

Reflected entropy and computable cross-norm negativity: Free theories and symmetry resolution

Clément Berthiere^{*} and Gilles Perez[†]

*Centre de Recherches Mathématiques, Université de Montréal, Montréal, Québec H3C 3J7, Canada
and Département de Physique, Université de Montréal, Montréal, Québec H3C 3J7, Canada*

 (Received 3 August 2023; accepted 8 September 2023; published 25 September 2023)

We investigate a separability criterion based on the computable cross-norm (CCNR), and a related quantity called the CCNR negativity. We introduce a *reflected* version of the CCNR negativity, and discuss its connection with other well-established entanglement-related quantities, namely the reflected entropy and the operator entanglement entropy. For free fermionic and bosonic theories, we derive exact formulas in terms of two-point correlation functions, which allow for systematic numerical investigations and, in principle, analytical treatments. For systems with a global $U(1)$ symmetry, we study the symmetry-resolved reflected entropy and CCNR negativity. We provide conformal field theory results for the charged moments in the case of adjacent intervals, finding perfect agreement with the numerics. We observe an equipartition of reflected entropies and CCNR negativities, both for free-fermions and free-boson models. The first charge-dependent corrections are conjectured for fermions and worked out from the conformal field theory calculations for bosons.

DOI: 10.1103/PhysRevD.108.054508

I. INTRODUCTION

A fundamental problem [1,2] in quantum information theory is the detection of entanglement for general quantum states. Since the pioneering work of Werner [3] concerning the problem of separability of mixed states, several criteria have been proposed to decide whether a given quantum state is entangled or not. Among them stand out the positive partial transposition (PPT) criterion [4,5] and the computable cross-norm or realignment (CCNR) criterion [6,7], providing simple necessary conditions for separability. These two independent criteria are based on permutations of density matrix elements [8,9], and neither is stronger in general.

Based on the PPT criterion, the PPT logarithmic negativity [10–12] is a well-established entanglement measure for bipartite mixed states, in particular in the context of quantum many-body systems [13–18]. Recently, a CCNR negativity based on the CCNR criterion has been discussed in conformal field theory (CFT) [19], holography [20] and topological systems [21].

Entanglement plays a prominent role in quantum many-body systems [22–24], particularly in relation to critical properties [25–34] and topological order [35–37]. In the context of quantum many-body systems with a global conserved charge, an important issue is to understand how each symmetry sector contributes to the total entanglement. This *symmetry resolution* of entanglement is the object of intense research, both theoretical and experimental [38–45]. Symmetry-resolved entanglement entropies have notably been investigated in the context of critical systems [39–41,46–53], integrable field theories and lattice models [54–59], topological phases [60–65] and quantum many-body systems out of equilibrium [66–74]. In addition, other entanglement-related quantities possess a meaningful symmetry resolution [45,75–82]. In this paper, we investigate the symmetry resolution of two related quantities, the CCNR negativity and reflected entropies, in the context of CFT and free theories.

This paper is organized as follows. We begin in Sec. II with the definition of the (Rényi) CCNR negativity, and its *reflected* generalization that we introduce. We discuss the relations between the Rényi CCNR negativities and two other important quantities, namely the reflected entropy and the operator entanglement entropy. In Sec. III, we derive general formulas for the Rényi reflected entropies and CCNR negativities, valid for free fermionic and bosonic fields in arbitrary dimensions. The resulting expressions are fully determined in terms of the two-point correlation functions, making them suitable for lattice calculations. We study the symmetry resolution of reflected

^{*}clement.berthiere@umontreal.ca

[†]gilles.perez@umontreal.ca

Published by the American Physical Society under the terms of the Creative Commons Attribution 4.0 International license. Further distribution of this work must maintain attribution to the author(s) and the published article's title, journal citation, and DOI. Funded by SCOAP³.

entropy and CCNR negativity in Sec. IV. We present CFT results for two adjacent regions, which we verify against numerical calculations for free fermions and bosons. We conclude in Sec. V with a summary of our main results, and give an outlook on future study. Finally, some technical details can be found in the three appendices that complete this work.

II. QUANTITIES OF INTEREST

In this section we define the CCNR negativity, the reflected entropy and the operator entanglement entropy. We discuss various connections between these quantities and their important properties.

A. Definition of the CCNR negativity

Let ρ_{AB} be the density matrix of a bipartite system $A \cup B$ in a general state. By definition, ρ_{AB} is a positive semi-definite Hermitian operator with trace one. We assume that the total Hilbert space factorizes as the product of local Hilbert spaces $\mathcal{H}_A \otimes \mathcal{H}_B$, and define $\{|a\rangle\}$ and $\{|b\rangle\}$ to be orthonormal basis states of \mathcal{H}_A and \mathcal{H}_B , respectively. With this choice of bases, the density matrix reads

$$\rho_{AB} = \sum_{a,a'} \sum_{b,b'} \langle ab | \rho_{AB} | a'b' \rangle | ab \rangle \langle a'b' |, \quad (1)$$

where $|ab\rangle \equiv |a\rangle \otimes |b\rangle$. We introduce the realignment matrix of ρ_{AB} , denoted $R \equiv R(\rho_{AB})$ as

$$R = \sum_{a,a'} \sum_{b,b'} \langle ab | \rho_{AB} | a'b' \rangle | aa' \rangle \langle bb' | \quad (2)$$

by swapping basis elements $|b\rangle$ and $|a'\rangle$ of ρ_{AB} . By definition, R is not necessarily a square matrix, though the product RR^\dagger is. The latter operator acts on $\mathcal{H}_A \otimes \mathcal{H}_A$,

$$RR^\dagger = \sum_{a,a'} \sum_{\tilde{a},\tilde{a}'} \text{Tr}_{\mathcal{H}_B} (\langle a | \rho_{AB} | a' \rangle \langle \tilde{a}' | \rho_{AB} | \tilde{a} \rangle) | aa' \rangle \langle \tilde{a}\tilde{a}' |. \quad (3)$$

For separable states, $\|R\| = \text{Tr}(\sqrt{RR^\dagger}) \leq 1$, which constitutes the CCNR separability criterion [6,7]. A state is thus guaranteed to be entangled if $\|R\| > 1$.

Analogously to the PPT logarithmic negativity, the CCNR negativity is defined as

$$\mathcal{E} = \log \text{Tr}(\sqrt{RR^\dagger}). \quad (4)$$

The trace norm $\|R\|$ is challenging to compute in practice. Fortunately, a replica formulation of (4) can be implemented [19]. The replica method relates the trace norm of R to the moments of RR^\dagger ,

$$Z_n = \text{Tr}(RR^\dagger)^n, \quad (5)$$

where $\|R\|$ is obtained by analytic continuation from integer values n to $1/2$. A Rényi generalization of (4) can be defined as

$$\mathcal{E}_n = \log \text{Tr}(RR^\dagger)^n, \quad (6)$$

such that $\mathcal{E} = \lim_{n \rightarrow 1/2} \mathcal{E}_n$. For $n = 1$ we retrieve the purity $\text{Tr}(RR^\dagger) = \text{Tr}\rho_{AB}^2$ such that $\mathcal{E}_1 = -S_2(\rho_{AB})$ where $S_n(\rho) = (1-n)^{-1} \log \text{Tr}\rho^n$ is the Rényi entropy of ρ .

For pure states, it is straightforward to show that

$$\mathcal{E}_n = 2 \log \text{Tr}\rho_A^n = 2(1-n)S_n(\rho_A), \quad (7)$$

where $\rho_A = \text{Tr}_{\mathcal{H}_B} \rho_{AB}$, such that the CCNR negativity is the Rényi entropy of order $1/2$, $\mathcal{E} = S_{1/2}(\rho_A)$. For simple separable states, $\rho_{AB} = \sum_k p_k \rho_A^{(k)} \otimes \rho_B^{(k)}$ where $\rho_B^{(k)}$ are projectors with orthogonal support, we have

$$\mathcal{E}_n = \log \left(\sum_k p_k^{2n} \right), \quad (8)$$

which yields a vanishing CCNR negativity for $n = 1/2$ since $\sum_k p_k = 1$, in accordance with the criterion $\mathcal{E} \leq 0$ for separable states.

B. A generalization of the realignment matrix

We may generalize the realignment matrix of ρ_{AB} to that of $\rho_{AB}^{m/2}$ with $m \in \mathbb{Z}^+$ as

$$R_m = \sum_{a,a'} \sum_{b,b'} \langle ab | \rho_{AB}^{m/2} | a'b' \rangle | aa' \rangle \langle bb' |. \quad (9)$$

For $m = 2$ it reduces to R . Similarly to the special case $m = 2$ discussed above, we introduce the moments

$$Z_{m,n} = \text{Tr}(R_m R_m^\dagger)^n, \quad (10)$$

and define the (m, n) -Rényi CCNR negativity as

$$\mathcal{E}_{m,n} = \log \text{Tr}(R_m R_m^\dagger)^n, \quad (11)$$

such that $\mathcal{E}_{2,n} \equiv \mathcal{E}_n$. For pure states, it still holds that $\mathcal{E}_{m,n} = 2(1-n)S_n(\rho_A)$, irrespective of m . Moreover, we have the relation $\text{Tr}(R_m R_m^\dagger) = \text{Tr}\rho_{AB}^m$ such that $\mathcal{E}_{m,1} = (1-m)S_m(\rho_{AB})$.

C. Relation to reflected entropy and operator entanglement entropy

In this subsection, we discuss the relations between three information theoretic quantities of recent interest: the CCNR negativity, the reflected entropy and the operator entanglement entropy.

The reflected entropy [83] has attracted much attention [84–96] since its introduction as a correlation measure for mixed states in the holographic context. However, it has recently been shown that the reflected entropy is not a measure of physical correlations in general, since it is not monotonically nonincreasing under partial trace [97]. Nevertheless, this quantity has a meaningful relationship to entanglement, in particular with tripartite entanglement [98,99] (see also Appendix C), and with mutual information [88,100]. To define the reflected entropy, we consider a purification of ρ_{AB}^m with $m \in 2\mathbb{Z}^+$, denoted $|\Omega_m\rangle$, such that $\text{Tr}_{\mathcal{H}_A \otimes \mathcal{H}_B}(|\Omega_m\rangle\langle\Omega_m|) = \frac{1}{\text{Tr}\rho_{AB}^m} \rho_{AB}^m$. It can be constructed using the Choi-Jamiołkowski isomorphism [101,102] as

$$|\Omega_m\rangle = \frac{1}{\sqrt{\text{Tr}\rho_{AB}^m}} \sum_{a,a'} \sum_{b,b'} \langle ab|\rho_{AB}^{m/2}|a'b'\rangle |ab\rangle \otimes |a'b'\rangle, \quad (12)$$

by doubling the original Hilbert space. A replica formulation of the reflected entropy gives a practical handle for computations. This replica trick involves two replica indices, m and n . The latter represents the usual Rényi index while the former is related to the purification. One then defines the reflected density matrix $\rho_{AA}^{(m)} = \text{Tr}_{\mathcal{H}_B \otimes \mathcal{H}_B}(|\Omega_m\rangle\langle\Omega_m|)$, and introduces the (m, n) -Rényi reflected entropy as

$$S_{m,n}^R = \frac{1}{1-n} \log \text{Tr}(\rho_{AA}^{(m)})^n. \quad (13)$$

The n -Rényi reflected entropy is recovered by analytic continuation $m \rightarrow 1$, and the reflected entropy by further taking the limit $n \rightarrow 1$.

The reflected density matrix $\rho_{AA}^{(m)}$, constructed using the Choi-Jamiołkowski isomorphism as described above, can also be defined using the realignment matrix of $\rho_{AB}^{m/2}$ [see (9)] as

$$\rho_{AA}^{(m)} = \frac{1}{\text{Tr}\rho_{AB}^m} R_m R_m^\dagger. \quad (14)$$

Comparing the (m, n) -Rényi CCNR negativity in (11) with the (m, n) -Rényi reflected entropy in (13), we have

$$\mathcal{E}_{m,n} = (1-n)S_{m,n}^R + n(1-m)S_m(\rho_{AB}), \quad (15)$$

such that setting $m = 2$ we get the Rényi CCNR negativity \mathcal{E}_n . The (m, n) -Rényi CCNR negativity can be viewed as the unnormalized (m, n) -Rényi reflected entropy. With expression (15), it can be checked that, for topological systems, the results of [21] on CCNR negativity match earlier work on reflected entropy [87].

The operator entanglement entropy of a density matrix [45,103–105], denoted E_n (for its Rényi generalization), is the (Rényi) Shannon entropy of the squared probability distribution values of the operator Schmidt decomposition

of that density matrix. It is direct to realize that the reflected entropy is the operator entanglement entropy of $\sqrt{\rho_{AB}}$, that is

$$E_n = \frac{1}{1-n} \log \text{Tr}(\rho_{AA}^{(2)})^n = S_{2,n}^R, \quad (16)$$

implying that $\mathcal{E}_n = (1-n)E_n - nS_2(\rho_{AB})$.

D. CCNR negativity is not a correlation measure

Here we point out that, contrary to what has been commonly stated in the recent literature, the CCNR negativity, similarly to the reflected entropy [97], is not a correlation measure.

Beside the nonnegativeness required of a bipartite correlation measure $\mathcal{C}(A:B)$, which is clearly not satisfied by the (Rényi) CCNR negativity, one fundamental requirement is that correlation cannot increase under local operations. This means that a correlation measure should not increase under the local discarding of information, i.e. the following inequality must hold

$$\mathcal{C}(A:B \cup C) \geq \mathcal{C}(A:B). \quad (17)$$

A quantity satisfying this inequality is said to be monotonically nonincreasing under partial trace. We prove in Appendix A that the Rényi CCNR negativity is *not* monotonically nonincreasing under partial trace in the range $n \in (0, 1) \cup (1, \infty)$, by providing counterexamples to inequality (17). As a byproduct, we prove that the (m, n) -Rényi reflected entropy $S_{m,n}^R$, which reduces to the Rényi operator entanglement entropy E_n for $m = 2$, is not monotonically nonincreasing under partial trace in the range $m > 0$ and $n \in (0, 2)$. We thus conclude that the CCNR negativity and operator entanglement entropies are not correlation measures.

Nevertheless, the CCNR negativity provides a useful entanglement witness, which is computable for quantum many-body states and exhibits new universal features [19]. Moreover, the operator entanglement entropy is a key quantifier of the complexity of a reduced density matrix, notably in out-of-equilibrium situations, and it can be measured experimentally [45].

III. REFLECTED ENTROPIES AND CCNR NEGATIVITY FOR FREE THEORIES

In this section, we compute the (m, n) -Rényi reflected entropies and CCNR negativities for Gaussian (free) fermionic and bosonic systems in any dimension from the two-point correlation functions of the theory. Doing so, we generalize the results of [85,88] for the $m = 1$ reflected entropy to arbitrary m .

A. Free fermions

Consider the (diagonalized) fermionic Gaussian state

$$\begin{aligned}\rho_{AB} &= \bigotimes_k \frac{e^{-\epsilon_k c_k^\dagger c_k}}{1 + e^{-\epsilon_k}}, \\ &= \bigotimes_k [(1 - n_k)|0_k\rangle\langle 0_k| + n_k|1_k\rangle\langle 1_k|],\end{aligned}\quad (18)$$

where $e^{-\epsilon_k} = n_k/(1 - n_k)$ with n_k being the occupation number at a given wave vector k , and the c_k s are fermionic operators satisfying $\{c_i, c_j^\dagger\} = \delta_{ij}$. The generalized purification $|\Omega_m\rangle$, see (12), can be constructed as

$$|\Omega_m\rangle = \frac{1}{\sqrt{\text{Tr}\rho_{AB}^m}} \bigotimes_k [(1 - n_k)^{m/2} + n_k^{m/2} c_k^\dagger \tilde{c}_k^\dagger] |0_k\rangle |\tilde{0}_k\rangle, \quad (19)$$

where \tilde{c}_k are copies of c_k introduced in the vectorization process. The correlation matrix of the state $|\Omega_m\rangle$ reads

$$\begin{aligned}\mathbf{C}_{kk'}^{(m)} &= \langle \Omega_m | \begin{pmatrix} c_k^\dagger \\ \tilde{c}_k \end{pmatrix} (c_{k'} \quad \tilde{c}_{k'}^\dagger) | \Omega_m \rangle, \\ &= \frac{\delta_{kk'}}{n_k^m + (1 - n_k)^m} \begin{pmatrix} n_k^m & (n_k(1 - n_k))^{m/2} \\ (n_k(1 - n_k))^{m/2} & (1 - n_k)^m \end{pmatrix}.\end{aligned}\quad (20)$$

Performing a Fourier transform leads to the correlation matrix in the spatial basis

$$\mathbf{C}^{(m)} = \begin{pmatrix} \frac{C_{AB}^m}{C_{AB}^m + (1 - C_{AB})^m} & \frac{(C_{AB}(1 - C_{AB}))^{m/2}}{C_{AB}^m + (1 - C_{AB})^m} \\ \frac{(C_{AB}(1 - C_{AB}))^{m/2}}{C_{AB}^m + (1 - C_{AB})^m} & \frac{(1 - C_{AB})^m}{C_{AB}^m + (1 - C_{AB})^m} \end{pmatrix}, \quad (21)$$

where C_{AB} is the two-point correlation matrix for ρ_{AB} . Introducing $\mathbf{C}_A^{(m)}$ as the restriction of $\mathbf{C}^{(m)}$ to subsystem A, the Rényi reflected entropy $S_{m,n}^R$ is obtained as

$$S_{m,n}^R = \frac{1}{1 - n} \text{Tr} \log [(\mathbf{C}_A^{(m)})^n + (1 - \mathbf{C}_A^{(m)})^n]. \quad (22)$$

For $m = 1$, this formula reduces to that for the n -Rényi reflected entropy derived in [85], while we recover the operator entanglement entropy [45] for $m = 2$. The (m, n) -Rényi CCNR is obtained using (15), explicitly

$$\begin{aligned}\mathcal{E}_{m,n} &= \text{Tr} \log [(\mathbf{C}_A^{(m)})^n + (1 - \mathbf{C}_A^{(m)})^n] \\ &\quad + n \text{Tr} \log [C_{AB}^m + (1 - C_{AB})^m].\end{aligned}\quad (23)$$

For $n = 1$, we recover $\mathcal{E}_{m,1} = \log \text{Tr}\rho_{AB}^m$. Moreover, for ρ_{AB} pure we get $\mathcal{E}_{m,n} = 2 \log \text{Tr}\rho_A^m$, in agreement with (7). Indeed, in that case we have $C_{AB}^2 = C_{AB}$ such that

$\mathbf{C}_A^{(m)} = C_A \oplus (1 - C_A)$, where C_A is the correlation matrix C_{AB} restricted to region A.

B. Free bosons

For Gaussian scalar theories invariant under time reflection, the relevant two-point functions to consider are that of the scalar and the conjugate momentum. Let ϕ_i and π_i , $i = 1, \dots, N$, be a system of scalars and conjugate momenta acting on the Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B$. These Hermitian operators satisfy the canonical commutation relations $[\phi_i, \pi_j] = i\delta_{ij}$ and $[\phi_i, \phi_j] = [\pi_i, \pi_j] = 0$.

Given a density matrix ρ_{AB} acting on $\mathcal{H}_A \otimes \mathcal{H}_B$, a purification $|\Omega_m\rangle$ of ρ_{AB}^m can be constructed using the Choi-Jamiolkowski isomorphism by doubling the original Hilbert space of ρ_{AB} and extending accordingly the bosonic algebra with additional operators $\tilde{\phi}_i$ and $\tilde{\pi}_i$ acting on the second copy of $\mathcal{H}_A \otimes \mathcal{H}_B$. We refer the reader to [88] for details on the doubling of Hilbert space for scalar fields. As conveniently done in [88], let us define $\Psi_i^0 \equiv \phi_i$, $\Psi_i^1 \equiv \pi_i$, and similarly for $\tilde{\Psi}_i^a$, $a = 0, 1$. We find that the correlators for $|\Omega_m\rangle$ read

$$\begin{aligned}\langle \Omega_m | \Psi_{i_1}^{a_1} \dots \Psi_{i_p}^{a_p} \tilde{\Psi}_{j_1}^{b_1} \dots \tilde{\Psi}_{j_q}^{b_q} | \Omega_m \rangle \\ = \frac{(-1)^{\sum_q b_q}}{\text{Tr}\rho_{AB}^m} \text{Tr}(\rho_{AB}^{m/2} \Psi_{i_1}^{a_1} \dots \Psi_{i_p}^{a_p} \rho_{AB}^{m/2} \tilde{\Psi}_{j_q}^{b_q} \dots \tilde{\Psi}_{j_1}^{b_1}).\end{aligned}\quad (24)$$

Organizing the scalars in a single field Φ where $\Phi_i \equiv \phi_i$ and $\Phi_{i+N} \equiv \tilde{\phi}_i$ with $i = 1, \dots, N$, and similarly for the momenta Π_i , we are interested in the following two-point correlation functions:

$$\Phi_{kl}^{(m)} = \langle \Omega_m | \Phi_k \Phi_l | \Omega_m \rangle, \quad \Pi_{kl}^{(m)} = \langle \Omega_m | \Pi_k \Pi_l | \Omega_m \rangle, \quad (25)$$

with $k, l = 1, \dots, 2N$. These two-point functions have been computed in [88] using (24) derived for $m = 1$. They depend on the density matrix ρ_{AB} for the first N scalars only. Performing the traces, we obtain

$$\Phi^{(m)} = \begin{pmatrix} f_m(XP)X & g_m(XP)X \\ g_m(XP)X & f_m(XP)X \end{pmatrix}, \quad (26)$$

$$\Pi^{(m)} = \begin{pmatrix} Pf_m(XP) & -Pg_m(XP) \\ -Pg_m(XP) & Pf_m(XP) \end{pmatrix}, \quad (27)$$

where we defined

$$\begin{aligned}f_m(M) &= \frac{1(\sqrt{M} + 1/2)^m + (\sqrt{M} - 1/2)^m}{2(\sqrt{M} + 1/2)^m - (\sqrt{M} - 1/2)^m} \sqrt{M}^{-1}, \\ g_m(M) &= \frac{(M - 1/4)^{m/2}}{(\sqrt{M} + 1/2)^m - (\sqrt{M} - 1/2)^m} \sqrt{M}^{-1},\end{aligned}\quad (28)$$

and $X_{ij} = \text{Tr}(\rho_{AB}\phi_i\phi_j)$ and $P_{ij} = \text{Tr}(\rho_{AB}\pi_i\pi_j)$ are the usual two-point correlation functions. The (m, n) -Rényi reflected entropy follows from the correlation matrix

$$\mathbf{C}_A^{(m)} = \sqrt{\Phi_A^{(m)} \Pi_A^{(m)}}, \text{ i.e.}$$

$$S_{m,n}^R = \frac{1}{n-1} \text{Tr} \log [(\mathbf{C}_A^{(m)} + 1/2)^n - (\mathbf{C}_A^{(m)} - 1/2)^n]. \quad (29)$$

For $m = 1$, this formula reduces to that for the n -Rényi reflected entropy derived in [88]. For $m = 2$, we have the operator entanglement entropy which was considered for free fermions only in [45]. The (m, n) -Rényi CCNR is obtained using (15), explicitly

$$\begin{aligned} \mathcal{E}_{m,n} = & -\text{Tr} \log [(\mathbf{C}_A^{(m)} + 1/2)^n - (\mathbf{C}_A^{(m)} - 1/2)^n] \\ & - n \text{Tr} \log [(C_{AB} + 1/2)^m - (C_{AB} - 1/2)^m], \end{aligned} \quad (30)$$

where $C_{AB} = \sqrt{XP}$. For ρ_{AB} pure, we recover $\mathcal{E}_{m,n} = 2 \log \text{Tr} \rho_A^n$ since $C_{AB} = 1/2$ such that $\mathbf{C}_A^{(m)} = C_A \oplus C_A$, with $C_A = \sqrt{X_A P_A}$.

IV. SYMMETRY RESOLUTION

In this section we investigate the symmetry resolution of the reflected entropies and the CCNR negativity in the context of quantum many-body systems with a global conserved $U(1)$ charge Q .

A. Definitions

Let us consider a state ρ_{AB} which commutes with a global charge $Q = Q_A + Q_B$, and we further assume $Q = Q^T$. Here, Q_A and Q_B are the local charges for each subsystem. As we show in Appendix B (see also [20,45]), the reflected density matrix $\rho_{AA}^{(m)}$ commutes with the charge imbalance operators of the two copies of \mathcal{H}_A ,

$$[Q_A, \rho_{AA}^{(m)}] = 0, \quad (31)$$

where

$$Q_A = Q_A \otimes \mathbb{I}_A - \mathbb{I}_A \otimes Q_A. \quad (32)$$

The commutation relation (31) implies that $\rho_{AA}^{(m)}$ has a block-diagonal structure where the blocks pertain to different eigenvalues q of Q_A . We have

$$\rho_{AA}^{(m)} = \bigoplus_q (p_q^{(m)} \rho_{AA}^{(m)}(q)) \quad (33)$$

with $\sum_q p_q^{(m)} = 1$ and $\text{Tr}(\rho_{AA}^{(m)}(q)) = 1$. Here $p_q^{(m)}$ is the probability of measuring the eigenvalue q of the charge Q_A in the state $\rho_{AA}^{(m)}$ and $\rho_{AA}^{(m)}(q)$ is the reflected density matrix of the charge sector q .

The symmetry-resolved (m, n) -Rényi reflected entropies are defined as

$$S_{m,n}^R(q) = \frac{1}{1-n} \log \text{Tr}(\rho_{AA}^{(m)}(q))^n. \quad (34)$$

Using (16), the symmetry-resolved operator entanglement entropies read

$$E_n(q) = S_{2,n}^R(q), \quad (35)$$

and this definition corresponds to that introduced in [45].

Similarly as for the standard entanglement entropy, there is a simple relation between the total reflected entanglement entropy and the symmetry-resolved ones,

$$S_{m,1}^R = \sum_q p_q^{(m)} S_{m,1}^R(q) - \sum_q p_q^{(m)} \log p_q^{(m)}. \quad (36)$$

By analogy with entanglement entropy [42], we call the first term the configurational reflected entropy, and the second one the number reflected entropy. There is a similar, but more cumbersome, relation for reflected entropies with Rényi index $n \neq 1$ that we do not reproduce here.

Combining the relation (14) and the commutation relation (31), it follows that $[Q_A, R_m R_m^\dagger] = 0$, and hence $R_m R_m^\dagger$ also has a block-diagonal structure. However, since the matrix $R_m R_m^\dagger$ does not have unit trace, there is no natural (normalized) definition for the symmetry-resolved (m, n) -CCNR negativity. By analogy with (15), we choose to define it as

$$\mathcal{E}_{m,n}(q) = (1-n) S_{m,n}^R(q) + n(1-m) S_m(\rho_{AB}), \quad (37)$$

and the symmetry-resolved CCNR negativity is obtained by setting $m = 2$ and taking the limit $n \rightarrow 1/2$. The relation between total and symmetry-resolved CCNR negativities reads

$$e^{\mathcal{E}_{m,n}} = \sum_q (p_q^{(m)})^n e^{\mathcal{E}_{m,n}(q)}, \quad (38)$$

similarly to the charge-imbalance-resolved PPT logarithmic negativity [70,75,76].

B. Charged moments

Extracting the contribution of each charge sector to the total entanglement is a challenging task, since, in principle, it requires the knowledge of the total density matrix. Fortunately, a way to circumvent this issue was proposed in [39], where the idea is to compute so-called charged moments and their Fourier transform. We adapt these ideas to the reflected density matrix and define the corresponding charged moments as

$$Z_{m,n}^R(\alpha) = \text{Tr}((\rho_{AA}^{(m)})^n e^{i\alpha Q_A}). \quad (39)$$

Taking the Fourier transform yields the symmetry-resolved moments,

$$\mathcal{Z}_{m,n}^R(q) = \int_{-\pi}^{\pi} \frac{d\alpha}{2\pi} e^{-iq\alpha} Z_{m,n}^R(\alpha), \quad p_q^{(m)} = \mathcal{Z}_{m,1}^R(q), \quad (40)$$

and the symmetry-resolved Rényi reflected entropies are

$$S_{m,n}^R(q) = \frac{1}{1-n} \log \left(\frac{\mathcal{Z}_{m,n}^R(q)}{(\mathcal{Z}_{m,1}^R(q))^n} \right). \quad (41)$$

Taking $m = 2$ in (41) yields the symmetry-resolved operator entanglement entropies [45], while taking the limit $m \rightarrow 1$ yields the symmetry-resolved reflected entropies.

For the CCNR Rényi negativity, the natural charged moments to consider are [20]

$$\begin{aligned} Z_{m,n}(\alpha) &= \text{Tr}((R_m R_m^\dagger)^n e^{i\alpha Q_A}), \\ &= Z_{m,n}^R(\alpha) (\text{Tr} \rho_{AB}^m)^n, \end{aligned} \quad (42)$$

and their Fourier transform yields the symmetry-resolved CCNR moments

$$\mathcal{Z}_{m,n}(q) = \int_{-\pi}^{\pi} \frac{d\alpha}{2\pi} e^{-iq\alpha} Z_{m,n}(\alpha). \quad (43)$$

We have

$$\begin{aligned} \mathcal{E}_{m,n}(q) &= \log \left(\frac{\mathcal{Z}_{m,n}(q)}{(p_q^{(m)})^n} \right), \\ &= (1-n) S_{m,n}^R(q) + n(1-m) S_m(\rho_{AB}). \end{aligned} \quad (44)$$

C. Conformal field theory results

In the context of one-dimensional quantum critical system described by a CFT, a large body of results regarding entanglement measures can be obtained by means of the replica trick [13,23,26] and twist fields correlation functions [106]. Reflected entropies and the CCNR partition functions $Z_{m,n}$, see (10), can also be computed using similar methods [19,83].

We take the regions A and B to be adjacent segments of respective lengths ℓ_a and ℓ_b , see Fig. 1. Introducing the relevant twist operator conformal weights [83,106]

$$\begin{aligned} h_n &= \frac{c}{24} \left(n - \frac{1}{n} \right), \\ h'_{m,n} &= n h_m = n \frac{c}{24} \left(m - \frac{1}{m} \right), \end{aligned} \quad (45)$$

where c is the central charge of the CFT, the partition function $Z_{2,n}$ for two adjacent intervals can be expressed as [19]

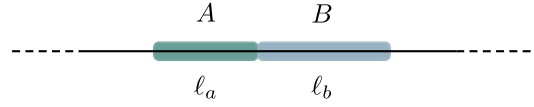


FIG. 1. Two adjacent intervals A and B of length ℓ_a and ℓ_b , respectively, in an infinite system.

$$Z_{2,n} \propto (\ell_a \ell_b)^{-4h_n} (\ell_a + \ell_b)^{4h_n - 4h'_{2,n}}, \quad (46)$$

and for the more generic reflected case we find

$$Z_{m,n} \propto (\ell_a \ell_b)^{-4h_n} (\ell_a + \ell_b)^{4h_n - 4h'_{m,n}}. \quad (47)$$

To investigate charged CCNR moments, we follow the approach developed in the context of charged [107–109] and symmetry-resolved Rényi entropies [39,41,55], where charged moments correspond to partition functions on multisheeted Riemann surfaces with twisted boundary conditions, such that the total phase accumulated upon going through the entire surface is α . Entanglement-related quantities are then obtained as correlation functions of modified twist fields. In particular, the partition function $Z_{2,n}(\alpha)$ corresponds to a correlation function of modified twist fields in $2n$ copies of the original theory. These are operators obtained by the fusion of standard twist fields of weight h_n with operators generating the flux α , with weight h_α . The resulting composite twist fields behave as primaries with weight

$$h_n(\alpha) = h_n + \frac{h_\alpha}{n}. \quad (48)$$

As an example, for the ground state of a Luttinger liquid with parameter K , the weight h_α reads [39,41]

$$h_\alpha = \frac{K}{2} \left(\frac{\alpha}{2\pi} \right)^2. \quad (49)$$

To maintain charge neutrality in the correlation function corresponding to $Z_{2,n}(\alpha)$, fields with charge α are inserted on odd copies, and fields with opposite charge are inserted on even copies. In the case of adjacent intervals, twist fields at the ends of the interval $A \cup B$ connect copies of different parities, and therefore fields with opposite charge cancel. However, at the interface between A and B , odd and even copies are decoupled [19] and hence the only charge dependence arises from that part. Generalizing the argument to arbitrary reflected index m , we find

$$Z_{m,n}(\alpha) \propto \left(\frac{\ell_a \ell_b}{\ell_a + \ell_b} \right)^{-4h_n(\alpha)} (\ell_a + \ell_b)^{-4nh_m(0)}, \quad (50)$$

where we assumed $h_\alpha = \bar{h}_\alpha$ and $h_m(0) = h_m$. We note that the reflected moments $Z_{m,n}^R(\alpha)$ are simply obtained by

setting $h'_{m,n} \rightarrow 0$ in (47), and the charged reflected moments are by retaining the first factor only in (50),

$$Z_{m,n}^R(\alpha) \propto \left(\frac{\ell_a \ell_b}{\ell_a + \ell_b} \right)^{-4h_n(\alpha)}. \quad (51)$$

For $\alpha = 0$, we recover the known result for the reflected moments [83]. In particular, the reflected entropies do not depend on m at leading order for adjacent regions.

D. Symmetry resolution for fermions

For critical free fermions ($c = 1$, $K = 1$) in one dimension, we have [39,41]

$$h_\alpha = \frac{1}{2} \left(\frac{\alpha}{2\pi} \right)^2, \quad (52)$$

finding

$$Z_{m,n}(\alpha) \propto \left(\frac{\ell_a \ell_b}{\ell_a + \ell_b} \right)^{-4h_n^f(\alpha)} (\ell_a + \ell_b)^{-4nh_m^f(0)} \quad (53)$$

with

$$h_n^f(\alpha) = \frac{1}{24} \left(n - \frac{1}{n} \right) + \frac{1}{2n} \left(\frac{\alpha}{2\pi} \right)^2. \quad (54)$$

To check our CFT predictions, we consider the one-dimensional tight-binding model with Hamiltonian

$$H = - \sum_{j=1}^L (c_{j+1}^\dagger c_j + \text{H.c.}). \quad (55)$$

Here, c_j^\dagger, c_j are fermion creation and annihilation operators, L is the length of the chain and we assume periodic boundary conditions. This model has a $U(1)$ symmetry generated by the charge operator $Q = \sum_j c_j^\dagger c_j$, and its diagonalization is standard. The ground-state two-point correlation functions in the large- L limit reads

$$C_{ij} = \frac{\sin(\frac{\pi}{2}(i-j))}{\pi(i-j)}. \quad (56)$$

Using results from Sec. III A and standard free-fermion techniques [39,110,111], we express the charged moments $Z_{m,n}(\alpha)$ defined in (42) in terms of the two-point correlation matrix as

$$\log Z_{m,n}(\alpha) = \text{Tr} \log [(C_A^{(m)})^n e^{i\alpha} + (1 - C_A^{(m)})^n] + n \text{Tr} \log [C_{AB}^m + (1 - C_{AB}^m)^m]. \quad (57)$$

We note that the logarithmic charged moments have a trivial imaginary part, $\text{Im}(\log Z_{m,n}(\alpha)) = \alpha \ell_a$. We thus redefine the fermionic charged moments as

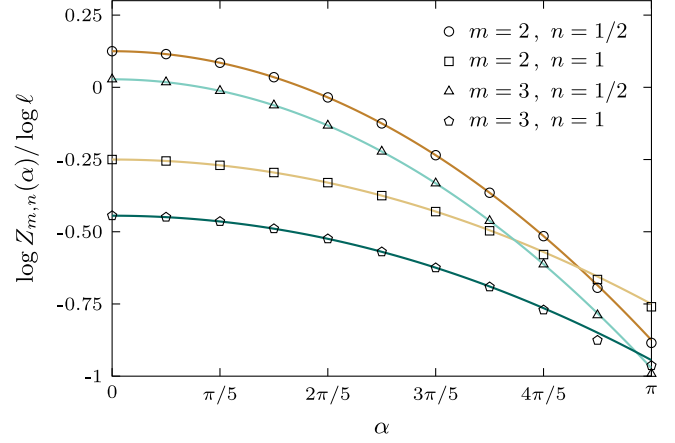


FIG. 2. Charged CCNR moments for free critical fermions on the infinite chain for two adjacent intervals of same length ℓ . The solid lines correspond to the analytical prediction $-4(h_n^f(\alpha) + nh_m^f(0))$ where $h_n^f(\alpha)$ is given in (54).

$$Z_{m,n}(\alpha) \rightarrow e^{-i\alpha \ell_a} Z_{m,n}(\alpha) \quad (58)$$

so that $\log Z_{m,n}(\alpha)$ is real. This redefinition is merely a convention which sets the average value of the charge imbalance to zero.

With all this in hand, we now investigate the charged CCNR moments $Z_{m,n}(\alpha)$ for adjacent intervals of equal length ℓ embedded in an infinite chain. In the limit of large ℓ we expect

$$\log Z_{m,n}(\alpha) = -4(h_n^f(\alpha) + nh_m^f(0)) \log \ell + \dots \quad (59)$$

where $h_n^f(\alpha)$ is given in (54). We compare this CFT prediction with exact numerical diagonalization in Fig. 2 and find a perfect agreement. Each numerical point is obtained by fitting the leading coefficient of $\log Z_{m,n}(\alpha)$ for increasing values of ℓ and fixed m, n, α .

Computing the Fourier transform of (53) yields the symmetry-resolved CCNR and reflected moments as

$$\mathcal{Z}_{m,n}^{(R)}(q) = Z_{m,n}^{(R)}(0) \sqrt{\frac{n\pi}{2 \log(\frac{\ell_a \ell_b}{\ell_a + \ell_b})}} \exp\left(-\frac{n\pi^2 q^2}{2 \log(\frac{\ell_a \ell_b}{\ell_a + \ell_b})}\right), \quad (60)$$

valid both for $\mathcal{Z}_{m,n}(q)$ and $\mathcal{Z}_{m,n}^{(R)}(q)$, since these charged moments have same α dependence. Expression (60) being exact at leading order, we do not check it numerically.

As both symmetry-resolved CCNR negativity $\mathcal{E}_{m,n}(q)$ and reflected entropies $S_{m,n}^R(q)$ have the same charge dependence, see (37), we focus our discussion on the latter. Combining (41) and (60), we find

$$S_{m,n}^R(q) = S_{m,n}^R - \frac{1}{2} \log \log \left(\frac{\ell_a \ell_b}{\ell_a + \ell_b} \right) + \frac{1}{2} \log \frac{\pi n^{1/(1-n)}}{2} + \dots, \quad (61)$$

where $S_{m,n}^R$ is the total (m, n) -Rényi reflected entropy. This result implies that, at leading order, the symmetry-resolved reflected entropies do not depend on the charge sector. This phenomena has been observed for symmetry-resolved entanglement entropies in numerous physical situations, and has been dubbed *equipartition of entanglement* [40]. Of course, (61) is a CFT prediction and thus holds in the large- ℓ limit. At finite size, we expect a subleading q -dependent correction. For the entanglement entropies this correction is of order $(q/\log \ell)^2$, see [41], and we expect a similar behavior for the reflected entropies. It is an important open problem to perform exact finite-size calculation of symmetry-resolved reflected entropies to determine the charge-dependent corrections. We leave this issue for further investigations.

E. Symmetry resolution for complex bosons

Consider a complex scalar field φ of mass μ with Lagrangian density $\mathcal{L} = \partial_\nu \varphi^\dagger \partial^\nu \varphi - \mu^2 \varphi^\dagger \varphi$. This Lagrangian exhibits a $U(1)$ symmetry, i.e. it is invariant under phase transformations of the field $\varphi \rightarrow e^{i\theta} \varphi$. The complex scalar field can be written in terms of two real ones defining $\varphi = (\varphi_1 + i\varphi_2)/\sqrt{2}$. Its Hamiltonian H is the sum of two identical Hamiltonians corresponding to each real scalars, $H = \sum_{i=1,2} \frac{1}{2} \int dx (\pi_i^2 + (\partial_x \varphi_i)^2 + \mu^2 \varphi_i^2)$. Introducing the creation and annihilation operators $a_i^\dagger(p), a_i(p)$ with momentum p for each real scalars $i = 1, 2$, the Hamiltonian and the conserved charge Q associated to the $U(1)$ symmetry can be written in terms of particles and antiparticles modes operators $a_p = (a_1(p) + ia_2(p))/\sqrt{2}$ and $b_p = (a_1^\dagger(p) + ia_2^\dagger(p))/\sqrt{2}$:

$$H = \int \frac{dp}{2\pi} \omega_p (a_p^\dagger a_p + b_p^\dagger b_p), \quad (62)$$

$$Q = \int \frac{dp}{2\pi} (a_p^\dagger a_p - b_p^\dagger b_p), \quad (63)$$

where $\omega_p^2 = \mu^2 + p^2$. In real space, the value of the conserved charge in a given subsystem A reads

$$Q_A = \int_A dx (a(x)^\dagger a(x) - b(x)^\dagger b(x)) \equiv N_A^a - N_A^b, \quad (64)$$

which is the difference between the number of particles and the number of antiparticles.

Using (64), the charge imbalance operators \mathcal{Q}_A of the two copies of \mathcal{H}_A that commutes with the reflected density matrix can be written as $\mathcal{Q}_A = \mathcal{N}_A^a - \mathcal{N}_A^b$, where

$$\mathcal{N}_A^a = N_A^a \otimes \mathbb{I}_A - \mathbb{I}_A \otimes N_A^a, \quad (65)$$

and similarly for \mathcal{N}_A^b . Since the Hamiltonian of a complex scalar field is identical to that of two decoupled real scalar fields, the density matrix factorizes accordingly. The charged moments of the reflected density matrix for the complex scalar field can be written in terms of those for a single scalar field, i.e.

$$Z_{m,n}^R(\alpha) = |\text{Tr}((\rho_{AA}^{(m)})^n e^{i\alpha \mathcal{N}_A})|^2, \quad (66)$$

where we used (65), and $\rho_{AA}^{(m)}$ corresponds to the reflected density matrix for a single scalar field.

On the lattice, the (complex) scalar field is realized by the (complex) harmonic chain. Using the results of Sec. III for bosonic states, the charged moments for the complex harmonic chain are obtained as

$$Z_{m,n}^R(\alpha) = \det |(\mathbf{C}_A^{(m)} + 1/2)^n - e^{i\alpha} (\mathbf{C}_A^{(m)} - 1/2)^n|^{-2}, \quad (67)$$

where the correlation matrix $\mathbf{C}_A^{(m)}$ is defined in terms of the two-point functions of the real harmonic chain.

Let us now focus on adjacent subsystems A and B , of respective lengths ℓ_a and ℓ_b . In the critical (massless) limit $\mu \rightarrow 0$, the charged reflected moments take the same form as in CFT, i.e.

$$Z_{m,n}^R(\alpha) \propto \left(\frac{\ell_a \ell_b}{\ell_a + \ell_b} \right)^{-4h_n^b(\alpha)}, \quad (68)$$

only with a nonstandard weight $h_n^b(\alpha)$, where a linear term in α is present (see, e.g., [54,55] for similar findings concerning the charged entanglement entropies),

$$h_n^b(\alpha) = \frac{1}{12} \left(n - \frac{1}{n} \right) + \frac{|\alpha|}{4\pi n} - \frac{1}{2n} \left(\frac{\alpha}{2\pi} \right)^2. \quad (69)$$

Similarly, the charged CCNR moments are given by

$$Z_{m,n}(\alpha) \propto \left(\frac{\ell_a \ell_b}{\ell_a + \ell_b} \right)^{-4h_n^b(\alpha)} (\ell_a + \ell_b)^{-4nh_m^b(0)}. \quad (70)$$

As a check of our formulas, we have numerically computed the charged CCNR moments for the complex boson in the critical regime $\mu \ll 1$. We take two adjacent regions of same length ℓ on the infinite chain. From (70), we expect $\log Z_{m,n}(\alpha) = -4(h_n^b(\alpha) + nh_m^b(0)) \log \ell + \dots$. We report in Fig. 3 our numerical results for the leading logarithmic coefficient of $\log Z_{m,n}(\alpha)$ as a function of α for

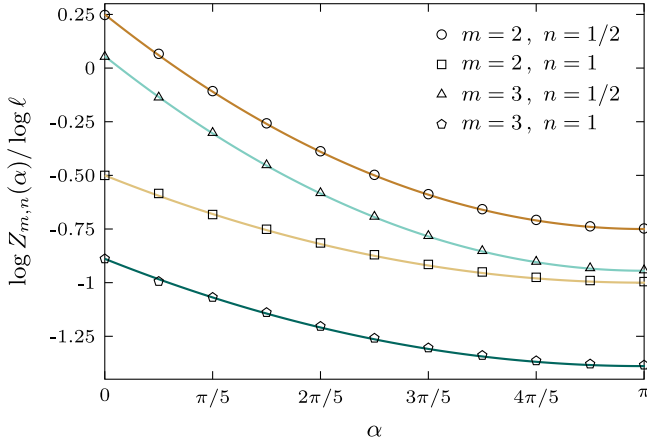


FIG. 3. Charged CCNR moments for the free complex boson on the line in the critical regime $\mu \ll 1$ for two adjacent intervals of same length ℓ . The solid lines correspond to the CFT prediction $-4(h_n^b(\alpha) + nh_m^b(0))$, where $h_n^b(\alpha)$ is given in (69).

different m and n . We observe a perfect agreement between (70) and the numerics.

We are interested in the Fourier transform of (68). In the saddle point approximation, the term in α^2 in (69) can be neglected, yielding

$$Z_{m,n}^R(q) = Z_{m,n}^R(0) \frac{n \log\left(\frac{\ell_a \ell_b}{\ell_a + \ell_b}\right)}{n^2 \pi^2 q^2 + \log^2\left(\frac{\ell_a \ell_b}{\ell_a + \ell_b}\right)}. \quad (71)$$

Similarly as for free fermions, we only investigate the symmetry-resolved reflected entropies since they have the same charge dependence as the symmetry-resolved CCNR negativities. Combining (41) and (71) we find

$$S_{m,n}^R(q) = S_{m,n}^R - \log \log\left(\frac{\ell_a \ell_b}{\ell_a + \ell_b}\right) + \frac{\log n}{1-n} + \frac{n\pi^2 q^2}{\log^2\left(\frac{\ell_a \ell_b}{\ell_a + \ell_b}\right)} + \dots, \quad (72)$$

where the subleading terms are of order $(\log \ell)^{-3}$. We thus observe an equipartition of the reflected entropies for free bosons. However, as opposed to the free-fermion case, the CFT prediction yields the first charge-dependent subleading correction, which is of order $(q/\log \ell)^2$. These results are very similar to those obtained for the standard symmetry-resolved entropies for free bosons [54,55].

V. CONCLUSION

In this paper, we have investigated the (m, n) -Rényi reflected entropies and CCNR negativities for CFTs and free theories. We first defined a reflected version of the CCNR negativity, denoted $\mathcal{E}_{m,n}$, and discussed its connections with other quantities of interest, namely the

reflected entropies and the operator entanglement entropies. On general ground, we have shown that the CCNR negativity and operator entanglement entropy are not monotonically nonincreasing under partial trace; hence, they are not correlation measures for mixed state. Nevertheless, these quantities present interesting universal properties. We have derived general formulas for the (m, n) -Rényi reflected entropies and CCNR negativities, valid for free fermionic and scalar fields in arbitrary dimensions. The resulting expressions are fully determined in terms of the fields' correlators, making them suitable for lattice calculations.

The main part of this work concerned the symmetry resolution of reflected entropies and CCNR negativities in the context of quantum many-body systems with a global $U(1)$ conserved charge. For two adjacent regions in an infinite system described by a $(1+1)$ -dimensional CFT, we have obtained the charged reflected moments and CCNR moments, see (50) and (51), focusing then on the former as they share the same charge dependence. Since the exact form of these charged moments depends on the details of the underlying theory, we have considered both a free fermionic model and a free bosonic model. Using our general formulas for the reflected entropies and CCNR negativities in free theories, we have tested our CFT predictions, finding perfect agreement in both cases. Our numerical investigations also confirm the CFT results of [19] for the (uncharged) CCNR negativity, for two new examples—free fermions and bosons—which were not considered in [19]. For free fermions, the symmetry-resolved reflected entropies do not depend on the charge sector up to subleading terms, which we conjecture to be of order $(q/\log \ell)^2$, thereby implying equipartition, similarly to the entanglement entropy. For free bosons, the CFT prediction yields the first charge-dependent subleading correction, which is of order $(q/\log \ell)^2$. Note that we worked with infinite systems for simplicity, but our results can easily be generalized to periodic systems with finite size L by replacing each subregion length ℓ_i by $\frac{L}{\pi} \sin \frac{\pi \ell_i}{L}$.

There are several future avenues worth exploring. An important direction would be to study the symmetry resolution of reflected entropies and CCNR negativities for disjoint subsystems. The CFT calculation in this case is more involved than for adjacent regions, and we expect would yield new insights on multipartite entanglement. One could also generalize our findings to finite temperature states, and to quench dynamics. Note that the symmetry-resolved operator entanglement entropy was investigated in [45] in out-of-equilibrium situations, and we expect similar results for the CCNR negativities and reflected entropies, in particular the presence of an entanglement barrier. It would be interesting to derive new exact lattice results for the reflected entropies and CCNR negativities, both in and out of equilibrium. In higher dimensions, the study of skeletal regions [112], i.e. regions A, B that have

no volume, could lead to new analytical results for reflected entropies and CCNR negativities. Finally, it is worth investigating reflected entropies and CCNR negativities for other mixed quantum states, such as Rokhsar-Kivelson states [113,114] whose Hilbert space is spanned by the configurations of an underlying statistical model (see Refs. [115,116] for recent developments on their entanglement structure).

ACKNOWLEDGMENTS

We warmly thank Sara Murciano, Filiberto Ares and Jérôme Dubail for useful discussions and sharing some preliminary results. We also thank Pablo Bueno and Riccarda Bonsignori for comments on the manuscript. C.B. was supported by a CRM-Simons Postdoctoral Fellowship at the Université de Montréal. G.P. holds an FRQNT and a CRM-ISM Postdoctoral Fellowship, and acknowledges support from the Mathematical Physics Laboratory of the CRM.

APPENDIX A: PROOF OF NONMONOTONICITY OF CCNR NEGATIVITY AND OPERATOR ENTANGLEMENT ENTROPY

In this appendix, we show that the CCNR Rényi negativity and the Rényi operator entanglement entropy are not monotonically nonincreasing under partial trace. To do so, it suffices to find counterexamples to the inequality (17), as was done in [97] for the reflected entropy $S_{1,n}^R$.

Let us consider the Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C$ of two qutrit A and B and one qubit C . The counterexample states, labeled by a single parameter p , are given by [97]

$$\begin{aligned} \rho_{ABC} = & \frac{1}{4p+2} (p(|000\rangle\langle 000| + |110\rangle\langle 110| \\ & + |200\rangle\langle 200| + |210\rangle\langle 210|) \\ & + |020\rangle\langle 020| + |121\rangle\langle 121|), \end{aligned} \quad (\text{A1})$$

and we further define $\rho_{AB} = \text{Tr}_{\mathcal{H}_C} \rho_{ABC}$. It is straightforward to obtain the reflected densities $\rho_{AA(C)}^{(m)}$ and $\rho_{AA}^{(m)}$ on $\mathcal{H}_A \otimes \mathcal{H}_A$ from the purifications of ρ_{ABC}^m and ρ_{AB}^m , respectively [see (12)]. They are given by

$$\begin{aligned} \rho_{AA(C)}^{(m)} = & \frac{1}{4p^m+2} (p^m(|00\rangle\langle 00| + 2|22\rangle\langle 22| + |00\rangle\langle 22| \\ & + |22\rangle\langle 00| + |11\rangle\langle 11| + |11\rangle\langle 22| + |22\rangle\langle 11|) \\ & + |00\rangle\langle 00| + |11\rangle\langle 11|), \end{aligned} \quad (\text{A2})$$

$$\rho_{AA}^{(m)} = \rho_{AA(C)}^{(m)} + \frac{1}{4p^m+2} (|00\rangle\langle 11| + |11\rangle\langle 00|). \quad (\text{A3})$$

Their corresponding (nonzero) eigenvalues are

$$\begin{aligned} \text{Spec}(\rho_{AA(C)}^{(m)}) &= \left\{ \frac{p^m+1}{4p^m+2}, \frac{1+3p^m \pm \sqrt{9p^{2m}-2p^m+1}}{2(4p^m+2)} \right\}, \end{aligned} \quad (\text{A4})$$

$$\begin{aligned} \text{Spec}(\rho_{AA}^{(m)}) &= \left\{ \frac{p^m}{4p^m+2}, \frac{2+3p^m \pm \sqrt{9p^{2m}-4p^m+4}}{2(4p^m+2)} \right\}. \end{aligned} \quad (\text{A5})$$

Using these expressions, one can compute $S_{m,n}^R$ and $\mathcal{E}_{m,n}$ defined in (13) and (15), respectively, for any value of p and m, n . For large p we find that

$$\Delta S_{m,n}^R = \begin{cases} \frac{1}{1-n} \frac{1-2^n}{1+3^n} \left(\frac{2}{3}\right)^n \frac{1}{p^{nm}} + \dots, & n < 1 \\ -\frac{1}{6} \frac{\log p^m}{p^m} + \dots, & n = 1, \\ \frac{1}{1-n} \frac{1-3^{n-2}}{1+3^n} \frac{n}{p^m} + \dots, & n > 1 \end{cases} \quad (\text{A6})$$

where $\Delta S_{m,n}^R \equiv S_{m,n}^R(A:B \cup C) - S_{m,n}^R(A:B)$ and the ellipsis denotes terms strictly smaller than $1/p^m$ in the limit $p \rightarrow \infty$. The difference $\Delta S_{m,n}^R$ is thus negative for large p in the range $0 < n < 2$ and $m > 0$. It can be shown to be positive for $n \geq 2$ for any $p > 0$. This implies that the Rényi reflected entropy $S_{m,n}^R$, which reduces to the Rényi operator entanglement entropy E_n for $m=2$, is not monotonically nonincreasing under partial trace.

Since $\text{Tr} \rho_{ABC}^m = \text{Tr} \rho_{AB}^m$, we have the difference

$$\mathcal{E}_n(A:B \cup C) - \mathcal{E}_n(A:B) = \log \frac{\text{Tr}(\rho_{AA(C)}^{(2)})^n}{\text{Tr}(\rho_{AA}^{(2)})^n}, \quad (\text{A7})$$

where we set $m=2$. For $n \in (0, 1) \cup (1, \infty)$, there exists a critical value $p_0 \simeq 1.284210838876$, solution to the equation $\partial_n (\text{Tr}(\rho_{AA(C)}^{(2)})^n / \text{Tr}(\rho_{AA}^{(2)})^n)|_{n=1} = 0$, for which we have $\mathcal{E}_n(A:B \cup C) - \mathcal{E}_n(A:B) < 0$, implying that the CCNR Rényi negativity is not monotonically nonincreasing under partial trace.

APPENDIX B: COMMUTATION AND CHARGE IMBALANCE

In this section, following [45], we prove the commutation relation (31) between the reflected density matrix $\rho_{AA}^{(m)}$ and the charge imbalance \mathcal{Q}_A .

Let us consider an operator \mathcal{O} which acts on $\mathcal{H}_A \otimes \mathcal{H}_B$ and commutes with the charge $Q = Q_A + Q_B$, namely $[\mathcal{O}, Q] = 0$. It follows that \mathcal{O} takes the form

$$\mathcal{O} = \sum_q \sum_j \lambda_j^{(q)} \mathcal{O}_{A,j}^{(q)} \otimes \mathcal{O}_{B,j}^{(-q)}, \quad (\text{B1})$$

where q labels the eigenvalues of Q ,

$$\text{Tr}\{(\mathcal{O}_{A,j}^{(q)})^\dagger \mathcal{O}_{A,j'}^{(q')}\} = \delta_{q,q'} \delta_{j,j'}, \quad [\mathcal{Q}_A, \mathcal{O}_{A,j}^{(q)}] = q \mathcal{O}_{A,j}^{(q)}, \quad (\text{B2})$$

and similarly for $\mathcal{O}_{B,j}^{(q)}$. We recast \mathcal{O} in components as

$$\mathcal{O} = \sum_{q,j} \sum_{a,a'} \sum_{b,b'} \lambda_j^{(q)} \langle a | \mathcal{O}_{A,j}^{(q)} | a' \rangle \langle b | \mathcal{O}_{B,j}^{(-q)} | b' \rangle | ab \rangle \langle a' b' |, \quad (\text{B3})$$

where $\{|a\rangle\}$ and $\{|b\rangle\}$ are orthonormal basis states of \mathcal{H}_A and \mathcal{H}_B . We choose the basis states to be eigenvectors of the charges,

$$\mathcal{Q}_A |a\rangle = q_a |a\rangle, \quad \mathcal{Q}_B |b\rangle = q_b |b\rangle. \quad (\text{B4})$$

The commutation relation (B2) together with (B4) implies

$$\langle a | \mathcal{O}_{A,j}^{(q)} | a' \rangle \propto \delta_{q_a - q_{a'}, q}, \quad \langle b | \mathcal{O}_{B,j}^{(-q)} | b' \rangle \propto \delta_{q_b - q_{b'}, -q}. \quad (\text{B5})$$

The vector version of \mathcal{O} reads

$$|\mathcal{O}\rangle = \sum_{q,j} \sum_{a,a'} \sum_{b,b'} \lambda_j^{(q)} \langle a | \mathcal{O}_{A,j}^{(q)} | a' \rangle \langle b | \mathcal{O}_{B,j}^{(-q)} | b' \rangle | ab \rangle \otimes | a' b' \rangle \quad (\text{B6})$$

and we have

$$\begin{aligned} & \text{Tr}_{\mathcal{H}_B \otimes \mathcal{H}_B} (|\mathcal{O}\rangle \langle \mathcal{O}|) \\ &= \sum_{q,j} \sum_{\substack{a,a' \\ A,A}} (\lambda_j^{(q)})^2 \langle a | \mathcal{O}_{A,j}^{(q)} | a' \rangle \langle A' | (\mathcal{O}_{A,j}^{(q)})^\dagger | A \rangle | a a' \rangle \langle A A' |, \end{aligned} \quad (\text{B7})$$

where we used the orthogonality relation (B2). Finally, with (B5) we conclude that each operator $|aa'\rangle \langle AA'|$ in the sum commutes with the charge imbalance

$$\mathcal{Q}_A = \mathcal{Q}_A \otimes \mathbb{I}_A - \mathbb{I}_A \otimes \mathcal{Q}_A, \quad (\text{B8})$$

and therefore

$$[\mathcal{Q}_A, \text{Tr}_{\mathcal{H}_B \otimes \mathcal{H}_B} (|\mathcal{O}\rangle \langle \mathcal{O}|)] = 0. \quad (\text{B9})$$

APPENDIX C: THE GENERALIZED MARKOV GAP

We consider a generalization of the Markov gap [99,117], defined as the difference of the m -reflected entropy with mutual information I

$$M_m \equiv S_{m,1}^R - I, \quad (\text{C1})$$

where we recall that mutual information between two subsystems A and B is given by $I(A:B) = S(\rho_A) + S(\rho_B) - S(\rho_{A \cup B})$. The reflected entropy $m=1$ is lower bounded by mutual information [83], hence the Markov gap is a nonnegative quantity $M \equiv M_1 \geq 0$. As argued in [117], a nonvanishing M indicates irreducible tripartite entanglement.

Using our results of Sec. III, we numerically compute M_m for two adjacent subsystems, both for free fermions and free bosons. In the thermodynamic limit, we find

$$M_m = \begin{cases} \frac{1}{3} \log 2m, & \text{fermions} \\ \frac{1}{3} \log 2m - \frac{1}{2} \log m, & \text{bosons} \end{cases}. \quad (\text{C2})$$

For $m=1$, we retrieve the expected universal CFT result [117] that $M = \frac{c}{3} \log 2$. However, for $m \neq 1$ we observe discrepancy between fermions and bosons. It would be interesting to understand if M_m is a meaningful quantity in general.

-
- [1] R. Horodecki, P. Horodecki, M. Horodecki, and K. Horodecki, Quantum entanglement, *Rev. Mod. Phys.* **81**, 865 (2009).
 - [2] O. Gühne and G. Tóth, Entanglement detection, *Phys. Rep.* **474**, 1 (2009).
 - [3] R. F. Werner, Quantum states with Einstein-Podolsky-Rosen correlations admitting a hidden-variable model, *Phys. Rev. A* **40**, 4277 (1989).
 - [4] A. Peres, Separability Criterion for Density Matrices, *Phys. Rev. Lett.* **77**, 1413 (1996).
 - [5] M. Horodecki, P. Horodecki, and R. Horodecki, Separability of mixed states: Necessary and sufficient conditions, *Phys. Lett. A* **223**, 1 (1996).
 - [6] O. Rudolph, On the cross norm criterion for separability, *J. Phys. A* **36**, 5825 (2003).
 - [7] K. Chen and L.-A. Wu, A matrix realignment method for recognizing entanglement, *Quantum Inf. Comput.* **3**, 193 (2003).
 - [8] M. Horodecki, P. Horodecki, and R. Horodecki, Separability of mixed quantum states: Linear contractions and permutation criteria, *Open Syst. Inf. Dyn.* **13**, 103 (2006).
 - [9] Z. Liu, Y. Tang, H. Dai, P. Liu, S. Chen, and X. Ma, Detecting Entanglement in Quantum Many-Body Systems via Permutation Moments, *Phys. Rev. Lett.* **129**, 260501 (2022).

- [10] K. Zyczkowski, P. Horodecki, A. Sanpera, and M. Lewenstein, Volume of the set of separable states, *Phys. Rev. A* **58**, 883 (1998).
- [11] J. Eisert and M. B. Plenio, A comparison of entanglement measures, *J. Mod. Opt.* **46**, 145 (1999).
- [12] G. Vidal and R. F. Werner, Computable measure of entanglement, *Phys. Rev. A* **65**, 032314 (2002).
- [13] P. Calabrese, J. L. Cardy, and E. Tonni, Entanglement Negativity in Quantum Field Theory, *Phys. Rev. Lett.* **109**, 130502 (2012).
- [14] P. Calabrese, J. L. Cardy, and E. Tonni, Entanglement negativity in extended systems: A field theoretical approach, *J. Stat. Mech.* (2013) P02008.
- [15] C. Castelnovo, Negativity and topological order in the toric code, *Phys. Rev. A* **88**, 042319 (2013).
- [16] Y. A. Lee and G. Vidal, Entanglement negativity and topological order, *Phys. Rev. A* **88**, 042318 (2013).
- [17] X. Wen, S. Matsuura, and S. Ryu, Edge theory approach to topological entanglement entropy, mutual information and entanglement negativity in Chern-Simons theories, *Phys. Rev. B* **93**, 245140 (2016).
- [18] J. Angel-Ramelli, C. Berthiere, V. G. M. Puletti, and L. Thorlacius, Logarithmic negativity in quantum lifshitz theories, *J. High Energy Phys.* **09** (2020) 011.
- [19] C. Yin and Z. Liu, Universal Entanglement and Correlation Measure in Two-Dimensional Conformal Field Theory, *Phys. Rev. Lett.* **131601** (2023).
- [20] A. Milekhin, P. Rath, and W. Weng, Computable cross norm in tensor networks and holography, [arXiv:2212.11978](https://arxiv.org/abs/2212.11978).
- [21] C. Yin and S. Liu, Mixed state entanglement measures in topological orders, *Phys. Rev. B* **108**, 035152 (2023).
- [22] L. Amico, R. Fazio, A. Osterloh, and V. Vedral, Entanglement in many-body systems, *Rev. Mod. Phys.* **80**, 517 (2008).
- [23] P. Calabrese and J. L. Cardy, Entanglement entropy and conformal field theory, *J. Phys. A* **42**, 504005 (2009).
- [24] N. Laflorencie, Quantum entanglement in condensed matter systems, *Phys. Rep.* **646**, 1 (2016).
- [25] G. Vidal, J. I. Latorre, E. Rico, and A. Kitaev, Entanglement in Quantum Critical Phenomena, *Phys. Rev. Lett.* **90**, 227902 (2003).
- [26] P. Calabrese and J. L. Cardy, Entanglement entropy and quantum field theory, *J. Stat. Mech.* (2004) P06002.
- [27] N. Laflorencie, E. S. Sorensen, M.-S. Chang, and I. Affleck, Boundary Effects in the Critical Scaling of Entanglement Entropy in 1D Systems, *Phys. Rev. Lett.* **96**, 100603 (2006).
- [28] J. Eisert, M. Cramer, and M. B. Plenio, Area laws for the entanglement entropy—a review, *Rev. Mod. Phys.* **82**, 277 (2010).
- [29] I. Affleck, N. Laflorencie, and E. S. Sorensen, Entanglement entropy in quantum impurity systems and systems with boundaries, *J. Phys. A* **42**, 504009 (2009).
- [30] P. Bueno, R. C. Myers, and W. Witczak-Krempa, Universality of Corner Entanglement in Conformal Field Theories, *Phys. Rev. Lett.* **115**, 021602 (2015).
- [31] D. V. Fursaev and S. N. Solodukhin, Anomalies, entropy and boundaries, *Phys. Rev. D* **93**, 084021 (2016).
- [32] H. Casini, I. S. Landea, and G. Torroba, The g-theorem and quantum information theory, *J. High Energy Phys.* **10** (2016) 140.
- [33] C. Berthiere, Boundary-corner entanglement for free bosons, *Phys. Rev. B* **99**, 165113 (2019).
- [34] C. Berthiere and W. Witczak-Krempa, Relating bulk to boundary entanglement, *Phys. Rev. B* **100**, 235112 (2019).
- [35] A. Kitaev and J. Preskill, Topological Entanglement Entropy, *Phys. Rev. Lett.* **96**, 110404 (2006).
- [36] M. Levin and X.-G. Wen, Detecting Topological Order in a Ground State Wave Function, *Phys. Rev. Lett.* **96**, 110405 (2006).
- [37] B. Zeng, X. Chen, D.-L. Zhou, and X.-G. Wen, Quantum information meets quantum matter—from quantum entanglement to topological phase in many-body systems, [arXiv:1508.02595](https://arxiv.org/abs/1508.02595).
- [38] N. Laflorencie and S. Rachel, Spin-resolved entanglement spectroscopy of critical spin chains and Luttinger liquids, *J. Stat. Mech.* (2014) P11013.
- [39] M. Goldstein and E. Sela, Symmetry-Resolved Entanglement in Many-Body Systems, *Phys. Rev. Lett.* **120**, 200602 (2018).
- [40] J. C. Xavier, F. C. Alcaraz, and G. Sierra, Equipartition of the entanglement entropy, *Phys. Rev. B* **98**, 041106(R) (2018).
- [41] R. Bonsignori, P. Ruggiero, and P. Calabrese, Symmetry resolved entanglement in free fermionic systems, *J. Phys. A* **52**, 475302 (2019).
- [42] A. Lukin, M. Rispoli, R. Schittko, M. E. Tai, A. M. Kaufman, S. Choi, V. Khemani, J. Léonard, and M. Greiner, Probing entanglement in a many-body localized system, *Science* **364**, 256 (2019).
- [43] V. Vitale, A. Elben, R. Kueng, A. Neven, J. Carrasco, B. Kraus, P. Zoller, P. Calabrese, B. Vermersch, and M. Dalmonte, Symmetry-resolved dynamical purification in synthetic quantum matter, *SciPost Phys.* **12**, 106 (2022).
- [44] A. Neven, J. Carrasco, V. Vitale, C. Kokail, A. Elben, M. Dalmonte, P. Calabrese, P. Zoller, B. Vermersch, R. Kueng *et al.*, Symmetry-resolved entanglement detection using partial transpose moments, *npj Quantum Inf.* **7**, 1 (2021).
- [45] A. Rath, V. Vitale, S. Murciano, M. Votto, J. Dubail, R. Kueng, C. Branciard, P. Calabrese, and B. Vermersch, Entanglement barrier and its symmetry resolution: Theory and experimental observation, *PRX Quantum* **4**, 010318 (2023).
- [46] R. Bonsignori and P. Calabrese, Boundary effects on symmetry resolved entanglement, *J. Phys. A* **54**, 015005 (2020).
- [47] H. Barghathi, C. M. Herdman, and A. Del Maestro, Rényi Generalization of the Accessible Entanglement Entropy, *Phys. Rev. Lett.* **121**, 150501 (2018).
- [48] H. Barghathi, E. Casiano-Diaz, and A. Del Maestro, Operationally accessible entanglement of one-dimensional spinless fermions, *Phys. Rev. A* **100**, 022324 (2019).
- [49] S. Murciano, P. Ruggiero, and P. Calabrese, Symmetry resolved entanglement in two-dimensional systems via dimensional reduction, *J. Stat. Mech.* (2020) 083102.
- [50] B. Estienne, Y. Ikhlef, and A. Morin-Duchesne, Finite-size corrections in critical symmetry-resolved entanglement, *SciPost Phys.* **10**, 54 (2021).

- [51] F. Ares, P. Calabrese, G. Di Giulio, and S. Murciano, Multi-charged moments of two intervals in conformal field theory, *J. High Energy Phys.* **09** (2022) 051.
- [52] F. Ares, S. Murciano, and P. Calabrese, Symmetry-resolved entanglement in a long-range free-fermion chain, *J. Stat. Mech.* (2022) 063104.
- [53] G. Di Giulio, R. Meyer, C. Northe, H. Scheppach, and S. Zhao, On the boundary conformal field theory approach to symmetry-resolved entanglement, [arXiv:2212.09767](https://arxiv.org/abs/2212.09767).
- [54] S. Murciano, G. Di Giulio, and P. Calabrese, Symmetry resolved entanglement in gapped integrable systems: A corner transfer matrix approach, *SciPost Phys.* **8**, 046 (2020).
- [55] S. Murciano, G. Di Giulio, and P. Calabrese, Entanglement and symmetry resolution in two dimensional free quantum field theories, *J. High Energy Phys.* **08** (2020) 073.
- [56] D. X. Horváth and P. Calabrese, Symmetry resolved entanglement in integrable field theories via form factor bootstrap, *J. High Energy Phys.* **11** (2020) 131.
- [57] D. X. Horváth, L. Capizzi, and P. Calabrese, U(1) symmetry resolved entanglement in free 1 + 1 dimensional field theories via form factor bootstrap, *J. High Energy Phys.* **05** (2021) 197.
- [58] L. Capizzi, O. A. Castro-Alvaredo, C. De Fazio, M. Mazzoni, and L. Santamaría-Sanz, Symmetry resolved entanglement of excited states in quantum field theory. Part I. Free theories, twist fields and qubits, *J. High Energy Phys.* **12** (2022) 127.
- [59] L. Capizzi, D. X. Horváth, P. Calabrese, and O. A. Castro-Alvaredo, Entanglement of the 3-state Potts model via form factor bootstrap: Total and symmetry resolved entropies, *J. High Energy Phys.* **05** (2022) 113.
- [60] D. Azses and E. Sela, Symmetry-resolved entanglement in symmetry-protected topological phases, *Phys. Rev. B* **102**, 235157 (2020).
- [61] K. Monkman and J. Sirker, Operational entanglement of symmetry-protected topological edge states, *Phys. Rev. Res.* **2**, 043191 (2020).
- [62] B. Oblak, N. Regnault, and B. Estienne, Equipartition of entanglement in quantum Hall states, *Phys. Rev. B* **105**, 115131 (2022).
- [63] D. X. Horváth, S. Fraenkel, S. Scopa, and C. Rylands, Charge-resolved entanglement in the presence of topological defects, [arXiv:2306.15532](https://arxiv.org/abs/2306.15532).
- [64] M. Fossati, F. Ares, and P. Calabrese, Symmetry-resolved entanglement in critical non-Hermitian systems, *Phys. Rev. B* **107**, 205153 (2023).
- [65] K. Monkman and J. Sirker, Symmetry-resolved entanglement: General considerations, calculation from correlation functions, and bounds for symmetry-protected topological phases, [arXiv:2307.05820](https://arxiv.org/abs/2307.05820).
- [66] N. Feldman and M. Goldstein, Dynamics of charge-resolved entanglement after a local quench, *Phys. Rev. B* **100**, 235146 (2019).
- [67] S. Fraenkel and M. Goldstein, Entanglement measures in a nonequilibrium steady state: Exact results in one dimension, *SciPost Phys.* **11**, 085 (2021).
- [68] G. Perez, R. Bonsignori, and P. Calabrese, Quasiparticle dynamics of symmetry-resolved entanglement after a quench: Examples of conformal field theories and free fermions, *Phys. Rev. B* **103**, L041104 (2021).
- [69] G. Perez, R. Bonsignori, and P. Calabrese, Exact quench dynamics of symmetry resolved entanglement in a free fermion chain, *J. Stat. Mech.* (2021) 093102.
- [70] G. Perez, R. Bonsignori, and P. Calabrese, Dynamics of charge-imbalance-resolved entanglement negativity after a quench in a free-fermion model, *J. Stat. Mech.* (2022) 053103.
- [71] G. Perez and R. Bonsignori, Analytical results for the entanglement dynamics of disjoint blocks in the XY spin chain, *J. Phys. A* **55**, 505005 (2022).
- [72] L. Piroli, E. Vernier, M. Collura, and P. Calabrese, Thermodynamic symmetry resolved entanglement entropies in integrable systems, *J. Stat. Mech.* (2022) 073102.
- [73] S. Scopa and D. X. Horváth, Exact hydrodynamic description of symmetry-resolved Rényi entropies after a quantum quench, *J. Stat. Mech.* (2022) 083104.
- [74] S. Murciano, P. Calabrese, and V. Alba, Symmetry-resolved entanglement in fermionic systems with dissipation, [arXiv:2303.12120](https://arxiv.org/abs/2303.12120).
- [75] E. Cornfeld, M. Goldstein, and E. Sela, Imbalance entanglement: Symmetry decomposition of negativity, *Phys. Rev. A* **98**, 032302 (2018).
- [76] S. Murciano, R. Bonsignori, and P. Calabrese, Symmetry decomposition of negativity of massless free fermions, *SciPost Phys.* **10**, 111 (2021).
- [77] H. Gaur and U. A. Yajnik, Charge imbalance resolved Rényi negativity for free compact boson: Two disjoint interval case, *J. High Energy Phys.* **02** (2023) 118.
- [78] L. Capizzi and P. Calabrese, Symmetry resolved relative entropies and distances in conformal field theory, *J. High Energy Phys.* **10** (2021) 195.
- [79] S. Murciano, P. Calabrese, and L. Piroli, Symmetry-resolved Page curves, *Phys. Rev. D* **106**, 046015 (2022).
- [80] G. Perez, Symmetry-resolved Rényi fidelities and quantum phase transitions, *Phys. Rev. B* **106**, 235101 (2022).
- [81] F. Ares, S. Murciano, and P. Calabrese, Entanglement asymmetry as a probe of symmetry breaking, *Nat. Commun.* **14**, 2036 (2023).
- [82] G. Di Giulio and J. Erdmenger, Symmetry-resolved modular correlation functions in free fermionic theories, *J. High Energy Phys.* **07** (2023) 058.
- [83] S. Dutta and T. Faulkner, A canonical purification for the entanglement wedge cross-section, *J. High Energy Phys.* **03** (2021) 178.
- [84] H.-S. Jeong, K.-Y. Kim, and M. Nishida, Reflected entropy and entanglement wedge cross section with the first order correction, *J. High Energy Phys.* **12** (2019) 170.
- [85] P. Bueno and H. Casini, Reflected entropy, symmetries and free fermions, *J. High Energy Phys.* **05** (2020) 103.
- [86] T. Li, J. Chu, and Y. Zhou, Reflected entropy for an evaporating black hole, *J. High Energy Phys.* **11** (2020) 155.
- [87] C. Berthiere, H. Chen, Y. Liu, and B. Chen, Topological reflected entropy in Chern-Simons theories, *Phys. Rev. B* **103**, 035149 (2021).
- [88] P. Bueno and H. Casini, Reflected entropy for free scalars, *J. High Energy Phys.* **11** (2020) 148.

- [89] C. Akers, T. Faulkner, S. Lin, and P. Rath, The Page curve for reflected entropy, *J. High Energy Phys.* **06** (2022) 089.
- [90] B. Chen, Y. Liu, and B. Yu, Reflected entropy in AdS_3/WCFT , *J. High Energy Phys.* **12** (2022) 008.
- [91] M. J. Vasli, M. R. Mohammadi Mozaffar, K. Babaei Velni, and M. Sahraei, Holographic study of reflected entropy in anisotropic theories, *Phys. Rev. D* **107**, 026012 (2023).
- [92] Y. Lu and J. Lin, The Markov gap in the presence of islands, *J. High Energy Phys.* **03** (2023) 043.
- [93] M. Afrasiar, J. K. Basak, A. Chandra, and G. Sengupta, Reflected entropy for communicating black holes. Part I. Karch-Randall braneworlds, *J. High Energy Phys.* **02** (2023) 203.
- [94] R. Sohal and S. Ryu, Entanglement in tripartitions of topological orders: A diagrammatic approach, *Phys. Rev. B* **108**, 045104 (2023).
- [95] J. K. Basak, D. Giataganas, S. Mondal, and W.-Y. Wen, Reflected entropy and Markov gap in non-inertial frames, [arXiv:2306.17490](https://arxiv.org/abs/2306.17490).
- [96] C. Berthiere, B. Chen, and H. Chen, Reflected entropy and Markov gap in Lifshitz theories, [arXiv:2307.12247](https://arxiv.org/abs/2307.12247).
- [97] P. Hayden, M. Lemm, and J. Sorce, Reflected entropy: Not a correlation measure, *Phys. Rev. A* **107**, L050401 (2023).
- [98] C. Akers and P. Rath, Entanglement wedge cross sections require tripartite entanglement, *J. High Energy Phys.* **04** (2020) 208.
- [99] P. Hayden, O. Parrikar, and J. Sorce, The Markov gap for geometric reflected entropy, *J. High Energy Phys.* **10** (2021) 047.
- [100] H. A. Camargo, L. Hackl, M. P. Heller, A. Jahn, and B. Windt, Long Distance Entanglement of Purification and Reflected Entropy in Conformal Field Theory, *Phys. Rev. Lett.* **127**, 141604 (2021).
- [101] A. Jamiołkowski, Linear transformations which preserve trace and positive semidefiniteness of operators, *Rep. Math. Phys.* **3**, 275 (1972).
- [102] M.-D. Choi, Completely positive linear maps on complex matrices, *Linear Algebra Appl.* **10**, 285 (1975).
- [103] P. Zanardi, Entanglement of quantum evolutions, *Phys. Rev. A* **63**, 040304 (2001).
- [104] T. Prosen and I. Pižorn, Operator space entanglement entropy in a transverse Ising chain, *Phys. Rev. A* **76**, 032316 (2007).
- [105] J. Dubail, Entanglement scaling of operators: A conformal field theory approach, with a glimpse of simulability of long-time dynamics in $1 + 1d$, *J. Phys. A* **50**, 234001 (2017).
- [106] J. L. Cardy, O. A. Castro-Alvaredo, and B. Doyon, Form factors of branch-point twist fields in quantum integrable models and entanglement entropy, *J. Stat. Phys.* **130**, 129 (2008).
- [107] A. Belin, L.-Y. Hung, A. Maloney, S. Matsuura, R. C. Myers, and T. Sierens, Holographic charged Rényi entropies, *J. High Energy Phys.* **12** (2013) 059.
- [108] J. S. Dowker, Conformal weights of charged Rényi entropy twist operators for free scalar fields in arbitrary dimensions, *J. Phys. A* **49**, 145401 (2016).
- [109] J. S. Dowker, Charged Rényi entropies for free scalar fields, *J. Phys. A* **50**, 165401 (2017).
- [110] I. Peschel, Calculation of reduced density matrices from correlation functions, *J. Phys. A* **36**, L205 (2003).
- [111] I. Peschel and V. Eisler, Reduced density matrices and entanglement entropy in free lattice models, *J. Phys. A* **42**, 504003 (2009).
- [112] C. Berthiere and W. Witczak-Krempa, Entanglement of Skeletal Regions, *Phys. Rev. Lett.* **128**, 240502 (2022).
- [113] D. S. Rokhsar and S. A. Kivelson, Superconductivity and the Quantum Hard-Core Dimer Gas, *Phys. Rev. Lett.* **61**, 2376 (1988).
- [114] S. A. Kivelson, D. S. Rokhsar, and J. P. Sethna, Topology of the resonating valence-bond state: Solitons and high- T_c superconductivity, *Phys. Rev. B* **35**, 8865 (1987).
- [115] C. Boudreault, C. Berthiere, and W. Witczak-Krempa, Entanglement and separability in continuum Rokhsar-Kivelson states, *Phys. Rev. Res.* **4**, 033251 (2022).
- [116] G. Perez, C. Berthiere, and W. Witczak-Krempa, Separability and entanglement of resonating valence-bond states, *SciPost Phys.* **15**, 066 (2023).
- [117] Y. Zou, K. Siva, T. Soejima, R. S. K. Mong, and M. P. Zaletel, Universal Tripartite Entanglement in One-Dimensional Many-Body Systems, *Phys. Rev. Lett.* **126**, 120501 (2021).