $g_{\mu\nu}$, $u_\mu u_\nu$, $b_\mu b_\nu$ and $b_\mu u_\nu + u_\mu b_\nu$ (with only $u_\mu$ and $b_\mu$ at hand, there is no more symmetric tensors—including also those involving explicitly the remaining tensor $\epsilon_{\alpha\beta\mu\nu}$—that would be independent of the previous tensors). The conditions determine the coefficients of this combination uniquely. One obtains as a solution

$$T^b_{\mu\nu} = \rho_b\left(\frac{u_\mu u_\nu}{c^2}\right) + p_b\left(\frac{u_\mu u_\nu}{c^2} + g_{\mu\nu}\right) - 2\tau_b\frac{b_\mu b_\nu}{b^2},$$
$$b^2 = b_\alpha b^\alpha, \qquad b_\alpha u^\alpha = 0, \qquad \rho_b = p_b = \tau_b = \frac{b^2}{8\pi}.$$

The above reasoning led us to a covariant form of magnetic stress-energy tensor equivalent to that known from the relativistic magnetohydrodynamical theory [25].

Furthermore, the field $b_\mu$ may induce material magnetic moments. Depending on the properties of matter, the induced magnetic moment $m_\mu$ per volume can be related to $b_\mu$ in a nontrivial fashion. The simplest model of magnetization assumes local isotropy of the material, that is, $m_\mu = \chi b_\mu$ with $\chi$ being a scalar. Similar as before, the contribution of the magnetization to the stress-energy tensor must be a linear combination $T^\chi_{\mu\nu}$ of independent symmetric tensors. Since $m_\mu$ and $b_\mu$ are collinear in the present case, again there are only 4 such tensors: $g_{\mu\nu}$, $u^\mu u^\nu$, $b^\mu b^\nu$, and $u^\mu b^\nu + u^\nu b^\mu$. The coefficients of this combination are found by the requirement that $T^\chi_{\mu\nu}u^\nu = 0$ (no associated additional material currents) and $T^\chi_{\mu\nu}b^\nu = 0$ (the spatial stresses are transverse to $b_\mu$). By solving the conditions, it follows that the simple magnetization field contributes to the interaction part in the stress-energy tensor through

$$\frac{1}{8\pi\chi}T^\chi_{\mu\nu} \propto p_b\left(\frac{u_\mu u_\nu}{c^2} + g_{\mu\nu}\right) - \tau_b\frac{b_\mu b_\nu}{b^2}$$

times a constant number.

The tensors $T^b$ and $T^\chi$ are in some sense complementary one to another: the vectors $b_\mu$ and $u_\mu$ span the eigenspace of $T^b$ and the kernel of $T^\chi$. Of these two only $T^b$ contributes to the magnetic energy density $\rho_b$. Either $T^b$ and $T^\chi$ introduce locally anisotropic stresses: they consist of the isotropic pressure (equal to or proportional to $\rho_b$) and a longitudinal tension (reversing or canceling the effect of pressure in the longitudinal direction). In both cases the stresses are proportional to the magnetic field energy density. In effect, $T^b$ contributes a plane-isotropic transversal pressure and equal in magnitude longitudinal tension, while $T^\chi$ contributes only a plane-isotropic transversal tension. It is clear that when $m^\mu$ is assumed proportional to $b^\mu$ and depending on the sign of $\chi$, the only role of magnetization *is to increase or reduce the effect of magnetic pressure in the transversal directions, while keeping it unaffected in the longitudinal direction.*

### B. Total gravitating stress-energy tensor

The matter stress-energy tensor will be taken to be that of a fully isotropic perfect fluid

$$T^{m}_{\mu\nu} = \rho_m u_\mu u_\nu + p_m\left(\frac{u_\mu u_\nu}{c^2} + g_{\mu\nu}\right).$$

From the structure of the magnetic stress-energy tensors it is clear that the gravitational effect of magnetic field can be described in terms of the stress-tensor of an isotropic perfect fluid augmented with additional anisotropic term $b_\mu b_\nu$ that introduces substantial tension in the direction of the local field $b^\mu$. The total stress-energy tensor reads

$$T_{\mu\nu} = \rho u_\mu u_\nu + p\left(\frac{u_\mu u_\nu}{c^2} + g_{\mu\nu}\right) - \tau\frac{b_\mu b_\nu}{b^2},$$
$$\rho = \rho_m + \rho_b/c^2, \quad p = p_m + (1+\eta)p_b, \quad \tau = \tau_b(2+\eta),$$

with $p_b$, $\tau_b$, $\rho_b$ defined as earlier. Here, the parameter $\eta$ can be interpreted as a fraction of the magnetization pressure to the magnetic pressure ($\eta$ is equal to $8\pi\chi$ times a constant number, depending on notational conventions). When the parameter $\chi$ is small, the magnetization effect can be neglected by setting $\eta = 0$.

### C. Approximation of spherical symmetry

The isotropic, ideal fluid part of the electromagnetic contribution to the above $T_{\mu\nu}$ through $\rho$ and $p$, is consistent with the spherical symmetry. In contrast, the last, anisotropic term is inconsistent with this symmetry, unless $b^\mu$ is purely radial. One cannot simply discard the part of the anisotropic term violating this symmetry, because its magnitude is comparable with that of the electromagnetic pressure. The Einstein tensor with spherically symmetric metric has off-diagonal components all vanishing, so the corresponding components of the electromagnetic stress-tensor must also be vanishing.

For consistency with the spherical symmetry, one can perform various sorts of averaging of $T^{\mu\nu}$. Here, two most natural are given. The first is the *spherical averaging at a point* ($\vartheta$, $\phi$) over all directions of local magnetic fields, assuming these directions are equally probable in a volume element comprising some vicinity of that point. The averaging is to be made by performing integration over spherical angles $\lambda$ and $\psi$ in a local frame spanned at a given point by unit vectors $\hat{e}_r$, $\hat{e}_\vartheta$, $\hat{e}_\phi$, in which $b_\mu$ has been decomposed into components $b_i = b_\mu \hat{e}_i{}^\mu$ so that $\vec{b}(\psi,\lambda) = \sqrt{b^\alpha b_\alpha}[\cos\psi\,\hat{e}_r + \sin\psi(\hat{e}_\vartheta\cos\lambda + \hat{e}_\phi\sin\lambda)]$ in that frame. In this case, one obtains an effective stress-energy tensor of isotropic perfect fluid with the equation of state

$$p_r = p_\vartheta = p_\phi = (1+2\eta)\rho_b/3 \tag{1}$$

(for $\eta = 0$ one recovers the equation of state of ultra-relativistic gas). The idea of isotropic perfect fluid seems to be a standard approximation in the context of magnetized spherical compact stars, including models with modified gravitation Lagrangians [26]. It was already considered in the literature by Adam in 1986 [27] assuming effective pressure $(1/3)(p_\parallel + 2p_\perp) = (1/3)\rho_b$ in the context of magnetized white dwarfs, with $\rho_b$ dependent only of the radius. Recently, this idea is used in various contexts and with various assumptions about the magnetic pressure, e.g., equal to $\rho_b$ (in the context of neutron stars) [26], $(1/3)\rho_b$ (white dwarfs) [11] and $(-)\rho_b$ (strange magnetars) [13]. The magnetic pressure (or tension, depending on the sign) is assumed to be a fraction of the material pressure—similarly as it is assumed for the radiation pressure in modeling the main-sequence ordinary stars [28]—or a function of matter density. It is customary to parametrize the magnetic energy density inside a compact star with the particle concentration or the material energy density; this allows to impose easily the constraints on the magnetic field magnitude in the core and at the boundary surface [29]. In effect this all leads to a hydrostatic equilibrium equation as if for an ordinary star under spherical symmetry. Because the above averaging assumed local isotropy, the anisotropy of magnetic stresses has been erased out. The anisotropy of stresses characteristic of magnetic fields could be, nevertheless, recovered by hand with the help of an additional function $\chi(\rho_b)$ in such a way that the trace of the stresses is not changed, hence

$$p_r = (1+2\eta)\rho_p/3 - 2\chi(\rho_p),$$
$$p_\vartheta = p_\phi = (1+2\eta)\rho_p/3 + \chi(\rho_b).$$

The second kind of local averaging, respecting the global spherical symmetry of the gravitating system as a whole, is the *weighted axial averaging at a point* with respect to the local radial direction $\hat{e}_r$. Now, various $\psi$ can have various weights, and the appropriate averaging integral is

$$(F)_{\mathrm{mean}} = \frac{1}{N}\int_0^{2\pi} \mathrm{d}\lambda \int_0^{\pi} F(\psi,\lambda)w(\psi)\sin\psi\,\mathrm{d}\psi,$$
$$N = 2\pi\int_0^{\pi} \mathrm{d}\psi\, w(\psi)\sin\psi, \qquad w(\psi) > 0,$$

where $w(\psi)$ is a weighting function (the isotropic case is recovered for $w(\psi) = 1$). Denoting $\alpha = A_{\mathrm{mean}}$ for $A(\psi,\lambda) = \cos^2\psi$ (then $0 < \alpha < 1$), the general result is again an effective diagonal magnetic stress tensor with

$$p_r = ((1-2\alpha) + \eta(1-\alpha))\rho_b,$$
$$p_\vartheta = p_\phi = (\alpha + \eta(1+\alpha)/2)\rho_b,$$

with the reservation that $\rho_b$ and $\alpha$, $\eta$ have to be functions of the radial variable only:

$$\rho_b = \rho_b(r), \qquad \alpha = \alpha(r), \qquad \eta = \eta(r).$$

For α closer to 0 the directions of magnetic field lines are more transversal, for α closer to 1 they are more aligned in the radial direction, and for $\alpha = 1/3$ they are isotropic (in all cases local orientations along the field directions are still irrelevant). In the limit $\alpha \to 1$ the magnetic stress tensor is described by the plane-isotropic transverse pressure and the radial tension

$$p_r = -\rho_b, \qquad p_\vartheta = p_\phi = (1+\eta)\rho_b. \tag{2}$$

It should be noticed that the effective stress tensor defined by Eq. (2) is formally the same as for a purely radial hedgehog magnetic field with a unique orientation and with a given distribution of (effective) magnetic charge uniquely determined by the given magnetic energy density. The effective stress tensor Eq. (2) (as well as that for any α) is a particular case of a spherically-symmetric anisotropic fluid [30] for which the radial and tangential pressure are independent. It should be also stressed that the averaging assumption is not true when there are globally ordered large-scale stellar magnetic fields. This means that the above approach can be considered only as a coarse-grained approximation, nonetheless, correctly accounting for the magnetic energy amount on any separate spherical shell.

### D. Effective magnetic charges

The above construction in Sec. II C shows that the non-spherical contribution due to the directional character of the originally sourceless magnetic field, when negligible in a given context, can be eliminated on the level of the magnetic stress-energy tensor by a suitable averaging consistent with the assumed symmetry or by other means, and one can use an approximate effective stress-energy tensor instead, which is simpler to tackle with. Once the effective tensor is defined and plugged into the spherically symmetric Einstein equations as a source of the gravitational field, it becomes quite irrelevant the source of the original complicated magnetic field that has been used to produce the effective tensor.

The problem arises when one decides, nevertheless, to interpret the effective stress-energy tensor back in terms of a corresponding substitute magnetic field, consistently with the assumed symmetry and without introducing effective magnetic charge density at the same time. Surely, this will not go in the approximation defined by Eq. (2) if one sets

$$b^\mu_{\rm eff}(r) = \pm(\hat{e}_r)^\mu\sqrt{8\pi\rho_b(r)},$$

giving rise to the effective magnetic charge density

$$\rho_{\rm eff}(r) = \frac{1}{4\pi r^2\sqrt{g_{rr}(r)}}\left(r^2\sqrt{8\pi\rho_b(r)}\right)_{,r}.$$

After Adam [27], one can define a substitute field magnitude by averaging over spherical shells the magnitude squared of the original field (presumably, with arbitrarily curved or twisted lines):

$$\tilde{b}^2(r) = \int \mathrm{d}\Omega b^2(r,\Omega), \qquad \rho_b \equiv \tilde{b}^2(r)/8\pi, \qquad p_b = \rho_b/3.$$

Such identified energy density and (isotropic) pressure are then to be added to the density and pressure of matter regarded as a locally isotropic perfect fluid. This agrees with the approximation defined in Eq. (1). Similarly, in all other sorts of spherical symmetry approximations, the information about electromagnetic field is reduced to a single scalar—the mean magnetic field magnitude—that defines both the energy density and pressures on the diagonal of the effective magnetic stress-energy tensor.

Upon averaging as presented in Sec. II C, the directional information about the field has been lost. One can construct a number of substitute fields that account for the same $\tilde{b}^2(r)$, however with some effective magnetic charge density, because spherical symmetry is incompatible with Maxwell fields of magnetic type (except for Dirac's point monopole). The description of magnetized stars in the approximation of spherical symmetry requires substantial modification of the original stress tensor and these changes can be described by effective charges. As will be illustrated below, the scale of effective charges is determined by the scale of magnetic energy density (and thus also by the scale of magnetic stresses), which can be comparable with the energy density of matter.

To give an example, take an ordinary dipole in Minkowski spacetime as the original sourceless and non-spherically symmetric field with off-diagonal stresses. Then $b^2(r,\vartheta) = \frac{\mu^2}{r^6}(1+3\cos^2\vartheta)$, hence $\tilde{b}^2(r) = 2\mu^2/r^6$, which gives the mean energy density $\tilde{\rho}_b = \mu^2/4\pi r^6$. Another sourceless field with the same mean energy density is the longitudinal field $\sqrt{2}\mu\hat{e}_\phi/r^3$. Even though the field alone is not spherically symmetric, the energy density already is (the Adam integration needs not to be carried out in this case), there is no off-diagonal stresses unlike before, however the diagonal terms still violate spherical symmetry: $p_r = p_\vartheta = -p_\phi = \tilde{\rho}_b$.

The above mean energy density could be interpreted, consistently with the spherical symmetry, as that of a substitute purely radial field $\vec{b} = \tilde{b}(r)\hat{e}_r$ with the (nonintegrable at $r=0$) effective charge density $\rho_{\rm eff} = -\frac{\sqrt{2}}{4\pi}\mu/r^4$, and the corresponding transversal pressures and radial tension: $p_\vartheta = p_\phi = -p_r = \tilde{\rho}_b$. Since total flux of $\vec{b}$ through the sphere at infinity is 0, a regularized field $\vec{b}_\epsilon = \frac{r\mu\sqrt{2}}{(r^2+\epsilon^2)^2}\vec{e}_r$ with $\rho_{\epsilon\rm eff} = \frac{\mu\sqrt{2}(3\epsilon^2-r^2)}{4\pi(r^2+\epsilon^2)^3}$ integrable to 0 can be taken instead, not to clash with the Gauss integral theorem for arbitrary small $\epsilon$. Then the charge $Q_m = \frac{3\sqrt{6}\mu}{16\epsilon}$ enclosed by the sphere of radius $\sqrt{3}\epsilon$ is canceled by the opposite charge $-Q_m$ exterior to that sphere, both infinite in the limit $\epsilon \to 0$ (a similar screening phenomenon of singular compensating sources can be observed in other contexts—see, e,g., [31]).

These two example fields above are extremely nonspherical, however they were used to illustrate the concept of effective charges that appear as a result of some kind of averaging under the assumed symmetry. What is important, it that the amount of magnetic energy $E_\epsilon = \frac{\mu^2}{3r^3}(1 - \frac{12}{5}\epsilon^2/r^2 + \ldots)$ exterior to any sphere of radius $r \gg \epsilon$ and associated with the original field is determined by the amount of effective charge $Q_\epsilon = -\frac{\sqrt{2}\mu}{r}(1 - 2\epsilon^2/r^2 + \ldots)$ associated with the substitute hedgehog field and generated in the same region: $Q_\epsilon^2 \approx 6rE_\epsilon$, with the energy per arbitrary spherical shell the same as for the original field without charges. In other words, the effective charge distribution is right that to produce the required amount of magnetic energy in spherical shells through the original field.

## III. APPROXIMATION OF EXACT SPHERICAL SYMMETRY

As have been already noticed in Sec. II C, the axial averaging method led to the limiting diagonal stress tensor defined by Eq. (2) which is in effect the same as due to hedgehog magnetic field under exact spherical symmetry. Exact spherical symmetry means that the Lie derivative of the electromagnetic field vanishes with respect to any of the three Killing vectors of the spherical symmetry. One can show for a general spherically symmetric spacetime (with $r$ as the areal radius variable), that such a field must be of the form $F_{tr} = -F_{rt} = u(r)$, $F_{\vartheta\phi} = -F_{\phi\vartheta} = v(r)\sin\vartheta$ with other components of the Faraday tensor vanishing (similarly, $w^t = u(r)$, $w^r = v(r)$ and $w^\vartheta = w^\phi = 0$ for vectors). This means coexistence of purely radial magnetic and electric fields, varying only with $r$.

Unlike for electric fields, outside a star the energy density of large-scale ordered magnetic fields can still be large, however it must be due to sourceless magnetic fields with energy density falling-off like $r^{-6}$ (in agreement with vanishing integrated charge), that is, faster than the energy density of magnetic monopole, which falls off like $r^{-4}$. However, this concerns only the asymptotic behavior at a very large distance from the star. In the close vicinity of the star the magnetic energy density can be comparable with that below the stellar surface.

On the side of the gravitational equilibrium equation under the approximation of spherical symmetry, the magnetic field is described through a predefined mean value energy density $\tilde{b}^2(r)/8\pi$ accounting for that observed inside real stars, and as far as one is interested only in the interior mean magnetic energy density accounting for that inside real stars or in its close vicinity, this approximation seems acceptable. However, this difference in the electromagnetic energy behavior at very large distances may lead to some discrepancy between the total estimated mass.

### A. Magnetically charged exterior and interior solution

The Reissner-Nordström black hole metric

$$\mathrm{d}s^2 = -c^2 f(r)\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{f(r)} + r^2\mathrm{d}\Omega^2,$$

$$f(r) \equiv 1 - \frac{2Gm}{rc^2} + \frac{Gq_e^2}{c^4 r^2}$$

describes the exterior of a spherically symmetric electrically charged and massive body. In addition to the mass $m$ and electric charge $q_e$, in Einstein-Maxwell theory one can also consider a point magnetic charge $q_m$, because it is located at the gravitational singularity. This amounts to replacing the previous $f$ with a new one

$$f(r) \equiv 1 - \frac{2Gm}{rc^2} + \frac{G(q_e^2 + q_m^2)}{c^4 r^2}.$$

The associated electric and magnetic fields are purely radial, with physical values $E_r \equiv F_{\mu\nu} e_t^\mu e_r^\nu = q_e/r^2$ and $B_r \equiv F_{\mu\nu} e_\vartheta^\mu e_\phi^\mu = q_m/r^2$, as determined with respect to the orthonormal tetrad $e_i^\mu$ of the local static observer.

It is interesting that the electric and magnetic part contribution to the gravitational field are indistinguishable from the standpoint of an electromagnetically neutral particle moving in the gravitational field. In this context it is unimportant whether the electromagnetic contribution to the gravitational stellar mass is due to electric or magnetic charge or a mixture of them.

Magnetic charges as such, even if effective, could not be considered as stellar constituents consistently with Maxwell equations, except for a single point magnetic charge at the center of spherical symmetry (e.g., magnetically charged black hole) or in form of a split monopole [32], (however with additional effective currents of which gravitational influence might also be important). Given that magnetic monopoles have not yet been observed, the magnetic charge density, if introduced, should be regarded as an effective substitute used to model the gravitational influence of the intense magnetic field on the stellar structure. With this reservation, considering magnetically charged sources is possible, however, in an extended Maxwell theory with two gauge potentials [33], as will be discussed below.

The magnetic charge concept is unavoidable when considering an electrically charged particle consistently with the laws of quantum mechanics and ordinary Maxwell theory. A point-like elementary magnetic charge with unique and quantized magnitude (and unspecified mass) is possible at the expense of introducing an unobservable singularity string emanating from the pole position point, as predicted by Dirac in 1931 [14,34,35]. The quantization of magnetic charge can be viewed as the condition for the singularity string in the vector potential to be unobservable. Magnetic poles can be introduced also in terms of field-theoretical concepts. They appear in non-Abelian gauge field theories via a mechanism similar to that of a monopole discovered independently by 't Hooft [15] and Polyakov [24]. They are spherically symmetric massive spatially extended solitonic solutions characterized by a topological charge that in the low energy regime of unified theories of interactions can be reinterpreted in terms of magnetic charge of a U(1) gauge potential, equivalent to one of a genuine Maxwell theory. The resulting magnetic field is asymptotically that of Dirac's monopole. Currently, no magnetic monopoles have been found despite many attempts to detect them. The discussion on the theoretical and experimental status of magnetic monopoles can be found in [18,19]. A comprehensive review on various aspects of the theory of magnetic monopoles is presented in [36]. Recent advances in the physics of magnetic monopoles are discussed in [20]. The issue of (non-)existence of magnetic monopoles from the theoretical standpoint is briefly addressed and relevant original literature cited in [37].

An alternative formulation of electrodynamics due to the Cabibbo and Ferrari idea [33], in which both the electric and magnetic field may be expressed in terms of a pair of gauge potentials [38], allows to regard electric and magnetic charges on a similar footing as gauge charges. This theory can be expressed in terms of two antisymmetric fields $F_{\mu\nu}$ and $G_{\mu\nu}$, with electric vector $F^{i0} - {}^*G^{i0}$ and magnetic pseudo- vector $G^{i0} + {}^*F^{i0}$ (here, the star denotes the Hodge dual). The stress-energy tensor involves two terms for the two tensor fields separately, which are of the same form as the stress-energy tensor in Maxwell theory, and additional cross-terms ensuring that the two kind of charges may interact with each other [38]. In this theory, one can consider a situation in which electric charges are absent, hence one is left with a theory similar to ordinary Maxwell theory with only magnetically charged sources. This is what is assumed in what follows.

The interior metric of a spherically symmetric distribution of magnetically charged static star must agree with the magnetized Reissner-Nordstrom exterior solution at the radius of vanishing pressure of stellar material. The matching condition allows to identify the mass of the star, including the electromagnetic contribution to the mass. This correspondence suggest to parametrize the metric components in terms of a formal (for the time being) mass function $M(r)$ such that, for $r < R$:

$$g_{tt} = -c^2 \exp(2\Phi(r)), \qquad g_{rr} = 1 - \frac{2GM(r)}{rc^2} + \frac{GQ_m^2(r)}{c^4 r^2}, \tag{3}$$

where $Q_m(r)$ is the total magnetic charge enclosed within a sphere of areal radius $r$, and $\Phi(r)$ determines the redshift formula. For $r > R$ the metric is that of Reissner-Nordström with constant parameters $M = M(R)$ and $Q_m = Q_m(R)$. Given an equation of state relating proper mass density $\rho(r)$ and partial pressure $p(r)$ of a perfect fluid and their functional relation to the magnetic field, the structure functions $M(r)$ and $Q(r)$ define the equilibrium solution. The physical magnetic field strength $b(r)$ is related to the effective charge $Q_m(r)$ through

$$b(r) = \frac{Q_m(r)}{r^2}.$$

This simple relation between $b(r)$ and $Q_m(r)$ was the motivation to express here the equations in terms of $b$ instead of the $G_{rt}$ component of the electromagnetic tensor unsuitable to correctly measure the physical field strength in the strong field regime ($G_{rt}(r) = b(r)\sqrt{-g_{tt}g_{rr}}$).

The matching conditions with the external magnetically charged Reissner-Nordström metric in the adopted spherical coordinates require the continuity of $g_{tt}$ and $g_{rr}$ at the boundary point $r = R$ defined by the condition of vanishing pressure $p = 0$. This allows to identify the total mass $M$ and the total effective magnetic charge $Q_m = R^2 b_R$ corresponding to the value of magnetic field $b_R$ measured at the boundary. The structure function $M(r)$, although overlapping with the mass at the matching point with the exterior Reissner-Nordström metric, is not necessarily the true mass function in the star interior.

### B. The issue of mass definition in the presence of gravitationally strong electromagnetic field

The identification of the true mass function is difficult in general relativity. A possible choice is the notion of mass provided by Komar integral [39] when the spacetime possesses a timelike Killing vector. This integral can be regarded as a generalization of the Gauss theorem. In the Newtonian gravitation, the theorem is used to relate the mass bounded by a closed surface, with the flux of the gravitational field through that surface. The Komar integral can be represented as an integral of a surface two-form involving the timelike Killing vector $\xi$:

$$-\frac{c}{16\pi G}\oint\!\!\oint_{\partial V} \sqrt{-\det g}\,\nabla^\alpha \xi^\beta \epsilon_{\alpha\beta\mu\nu} dx^\mu \wedge dx^\nu \tag{4}$$

One may prefer this notation as more convenient in the context of differential forms calculus than another notation used for the same integral in [40]. The integration in Eq. (4) is taken over a closed surface $\partial V$ bounding a three-dimensional spatial region $V$. Now, by applying the Stokes theorem and making use of Einstein equations, the Komar integral can be recast as a volume integral over the interior of region $V$:

$$\frac{2}{c^3}\iiint_V\left(T_{\mu\nu}-\frac{1}{2}Tg_{\mu\nu}\right)n^\mu\xi^\nu \mathrm{d}V,$$

where d$V$ is the proper volume element of the constant-time surface and $n^\mu$ is a unit vector normal to that surface. The result of applying the Komar mass formula to the metric tensor Eq. (3) over a region $V$ bounded by a sphere of areal radius $r$ allows us to define a mass function:

$$M_K(r)=\int_0^r\frac{4\pi r^2e^{\Phi(\tilde r)}\mathrm{d}\tilde r\left(\rho(\tilde r)+\frac{3p(\tilde r)}{c^2}+\frac{b^2(\tilde r)}{4\pi c^2}\right)}{\sqrt{1-\frac{2GM(\tilde r)}{c^2\tilde r}+\frac{G}{c^4}\tilde r^2b^2(\tilde r)}},$$

which gives

$$M_K(r)=\frac{1}{c^2}R^3b^2(R)\left(1-\frac{R}{r}\right)+M_K(R)\quad \text{for } r>R.$$

The above integral agrees with the Tolman energy—in the absence of electromagnetic field one gets an expression equivalent to one for the energy of a sphere of perfect fluid as obtained by Tolman [41], which in the Newtonian limit and when the pressure is low, consists of the proper energy of the sphere's material and of the potential gravitational energy. The proportionality factor in the magnetic field contribution can be made clear in terms of an energy density $u=b^2/8\pi$, a radial tension $p_r=-u$ and transversal pressures $p_\vartheta=p_\phi=u$ (as it results from the form of the stress-energy tensor for a purely radial magnetic or electric field shown later in Sec. III C), which sum up to $2u=b^2/4\pi$, in analogy to the energy density and pressures of a fluid which sum up to $\rho c^2+3p$.

Let us see what this definition of mass $M_K$ leads to in the case of pure electric or magnetic field. In the limit of extremely weak field, consider a uniformly charged ball of radius $a$ and total charge $Q$, which means that $Q(r)=Q\cdot(r/a)^3$ for $0<r<a$ and $Q(r)=Q$ for $r>0$. Then $M_K(r)=\frac{1}{5}\frac{Q^2}{ac^2}(r/a)^5$ for $0<r<a$, $M_K(r)=\frac{Q^2}{ac^2}(\frac{6}{5}-\frac{a}{r})$ for $r>a$. The total Komar mass is $M_K(\infty)=2\frac{3}{5}\frac{Q^2}{ac^2}$, that is, twice the textbook value obtained from the integrated electromagnetic energy density. This means that the gravitational mass is twice greater than the electromagnetic mass. This result is striking, however it can be understood with a simple example. Consider a uniform ultrarelativistic gas or the photon gas for which $p=\frac{1}{3}\rho c^2$, enclosed inside the same ball of radius $a$. Then $\rho+3p/c^2=2\rho$ inside the ball and 0 outside the ball. In Minkowski spacetime the integration over whole space gives $M_K=2\rho V$, where $V$ is the ball volume. This is twice the total integrated energy density of that gas.

Another definition of mass of a star with strong electromagnetic field comes from studying the gravitational effect of the star on an electromagnetically neutral test particle in orbit around the star. In the limit of Newtonian mechanics of the Reissner–Nordström metric, at a large distance, the particle has the gravitational potential energy $U(r)=-\frac{GM}{r}+\frac{GQ_m^2}{2c^2r^2}$, which means that a noncircular bound orbit is not an ellipse but a rosette. The effective gravitational stellar mass, as determined in the Newtonian regime for a circular orbit of radius $a$, would be $M_a=M-\frac{Q_m^2}{2c^2a}$. Hence, the mass ascertained at the infinity in the Newtonian weak field limit overlaps with the stellar mass $M_\infty=M\equiv M(R)$ the same as following from the matching condition of the interior and exterior metric.

As one can infer, the gravitational equilibrium equations (as given later in Sec. III C) resulting from Einstein equations with the metric parametrization Eq. (3) adopted here to conform with the external Reissner–Nordström metric, lead for $r>R$ to the mass function $M(r)=M$ constant outside the star (the effective charge does not accumulate exterior to the star), where

$$M=\frac{1}{2c^2}R^3b^2(R)+\int_0^R4\pi r^2\mathrm{d}r\left(\rho(r)+\frac{1}{c^2}\frac{b^2(r)}{8\pi}\right).\qquad(5)$$

The integral in Eq. (5) includes the contribution from the matter energy density and from the energy density of the electromagnetic field inside the star (there is no explicit contribution from pressures and tensions in contrast to Tolman mass). The free term outside the integration sign in Eq. (5) coincides with the electromagnetic energy of the monopole field integrated outside the star out to infinity. The contribution of the free term to the total mass is of the order

$$\frac{1}{2c^2}R^3b^2(R)\approx\left[\frac{b}{10^{18}\ \mathrm{Gs}}\right]^2\left[\frac{R}{10\ \mathrm{km}}\right]^3\cdot 0.280M_\odot.$$

It is important, that the definition of total mass Eq. (5), when applied to the example of uniform charged ball discussed earlier, leads in the weak field limit to $M(R)=\frac{3}{5}\frac{Q_m^2}{c^2a}$, which agrees with the classical result in electrodynamics for the total electromagnetic energy (the Komar-Tolman mass was twice greater). This coincidence can be used in favor of the definition of the star total mass, as defined in Eq. (5), rather than in favor of the more natural from another perspective Komar-Tolman mass.

The above discussion can be summarized as follows: *the active gravitational mass of a star as ascertained by electromagnetically neutral distant test body in orbit around the star, consists of the active gravitational mass of the material forming the star and of the gravitational active* *mass of the whole electromagnetic field associated with that star (including the field in the star exterior).*

The electromagnetic field density is nonzero outside the star, hence the amount of electromagnetic mass enclosed within concentric spheres should grow with the radius also in the star exterior. In the limit of infinite radius one obtains a net contribution to the whole gravitating mass as detectable based on the motions of very remote electromagnetically neutral objects. However, what is observed contrary to intuition, the mass function no longer increases outside the star surface, and equals the mass determined by the matching condition at the star surface, already including the whole electromagnetic energy present outside the star.

From the above discussion it also follows, that this is $M(R)$ as defined in Eq. (5) which should be identified as the total stellar mass and therefore it should appear on the $M$-$R$ diagram. Furthermore, $M(r)$ is a formal metric structure function for $r < R$ with the dimension of mass that only coincides with the stellar mass at $r = R$, however $M(r)$ may not necessarily be a reliable mass function in the stellar interior. Definition of mass function is notoriously difficult in general relativity theory.

### C. Equations of the gravitational equilibrium

As observed in [5] the magnetization of matter is negligible even at the field strength of $10^{19}$ Gs—the magnetic pressure is at least one order of magnitude higher than the magnetization pressure. A similar conclusion was arrived at in [10]. The magnetization pressure did not affect the results obtained for white dwarfs exhibiting highly super-Chandrasekhar masses in the presence of strong magnetic fields [42], magnetization was neglected also in [11]. Therefore, the magnetization tensor, which would otherwise contribute to the matter tensor, will be neglected also here by setting $\eta = 0$. Similarly as in [12], it will be assumed that realistic fields only minimally affect the spherical shape of a compact star.

Assuming the limiting exact spherically-symmetric model Eq. (2) with $\eta = 0$, the physical components of the total stress tensor (in the local orthonormalized basis) are then simply:

$$T_{\hat{t}\hat{t}} = c^2\rho + \frac{b^2}{8\pi}, \qquad T_{\hat{r}\hat{r}} = p - \frac{b^2}{8\pi},$$
$$T_{\hat{\vartheta}\hat{\vartheta}} = T_{\hat{\phi}\hat{\phi}} = p + \frac{b^2}{8\pi}$$

[the effect of magnetization in this case would be to increase the transversal pressures by a fraction of the electromagnetic density, cf. Eq. (2)]. The form of the above stress tensor without off-diagonal terms and expressed by electromagnetic density and electromagnetic pressure scalars, does not differ from the form of stress-energy tensors assumed in the context of magnetized compact stars in the literature (e.g., like that with α parameter in Sec. II C). As in the literature, one can assume arbitrary functional forms relating the electromagnetic field energy density to the density of matter (however, without paying attention to the effective charges that are produced this way—see the discussion in Sec. II D).

The profiles of magnetic field amplitude and matter density in the star interior are not exactly known; nonuniform magnetic field, lead to stronger field estimates in the core [10]. In what follows, the proper magnetic charge density (as defined in the fluid rest frame) will be assumed proportional to the proper energy density of matter (similarly as once assumed for degenerate Fermi gas [43]), which leads to:

$$Q'_m(r) = \frac{4\pi r^2}{\sqrt{g_{rr}}} K\rho(r),$$

with $K$ being the magnetic charge–to–mass ratio. As discussed earlier in Sec. II D, the magnetic charge is effective—it appears as the consequence of the exact spherical symmetry assumption.

The proportionality of the charge density and the material energy density seems quite reasonable an assumption as can be seen in the context of models of electrically charged stars [44]. As already noticed in Sec. I B, the charge-to-mass ratio of the order of $\sqrt{G}$ required for a substantial gravitational effect to occur, is negligible in comparison with the charge-to-mass ratio of charged nucleons, or hypothetical magnetic charges. In the case of electric charges, a charge-to-mass ratio that low allowed to safely assume that the equation of state of nuclear matter was not much affected [44]. A continuous distribution of electric charge was considered in the literature in the context of compact star models, however together with a thin layer of opposite charge making the star electrically neutral at large radii [45]. Since magnetic fields are large-scale fields present also outside compact stars, the screening layer of compensating opposite magnetic charges in the spherical model will be ignored.

Using the solar mass $M_\odot$, the speed of light $c$ and the gravitational constant $G$ as quantities providing three independent mechanical units, here introduced are relativistic units of length $r_u \coloneqq \frac{GM_\odot}{c^2} \sim 1.477$ km, pressure or energy density $p_u \coloneqq \frac{M_\odot c^2}{4\pi r_u^3} \sim 27.55\,\mathrm{GeV/fm^3} \sim 4.415 \times 10^{37}\,\mathrm{erg/cm^3}$, and of the electric or magnetic charge $q_u \coloneqq M_\odot\sqrt{G}$ (with $q_u/r_u^2 \sim 2.355 \times 10^{19}$ Gs as the derived unit of magnetic strength). In this units it is convenient to use dimensionless counterparts of the following dimensional quantities: $x \coloneqq r/r_u$ as the radial variable (areal radius in spherical geometry), cumulative mass function $\mathcal{M} \coloneqq M(r)/M_\odot$, cumulative function of the substitute magnetic charge $\mathcal{Q} \coloneqq Q_m(r)/q_u$, local isotropic pressure $\mathcal{P} \coloneqq p(r)/p_u$, baryon number density $\nu \coloneqq n_B(r) r_u^3$, and the charge-to-mass ratio

$\kappa \coloneqq K/\sqrt{G}$. The equations of the gravitational equilibrium of a magnetized star can now be put in a dimensionless form

$$\mathcal{M}' = \left(1 + \kappa\Gamma\frac{\mathcal{Q}}{x}\right)x^2x'\mathcal{R}, \qquad \mathcal{Q}' = \kappa\Gamma x^2x'\mathcal{R},$$
$$\mathcal{N}' = \Gamma x^2 x'\nu$$
$$x' = -x^2\left(\Gamma^2(\mathcal{R}+\mathcal{P})\left[\mathcal{M}+\mathcal{P}x^3 - \frac{\mathcal{Q}^2}{x}\right] - \kappa\Gamma\mathcal{Q}\mathcal{R}\right)^{-1},$$

where the prime sign denotes differentiation with respect to pressure $\mathcal{P}$. The function $\Gamma$ is the relativistic factor in the volume element $4\pi\Gamma r^2 \mathrm{d}r$ in spherical coordinates with $r$ as the areal radius:

$$\Gamma = \left(1 - \frac{2\mathcal{M}}{x} + \frac{\mathcal{Q}^2}{x^2}\right)^{-1/2}.$$

In the above equations there is a single free parameter $\kappa$ and four variables $x$, $\mathcal{M}$, $\mathcal{Q}$, $\mathcal{N}$ which will be regarded as functions dependent of the independent pressure variable $\mathcal{P}$. It should be stressed, that the mass function $\mathcal{M}$, in addition to stellar material, includes the contribution from magnetic field. The value of $\mathcal{M}$ at the stellar boundary is precisely the gravitational active mass felt by an asymptotic test body in orbit around the star, as discussed in Sec. III B.

By specifying the equation of state $\mathcal{R}(\mathcal{P})$, the numerical value of the central pressure $\mathcal{P}_c$ and of the parameter $K$, these equations are numerically integrated unless $\mathcal{P} = 0$, then the radius $R$, mass $M(R)$ and surface magnetic field $B(R)$ of the magnetized star are determined from

$$R = r_u x(0), \qquad M(R) = M_\odot\mathcal{M}(0), \qquad B(R) = \frac{q_u}{r_u^2}\frac{\mathcal{Q}(0)}{x^2(0)}$$

(the limit of a star without magnetic field is obtained for $\kappa = 0$). The singular point of the system of differential equations at $x = 0$ for $\mathcal{P} = \mathcal{P}_c$ should be avoided by the numerical integrator. For this reason, one should map the initial condition to a neighboring regular point $\mathcal{P}_i = (1-\epsilon)\mathcal{P}_c$ with $\epsilon$ small enough, and only then start the integration. In the leading order of approximation, the initial conditions mapped to $\mathcal{P}_i$ are: $x_i = \sqrt{\epsilon}\sqrt{6\mathcal{P}_c/(3\mathcal{P}_c^2 + 4\mathcal{P}_c\mathcal{R}_c + (1-\kappa^2)\mathcal{R}_c^2)}$, $\mu_i = (\mathcal{R}_c/3)x_i^3$, $\mathcal{Q}_i = \kappa\mu_i$, $\mathcal{N}_i = x_i^3\nu(\mathcal{P}_c)/3$, $\mathcal{R}_c \coloneqq \mathcal{R}(\mathcal{P}_c)$. In fact, here was used a higher order series expansion in the parameter $\epsilon$ with its value chosen so as the series cutoff error be negligible compared with the accuracy of the numerical integrator (this was done here separately for a neutron and a quark star, as then higher derivatives of the equation of state $\mathcal{R}(\mathcal{P})$ appear). In contrast to the method of integration with $x$ as the independent variable (for which the stellar radius, and so the integration region, is not *a priori* specified), with the present method the integration region is specified by the interval $(\mathcal{P}_c, 0)$, however a singularity at the stellar surface $\mathcal{P} = 0$ /if there is a boundary/ may appear, which one should take additional care of, for equations of state for which $\mathcal{R}(0) = 0$ (for example, $\mathcal{R}(\mathcal{P}) = \mathcal{P}^n$ with $n > 0$).

### D. Formal resemblance to models with radial electric field

Under exact spherical symmetry, the static model with effective magnetic charges is formally identical with the static model with electric charges. In this symmetry, the gravitational effects associated with the presence of strong electric fields can be expected also to occur in the presence of strong magnetic fields.

Both models have their pros and cons. Intense, large-scale interior electric fields due to the electric charge separation or induced by the presence of large amounts of net electric charges are rather not to be expected—matter must be electrically neutral on the macroscopic scales, much lower than the star radius. In contrast, large-scale magnetic fields are present inside and outside compact stars, however they are induced by stationary electric currents or other magnetization effects (excluding magnetic monopoles) of macroscopically electrically neutral matter. In other words, both of the static spherical models with electric and magnetic fields are excluded in the context of compact stars, because matter must be magnetically and electrically neutral.

However, this argument does not apply if one tries to model the gravitational effect of strong electric and magnetic field as such due to their energy density and pressure, disregarding at the same time the effects responsible for the generation mechanisms of such fields, which cannot be described in static and spherically symmetric space-time (eg. massive and rotating aligned magnetic dipole with a net angular momentum and its associated stationary circular source electric currents in a strong gravitational field regime, must be described in an axisymmetric stationary spacetime, cf. even for a much more simple case of electrically charged magnetic dipole [46]).

The idea of relativistic electrically charged perfect fluid can be traced back to Bekenstein [47]. In his work, the analog of Oppenheimer-Volkoff equation [48] with electrostatic fields appears in the context of the spherical gravitational collapse of electrically charged matter. It is not *a priori* clear from this equation whether it is the Coulombic repulsion or the gravitational attraction due to the corresponding electromagnetic stresses that prevails in the strong field regime. The equilibrium equation was solved assuming a power-law ansatz for the electric charge distribution in the incompressible fluid [49]. It turned out that solutions are possible with a radius lower than the Buchdahl [50] limiting value of 9/8 of the Schwarzschild radius (saturated by the incompressible neutral matter), in this respect see also [51]. Charged (not necessarily perfect fluid) systems can be also studied as such, which may lead to new interesting interior exact solutions, e.g., [52,53].

The idea of charged systems was also applied to study compact stellar objects. However, Bekenstein [47] found unnecessary considering such generalization for the study of neutron star structure. The reason for this is twofold. First, for the electric field to have a gravitational effect, its energy density would have to approach the values of the core pressures of $10^{34}$–$10^{36}$ dyne cm$^{-2}$. But the required electric fields would have to exceed by a factor $10^5$ the critical field of $10^{16}$ volt cm$^{-1}$ for pair creation, and therefore the electric field would destroy itself. Second, due to its high conductivity, the core could not sustain large electric fields which would cause a rapid transfer of the charge to the surface.

In order to see how the presence of various amount of net electric charge would affect the structure of a compact system (without worrying about mechanisms that could generate the required amount of charge) a completely degenerate Fermi gas with a charge density proportional to the matter density was investigated in the context of neutron stars in [43]. It turned out, that the critical mass characteristic of degenerate configurations can be substantially altered (even doubled) in the presence of a net charge, however, exceeding much the values admitted by known at that time charge generation mechanism. As remarked therein, such calculations should, nevertheless, be taken to see to what degree the presence of various amount of net charge affects the stellar structure.

Ultrastrong electric fields of $10^{18}$ volt cm$^{-1}$, or even much beyond this value, are expected in an extremely thin dipolar layer region formed on the surfaces of strange quark matter stars. The corresponding electromagnetic stresses are comparable with the energy density of strange matter itself, in the result of which the predicted masses and radii of strange stars, as described by the MIT bag model equation of state, are appreciably increased [45]. Assuming that the electric charge density follows that of mass density of polytropic matter with some charge-to-mass parameter, it can be shown that any appreciable gravitational effect of electric fields would be seen only with extremely large electric fields of $10^{21}$ volt cm$^{-1}$ and the corresponding integrated charge of $10^{20}$ Coulomb [44,54]. A similar conclusion could be reached by applying a simple dimensional analysis analogous to that presented in Sec. I B: comparing $GQ^2/(c^4r^2)$ and $2GM/(rc^2)$ to unity gives the electric field scale $Q/r^2 \sim 3.53 \times 10^{21}$ volt cm$^{-1}$ and the scale of charge $Q \sim 2M\sqrt{G} \sim 3.43 \times 10^{20}$ Coulomb at a unit solar mass. The amount of charge is huge compared with the electron escape effect of $Q_{\mathrm{esc}} = 100$ Coulomb per solar mass known for bound systems [55], that can be interpreted as a separation charge (roughly, $Q_{\mathrm{esc}} = GM_\odot m_p/q_e \sim 154.2$ Coulomb from the balance of the electric and the gravitational force in the weak field limit), however still reasonable—the charge-to-mass ratio is a tiny fraction $1.8 \times 10^{-18}$ of that for the proton.

## IV. RESULTS

The assumed magnetic field corresponds to an effective magnetic monopole charge with the proper volume density proportional to the proper energy density of matter forming the star. The equation of the gravitational equilibrium involves an additional term due to the presence of a large-scale magnetic field that modifies the mass-radius diagram. Below are considered two simple matter forms constituting the compact stars—strange quark matter described by the equation of state $\rho c^2 = 4B + 3p$ within the MIT bag model (and neglected strange quark mass), and neutron matter described by the UV14 + TNI equation of state [56]. This enables to see the difference in the influence of the magnetic field on the stellar structure for two families of stars. In the case of quark stars, we adopt for comparison three values for the bag constant: 70, 80 and 90 MeV/fm$^3$. The equations of state are compared in figure Fig. 1. The magnetic field changes the position of the maximum mass of a star in the diagram showing the family of compact star models parametrized with the central pressure on the radius–mass plane, as seen in figures Figs. 2 and 3. It can be noticed, that the influence of magnetic field is relatively stronger than the influence due to variations in the model parameters (e.g., of the bag constant in the case of quark star) or in the qualitative properties introduced by the equation of state (MIT bag model vs. UV14 + TNI neutron matter). This influence is substantial for strong magnetic fields of $10^{17}$–$10^{18}$ Gs for which the gravitational effect of the electromagnetic stress-energy is comparable with that of matter.

The maximum mass of a strange star with neglected magnetic field for $B = 90$ MeV/fm$^3$ is $1.6M_\odot$ at the corresponding radius 8.74 km. A decrease of bag constant

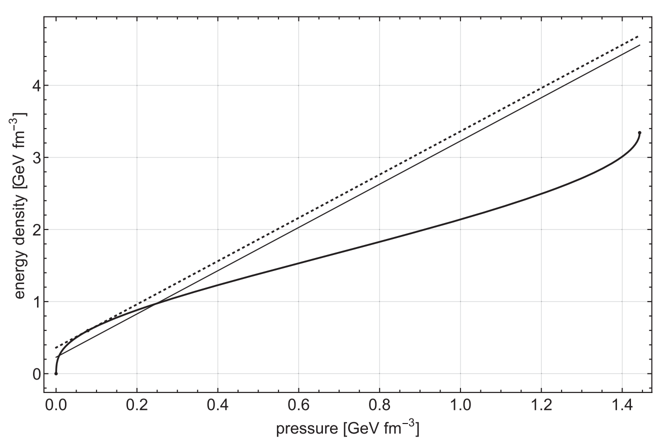


FIG. 1. The equation of state of neutron matter (UV14 + TNI) [*thick curve*] compared with the MIT-bag model equation of state with bag constant $B = 57$ MeV $\cdot$ fm$^{-3}$ and massless strange quark [*straight thin line*]. The tangent *dotted straight line* corresponds to $B \sim 90.3$ MeV $\cdot$ fm$^{-3}$. The pressure at the turning point in the plot of NS matter corresponds to $1.443 \sim$ MeV $\cdot$ fm$^{-3}$.

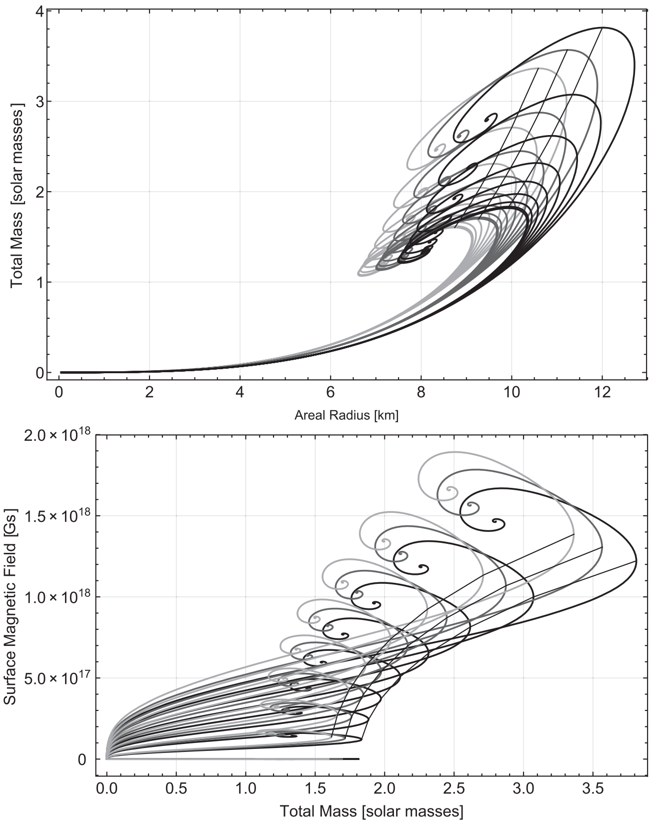


FIG. 2. The characteristics of magnetized strange quark stars in the studied model. Top panel: total mass vs radius diagram; and bottom panel: the mass vs surface magnetic field diagram; for SQM magnetized stars for different values of the $K$ ratio parameter (three bag constants are shown in gray shades: 70, 80, and 90 $\mathrm{MeV \cdot fm^{-3}}$, respectively, from dark gray to light gray). The chord lines in both figures connect maximum mass positions on the diagrams, with the lowest mass value corresponding to no magnetic field and increasing with the $K$ ratio parameter (which changes starting from 0 out to $0.9\sqrt{G}$ in steps of $0.1\sqrt{G}$).

to $B = 70\ \mathrm{MeV/fm^3}$ increases the maximum mass to $1.82 M_\odot$ at the increased radius 9.91 km. With the surface magnetic field of $5 \times 10^{17}$ Gs, the maximum mass for $B = 70\ \mathrm{MeV/fm^3}$ is $2.11 M_\odot$ at the radius 10.3 km. With the magnetic field increased to $1.22 \times 10^{18}$ Gs the maximum mass strongly increases further to $3.81 M_\odot$ at the radius 12 km. For the neutron star with neglected magnetic field, the maximum mass is $1.83 M_\odot$ at the radius 9.31 km, which are values similar to those for the quark star. With the magnetic field of $5.5 \times 10^{17}$ Gs the maximum mass increases to $2.05 M_\odot$ at the radius 9.59 km, and with still stronger surface magnetic field of $1.14 \times 10^{18}$ Gs the maximum mass increases to $3.06 M_\odot$ at the radius 10.81 km. This shows that in the presence of strong stellar magnetic fields the masses of compact stars can reach values exceeding those normally expected for compact stars.

The observed effect of magnetic fields on the mass of a quark star, as depicted in Fig. 4, is comparable to that

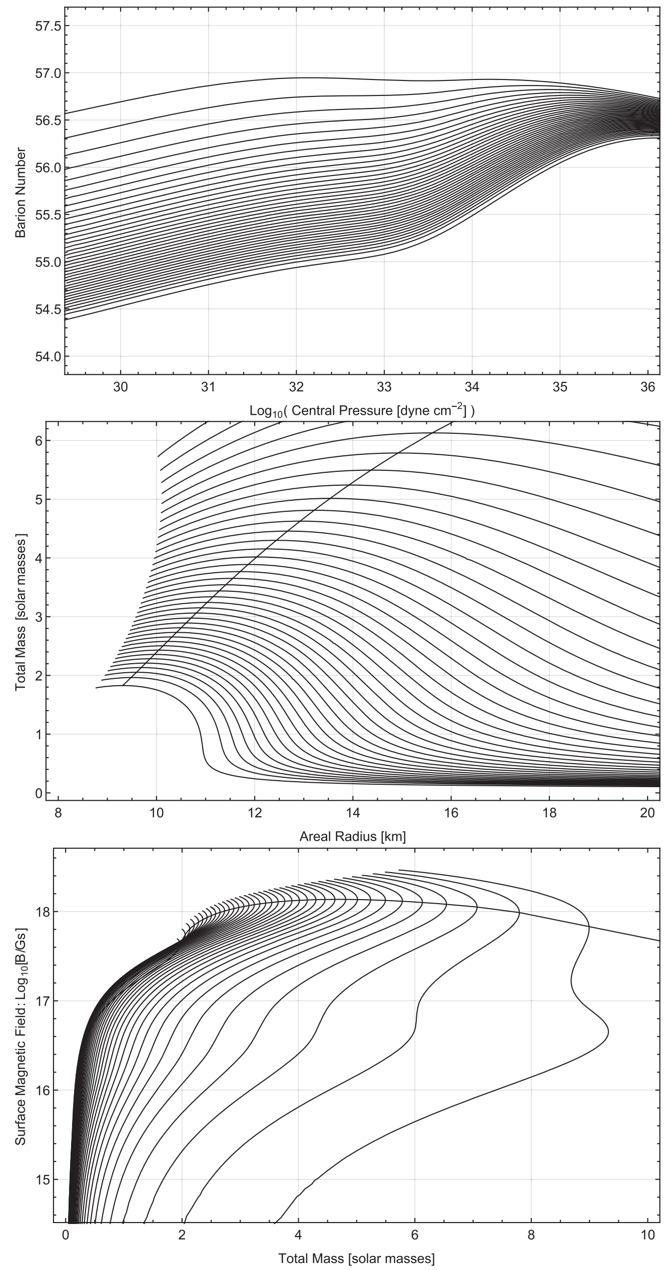


FIG. 3. The characteristics of magnetized neutron stars in the studied model. Top panel: total baryon number vs central pressure diagram, Middle panel: total mass vs radius diagram and bottom panel: the surface magnetic field vs total mass diagram. The curves in the figures represent the same sequence of magnetized neutron stars shown for several values of the $K$ ratio parameter (here, $K = s^{1/3}\sqrt{G}$ with $s$ starting from 0 in steps of 0.025 out to $s_{\max} = 0.95$ corresponding to $K_{\max} \sim 0.983\sqrt{G}$). The chord lines in the middle and bottom diagrams connect the maximum mass positions on the curves, with the lowest mass value corresponding to $K = 0$ (no magnetic field) and increasing with $K$.

obtained in [12]. For the core magnetic field of $\sim 4 \times 10^{17}$ Gs, the maximum mass obtained in that work is $\sim 2.2 M_\odot$ at a bag constant 60 $\mathrm{MeV \cdot fm^{-3}}$, while in our model of magnetized quark star with a similar bag constant

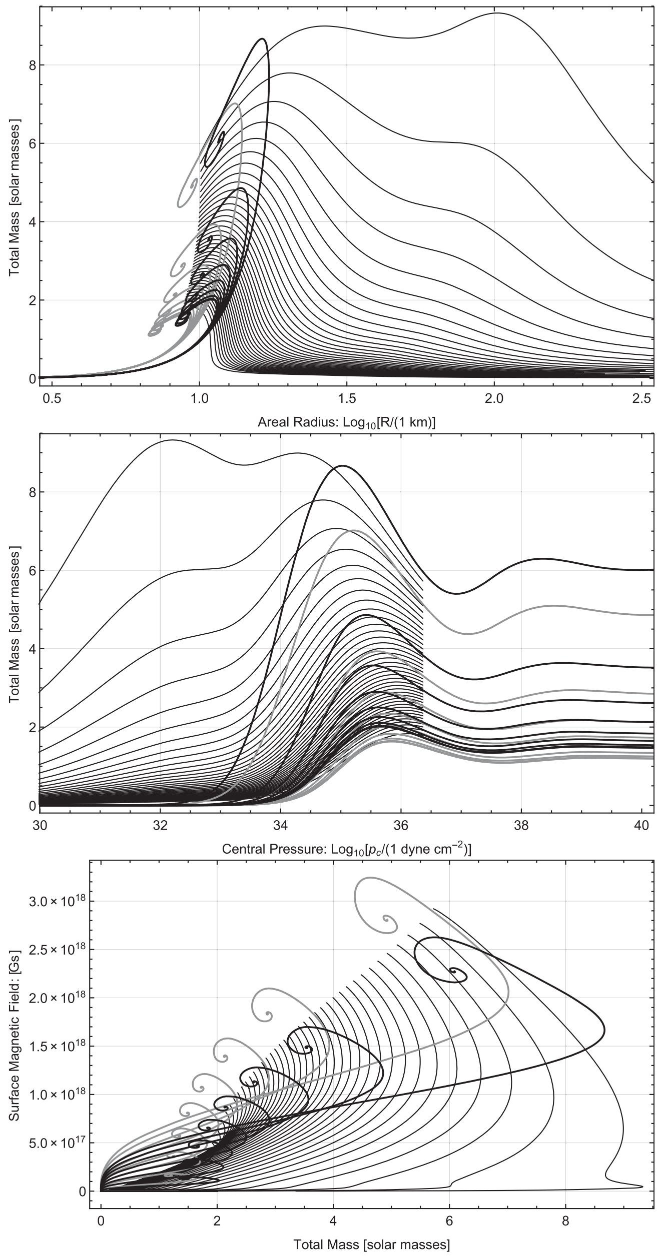


FIG. 4. Comparison of results for quark stars and neutron stars. Characteristics of strange quark stars and neutron stars for various $K$-ratio parameters. Top panel: mass–radius diagrams; Middle panel: central pressure–mass diagrams; Bottom panel: mass–surface magnetic fields diagrams. In all panels, neutron stars are represented with thin lines corresponding to the lines shown in Fig. 3 (with $K_{\max} \sim 0.983\sqrt{G}$). Quark stars are represented in all panels with thick black lines for a bag constant 57 $\mathrm{MeV} \cdot \mathrm{fm}^{-3}$ and with thick gray lines for a bag constant 87 $\mathrm{MeV} \cdot \mathrm{fm}^{-3}$ (in the case of quark stars, the lines correspond to the $K$-ratio parameters that changes from $1\sqrt{G}$ down to 0 in steps of $0.025\sqrt{G}$).

57 $\mathrm{MeV} \cdot \mathrm{fm}^{-3}$ the maximum mass $\sim 2.2 M_\odot$ is obtained for the surface magnetic field of about $4 \times 10^{17}$ Gs. The maximum masses obtained here in the range 4–$8 M_\odot$ corresponding to surface magnetic fields of the order of $10^{18}$ Gs are closer to the masses expected for black holes than for compact objects. Comparably high maximum masses were obtained in [44] for compact stars described by the polytropic equation of state in the presence of a strong electric field. From the diagrams of total mass against central pressure presented there, it follows that the maximum mass at the charge-to-mass ratio of $\sim 0.87G$ is about $4M_\odot$, while the maximum mass of a magnetized neutron star obtained in the present model at the same charge–to–mass ratio is about $4.5M_\odot$.

The gravitational effect of extremely strong magnetic fields of $10^{17}$–$10^{18}$ Gs is more pronounced for quark stars as compared with neutron stars. An important feature that distinguishes strange stars from neutron stars is that the effects associated with the magnetic field disappear rapidly as the mass and radius of the strange star decrease. The situation is different for neutron stars—here the gravitational effect of magnetic field is particularly visible for stars with masses lower than the maximum mass. For example, as seen in the top panel in figure Fig. 4, a neutron star of mass $0.5M_\odot$ (at a radius of 11 km) in the absence of magnetic field, would attain a mass of about $1M_\odot$ (at a similar radius of 11 km) in the presence of magnetic field of $8 \times 10^{16}$ Gs, and if the field were of the order of $10^{18}$ Gs the mass would exceed $2M_\odot$.

It is interesting to see that magnetic fields, even as strong as $10^{18}$ Gs, neither change qualitatively the mass-radius profile nor the central pressure-mass profile for quark stars. The only effect of the enhanced magnetic field is the increase in mass and radius for the most massive quark stars, as can be noticed in Fig. 2. This lack of qualitative change will become clear later in Sec. V—it can be shown that the characteristic of quark stars scaling of masses and radii with the bag constant, still holds in the presence of magnetic field in the considered model.

In Fig. 3 analogous results are shown for neutron stars. Here, the situation is different—an increase in the magnetic field leads to a qualitative change in the shapes of profiles. At the highest magnetic field values, a second maximum appears on the central pressure-mass diagram with a branch of probably instable stellar configurations.

An interesting difference between quark and neutron stars can be seen in Fig. 5, where the magnetic field profile in a star is plotted as a function of the radial variable at a fixed value of the $K$-ratio parameter. For all of quark stars enumerated with the central pressure, below the critical value at the given $K$ (in the stability region), the magnetic field strength profile increases with the radial variable attaining the maximal value at the surface of the star. Using the analogy to the uniformly charged sphere, one can say that the degree of accumulation of the charge prevails the degree of attenuation of the resulting field as the distance from the center increases. This situation only changes for central pressures above the critical value at a given $K$, in which case one can expect that stars are instable against

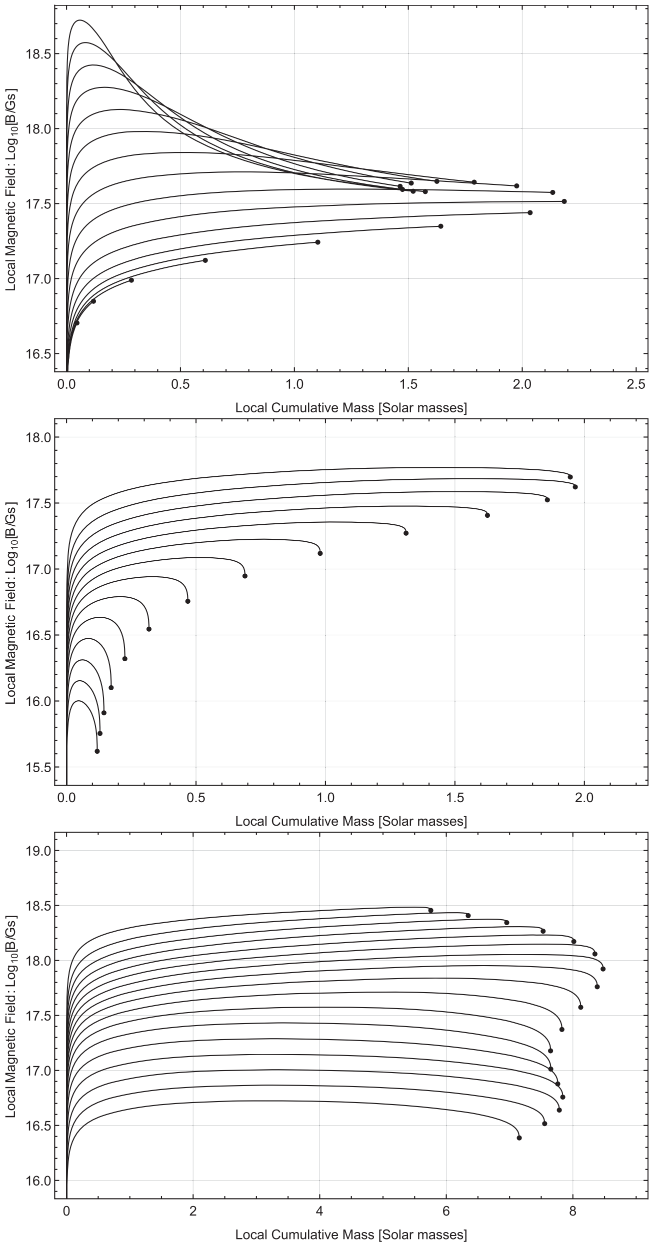


FIG. 5. Magnetic field profile lines in stellar interiors shown as a function of local cumulative mass variable for various central pressures of $10^s$ dyne $\cdot$ cm$^{-2}$ in steps of $\Delta s = 0.3$ (the higher $s$ the higher magnetic field/more elevated line). Top panel: a strange quark star with bag constant $B = 57$ MeV $\cdot$ fm$^{-3}$, $K = 0.3\sqrt{G}$, starting with $s = 38.3$ down to 33.5. Middle panel: a neutron star for $K = 0.3\sqrt{G}$, starting with $s = 36.3$ down to 32.4. Bottom panel: a neutron star for $K = 0.98\sqrt{G}$, starting with $s = 36.3$ down to 31.5.

radial perturbations. Then the magnetic field strength profile initially grows with the radial variable, attains its maximum value somewhere inside the star, and then the field strength decreases as one approaches the star surface.

A different behavior can be observed for neutron stars as seen in the middle and bottom panels in Fig. 5. The region

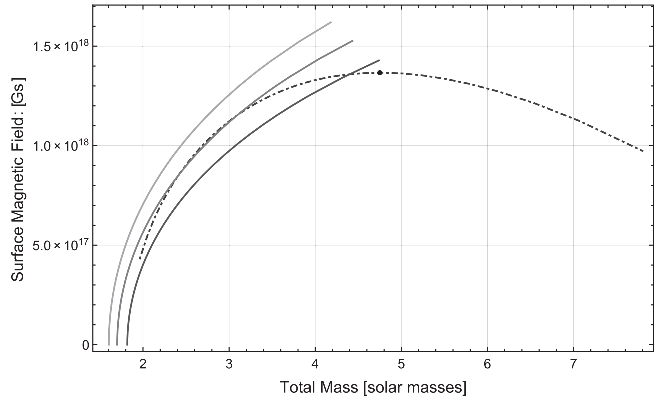


FIG. 6. Surface magnetic field vs total mass diagram for maximum-mass compact stars. Each line is parametrized with the $K$ ratio parameter growing along every line in the direction from the left to the right. The maximum-mass neutron star is shown with the *dot-dashed line* (this is the crossing line in the bottom panel of Fig. 3). On this line the maximum surface field of $1.367 \times 10^{18}$ Gs is attained at the total mass $4.75 M_\odot$. The other three *solid lines* are the maximum-mass strange quark stars corresponding to three bag constants 70, 80 and 90 MeV $\cdot$ fm$^{-3}$, shown in gray shades—from darker to lighter, respectively (these are the three crossing lines of which parts are shown in the bottom panel of Fig. 2).

with the magnetic field strength decreasing with the radial variable (and thus with the mass function value), after reaching its maximum somewhere inside the star, is already observed for central pressures below the critical value for a given $K$. This behavior can be also observed for neutron stars with extremely high total masses (i.e., with the largest magnetic fields) as seen in the bottom panel of Fig. 5. However, in this case this behavior occurs also for central pressures beyond a critical value for which one can expect that stars are instable against radial perturbations.

Figure 6 shows the lines of surface magnetic field of maximum mass stars parametrized with the $K$-ratio. For quark stars the maximum mass grows with the surface magnetic field. For neutron stars the surface magnetic field of maximum mass stars cannot exceed the limiting value of $\sim 1.37 \times 10^{18}$ Gs. This result is interesting as it sets the upper limit for magnetic fields for the maximum mass neutron stars. This limit agrees as to the order of magnitude with the expectation based on a simple virial argument [6].

The interesting feature of magnetized neutron stars is the onset of new maxima on the pressure–mass diagram as seen in the middle panel in figure Fig. 4 for stronger magnetic fields. The classical criterion of stability of stars requires that the mass of a stable star should grow with the central pressure—the derivative of total mass with respect to the central pressure should be positive. For small $K$ there is only one maximum at some limiting central pressure above which the star is instable, like for ordinary star without magnetic field. When $K$ increases, a new maximum emerges below the limiting pressure and with it a region of central pressures appears where the derivative is negative. This suggests that sufficiently strong magnetic fields may render the star instable.

## V. SCALING OF QUARK STAR STRUCTURE

Witten [57] found that the properties of a stable quark star of maximum mass, as described within the simple quark star model with neglected quark masses, scale with the bag constant $B$. Namely, the maximum mass scales with $B$ as $M(B) = M(B_0)\sqrt{B_0/B}$, at the same time the radius of the star scales as $R(B) = R(B_0)\sqrt{B_0/B}$ and the central density as $\rho_c(B) = \rho_c(B_0)(B/B_0)$, where $M(B_0)$, $R(B_0)$, $\rho_c(B_0)$ are the values obtained from numerical integration of the equations of stellar structure at some arbitrary reference bag constant $B_0$. The scaling relations hold to a very good approximation also for maximum mass stars in other models of strange quark stars [58].

It turns out, that similar scaling relations can be observed for the present model with magnetic field. Remarkably, by applying this scaling to the mass-radius diagrams shown in the top panel of figure Fig. 2, it turns out that all curves corresponding in this diagram to the same value of the $K$-ratio parameter can be made to overlap (exactly to within numerical inaccuracies) with any other curve with the same $K$, in particular, the one corresponding to some reference value $B_0$ of the bag constant. Moreover, a similar scaling can be observed for the diagram of surface magnetic field versus radius, shown in the bottom panel of figure Fig. 2 or for other such diagrams. This observation suggests that there are $K$-dependent scaling relations which connect all stars with the same $K$ and various $B$, thus not only the particular maximum mass stars.

To verify this expectation, one can investigate scaling properties of the equations for the gravitational equilibrium of magnetized stars as given in section Sec. III C. To this end one has to assume $\mathcal{R} = 3\mathcal{P} + 4\beta$ and $\nu = (\beta + \mathcal{P})^{3/4}$ in these equations, respectively, for the energy density and particle number density, as required by the form of the quark matter considered. Next, one can assume a general scaling transformation law for all quantities and functions in these equations: starting with $\mathcal{P} = \tilde{\mathcal{P}}\lambda$ for the independent variable, $\mathcal{N}(\mathcal{P}) = \lambda^n \tilde{\mathcal{N}}(\lambda^{-1}\mathcal{P})$, $x(\mathcal{P}) = \lambda^r \tilde{x}(\lambda^{-1}\mathcal{P})$, $\mathcal{Q}(\mathcal{P}) = \lambda^q \tilde{\mathcal{Q}}(\lambda^{-1}\mathcal{P})$, $\mathcal{M}(\mathcal{P}) = \lambda^m \tilde{\mathcal{M}}(\lambda^{-1}\mathcal{P})$ for the dependent variables (supplemented with the resulting scaling relations for the derivatives, e.g., $x'(\mathcal{P}) = \lambda^{r-1}\tilde{x}'(\tilde{\mathcal{P}})$, etc) with exponents $n$, $r$, $q$, $m$ being the unknown homogeneity degrees, and finally $\beta = \lambda^b \tilde{\beta}$ for the bag constant (leaving $\kappa$ unaffected by this transformation). On substituting these relations into the equilibrium equations and by requiring that one should end up with the same form of equations to be satisfied by the new quantities with the tilde sign, one is left with several conditions to be satisfied by the exponents. This way one is led to the following result: $b = 1$, $n = -3/4$, $r = -1/2$, $q = -1/2$, $m = -1/2$. Being the symmetry of the equations of structure, this scaling is deeply rooted in the model and is not only a mere symmetry of some global characteristics of the considered star. Coming back to the CGS units, denoting the tilded quantities as the reference values with subscript 0 and noticing that $\lambda = \beta/\tilde{\beta} = B/B_0$, this implies, in particular, the scaling relations:

$$P_c = P_{c0}\frac{B}{B_0}, \qquad N = N_0\left(\frac{B_0}{B}\right)^{+3/4}, \qquad R = R_0\left(\frac{B_0}{B}\right)^{1/2},$$

$$Q = Q_0\left(\frac{B_0}{B}\right)^{1/2}, \qquad M = M_0\left(\frac{B_0}{B}\right)^{1/2},$$

respectively, for the central pressure $P_c$ and the corresponding total baryon number $N$, radius $R$, total effective charge $Q$, and total mass $M$ (note that except for $P_c$ all the quantities concern the star surface defined by the scaling-invariant condition $P = 0$). From this and the defining relation $H(r) = Q(r)/r^2$ it follows that surface magnetic field $H$ scales as

$$H = H_0\left(\frac{B}{B_0}\right)^{1/2},$$

which means that the magnetic pressure or energy density scales the same way as the matter pressure, in agreement with expectations.

With the above scaling symmetries at hand, it would suffice to find characteristics of the star (for example, the total baryon number) only for some reference bag constant and various values of the parameter $K$. The characteristics for all other $B$ at the same $K$'s could be then easily found, without the necessity of performing additional numerical integration (the curves presented in the diagrams were prepared for different $B$ by independent integration, and this scaling indeed applies).

In Figs. 7 and 8 the scaling property is illustrated on the example of total baryon number and total mass of quark stars considered as functions of the central pressure. For the purpose of comparison of the results for various bag constants, it is convenient to express the baryon number in solar masses assuming some arbitrary mass (e.g., the proton mass) as the reference mass per baryon (this is not the genuine baryon mass that would be obtained by integrating the proper energy density over the proper volume element). For a given parameter $K$ the baryon number curves are different for different bag constants, as seen in the top panel of figure Fig. 7. On rescaling the values of central pressures $P_c$ and the values of baryon numbers $N_B$ according to the transformation rules $P_c \to (B_0/B)P_c$ and $N_B \to (B/B_0)^{3/4}N_B$ discussed above, the curves get overlapped with the one corresponding to the reference bag constant $B_0 = 60\ \mathrm{MeV}\cdot\mathrm{fm}^{-3}$ at the same $K$, as seen in the bottom panel of this figure.

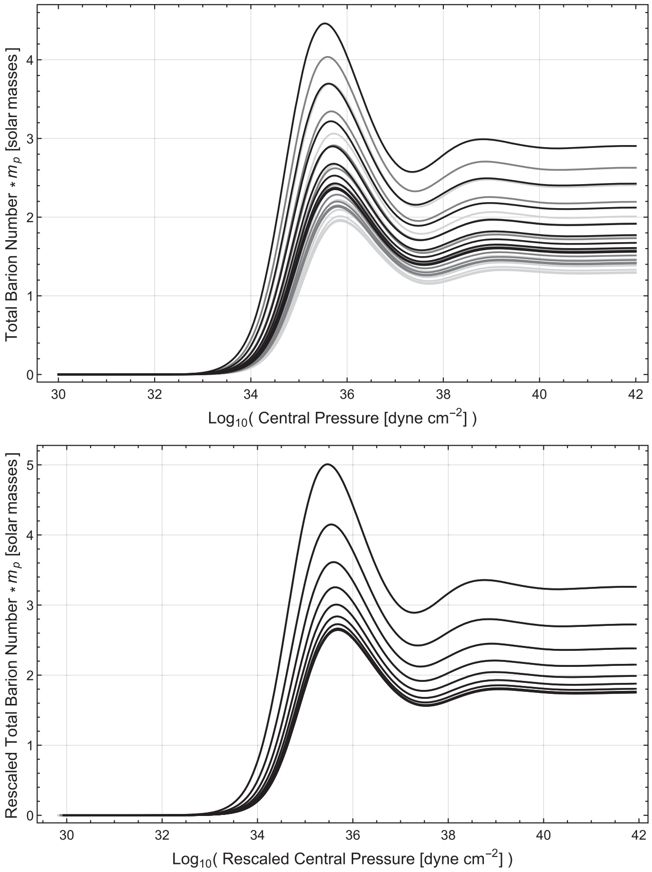


FIG. 7. Top panel: total baryon number for SQM magnetized stars for different values of the $K$ ratio parameter (which changes starting from 0 out to $0.9\sqrt{G}$ in steps of $0.1\sqrt{G}$) presented in gray shades for three bag constants: 70, 80, and 90 $\mathrm{MeV}\cdot\mathrm{fm}^{-3}$ (respectively, from dark gray to light gray). For each bag constant (gray shade), the curve with the lowest maximum baryon number corresponds to $K=0$ (no magnetic field) and the maximum increases with $K$. Note, that the baryon number has been expressed in conventional solar masses by adopting proton mass as a reference mass for a single baryon (this is not the genuine baryon mass). Bottom panel: the total baryon number $N_B$ rescaled to the reference bag constant $B_0 = 60\ \mathrm{MeV}\cdot\mathrm{fm}^{-3}$ according to the rescaling $P \to (B_0/B)P$, $N_B \to (B/B_0)^{3/4}N_B$—the curves with different bag constants (different shades in the top panel) got exactly overlapped with each other.

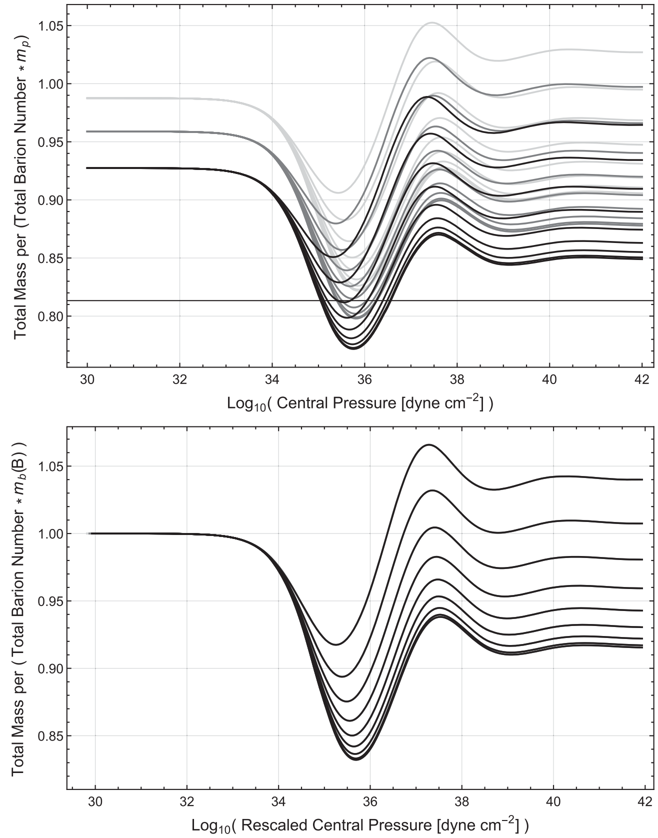


FIG. 8. Top panel: the total baryon number compared with the total mass (presented as a ratio $M/N_B$) for SQM magnetized stars shown for different values of the $K$ ratio parameter (which changes starting from 0 out to $0.9\sqrt{G}$ in steps of $0.1\sqrt{G}$) and presented in gray shades for three bag constants: 70, 80, and 90 $\mathrm{MeV}\cdot\mathrm{fm}^{-3}$ (respectively, from dark gray to light gray). For each bag constant (gray shade), the curve with the shallowest global minimum corresponds to $K=0$ (no magnetic field) and the minimum decreases with increasing $K$. Note, that in this ratio the baryon number has been conventionally multiplied by the proton mass as a reference mass so that the ratio becomes dimensionless and comparable with unity. Bottom panel: a diagram showing the ratio curves of the top panel rescaled according to $P \to (B_0/B)P$ (with the reference bag constant $B_0 = 60$) and $M/(N_B m_p) \to M/(N_B \cdot m_b(B))$ (with $m_b(B)$ defined in the text), so that the ratio becomes rescaling-invariant and equal to 1 in the limit of small central pressures for all $B$ and $K$—the curves with different bag constants (different shades in the top panel) got exactly overlapped with each other.

The behavior of the total mass $M$ can be compared with the behavior of baryon number $N_B$ by considering their ratio as a function of the central pressure. The result is shown in the top panel of figure Fig. 8 (with proton mass as the reference mass). However, the mass in the nonrelativistic limit should be equal to the baryon mass for almost uniform density stars, in which case one should take the energy per baryon $m_b(B) = (108\pi^2 B\hbar^3 c^{-5})^{1/4}$ as the reference baryon mass, which, in addition, has the same scaling property as the ratio $M/N_B$. In effect, the dimensionless function $M/(N_B m_B)$ is scaling-invariant and it is equal to 1 in the limit of vanishing mass (thus also independent of $K$ in this limit and, consequently, independent of the magnetic field in this limit). The rescaled diagrams are shown in the bottom panel of Fig. 8 assuming the reference bag constant $B_0 = 60\ \mathrm{MeV}\cdot\mathrm{fm}^{-3}$ as before. Now one can observe the real change in $M/(N_B m_B)$ due to the increase in value of parameter $K$ only, independent of $B$, and indirectly, the influence of magnetic field. Judging on this diagram, one can surmise that the dependence of the solutions on the magnetization parameter $K$ is more complicated that a mere rescaling.

## VI. SUMMARIZING REMARKS AND CONCLUSIONS

For the quark star the realistic large-scale magnetic fields cannot be spherically symmetric. In this work some systematizing remarks concerning the spherical symmetry approximation in the description of compact stars with magnetic field were given.

Magnetic stress-energy tensor, including also the simplest magnetization model, can be formed uniquely by algebraical requirements out of a magnetic vector defined by the static observer (Sec. II A). Then a local averaging respecting the global spherical symmetry leads to a general form of the total stress-energy tensor expressed by the magnetic field energy density and two radius-dependent parameters $\eta$ and $\alpha$, one characterizing the induced magnetization and the other the proportions of the principal magnetic stresses (Sec. II C) aligned with local radial and transversal directions.

This averaging erases to a large extent the information about the directional character of the original large-scale and sourceless magnetic field. So it should come as no surprise that this approximation is possible at the cost of introducing effective magnetic charges (Sec. II D). The density of these charges is right that to reproduce the required magnetic energy density (this was illustrated with a simple example). However, the effective charges should not be regarded as real, because they are the result of the adopted approximation.

The limiting model for $\alpha = 1$, which is exactly spherically symmetric (the symmetry concerns also the magnetic field), is formally identical with a model of magnetically charged spherical star. It may seem odd, at first glance, the idea of considering such a model. However, as a limit, the model is worth considering as another handy approximation within a class of considered models. Similar to any model with $0 < \alpha < 1$ (including models considered in the literature, e.g. $\alpha = 1/3$ [27]) it allows to introduce the gravitational effect of strong magnetic fields with a prescribed magnetic energy density into the description of a nearly spherical compact star and study the resulting changes. It is true that the effective magnetic charges appear in this approach, however, the magnetic field in the limiting case $\alpha = 1$ is not claimed here to be due to genuine magnetic monopoles, even though one can admit that considering the existence of magnetic monopoles is a legitimate question that is taken seriously in theoretical physics.

In this work also the problem of mass definition in the presence of magnetic fields was addressed as one of importance in the context of magnetized compact stars (Sec. III B). Two physically clear definitions of mass were compared, one based on the Gauss concept—the integrated flux of the gravitational field through a sphere at infinity—and the other which can be inferred based on the motion of electromagnetically neutral distant test bodies. In the general-relativistic context, these are the Komar-Tolman mass and the mass defined by a matching condition with the exterior metric. Both concepts of mass lead to different results in the presence of electromagnetic field, which can be seen on simple examples (in particular, for a uniform ball of massless electromagnetic charges the Komar mass would be twice the second kind of mass).

The gravitational effect of ultra–strong magnetic fields of $10^{17}$–$10^{18}$ Gs was studied in the framework (Sec. III C) of exact spherical symmetry approximation ($\alpha \to 1$) for two simple models of compact stars. Namely, the strange quark star (described by the simplest equation of state in the framework of MIT bag model) and a neutron star (described by the UV14 + TNI equation of state). The obtained results (Sec. IV) do not wander astray from what is known for other models of nearly spherical compact stars with strong electric or magnetic fields.

The gravitational effect of magnetic fields in the range $10^{12}$–$10^{15}$ Gs is negligible for compact stars. For gravitationally moderate fields of $10^{16}$–$10^{17}$ Gs the masses and radii of compact stars are changed at the level of a few percent. For this effect to be noticeable the field should be as large as $10^{17}$–$10^{18}$ Gs. Magnetic fields that strong alter the mass-radius relation to a degree comparable or greater than that introduced by modifying the equations of state (or their parameters, such as the bag constant) describing the material of compact stars. For gravitationally moderate fields it may become difficult to discriminate between compact stars with different equations of state, even if there are good observational data for the masses and sizes of these objects.

Magnetic fields even sa strong as $10^{18}$ Gs do not qualitatively affect the mass-radius and central pressure-mass diagrams for strange quark stars (for stars with the highest central pressures a substantial increase of the mass and radius is observed with growing field strength). For neutron stars, a qualitative change is observed for the highest magnetic fields—a second maximum appears on the central pressure-mass diagram with a region of probably instable stellar configurations. Remarkably, one can observe (Fig. 6) that the surface magnetic field for maximum mass neutron stars cannot exceed the limiting upper value of $1.37 \times 10^{18}$ Gs. This limit agrees as to the order of magnitude with the expectation based on a simple virial argument. One should expect that sufficiently strong magnetic fields may render the stars instable. A question that arises is to what extent the prediction of spherically symmetric models is changed for more realistic large-scale ordered magnetic fields that violate spherical symmetry.

In view of these results, it is found important to stress that extremely strong magnetic fields elevate masses of compact stars to values exceeding those normally expected for such objects (Fig. 4) and, therefore, ascribed rather to black holes or models with exotic equations of state. This has observational implications—finding a compact object with a mass of a few solar masses, with simultaneous strong evidence for the presence of the boundary surface, might point to extremely strong magnetic field present in such an object as another possibility worth of considering among other hypotheses such as exotic constituents of the star. However, it should be emphasized that this problem requires further research, as the most extreme magnetic fields affect the equation of state [59–62], softening the neutron star matter, with the result that maximum masses in the presence of ultra-strong magnetic fields could turn out to be much lower. In addition, the matters get complicated by the fact that the state functions of neutron and quark stars may respond differently to the magnetic field.

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