Anomaly cancellation in effective field theories from the covariant derivative expansion

Timothy Cohen,[1,2,3,*] Xiaochuan Lu,[4,3,†] and Zhengkang Zhang[5,‡]

[1]Theoretical Physics Department, CERN, 1211 Geneva, Switzerland
[2]Theoretical Particle Physics Laboratory, EPFL, 1015 Lausanne, Switzerland
[3]Institute for Fundamental Science, University of Oregon, Eugene, Oregon 97403, USA
[4]Department of Physics, University of California, San Diego, La Jolla, California 92093, USA
[5]Department of Physics, University of California, Santa Barbara, California 91106, USA

(Received 2 June 2023; accepted 8 September 2023; published 29 September 2023)

We extend our recently proposed formalism for calculating anomalies of global and gauge symmetries using the covariant derivative expansion to include a general class of operators that can appear in relativistic effective field theories (EFTs). This allows us to prove that EFT operators involving general scalar, vector, and tensor couplings to fermion bilinears only give rise to irrelevant anomalies, which can be removed by an appropriate choice of counterterms, thereby confirming the absence of new constraints from anomaly cancellation on the Standard Model EFT.

DOI: 10.1103/PhysRevD.108.056027

I. INTRODUCTION

Anomalies provide critical consistency conditions on gauge theories such as the Standard Model; see, e.g., Refs. [1,2] for reviews. Anomaly cancellation in the Standard Model itself is, of course, well understood. However, anomaly cancellation for effective field theories (EFTs) with higher-dimensional operators is a more subtle issue, which has received renewed interest recently in the context of the Standard Model effective field theory (SMEFT) [3,4] (see also Refs. [5–10] for earlier studies). As shown in these papers, demonstrating anomaly cancellation for SMEFT involves carefully accounting for the interplay of various interactions encoded in the higher-dimensional operators.

In this paper, we generalize the method developed in Ref. [11] for computing anomalies with the covariant derivative expansion (CDE) [12–16] to the case of EFTs. This will allow us to confirm that the anomaly cancellation condition is unchanged by the presence of a general class of higher-dimensional operators. More precisely, contributions to anomalies from higher-dimensional operators are in the form of the gauge variation of local operators. These are known as irrelevant anomalies and can be removed by the renormalization procedure with appropriate counterterms (see, e.g., Ref. [17] for a recent systematic study of such counterterms focused on renormalizable theories). In contrast, relevant anomalies are IR effects and are not affected by higher-dimensional operators. We will demonstrate this explicitly with a CDE calculation.

To see that anomalies *a priori* may depend on the detailed form of the interactions in the theory, let us briefly review its definition. We extract anomalies from the gauge variation of the bosonic effective action $W[G^\mu]$, defined as

$$e^{iW[G^\mu]} \equiv \int \mathcal{D}\chi\mathcal{D}\chi^\dagger e^{iS[\chi,\chi^\dagger,G^\mu]}. \tag{1.1}$$

Even when the classical action is gauge invariant, $S[\chi_\alpha,\chi_\alpha^\dagger,G_\alpha^\mu] = S[\chi,\chi^\dagger,G^\mu]$ (where subscript α denotes the gauge-transformed quantity), the bosonic effective action after integrating out the fermions may not be:

$$W[G_\alpha^\mu] \stackrel{?}{=} W[G^\mu]. \tag{1.2}$$

This possible discrepancy is due to the path integral measure $\mathcal{D}\chi\mathcal{D}\chi^\dagger$:

$$e^{iW[G_\alpha^\mu]} = \int \mathcal{D}\chi\mathcal{D}\chi^\dagger e^{iS[\chi,\chi^\dagger,G_\alpha^\mu]} = \int \mathcal{D}\chi_\alpha\mathcal{D}\chi_\alpha^\dagger e^{iS[\chi_\alpha,\chi_\alpha^\dagger,G_\alpha^\mu]} = \int \mathcal{J}_\alpha^{-1}\mathcal{D}\chi\mathcal{D}\chi^\dagger e^{iS[\chi,\chi^\dagger,G^\mu]} = e^{iW[G^\mu]}\langle\mathcal{J}_\alpha^{-1}\rangle_G, \tag{1.3}$$

or equivalently

$$W[G_\alpha^\mu] - W[G^\mu] = -i\log\langle\mathcal{J}_\alpha^{-1}\rangle_G = \mathcal{A}[\alpha] + \mathcal{O}(\alpha^2). \tag{1.4}$$

*tim.cohen@cern.ch
†xil224@ucsd.edu
‡zkzhang@ucsb.edu

We see that the anomaly functional $\mathcal{A}[\alpha]$ (which we will often just refer to as the anomaly), as defined by the first-order gauge variation of the bosonic effective action, is related to the expectation value of the Jacobian factor $\langle \mathcal{J}_\alpha^{-1} \rangle_G$. As emphasized by the subscript G, this expectation value may *a priori* depend on the details of the theory, namely what interactions it contains (just as expectation values of generic operators would). So one needs a general formalism to calculate the anomalies for theories with generic interactions.

In Ref. [11], we focused on the case of chiral fermions minimally coupled to gauge fields and introduced a regularization prescription—a generalized version of the classic Fujikawa's method [18–21]—to efficiently evaluate the anomaly in $d = 4$ spacetime dimensions using CDE. This approach leads to unambiguous evaluation results, in the form of a master formula for the anomaly functional $\mathcal{A}[\alpha]$ that integrates various known results about anomalies. In this paper, we extend this formalism to include a more general set of interactions in Lorentz-invariant EFTs such as SMEFT.

The rest of this paper is organized as follows. In Sec. II we present our parametrization of a general class of EFT operators, involving scalar, vector, and tensor couplings to fermion bilinears. In Sec. III we generalize the formalism in Ref. [11] and explain how to calculate the anomaly in such EFTs with CDE. We complete the detailed evaluation of the anomaly in Sec. IV and show that extra contributions from the interactions beyond minimal coupling are all irrelevant anomalies. Finally, in Sec. V we conclude and discuss some future directions.

II. PARAMETRIZATION OF A GENERAL EFT

We are interested in anomalies of both gauge and global symmetries in a general Lorentz-invariant EFT. As in Ref. [11], we introduce *auxiliary* gauge fields for all the global symmetries of interest. Putting them together with the *physical* gauge fields, we denote the whole collection by G_μ, which can be a sum over multiple (Abelian and/or non-Abelian) group sectors:

$$G_\mu \equiv \sum_a G_\mu^a t^a. \tag{2.1}$$

The (Hermitian) covariant derivative is

$$P_\mu \equiv iD_\mu = i\partial_\mu + G_\mu, \tag{2.2}$$

and the gauge field strength is given by

$$\begin{aligned} F_{\mu\nu} &= \sum_a F_{\mu\nu}^a t^a = -i[P_\mu, P_\nu] \\ &= (\partial_\mu G_\nu) - (\partial_\nu G_\mu) - i[G_\mu, G_\nu]. \end{aligned} \tag{2.3}$$

We consider a general theory of n left-handed Weyl fermions $\chi_1, \ldots, \chi_n$, with each χ_i transforming in an irreducible representation of the (global and gauge) symmetries. The theory may also contain an arbitrary number of scalar fields, collectively denoted as ϕ. The EFT Lagrangian we consider has the following general form:

$$\begin{aligned} \mathcal{L} = \mathcal{L}_{G,\phi} &+ \sum_{i=1}^n \chi_i^\dagger \bar{\sigma}^\mu P_\mu \chi_i \\ &+ \sum_{i,j=1}^n \left\{ \chi_i^\dagger \bar{\sigma}_\mu V_{ij}^\mu \chi_j + \left[\chi_i (S_{ij} + i\sigma_\mu \bar{\sigma}_\nu T_{ij}^{\mu\nu}) \chi_j + \text{H.c.} \right] \right\}. \end{aligned} \tag{2.4}$$

Here $\mathcal{L}_{G,\phi}$ collects the interactions that do not involve fermions. The rest of the first line encodes the minimal couplings between the fermions χ_i and gauge fields G_μ. In the second line, we parametrize an extended set of interactions with fermion bilinears, categorizing them into scalar, vector, and tensor interactions,

$$\chi_i S_{ij} \chi_j, \qquad \chi_i^\dagger \bar{\sigma}_\mu V_{ij}^\mu \chi_j, \qquad \chi_i i\sigma_\mu \bar{\sigma}_\nu T_{ij}^{\mu\nu} \chi_j, \tag{2.5}$$

where $S_{ij}[G,\phi]$, $V_{ij}^\mu[G,\phi]$, and $T_{ij}^{\mu\nu}[G,\phi]$ are functions made of G_μ, ϕ, and their derivatives and can have arbitrarily high operator dimensions. Note that due to the Clifford algebra $\sigma_\mu \bar{\sigma}_\nu + \sigma_\nu \bar{\sigma}_\mu = 2\eta_{\mu\nu} \mathbb{1}$, the $\mu \leftrightarrow \nu$ symmetric components of $T_{ij}^{\mu\nu}$ can be absorbed into the scalar interactions S_{ij}, so we define the tensor interactions to be antisymmetric, $T_{ij}^{\nu\mu} = -T_{ij}^{\mu\nu}$.

In the equations above, we have been using the standard two-component notation and have suppressed the spinor indices. Taking the scalar interactions, for example, if we write out the spinor indices and put the expression into matrix form, we have

$$\chi_i \chi_j = (\chi_i)^\alpha (\chi_j)_\alpha = (\chi_i)_\beta (-\epsilon^{\beta\alpha}) (\chi_j)_\alpha \to \chi_i^\mathsf{T} (-i\sigma^2) \chi_j, \tag{2.6}$$

which is symmetric under $i \leftrightarrow j$. Absorbing the i, j indices also into matrix form, we can write

$$\chi_i S_{ij} \chi_j \longrightarrow \chi^\mathsf{T} (-i\sigma^2 S) \chi, \tag{2.7a}$$

$$\chi_i^\dagger \bar{\sigma}_\mu V_{ij}^\mu \chi_j \longrightarrow \chi^\dagger (\bar{\sigma}^\mu V_\mu) \chi, \tag{2.7b}$$

$$\chi_i i\sigma_\mu \bar{\sigma}_\nu T_{ij}^{\mu\nu} \chi_j \longrightarrow \chi^\mathsf{T} (-i\sigma^2 i\sigma^\mu \bar{\sigma}^\nu T_{\mu\nu}) \chi. \tag{2.7c}$$

We see that without loss of generality we can require

$$(-i\sigma^2 S)^\mathsf{T} = -(-i\sigma^2 S) \Rightarrow S^\mathsf{T} = S, \tag{2.8a}$$

$$(-i\sigma^2 i\sigma^\mu\bar{\sigma}^\nu T_{\mu\nu})^\mathsf{T} = -(-i\sigma^2 i\sigma^\mu\bar{\sigma}^\nu T_{\mu\nu}) \Rightarrow T^\mathsf{T}_{\mu\nu} = T_{\nu\mu} = -T_{\mu\nu}. \tag{2.8b}$$

Furthermore, Hermiticity of the Lagrangian in Eq. (2.4) requires $V^\dagger_\mu = V_\mu$.

A general symmetry transformation of the fermions can be parametrized as

$$\chi \longrightarrow \chi_\alpha \equiv U_\alpha \chi = e^{i\alpha}\chi, \tag{2.9}$$

where, similar to the gauge fields in Eq. (2.1), $\alpha \equiv \alpha^a t^a$ is a sum over all symmetry group generators (across multiple sectors). For the Lagrangian in Eq. (2.4) to respect the (global and gauge) symmetries, we need the following transformation properties of various quantities:

$$P^\mu \longrightarrow P^\mu_\alpha = U_\alpha P^\mu U^\dagger_\alpha, \qquad \delta_\alpha P^\mu = \delta_\alpha G^\mu = i[\alpha, P^\mu], \tag{2.10a}$$

$$V^\mu \longrightarrow V^\mu_\alpha = U_\alpha V^\mu U^\dagger_\alpha, \qquad \delta_\alpha V^\mu = i[\alpha, V^\mu], \tag{2.10b}$$

$$S \longrightarrow S_\alpha = U^*_\alpha S U^\dagger_\alpha, \qquad \delta_\alpha S = -i(\alpha^\mathsf{T} S + S\alpha), \tag{2.10c}$$

$$T^{\mu\nu} \longrightarrow T^{\mu\nu}_\alpha = U^*_\alpha T^{\mu\nu} U^\dagger_\alpha, \qquad \delta_\alpha T^{\mu\nu} = -i(\alpha^\mathsf{T} T^{\mu\nu} + T^{\mu\nu}\alpha), \tag{2.10d}$$

$$\mathcal{L}_{G,\phi} \longrightarrow \mathcal{L}_{G,\phi}[G_\alpha, \phi_\alpha] = \mathcal{L}_{G,\phi}[G, \phi], \qquad \delta_\alpha \mathcal{L}_{G,\phi} = 0, \tag{2.10e}$$

where δ_α denotes the first-order (in α) gauge variation, e.g., $\delta_\alpha P^\mu \equiv (P^\mu_\alpha - P^\mu)|_{\mathcal{O}(\alpha)}$.

The Lagrangian in Eq. (2.4) incorporates the most general scalar, vector, and tensor couplings to fermion bilinears. For example, in SMEFT, the vector interactions V^μ_{ij} cover current-current operators such as[1]

$$(H^\dagger i \overleftrightarrow{D}_\mu H)(\bar{\psi}\gamma^\mu\psi), \quad |H|^2(H^\dagger i \overleftrightarrow{D}_\mu H)(\bar{\psi}\gamma^\mu\psi), \ldots, \tag{2.11}$$

where H is the Higgs doublet and ψ may represent any of the SM fermions $\psi \in \{q, u, d, \ell, e\}$, written in four-component notation here. The scalar interactions S_{ij} cover Yukawa-type operators such as

$$\bar{\ell} e H, \qquad |H|^2 \bar{\ell} e H, \ldots. \tag{2.12}$$

The tensor interactions $T^{\mu\nu}_{ij}$ cover dipole operators such as

$$\bar{\ell}\sigma^{\mu\nu} e B_{\mu\nu} H, \qquad |H|^2 \bar{\ell}\sigma^{\mu\nu} e B_{\mu\nu} H, \ldots, \tag{2.13}$$

where $\sigma^{\mu\nu} \equiv \frac{i}{2}[\gamma^\mu, \gamma^\nu]$. At dimension six, these include all but the four-fermion operators in the Warsaw basis [22]. In fact, all operators in SMEFT involving up to two powers of fermions and no derivatives acting on them, up to arbitrarily high dimensions, are captured by Eq. (2.4). Furthermore, as argued in Refs. [4,9], four-fermion operators can be captured by introducing auxiliary fields, and we believe this argument may be extended to operators with six and more fermions by perturbatively including interactions among such auxiliary fields. Therefore, our calculation in what follows should apply quite generally to all higher-dimensional SMEFT (as well as other relativistic EFT) operators[2] with no derivatives acting on the fermions.

III. REGULARIZING AND EVALUATING THE ANOMALY WITH CDE

To facilitate the calculation of anomalies, we first recast the fermionic interactions in Eq. (2.4) into the following matrix form:

$$\mathcal{L} = \mathcal{L}_{G,\phi} + \frac{1}{2}\begin{pmatrix} \chi^\dagger & \chi^\mathsf{T}(-i\sigma^2) \end{pmatrix} \hat{\boldsymbol{P}} \begin{pmatrix} \chi \\ i\sigma^2\chi^* \end{pmatrix}, \tag{3.1}$$

where

$$\hat{\boldsymbol{P}} \equiv \begin{pmatrix} \bar{\sigma}^\mu(i\partial_\mu + G_\mu + V_\mu) & S^\dagger - i\bar{\sigma}^\nu\sigma^\mu T^\dagger_{\mu\nu} \\ S + i\sigma^\mu\bar{\sigma}^\nu T_{\mu\nu} & \sigma^\mu(i\partial_\mu - G^\mathsf{T}_\mu - V^\mathsf{T}_\mu) \end{pmatrix}. \tag{3.2}$$

For the Lagrangian to be real and symmetry preserving, the matrix $\hat{\boldsymbol{P}}$ needs to be Hermitian, $\hat{\boldsymbol{P}}^\dagger = \hat{\boldsymbol{P}}$, and transform as

[1]While our example operators here are SMEFT operators written in the electroweak unbroken phase with the Higgs doublet H, our formalism applies equally well to effective operators written in the spontaneously broken phase, with an explicit vacuum expectation value v, the physical Higgs h, and the Goldstone fields. It is a matter of rewriting the effective operators in the Lagrangian and the local counterterms accordingly. The conclusion that higher-dimensional operators yield irrelevant anomalies still holds.

[2]Throughout this paper we use the term "higher-dimensional operators" to emphasize the relevance of our analysis for the infinite series of EFT operators, though technically the operators considered here include also renormalizable ones such as dimension-four Yukawa interactions [as part of S—see Eq. (2.12)].

$$\hat{\boldsymbol{P}} \longrightarrow \hat{\boldsymbol{P}}_\alpha = e^{i\boldsymbol{\alpha}}\hat{\boldsymbol{P}}e^{-i\boldsymbol{\alpha}}, \qquad \delta_\alpha \hat{\boldsymbol{P}} = i[\boldsymbol{\alpha}, \hat{\boldsymbol{P}}] \quad \text{with} \quad \boldsymbol{\alpha} \equiv \begin{pmatrix} \alpha & 0 \\ 0 & -\alpha^\mathsf{T} \end{pmatrix}. \tag{3.3}$$

Clearly, these properties of $\hat{\boldsymbol{P}}$ can be verified using its explicit expression in Eq. (3.2) together with the transformation properties given in Eq. (2.10).

As reviewed in Ref. [11] (see also Refs. [1,2]), anomalies can be derived from the gauge variation of the bosonic effective action obtained by integrating out the fermion fields. In the present case, the bosonic effective action depends on both gauge and scalar fields (i.e., we integrate out the fermions in the path integral while treating all bosonic fields as classical backgrounds). Formally, we have

$$e^{iW[G,\phi]} = \int \mathcal{D}\chi \mathcal{D}\chi^\dagger e^{iS[\chi,\chi^\dagger,G,\phi]} = e^{iS_{G,\phi}} (\det \hat{\boldsymbol{P}})^{1/2}. \tag{3.4}$$

Following Eq. (3.3), a gauge transformation yields

$$e^{iW[G_\alpha,\phi_\alpha]} = e^{iS_{G,\phi}} (\det \hat{\boldsymbol{P}}_\alpha)^{1/2} = e^{iS_{G,\phi}} [\det (e^{i\boldsymbol{\alpha}}\hat{\boldsymbol{P}}e^{-i\boldsymbol{\alpha}})]^{1/2}. \tag{3.5}$$

As in Eq. (1.4), the anomaly functional (or simply the anomaly) is defined by the first-order gauge variation of the bosonic effective action:

$$\begin{aligned} \mathcal{A}[\alpha] &\equiv \delta_\alpha W[G,\phi] = (W[G_\alpha,\phi_\alpha] - W[G,\phi])|_{\mathcal{O}(\alpha)} \\ &\simeq -\frac{i}{2} \mathrm{Tr} \log \left[\mathbf{1} + \frac{1}{\hat{\boldsymbol{P}}} (i\boldsymbol{\alpha}\hat{\boldsymbol{P}} - \hat{\boldsymbol{P}} i\boldsymbol{\alpha}) \right] \Big|_{\mathcal{O}(\alpha)} \\ &\simeq \frac{1}{2} \mathrm{Tr} \left[\frac{1}{\hat{\boldsymbol{P}}} (\boldsymbol{\alpha}\hat{\boldsymbol{P}} - \hat{\boldsymbol{P}}\boldsymbol{\alpha}) \right]. \end{aligned} \tag{3.6}$$

As explained in detail in Ref. [11], we use the notation "$\simeq$" to emphasize that the expressions are not exactly equal unless they are regularized in the same way.

To proceed further, we need to introduce a regulator. Following similar steps as in Sec. 3 of Ref. [11], we see that in the present case, each term in the expansion is proportional to

$$\begin{aligned} &\mathrm{tr} \begin{pmatrix} \sigma^{\mu_1}\bar{\sigma}^{\nu_1} \cdots \sigma^{\mu_n}\bar{\sigma}^{\nu_n} & \cdot \\ \cdot & \bar{\sigma}^{\mu_1}\sigma^{\nu_1} \cdots \bar{\sigma}^{\mu_n}\sigma^{\nu_n} \end{pmatrix} \\ &= \mathrm{tr} \left[\gamma^{\mu_1}\gamma^{\nu_1} \cdots \gamma^{\mu_n}\gamma^{\nu_n} \begin{pmatrix} \frac{1-\gamma^5}{2} & \cdot \\ \cdot & \frac{1+\gamma^5}{2} \end{pmatrix} \right]. \end{aligned} \tag{3.7}$$

We can therefore replace all the 2×2 Pauli matrices by 4×4 gamma matrices while freely inserting chirality projection factors $\frac{1\mp\gamma^5}{2} + \beta\frac{1\pm\gamma^5}{2}$, such that terms proportional to β will hit the opposite chirality projection operator when anticommuted to the right and vanish. To this end, let us define

$$\hat{\boldsymbol{P}}_\beta \equiv \begin{pmatrix} i\partial\!\!\!/ + G\!\!\!/\left(\frac{1-\gamma^5}{2} + \beta_G\frac{1+\gamma^5}{2}\right) + V\!\!\!/\left(\frac{1-\gamma^5}{2} + \beta_V\frac{1+\gamma^5}{2}\right) & S^\dagger\left(\frac{1+\gamma^5}{2} + \beta_S\frac{1-\gamma^5}{2}\right) + \sigma^{\mu\nu}T^\dagger_{\mu\nu}\left(\frac{1+\gamma^5}{2} + \beta_T\frac{1-\gamma^5}{2}\right) \\ S\left(\frac{1-\gamma^5}{2} + \beta_S\frac{1+\gamma^5}{2}\right) + \sigma^{\mu\nu}T_{\mu\nu}\left(\frac{1-\gamma^5}{2} + \beta_T\frac{1+\gamma^5}{2}\right) & i\partial\!\!\!/ - G\!\!\!/^\mathsf{T}\left(\frac{1+\gamma^5}{2} + \beta_G\frac{1-\gamma^5}{2}\right) - V\!\!\!/^\mathsf{T}\left(\frac{1+\gamma^5}{2} + \beta_V\frac{1-\gamma^5}{2}\right) \end{pmatrix}. \tag{3.8}$$

We clarify that $G\!\!\!/^\mathsf{T}$ and $V\!\!\!/^\mathsf{T}$ are defined as

$$G\!\!\!/^\mathsf{T} \equiv \gamma^\mu G^\mathsf{T}_\mu, \qquad V\!\!\!/^\mathsf{T} \equiv \gamma^\mu V^\mathsf{T}_\mu, \tag{3.9}$$

in which the gamma matrices are *not* transposed. Here β_G, β_S, β_V, β_T can all be different in principle, and we denote them collectively as β in the subscript of $\hat{\boldsymbol{P}}_\beta$ (and also $\mathcal{A}_\beta$ below). Replacing $\hat{\boldsymbol{P}} \to \hat{\boldsymbol{P}}_\beta$ in Eq. (3.6), we see that similar to Ref. [11], a damping factor $f(-\hat{\boldsymbol{P}}^2_\beta/\Lambda^2)$ emerges as a natural regulator, with any function $f(u)$ that satisfies the following conditions:

$$f(0) = 1; \qquad f(+\infty) = 0; \qquad \int_0^\infty \mathrm{d}u f(u) \quad \text{well defined}, \tag{3.10a}$$

$$u^n \frac{\mathrm{d}^n f}{\mathrm{d}u^n}\bigg|_{u=0} = u^n \frac{\mathrm{d}^n f}{\mathrm{d}u^n}\bigg|_{u\to+\infty} = 0 \quad \text{for } n \geq 1. \tag{3.10b}$$

The *regularized* anomaly is then defined by

$$\mathcal{A}^\Lambda_\beta[\alpha] \equiv \frac{1}{2} \mathrm{Tr} \left[f\left(-\frac{\hat{\boldsymbol{P}}^2_\beta}{\Lambda^2} \right) \hat{\boldsymbol{P}}^{-1}_\beta (\boldsymbol{\alpha}\hat{\boldsymbol{P}}_\beta - \hat{\boldsymbol{P}}_\beta\boldsymbol{\alpha}) \frac{1-\boldsymbol{\Gamma}^5}{2} \right], \tag{3.11}$$

where we have introduced the notation

$$\boldsymbol{\Gamma}^5 \equiv \begin{pmatrix} \gamma^5 & 0 \\ 0 & -\gamma^5 \end{pmatrix} \quad \text{satisfying} \quad \hat{\boldsymbol{P}}_\beta \boldsymbol{\Gamma}^5 = -\boldsymbol{\Gamma}^5 \hat{\boldsymbol{P}}_\beta. \tag{3.12}$$

Now using the cyclicity of the trace and commuting $\hat{\boldsymbol{P}}_\beta$ through $f(-\frac{\hat{\boldsymbol{P}}^2_\beta}{\Lambda^2})$, we obtain

$$\mathcal{A}^\Lambda_\beta[\alpha] = \frac{1}{2} \mathrm{Tr} \left[f\left(-\frac{\hat{\boldsymbol{P}}^2_\beta}{\Lambda^2} \right) \boldsymbol{\Gamma}^5 \boldsymbol{\alpha} \right]. \tag{3.13}$$

The *renormalized* anomaly is then defined by

$$\mathcal{A}_\beta[\alpha] \equiv \lim_{\Lambda\to\infty}\left(\mathcal{A}_\beta^\Lambda[\alpha] + \delta_\alpha \int \mathrm{d}^4x \mathcal{L}_{\mathrm{ct}}^\Lambda\right), \tag{3.14}$$

where $\mathcal{L}_{\mathrm{ct}}^\Lambda$ is the local counterterm Lagrangian. Since $\mathcal{A}_\beta^\Lambda[\alpha]$ may be quadratically divergent, we must include appropriate $\mathcal{O}(\Lambda^2)$ counterterms to make the renormalized anomaly finite in the limit $\Lambda \to \infty$. Meanwhile, the finite part of $\mathcal{L}_{\mathrm{ct}}^\Lambda$ defines the renormalization scheme. Generically $\mathcal{A}_\beta^\Lambda[\alpha]$ also contains $\mathcal{O}(1/\Lambda)$ terms, which we will suppress throughout the paper since they vanish when $\Lambda \to \infty$.

Equation (3.13) is a generalization of the minimal coupling (mc) case formula in Ref. [11]. To see the connection explicitly, we note that when $S = V_\mu = T_{\mu\nu} = 0$, $\hat{\boldsymbol{P}}_\beta$ becomes a block diagonal with the two blocks related by charge conjugation[3]:

$$\hat{\boldsymbol{P}}_\beta^{\mathrm{mc}} = \begin{pmatrix} \hat{P}_\beta & 0 \\ 0 & \bar{\hat{P}}_\beta \end{pmatrix} = \begin{pmatrix} i\partial\!\!\!/ + \gamma^\mu G_\mu\left(\frac{1-\gamma^5}{2} + \beta_G \frac{1+\gamma^5}{2}\right) & 0 \\ 0 & -i\partial\!\!\!/^{\mathsf{T}} - \gamma^\mu G_\mu^{\mathsf{T}}\left(\frac{1+\gamma^5}{2} + \beta_G \frac{1-\gamma^5}{2}\right) \end{pmatrix}, \tag{3.15}$$

where "charge conjugation" is the operation

$$\bar{\hat{P}}_\beta \equiv \gamma^0\gamma^2 \hat{P}_\beta^{\mathsf{T}} \gamma^0\gamma^2, \tag{3.16}$$

under which the gamma matrices transform as

$$\overline{\gamma^\mu} = -\gamma^\mu, \qquad \overline{\sigma^{\mu\nu}} = -\sigma^{\mu\nu}, \qquad \overline{\gamma^5} = \gamma^5. \tag{3.17}$$

It satisfies the expected properties

$$\bar{\bar{A}} = A, \qquad \mathrm{tr}(ABC\cdots) = \mathrm{tr}(\cdots \bar{C}\,\bar{B}\,\bar{A}). \tag{3.18}$$

Therefore, the two blocks in $\hat{\boldsymbol{P}}_\beta^{\mathrm{mc}}$ contribute equally, and Eq. (3.13) reduces to the result derived in Ref. [11]:

$$\mathcal{A}_\beta^{\Lambda,\mathrm{mc}}[\alpha] = \mathrm{Tr}\left[f\left(-\frac{\hat{P}_\beta^2}{\Lambda^2}\right)\gamma^5\alpha\right]. \tag{3.19}$$

It is useful to introduce an extended version of this charge conjugation operation. For a matrix $\boldsymbol{A}$ acting on the field multiplet space, we define

$$\bar{\boldsymbol{A}} \equiv \gamma^0\gamma^2 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \boldsymbol{A}^{\mathsf{T}} \gamma^0\gamma^2 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{3.20}$$

Clearly, the properties in Eq. (3.18) hold for this extended version as well, and one can also check that

$$\bar{\hat{\boldsymbol{P}}}_\beta = \hat{\boldsymbol{P}}_\beta, \qquad \bar{\boldsymbol{\alpha}} = -\boldsymbol{\alpha}, \qquad \overline{\boldsymbol{\Gamma}^5} = -\boldsymbol{\Gamma}^5. \tag{3.21}$$

The CDE evaluation of the functional trace in Eq. (3.13) proceeds in a similar way to the minimal coupling case detailed in Ref. [11]. After performing the loop integrals, we obtain

$$\mathcal{A}_\beta^\Lambda[\alpha] = \frac{i}{32\pi^2}\int \mathrm{d}^4x \left\{-\Lambda^2\left[\int_0^\infty \mathrm{d}u f(u)\right]\mathbf{tr}_0 + \frac{1}{6}(\mathbf{tr}_1 + \mathbf{tr}_2 + \mathbf{tr}_3)\right\}, \tag{3.22}$$

with a few (nonfunctional) traces over the field multiplet and internal indices (denoted by lowercase "tr"):

$$\mathbf{tr}_0 = \mathrm{tr}(\hat{\boldsymbol{P}}_\beta^2 \boldsymbol{\Gamma}^5\boldsymbol{\alpha}), \tag{3.23a}$$

$$\mathbf{tr}_1 = \mathrm{tr}(\hat{\boldsymbol{P}}_\beta^4 \boldsymbol{\Gamma}^5\boldsymbol{\alpha}), \tag{3.23b}$$

$$\mathbf{tr}_2 = -\frac{1}{2}\mathrm{tr}[(\hat{\boldsymbol{P}}_\beta^2\gamma^\mu\hat{\boldsymbol{P}}_\beta\gamma_\mu\hat{\boldsymbol{P}}_\beta + \hat{\boldsymbol{P}}_\beta\gamma^\mu\hat{\boldsymbol{P}}_\beta\gamma_\mu\hat{\boldsymbol{P}}_\beta^2)\boldsymbol{\Gamma}^5\boldsymbol{\alpha}], \tag{3.23c}$$

$$\mathbf{tr}_3 = -\frac{1}{2}\mathrm{tr}(\hat{\boldsymbol{P}}_\beta\gamma^\mu\hat{\boldsymbol{P}}_\beta^2\gamma_\mu\hat{\boldsymbol{P}}_\beta\boldsymbol{\Gamma}^5\boldsymbol{\alpha}). \tag{3.23d}$$

These are generalizations of tr_0–tr_3 (unbolded) in Ref. [11], although (obviously) their evaluation result is twice as large, $\mathbf{tr}_i = 2\mathrm{tr}_i$, in the minimal coupling case. It is also useful to note that the two terms in $\mathbf{tr}_2$ are related by charge conjugation and therefore equal,[4] which simplifies its calculation later on.

IV. HIGHER-DIMENSIONAL OPERATORS YIELD IRRELEVANT ANOMALIES

In this section, we complete the evaluation of the regularized anomaly $\mathcal{A}_\beta^\Lambda[\alpha]$ by computing the traces in Eq. (3.23). For general values of β_G, β_S, β_V, β_T, the calculation is very tedious and does not give new insights. The reason is that most β choices lead to results that do not satisfy the Wess-Zumino consistency condition [23], which

[3]Note that $(\partial_\mu)^{\mathsf{T}} = \overleftarrow{\partial}_\mu = -\partial_\mu$ upon integration by parts.

[4]Note that we need to use cyclic permutation inside the internal trace tr for this argument. See Appendix A of Ref. [11] for a detailed discussion about the legitimacy of such operations, which will be assumed throughout this paper.

means they do not correspond to consistent regularization schemes of the effective action and there is no meaningful notion of relevant vs irrelevant anomalies. This point has been discussed in detail in Ref. [11] in the minimal coupling case where only β_G is present; for example, $\beta_G = 0$ is the only choice that satisfies the Wess-Zumino consistency condition for the case of a nontrivial non-Abelian anomaly. Motivated by the results in Ref. [11], we will set all the β's to zero in the present analysis:

$$\beta_G = \beta_S = \beta_V = \beta_T = 0. \tag{4.1}$$

With this regularization scheme choice, we will show that all the additional contributions to the anomaly are irrelevant, namely

$$\mathcal{A}^{\Lambda}_{\beta=0}[\alpha] = \mathcal{A}^{\Lambda,\mathrm{mc}}_{\beta=0}[\alpha] - \delta_\alpha \int \mathrm{d}^4x \Delta\mathcal{L}^{\Lambda}_{\mathrm{ct}}. \tag{4.2}$$

This means that by appropriately adjusting the local counterterms (i.e., choosing the renormalization scheme), the renormalized anomaly defined in Eq. (3.14) is the same as that in the minimal coupling case

$$\mathcal{A}_{\beta=0}[\alpha] = \mathcal{A}^{\mathrm{mc}}_{\beta=0}[\alpha]. \tag{4.3}$$

Setting $\beta = 0$ significantly simplifies the presentation; we now have

$$\hat{\boldsymbol{P}}_{\beta=0} \equiv i\partial\!\!\!/ + \begin{pmatrix} G\!\!\!/ + V\!\!\!/ & S^\dagger + \sigma^{\mu\nu}T^\dagger_{\mu\nu} \\ S + \sigma^{\mu\nu}T_{\mu\nu} & -G\!\!\!/^{\mathsf{T}} - V\!\!\!/^{\mathsf{T}} \end{pmatrix} \frac{1-\boldsymbol{\Gamma}^5}{2}. \tag{4.4}$$

Nevertheless, the calculation including $S, V_\mu, T_{\mu\nu}$ all at once is still quite lengthy. So in what follows, we will work up to the full results gradually, adding one type of interactions at each step.

A. Vector interactions

We begin with the case of having vector interactions V_μ only, while setting S and $T_{\mu\nu}$ to zero. In this case, there is actually a shortcut. From the expression of $\hat{\boldsymbol{P}}_{\beta=0}$ in Eq. (4.4) we see that, instead of directly calculating the traces in Eq. (3.23), we can simply take the minimal coupling result and replace $G_\mu \to G_\mu + V_\mu$:

$$\mathcal{A}^{\Lambda}_{\beta=0}[\alpha]|_{S=T_{\mu\nu}=0} = \mathcal{A}^{\Lambda,\mathrm{mc}}_{\beta=0}[\alpha]|_{G_\mu \to G_\mu + V_\mu}. \tag{4.5}$$

In Ref. [11], we obtained the result for the minimal coupling case

$$\mathcal{A}^{\Lambda,\mathrm{mc}}_{\beta=0}[\alpha] = \int \mathrm{d}^4x \left\{ \frac{1}{48\pi^2} \varepsilon^{\mu\nu\rho\sigma} \mathrm{tr}\left[(\partial_\mu\alpha)(G_\nu F_{\rho\sigma} + iG_\nu G_\rho G_\sigma) \right] - \delta_\alpha \mathcal{L}^{\Lambda}_{\mathrm{ct},0} \right\}. \tag{4.6}$$

The first term is the standard result for the consistent anomaly. The second term, being the gauge variation of a local counterterm

$$\mathcal{L}^{\Lambda}_{\mathrm{ct},0} = \frac{1}{16\pi^2}\left[\Lambda^2 \int_0^\infty \mathrm{d}u f(u)\right] \mathrm{tr}(G^\mu G_\mu) + \frac{1}{96\pi^2} \mathrm{tr}\left[(\partial^\mu G_\mu)^2 - 2iF^{\mu\nu}G_\mu G_\nu + \frac{1}{2} G^\mu G^\nu G_\mu G_\nu \right], \tag{4.7}$$

is an irrelevant anomaly.

Upon making the substitution $G_\mu \to G_\mu + V_\mu$, we first note that the irrelevant term in Eq. (4.6) remains irrelevant, because the two operations "taking the gauge variation" and "substituting $G_\mu \to G_\mu + V_\mu$" commute with each other,

$$(\delta_\alpha \mathcal{L}^{\Lambda}_{\mathrm{ct},0})|_{G_\mu \to G_\mu + V_\mu} = \delta_\alpha (\mathcal{L}^{\Lambda}_{\mathrm{ct},0}|_{G_\mu \to G_\mu + V_\mu}), \tag{4.8}$$

due to the fact

$$\begin{aligned} \delta_\alpha (G_\mu + V_\mu) &= (\partial_\mu \alpha) + i[\alpha, G_\mu] + i[\alpha, V_\mu] \\ &= (\partial_\mu \alpha) + i[\alpha, G_\mu + V_\mu]. \end{aligned} \tag{4.9}$$

For the relevant part of $\mathcal{A}^{\Lambda,\mathrm{mc}}_{\beta=0}$ [first term in Eq. (4.6)], the substitution $G_\mu \to G_\mu + V_\mu$ produces additional terms that we need to track carefully. Using

$$F_{\mu\nu}|_{G_\mu \to G_\mu + V_\mu} = F_{\mu\nu} + (D_\mu V_\nu) - (D_\nu V_\mu) - i[V_\mu, V_\nu], \tag{4.10}$$

where $(D_\mu V_\nu) \equiv (\partial_\mu V_\nu) - i[G_\mu, V_\nu]$, we get

$$\begin{aligned} \mathcal{A}^{\Lambda}_{\beta=0}[\alpha]|_{S=T_{\mu\nu}=0} = \mathcal{A}^{\Lambda,\mathrm{mc}}_{\beta=0}[\alpha] &- \delta_\alpha \int \mathrm{d}^4x (\mathcal{L}^{\Lambda}_{\mathrm{ct},0}|_{G_\mu \to G_\mu + V_\mu} - \mathcal{L}^{\Lambda}_{\mathrm{ct},0}) - \int \mathrm{d}^4x \frac{1}{48\pi^2} \varepsilon^{\mu\nu\rho\sigma} \mathrm{tr}\{(\partial_\mu\alpha)[-(V_\nu G_{\rho\sigma} + G_{\rho\sigma}V_\nu) \\ &- i(G_\nu G_\rho V_\sigma + V_\nu G_\rho G_\sigma - G_\nu V_\rho G_\sigma) - 2V_\nu(D_\rho V_\sigma) \\ &- iV_\nu G_\rho V_\sigma + i(G_\nu V_\rho V_\sigma - V_\nu V_\rho G_\sigma) + iV_\nu V_\rho V_\sigma]\}. \end{aligned} \tag{4.11}$$

Using the gauge transformation properties of the various quantities

$$\delta_\alpha G_\mu = (\partial_\mu \alpha) + i[\alpha, G_\mu], \qquad \delta_\alpha G_{\mu\nu} = i[\alpha, G_{\mu\nu}], \tag{4.12a}$$

$$\delta_\alpha V_\mu = i[\alpha, V_\mu], \qquad \delta_\alpha (D_\mu V_\nu) = i[\alpha, (D_\mu V_\nu)], \tag{4.12b}$$

we can organize the terms beyond the first line in Eq. (4.11) into the gauge variation of the following local counterterm:

$$\begin{aligned}\Delta\mathcal{L}_{\text{ct}}^{(V)} = \frac{1}{48\pi^2}\varepsilon^{\mu\nu\rho\sigma}\text{tr}\Big[&-G_\mu(V_\nu F_{\rho\sigma} + F_{\rho\sigma}V_\nu) \\ &- iG_\mu G_\nu G_\rho V_\sigma - 2G_\mu V_\nu (D_\rho V_\sigma) \\ &- \frac{i}{2}G_\mu V_\nu G_\rho V_\sigma + iG_\mu G_\nu V_\rho V_\sigma + iG_\mu V_\nu V_\rho V_\sigma\Big].\end{aligned} \tag{4.13}$$

Note that when taking the gauge variation of the expression above, all the commutator terms generated through Eq. (4.12) cancel out, which leaves us with only terms proportional to $(\partial_\mu \alpha)$, reproducing the expression in Eq. (4.11). In summary, we have shown that

$$\begin{aligned}\mathcal{A}^\Lambda_{\beta=0}[\alpha]\Big|_{S=T_{\mu\nu}=0} = \mathcal{A}^{\Lambda,\text{mc}}_{\beta=0}[\alpha] - \delta_\alpha \int \text{d}^4x \Big(\mathcal{L}^\Lambda_{\text{ct},0}\Big|_{G_\mu\to G_\mu+V_\mu} \\ - \mathcal{L}^\Lambda_{\text{ct},0} + \Delta\mathcal{L}^{(V)}_{\text{ct}}\Big).\end{aligned} \tag{4.14}$$

We conclude that all additional contributions to the anomaly due to the vector interactions V_μ are irrelevant.

B. Vector and scalar interactions

In this subsection, we turn on both the scalar interactions S and vector interactions V_μ while keeping $T_{\mu\nu} = 0$. We will further include the tensor interactions $T_{\mu\nu}$ in the next subsection.

To calculate the traces in Eq. (3.23) in the presence of S and/or $T_{\mu\nu}$, it is useful to decompose $\hat{\boldsymbol{P}}_{\beta=0}$ in Eq. (4.4) as

$$\begin{aligned}\hat{\boldsymbol{P}}_{\beta=0} &= (S_L + \gamma_\mu V_L^\mu + \sigma_{\mu\nu}T_L^{\mu\nu})\frac{1-\gamma^5}{2} \\ &\quad + (S_R + \gamma_\mu V_R^\mu + \sigma_{\mu\nu}T_R^{\mu\nu})\frac{1+\gamma^5}{2} \\ &\equiv (S_L + V_L + T_L)\frac{1-\gamma^5}{2} \\ &\quad + (S_R + V_R + T_R)\frac{1+\gamma^5}{2},\end{aligned} \tag{4.15}$$

where we have introduced the notation

$$V_L = \begin{pmatrix} i\not{\partial} + \not{G} + \not{V} & 0 \\ 0 & i\not{\partial}\end{pmatrix}, \qquad S_L = \begin{pmatrix} 0 & 0 \\ S & 0\end{pmatrix},$$
$$T_L = \sigma_{\mu\nu}\begin{pmatrix} 0 & 0 \\ T^{\mu\nu} & 0\end{pmatrix}, \tag{4.16a}$$

$$V_R = \begin{pmatrix} i\not{\partial} & 0 \\ 0 & i\not{\partial} - \not{G}^\top - \not{V}^\top\end{pmatrix}, \qquad S_R = \begin{pmatrix} 0 & S^\dagger \\ 0 & 0\end{pmatrix},$$
$$T_R = \sigma_{\mu\nu}\begin{pmatrix} 0 & T^{\dagger\mu\nu} \\ 0 & 0\end{pmatrix}. \tag{4.16b}$$

These components satisfy the following relations under the (extended) charge conjugation defined in Eq. (3.20):

$$\bar{V}_{L/R} = V_{R/L}, \qquad \bar{S}_{L/R} = S_{L/R}, \qquad \bar{T}_{L/R} = T_{L/R}. \tag{4.17}$$

With the decomposition in Eq. (4.15), we can expand the traces in Eq. (3.23) into a set of terms, each being a product of the components

$$V_{L/R}\frac{1\mp\gamma^5}{2}, \qquad S_{L/R}\frac{1\mp\gamma^5}{2}, \qquad T_{L/R}\frac{1\mp\gamma^5}{2}. \tag{4.18}$$

The matrix structures of these components, their chiralities, and charge conjugation properties lead to simplifications of the calculation:

(i) For the Dirac trace to be nonzero, each term must have an even power of γ^μ matrices in total. Given the structures of the traces in Eq. (3.23), this implies that only terms with an even power of $V_{L/R}$ will contribute.

(ii) The matrix structure of $S_{L/R}$ tells us that

$$S_L(\cdots V_{L/R}\cdots)S_L = S_R(\cdots V_{L/R}\cdots)S_R = 0, \tag{4.19}$$

where $(\cdots V_{L/R}\cdots)$ does not contain any $S_{L/R}$ or $T_{L/R}$ factors. The same is true if we replace any of the $S_{L/R}$ in Eq. (4.19) with $T_{L/R}$.

(iii) The product of the chirality projection factors $\frac{1\mp\gamma^5}{2}$ will impose further selection rules.

(iv) Finally, one can make use of the charge conjugation properties in Eq. (4.17) to merge terms and simplify the result.

Now we apply these constraints to the case of this subsection, where $S, V_\mu \neq 0$ but $T_{\mu\nu} = 0$. It is easy to see that $\mathbf{tr}_0$ does not contain any S-dependent terms, while the nonzero terms in $\mathbf{tr}_1, \mathbf{tr}_2, \mathbf{tr}_3$ must have two powers of S and two powers of V with appropriate chirality combinations. Starting with $\mathbf{tr}_1$, we get

$$\begin{aligned}\mathbf{tr}_1^{(S^2V^2)} = \frac{1}{2}\text{tr}\Big[&(S_R V_L S_L V_R - V_L S_L V_R S_R \\ &+ S_L V_R S_R V_L - V_R S_R V_L S_L)\boldsymbol{\alpha}\Big],\end{aligned} \tag{4.20}$$

where terms containing one power of γ^5 have been dropped since $\mathrm{tr}(\gamma^\mu\gamma^\nu\gamma^5)=0$. We can use charge conjugation to further simplify this trace. Upon cyclic permutation the four terms in Eq. (4.20) combine in pairs and give

$$\begin{aligned}\mathbf{tr}_1^{(S^2V^2)} &= \mathrm{tr}[(S_RV_LS_LV_R - V_RS_RV_LS_L)\boldsymbol{\alpha}] \\ &= \mathrm{tr}(S_RV_LS_L[V_R,\boldsymbol{\alpha}]).\end{aligned} \tag{4.21}$$

The other two traces $\mathbf{tr}_2$ and $\mathbf{tr}_3$ admit similar simplifications. The general rule we follow is to rewrite half of the terms using charge conjugation such that the entire expression is proportional to the commutator $[V_R,\boldsymbol{\alpha}]$. After contracting the gamma matrices using $\gamma^\mu\gamma_\mu=4$, $\gamma^\mu\gamma^\nu\gamma_\mu=-2\gamma^\nu$, we find

$$\mathbf{tr}_2^{(S^2V^2)} = \mathrm{tr}\{(S_RV_RS_L - 2V_LS_RS_L - 2S_RS_LV_L)[V_R,\boldsymbol{\alpha}]\}, \tag{4.22a}$$

$$\begin{aligned}\mathbf{tr}_3^{(S^2V^2)} &= \mathrm{tr}(S_RS_L[V_R,\boldsymbol{\alpha}]) \\ &= \mathrm{tr}\{(S_RS_LV_R + V_RS_RS_L)[V_R,\boldsymbol{\alpha}]\}.\end{aligned} \tag{4.22b}$$

Combining the three traces above and substituting in the expressions for $S_{L,R}$ and $V_{L,R}$ from Eq. (4.16), we find that the additional contribution to the anomaly from scalar couplings is

$$\begin{aligned}\mathcal{A}^{\Lambda}_{\beta=0}[\alpha]|_{\mathcal{O}(S^2V^2)} = -\frac{1}{192\pi^2}\int \mathrm{d}^4x\,\mathrm{tr}\{[&S^\dagger(\not{G}^\mathsf{T}+\not{V}^\mathsf{T})S \\ &- S^\dagger S(\not{G}+\not{V}) \\ &- (\not{G}+\not{V})S^\dagger S + i(S^\dagger \overleftrightarrow{\not{D}}_V S)](\not{\partial}\alpha)\},\end{aligned} \tag{4.23}$$

where $(S^\dagger \overleftrightarrow{\not{D}}_V S) \equiv \gamma_\mu[S^\dagger(D_V^\mu S) - (D_V^\mu S^\dagger)S]$. Here we have defined a shifted covariant derivative D_V^μ that also contains the vector interactions V^μ:

$$D_V^\mu \equiv D^\mu|_{G^\mu\to G^\mu+V^\mu} = \partial^\mu - i(G^\mu+V^\mu). \tag{4.24}$$

Its action on $S, S^\dagger$ follows the same substitution:

$$\begin{aligned}(D_V^\mu S) &\equiv (D^\mu S)|_{G^\mu\to G^\mu+V^\mu}, \\ (D_V^\mu S^\dagger) &\equiv (D^\mu S^\dagger)|_{G^\mu\to G^\mu+V^\mu}.\end{aligned} \tag{4.25}$$

If desired, one could easily evaluate the Dirac trace $\mathrm{tr}(\gamma^\mu\gamma^\nu)=4\eta^{\mu\nu}$ in Eq. (4.23), but this is unnecessary for showing that it is an irrelevant anomaly.

To find the corresponding counterterm, we recall the gauge transformation of the scalar interactions $S[G_\mu,\phi]$ from Eq. (2.10):

$$S \longrightarrow S_\alpha = U_\alpha^* S U_\alpha^\dagger, \tag{4.26}$$

which leads to

$$S^\dagger S \longrightarrow U_\alpha S^\dagger S U_\alpha^\dagger, \qquad \delta_\alpha(S^\dagger S) = i[\alpha, S^\dagger S]. \tag{4.27}$$

Their covariant derivatives by definition transform in the same way. This remains true for the shifted covariant derivative D_V^μ defined in Eq. (4.24), and therefore we have

$$\delta_\alpha(S^\dagger \overleftrightarrow{\not{D}}_V S) = i[\alpha, (S^\dagger \overleftrightarrow{\not{D}}_V] S). \tag{4.28}$$

From the gauge transformation properties discussed above, together with those of G_μ and V_μ in Eq. (4.12), we can identify

$$\mathcal{A}^{\Lambda}_{\beta=0}[\alpha]|_{\mathcal{O}(S^2V^2)} = -\delta_\alpha\int \mathrm{d}^4x\,\Delta\mathcal{L}_{\mathrm{ct}}^{(S^2V^2)}, \tag{4.29}$$

where

$$\begin{aligned}\Delta\mathcal{L}_{\mathrm{ct}}^{(S^2V^2)} = \frac{1}{192\pi^2}\int \mathrm{d}^4x\,\mathrm{tr}\Big[&\frac{1}{2}S^\dagger(\not{G}^\mathsf{T}+\not{V}^\mathsf{T})S(\not{G}+\not{V}) \\ &- S^\dagger S(\not{G}+\not{V})(\not{G}+\not{V}) + i(S^\dagger \overleftrightarrow{\not{D}}_V S)\not{G}\Big].\end{aligned} \tag{4.30}$$

We therefore conclude that when both vector and scalar interactions are present, the additional contributions to the anomaly beyond the minimal coupling case are all irrelevant.

C. Vector, scalar, and tensor interactions

Finally, we also include the tensor interactions $T_{\mu\nu}$ alongside vector and scalar interactions in this subsection. The calculation proceeds in a similar way to the vector and scalar interactions case in the previous subsection; the gamma matrix algebra is slightly more tedious but it is straightforward.

Using the decomposition in Eq. (4.15), we immediately see that again, $\mathbf{tr}_0$ does not contain any $T_{\mu\nu}$-dependent terms. For $\mathbf{tr}_1, \mathbf{tr}_2, \mathbf{tr}_3$, the additional nonzero terms are of the form TSV^2 and T^2V^2. We examine them in turn below.

TSV^2 terms: Upon contraction of gamma matrices using $\gamma^\mu\gamma_\mu=4$, $\gamma^\mu\gamma^\nu\gamma_\mu=-2\gamma^\nu$, and noting $\gamma^\mu T_{L,R}\gamma_\mu=0$ (since $\gamma^\mu\gamma^\nu\gamma^\rho\gamma_\mu=4\eta^{\nu\rho}$ while $T_{L,R}$ involves the antisymmetric $\sigma^{\mu\nu}$), we find

$$\mathbf{tr}_1^{(TSV^2)} = \mathrm{tr}\{(T_RV_LS_L + S_RV_LT_L)[V_R,\boldsymbol{\alpha}](1+\gamma^5)\}, \tag{4.31a}$$

$$\begin{aligned}\mathbf{tr}_2^{(TSV^2)} = \mathrm{tr}\{(&T_RV_RS_L + S_RV_RT_L - 2V_LS_RT_L \\ &- 2T_RS_LV_L)[V_R,\boldsymbol{\alpha}](1+\gamma^5)\},\end{aligned} \tag{4.31b}$$

$$\mathbf{tr}_3^{(TSV^2)} = \mathrm{tr}\{T_R S_L [V_R^2, \boldsymbol{\alpha}](1+\gamma^5) + S_R T_L [V_R^2, \boldsymbol{\alpha}](1-\gamma^5)\}. \tag{4.31c}$$

To arrive at these equations we have combined terms that are related by charge conjugation and used cyclic permutation as in the previous subsection. We can further show that $\mathbf{tr}_3^{(TSV^2)} = 0$ because

$$[V_R^2, \boldsymbol{\alpha}] = (V_R^\mu [V_R^\nu, \boldsymbol{\alpha}] + [V_R^\mu, \boldsymbol{\alpha}] V_R^\nu)\gamma_\mu\gamma_\nu = ([V_R^\mu, [V_R^\nu, \boldsymbol{\alpha}]] + [V_R^\nu, \boldsymbol{\alpha}] V_R^\mu + [V_R^\mu, \boldsymbol{\alpha}] V_R^\nu)\gamma_\mu\gamma_\nu. \tag{4.32}$$

The expression in parentheses is symmetric in $\mu \leftrightarrow \nu$ (note that for $[V_R^\mu, [V_R^\nu, \boldsymbol{\alpha}]]$, only its upper-left block $(-\partial^\mu\partial^\nu\alpha)$ will eventually feed into the expressions), whereas the Dirac traces are antisymmetric:

$$\mathrm{tr}(\gamma_\mu\gamma_\nu\sigma_{\rho\tau}) = -\mathrm{tr}(\gamma_\nu\gamma_\mu\sigma_{\rho\tau}), \tag{4.33a}$$

$$\mathrm{tr}(\gamma_\mu\gamma_\nu\sigma_{\rho\tau}\gamma^5) = -\mathrm{tr}(\gamma_\nu\gamma_\mu\sigma_{\rho\tau}\gamma^5). \tag{4.33b}$$

Adding up $\mathbf{tr}_1$ and $\mathbf{tr}_2$ and substituting in the expressions for $S_{L,R}$, $V_{L,R}$, $T_{L,R}$ from Eq. (4.16), we obtain

$$\mathcal{A}^\Lambda_{\beta=0}[\alpha]|_{\mathcal{O}(TSV^2)} = -\frac{1}{192\pi^2}\int \mathrm{d}^4x \mathrm{tr}\{[(\sigma\cdot T^\dagger)(\not{G}^\mathsf{T} + \not{V}^\mathsf{T})S + S^\dagger(\not{G}^\mathsf{T} + \not{V}^\mathsf{T})(\sigma\cdot T) + 2i((\sigma\cdot T^\dagger)(\not{D}_V S) - (\not{D}_V S^\dagger)(\sigma\cdot T))](\not{\partial}\alpha)(1+\gamma^5)\}, \tag{4.34}$$

where we have introduced the shorthand notation

$$\sigma\cdot T \equiv \sigma_{\mu\nu}T^{\mu\nu}, \qquad \sigma\cdot T^\dagger \equiv \sigma_{\mu\nu}T^{\dagger\mu\nu}. \tag{4.35}$$

From the gauge transformation properties discussed earlier we see that

$$\mathcal{A}^\Lambda_{\beta=0}[\alpha]|_{\mathcal{O}(TSV^2)} = -\delta_\alpha \int \mathrm{d}^4x \Delta\mathcal{L}_{\mathrm{ct}}^{(TSV^2)} \tag{4.36}$$

is an irrelevant anomaly corresponding to the following local counterterm:

$$\Delta\mathcal{L}_{\mathrm{ct}}^{(TSV^2)} = \frac{1}{192\pi^2}\int \mathrm{d}^4x \mathrm{tr}\left\{\left[\frac{1}{2}((\sigma\cdot T^\dagger)(\not{G}^\mathsf{T} + \not{V}^\mathsf{T})S(\not{G}+\not{V}) + S^\dagger(\not{G}^\mathsf{T}+\not{V}^\mathsf{T})(\sigma\cdot T)(\not{G}+\not{V})) + 2i((\sigma\cdot T^\dagger)(\not{D}_V S) - (\not{D}_V S^\dagger)(\sigma\cdot T))\not{G}\right](1+\gamma^5)\right\}. \tag{4.37}$$

T^2V^2 terms: Finally, for the T^2V^2 terms, we find

$$\mathbf{tr}_1^{(T^2V^2)} = \mathrm{tr}\{T_R V_L T_L [V_R, \boldsymbol{\alpha}](1+\gamma^5)\}, \tag{4.38a}$$

$$\mathbf{tr}_2^{(T^2V^2)} = \mathrm{tr}\{T_R V_R T_L [V_R, \boldsymbol{\alpha}](1+\gamma^5)\}, \tag{4.38b}$$

$$\mathbf{tr}_3^{(T^2V^2)} = -\mathrm{tr}\{T_R\gamma_\mu T_L \gamma_\nu [V_R^\mu V_R^\nu, \boldsymbol{\alpha}](1+\gamma^5)\}, \tag{4.38c}$$

where we have used $\gamma^\mu T_L V_R \gamma_\mu = 2\gamma_\mu T_L V_R^\mu$ to simplify $\mathbf{tr}_3$. Further, since the Dirac traces involved are symmetric under the exchange of γ_μ and γ_ν,

$$\mathrm{tr}(\gamma_\mu\sigma_{\rho\tau}\gamma_\nu\sigma_{\kappa\lambda}) = \mathrm{tr}(\gamma_\nu\sigma_{\rho\tau}\gamma_\mu\sigma_{\kappa\lambda}), \tag{4.39a}$$

$$\mathrm{tr}(\gamma_\mu\sigma_{\rho\tau}\gamma_\nu\sigma_{\kappa\lambda}\gamma^5) = \mathrm{tr}(\gamma_\nu\sigma_{\rho\tau}\gamma_\mu\sigma_{\kappa\lambda}\gamma^5), \tag{4.39b}$$

we can freely interchange μ and ν in $\mathbf{tr}_3$ and obtain

$$\mathbf{tr}_3^{(T^2V^2)} = -\mathrm{tr}\{(T_R\gamma_\mu T_L V_R^\mu + V_R^\mu T_R \gamma_\mu T_L)[V_R, \boldsymbol{\alpha}](1+\gamma^5)\}. \tag{4.40}$$

Adding up all three traces and substituting in the expressions for $V_{L,R}$ and $T_{L,R}$ from Eq. (4.16), we get

$$\mathcal{A}^\Lambda_{\beta=0}[\alpha]|_{\mathcal{O}(T^2V^2)} = -\frac{1}{192\pi^2}\int \mathrm{d}^4x\, \mathrm{tr}\Big\{[(\sigma\cdot T^\dagger)(\not{G}^\mathsf{T} + \not{V}^\mathsf{T})(\sigma\cdot T) + (\sigma\cdot T^\dagger)\gamma_\mu(\sigma\cdot T)(G^\mu + V^\mu) + (G^\mu + V^\mu)(\sigma\cdot T^\dagger)\gamma_\mu(\sigma\cdot T) + i((\sigma\cdot T^\dagger)\overleftrightarrow{\not{D}}_V(\sigma\cdot T))](\not{\partial}\alpha)(1+\gamma^5)\Big\}, \tag{4.41}$$

where $((\sigma\cdot T^\dagger)\overleftrightarrow{\not{D}}_V(\sigma\cdot T)) = \sigma_{\rho\tau}\gamma_\mu\sigma_{\kappa\lambda}[T^{\dagger\rho\tau}(D_V^\mu T^{\kappa\lambda}) - (D_V^\mu T^{\dagger\rho\tau})T^{\kappa\lambda}]$. This again can be identified with the gauge variation of a local counterterm,

$$\mathcal{A}^\Lambda_{\beta=0}[\alpha]|_{\mathcal{O}(T^2V^2)} = -\delta_\alpha \int \mathrm{d}^4x \Delta\mathcal{L}_{\mathrm{ct}}^{(T^2V^2)}, \tag{4.42}$$

where

$$\Delta\mathcal{L}_{\mathrm{ct}}^{(T^2V^2)} = \frac{1}{192\pi^2}\int \mathrm{d}^4x\mathrm{tr}\left\{\left[\frac{1}{2}(\sigma\cdot T^\dagger)(\not{G}^\intercal + \not{V}^\intercal)(\sigma\cdot T)(\not{G}+\not{V}) + (\sigma\cdot T^\dagger)\gamma_\mu(\sigma\cdot T)\gamma_\nu(G^\mu+V^\mu)(G^\nu+V^\nu)\right.\right.$$
$$\left.\left. + i((\sigma\cdot T^\dagger)\overleftrightarrow{\not{D}}_V(\sigma\cdot T))\not{G}\right](1+\gamma^5)\right\}. \quad (4.43)$$

We therefore conclude that additional contributions to $\mathcal{A}^\Lambda_{\beta=0}[\alpha]$ remain irrelevant when tensor couplings are included.

D. Summary

To summarize, in this section we have completed the calculation of the regularized anomaly $\mathcal{A}^\Lambda_\beta[\alpha]$ in the presence of scalar, vector, and tensor couplings to fermion bilinears and found that, with the Wess-Zumino consistent scheme choice $\beta = 0$, the difference with respect to the minimal coupling case is an irrelevant anomaly,

$$\mathcal{A}^\Lambda_{\beta=0}[\alpha] = \mathcal{A}^{\Lambda,\mathrm{mc}}_{\beta=0}[\alpha] - \delta_\alpha\int \mathrm{d}^4x\Delta\mathcal{L}^\Lambda_{\mathrm{ct}}. \quad (4.44)$$

The corresponding local counterterm is

$$\Delta\mathcal{L}^\Lambda_{\mathrm{ct}} = \mathcal{L}^\Lambda_{\mathrm{ct},0}|_{G_\mu\to G_\mu+V_\mu} - \mathcal{L}^\Lambda_{\mathrm{ct},0} + \Delta\mathcal{L}^{(V)}_{\mathrm{ct}} + \Delta\mathcal{L}^{(S^2V^2)}_{\mathrm{ct}} + \Delta\mathcal{L}^{(TSV^2)}_{\mathrm{ct}} + \Delta\mathcal{L}^{(T^2V^2)}_{\mathrm{ct}}, \quad (4.45)$$

with $\mathcal{L}^\Lambda_{\mathrm{ct},0}$, $\Delta\mathcal{L}^{(V)}_{\mathrm{ct}}$, $\Delta\mathcal{L}^{(S^2V^2)}_{\mathrm{ct}}$, $\Delta\mathcal{L}^{(TSV^2)}_{\mathrm{ct}}$, and $\Delta\mathcal{L}^{(T^2V^2)}_{\mathrm{ct}}$ given by Eqs. (4.7), (4.13), (4.30), (4.37), and (4.43), respectively. This means that for the renormalized anomaly $\mathcal{A}_\beta[\alpha]$ defined in Eq. (3.14)

there exists a renormalization scheme where

$$\mathcal{A}_{\beta=0}[\alpha] = \mathcal{A}^{\mathrm{mc}}_{\beta=0}[\alpha]. \quad (4.46)$$

V. DISCUSSION AND FUTURE DIRECTIONS

In this paper, we generalized the CDE framework for computing anomalies in Ref. [11] to the case of relativistic EFTs with a general class of higher-dimensional operators. We systematically calculated the anomaly in this formalism and demonstrated explicitly that the additional contributions from higher-dimensional operators are irrelevant anomalies. This means, in particular, that the (relevant) anomaly cancellation condition in SMEFT including the aforementioned higher-dimensional operators is the same as that in the Standard Model.

Our calculation did not include higher-dimensional operators which involve derivatives acting on the fermions (beyond the kinetic term), such as

$$\epsilon^{ik}\epsilon^{jl}(H_iD_\mu H_j)(\ell_k^\intercal i\gamma^0\gamma^2D^\mu\ell_l), \qquad (H^\dagger D_\mu D_\nu H)(\bar{\ell}\gamma^\mu D^\nu\ell). \quad (5.1)$$

While there is no essential obstacle to incorporate them in our present formalism, the CDE calculation becomes more and more tedious with the inclusion of each derivative. Nevertheless, noting that the counterterms we found in Eqs. (4.30), (4.37), and (4.43) share similar structures, we are hopeful that there could be a more efficient framework that would make such a calculation more manageable and potentially also shed new light on the underlying structures of CDE. We plan to pursue this intriguing possibility in future work.

The master functional trace evaluated in this paper, Eq. (3.13), can also be relevant for certain EFT matching calculations, such as when integrating out heavy fermions that acquire masses from a Yukawa interaction via spontaneous symmetry breaking [24,25]. Modern EFT matching calculations are typically performed with dimensional regularization. However, we anticipate our regularization prescription, applied in exclusively $d = 4$ spacetime dimensions, should produce the same anomaly-related nondecoupling effects. We leave the exploration of this interesting question for future study.

ACKNOWLEDGMENTS

We thank Quentin Bonnefoy, Nathaniel Craig, Sungwoo Hong, Markus Luty, Aneesh Manohar, and Jérémie Quevillon for useful discussions. T. C. is supported by the U.S. Department of Energy under Grant No. DE-SC0011640. X. L. is supported by the U.S. Department of Energy under Grants No. DE-SC0009919 and No. DE-SC0011640. Z. Z. is supported by the U.S. Department of Energy under Grant No. DE-SC0011702. This work was performed in part at Aspen Center for Physics, which is supported by National Science Foundation Grant No. PHY-1607611.

[1] R. A. Bertlmann, *Anomalies in Quantum Field Theory* (Clarendon Press, 2001).
[2] A. Bilal, Lectures on anomalies, arXiv:0802.0634.
[3] Q. Bonnefoy, L. Di Luzio, C. Grojean, A. Paul, and A. N. Rossia, Comments on gauge anomalies at dimension-six in the Standard Model Effective Field Theory, J. High Energy Phys. 05 (2021) 153.
[4] F. Feruglio, A note on gauge anomaly cancellation in effective field theories, J. High Energy Phys. 03 (2021) 128.
[5] S. Marculescu and L. Mezincescu, Axial anomaly in non-renormalizable theories, Report No. IFIN-FT-162-1978.
[6] A. Manohar and G. W. Moore, Anomalous inequivalence of phenomenological theories, Nucl. Phys. **B243**, 55 (1984).
[7] J. Minn, J. Kim, and C.-k. Lee, Spinor loop anomalies with very general local fermion Lagrangians, Phys. Rev. D **35**, 1872 (1987).
[8] Y. Kim, P. Y. Pac, and H. K. Shin, Spinor loop anomalies in higher derivative theories, Phys. Rev. D **39**, 1251 (1989).
[9] J. Soto, Anomaly cancellation at finite cutoff, Phys. Rev. D **45**, 4621 (1992).
[10] J. A. Dixon, Anomalies, BRS Cohomology and Effective Theories, Phys. Rev. Lett. **67**, 797 (1991).
[11] T. Cohen, X. Lu, and Z. Zhang, Anomalies from the covariant derivative expansion, Phys. Rev. D **107**, 116015 (2023).
[12] M. K. Gaillard, The effective one loop Lagrangian with derivative couplings, Nucl. Phys. **B268**, 669 (1986).
[13] L.-H. Chan, Derivative Expansion for the One Loop Effective Actions With Internal Symmetry, Phys. Rev. Lett. **57**, 1199 (1986).
[14] O. Cheyette, Effective action for the standard model with large Higgs mass, Nucl. Phys. **B297**, 183 (1988).
[15] B. Henning, X. Lu, and H. Murayama, How to use the Standard Model effective field theory, J. High Energy Phys. 01 (2016) 023.
[16] B. Henning, X. Lu, and H. Murayama, One-loop matching and running with covariant derivative expansion, J. High Energy Phys. 01 (2018) 123.
[17] C. Cornella, F. Feruglio, and L. Vecchi, Gauge invariance and finite counterterms in chiral gauge theories, J. High Energy Phys. 02 (2023) 244.
[18] K. Fujikawa, Path Integral Measure for Gauge Invariant Fermion Theories, Phys. Rev. Lett. **42**, 1195 (1979).
[19] K. Fujikawa, Path integral for gauge theories with fermions, Phys. Rev. D **21**, 2848 (1980); Phys. Rev. D **22**, 1499(E) (1980).
[20] K. Fujikawa, On the evaluation of chiral anomaly in gauge theories with γ_5 couplings, Phys. Rev. D **29**, 285 (1984).
[21] K. Fujikawa and H. Suzuki, Path integrals and quantum anomalies, 10.1093/acprof:oso/9780198529132.001.0001 (2004).
[22] B. Grzadkowski, M. Iskrzynski, M. Misiak, and J. Rosiek, Dimension-six terms in the Standard Model Lagrangian, J. High Energy Phys. 10 (2010) 085.
[23] J. Wess and B. Zumino, Consequences of anomalous ward identities, Phys. Lett. **37B**, 95 (1971).
[24] E. D'Hoker and E. Farhi, Decoupling a fermion whose mass is generated by a Yukawa coupling: The general case, Nucl. Phys. **B248**, 59 (1984).
[25] E. D'Hoker and E. Farhi, Decoupling a fermion in the standard electroweak theory, Nucl. Phys. **B248**, 77 (1984).