


Dispersive determination of electroweak-scale masses

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We demonstrate that the Higgs boson mass can be extracted from the dispersion relation obeyed by the correlation function of two b -quark scalar currents. The solution to the dispersion relation with the input from the perturbative evaluation of the correlation function up to next-to-leading order in QCD and with the b -quark mass $m_b = 4.43$ GeV demands a specific Higgs mass 114 GeV. Our observation offers an alternative resolution to the long-standing fine-tuning problem of the Standard Model (SM); the Higgs mass is determined dynamically for the internal consistency of the SM. The similar formalism, as applied to the correlation function of two b -quark vector currents with the same m_b , leads to the Z boson mass 90.8 GeV. This solution exists only when the Z - and W -boson masses are proportionate, conforming to the Higgs mechanism of the electroweak symmetry breaking. We then consider the mixing between the $Q\bar{u}$ and $\bar{Q}u$ states for a fictitious heavy quark Q and a u quark through the $b\bar{b}$ channel, inspired by our earlier analysis of neutral meson mixing. Its dispersion relation, given the perturbative input from the responsible box diagrams and the same m_b , fixes the top-quark mass 176 GeV. It is highly nontrivial to predict the above electroweak-scale masses with at most 9% deviation from their measured values using the single parameter m_b . More accurate results are expected, as more precise perturbative inputs are adopted.

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I. INTRODUCTION

Recently we proposed the possibility that the parameters in the Standard Model (SM) are not free, but arranged properly to achieve the internal dynamical consistency [1]. This proposal was motivated by the emergence of a heavy (charm or bottom) quark mass in the dispersive analysis on decay widths of a heavy meson H_Q formed by a fictitious heavy quark Q with an arbitrary mass m_Q . A decay width, as the absorptive piece of a heavy meson matrix element of the four-quark effective operators, satisfies a dispersion relation, which imposes a stringent connection between high-mass and low-mass behaviors of a decay width. A solution to the dispersion relation with the input from heavy quark expansion (HQE) at high mass specifies the value of m_Q , which turns out to coincide with the physical c or b quark mass. Starting with massless final-state up and down quarks, we have shown that the solution for the decay $Q \rightarrow d\bar{u}$ ($Q \rightarrow c\bar{u}$) with the leading-order HQE input leads to the c - (b -)quark mass $m_c = 1.35$ ($m_b = 4.4$) GeV, given the binding energy $\bar{\Lambda} = 0.5$ (0.6) GeV of a heavy meson. Requiring that the dispersion relation for the $Q \rightarrow s\bar{u}$

($Q \rightarrow d\mu^+\nu_\mu$, $Q \rightarrow u\tau^-\bar{\nu}_\tau$) decay yields the same heavy quark mass, we obtained the strange quark (muon, τ lepton) mass $m_s = 0.12$ GeV ($m_\mu = 0.11$ GeV, $m_\tau = 2.0$ GeV). The above particle masses, close to the measured ones, support the aforementioned proposal.

This paper will further demonstrate that the electroweak-scale masses, i.e., the Higgs boson, Z -boson and top-quark masses, can also be determined in a similar manner. To extract the Higgs boson mass, we investigate the dispersion relation obeyed by the correlation function of two b -quark scalar currents, like those employed in QCD sum rules [2] for probing resonance properties (see [3], for instance). The solution to the dispersion relation, with the input from the perturbative evaluation of the correlation function up to next-to-leading order (NLO) in QCD and with the b -quark mass $m_b = 4.43$ GeV (slightly higher than that from [1]), demands a specific scalar mass 114 GeV, close to the measured one $m_H = (125.25 \pm 0.17)$ GeV [4]. This observation offers an alternative resolution to the long-standing fine-tuning problem of the SM; the Higgs boson must have this mass to make the internal consistency of the SM dynamics. The above formalism, as applied to the correlation function of two b -quark vector currents with the same m_b , generates the Z boson mass 90.8 GeV, close to the measured one $m_Z = (91.1876 \pm 0.0021)$ GeV [4]. We then consider the mixing between the $Q\bar{u}$ and $\bar{Q}u$ states for a fictitious heavy quark Q and a u quark through the $b\bar{b}$ channel, which was inspired by our earlier dispersive

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analysis on neutral meson mixing [5]. Its dispersion relation, given the perturbative input from the responsible box diagrams and the same m_b , fixes the top-quark mass 176 GeV, close to the measured one $m_t = (172.69 \pm 0.30)$ GeV [4]. The solved particle masses with the single parameter m_b , deviating from the measured values by at most 9%, are highly nontrivial. Our studies suggest that the masses over a huge hierarchy in the SM, from 0.1 GeV to 100 GeV, can be correlated to each other through appropriate dispersion relations.

II. HIGGS-BOSON MASS

The nonperturbative approach based on dispersion relations for physical observables was proposed in [6], and applied to the explorations of various quantities in [7–10]. Following the threads of the above references, we construct the dispersion relation obeyed by the two-point correlation function

$$\Pi(q^2) = i \int d^4x e^{iq \cdot x} \langle 0 | T[J(x)J(0)] | 0 \rangle, \quad (1)$$

with the momentum q being injected into the b -quark scalar current $J = \bar{b}b$. The momentum squared q^2 defines an invariant mass $m_S = \sqrt{q^2}$ of the scalar attaching to the current. The standard operator product expansion (OPE) implemented in QCD sum rules is reliable for Eq. (1) in the large m_S region. Inserting various states between the two operators $J(x)$ and $J(0)$, one realizes that information of any physical particle allowed to decay into a $b\bar{b}$ quark pair can be extracted from the above correlation function in principle, as the OPE is known precisely enough. It is thus clear that the $b\bar{b}$ final state, instead of lighter ones like $c\bar{c}$, is appropriate for extracting the Higgs boson mass, because no other scalar particles with masses above the B -meson-pair threshold decay into $b\bar{b}$.

It has been noticed [11,12] that the power corrections from heavy-quark condensates can be absorbed into those from the gluon condensates. The gluon condensate effects, starting with $\langle \alpha_s G^2 \rangle \sim O(0.1)$ GeV⁴ [13,14] and down by powers of $m_S > m_b$, are negligible. Hence, we keep only the leading term in the OPE, i.e., the perturbative contribution [15–17], which gives rise to the spectral function

$$\text{Im}\Pi^{\text{P}}(m_S) \propto \frac{(m_S^2 - 4m_b^2)^{3/2}}{m_S} \times \left[1 + \frac{\alpha_s(\mu)}{\pi} C_F \left(\frac{17}{4} + \frac{3}{2} \ln \frac{\mu^2}{m_S^2} \right) \right]. \quad (2)$$

The argument of the above function has been changed to m_S , and the suppressed constant prefactor is not crucial for the reasoning below. For the NLO correction, we include only that in QCD for illustration, which dominates over others from the electroweak interaction. Equation (2) is

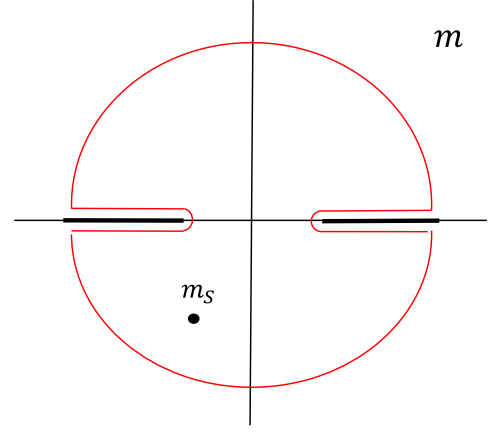


FIG. 1. Contour for Eq. (3), where the thick lines represent the branch cuts.

nothing but the width for a scalar decay into a $b\bar{b}$ quark pair without the initial particle density factor $1/(2m_S)$ and the relevant Yukawa coupling constant.

Viewing the presence of $1/m_S$ in Eq. (2), we consider the contour integration of $m_S^2 \Pi(m_S)$ in the complex m_S plane, instead of the m_S^2 plane, which contain different branching cuts [1]. This manipulation facilitates the derivation of a solution to the dispersion relation as seen later. The contour consists of two pieces of horizontal lines above and below the branch cut along the positive real axis, two pieces of horizontal lines above and below the branch cut along the negative real axis, and a circle C_R of large radius R as depicted in Fig. 1. We then have the identity

$$\frac{1}{2\pi i} \oint \frac{m^2 \Pi(m)}{m - m_S} dm = m_S^2 \Pi^{\text{P}}(m_S) = \frac{1}{2\pi i} \oint \frac{m^2 \Pi^{\text{P}}(m)}{m - m_S} dm. \quad (3)$$

The residue from the pole at $m = m_S$ marked in Fig. 1 has been replaced by its perturbative expression Π^{P} , since perturbation holds well for high scales. This residue is further written as the contour integral of $m_S^2 \Pi^{\text{P}}(m_S)$ based on the analyticity of perturbative calculations [5,9]. An intermediate advantage of considering the integrand $m^2 \Pi(m)$ is apparent; if there exists a singularity of $\Pi(m)$ in the low m region, its residue will be suppressed by a power of m^2/m_S^2 relative to the one in Eq. (3), and can be dropped.

Canceling the contributions from the large circle C_R on both sides of Eq. (3), which are approximated by the perturbative ones reliably, we arrive at

$$\int_{2m_b}^R \frac{m^2 \text{Im}\Pi(m)}{m - m_S} dm - \int_{-R}^{-2m_b} \frac{m^2 \text{Im}\Pi(m)}{m - m_S} dm = \int_{2m_b}^R \frac{m^2 \text{Im}\Pi^{\text{P}}(m)}{m - m_S} dm - \int_{-R}^{-2m_b} \frac{m^2 \text{Im}\Pi^{\text{P}}(m)}{m - m_S} dm. \quad (4)$$

The hadronic threshold $2m_B$, m_B being the B meson mass, and the quark-level threshold $2m_b$ have been assigned. The imaginary part of the perturbative correlation function is attributed to the logarithmic factor $\ln(-m^2 + 4m_b^2)$ from the b -quark loop, which can be split into $\ln(-m^2 + 4m_b^2) = \ln(m + 2m_b) + \ln(-m + 2m_b)$. The former (latter) gives the phase $i\pi$ ($-i\pi$) for $m < -2m_b$ ($m > 2m_b$) along the contour above the branch cut. This opposite sign has been reflected explicitly between the two terms on each side of Eq. (4). The unknown spectral function $\text{Im}\Pi(m)$, involving strong dynamics characterized by the scale m_B , will be solved from the above dispersion relation with the input of $\text{Im}\Pi^{\text{P}}(m)$. That is, Eq. (5) will be handled as an inverse problem [6].

Note that the input $\text{Im}\Pi^{\text{P}}(m)$ is an odd function in m , so is the corresponding solution $\text{Im}\Pi(m)$ from Eq. (4). The variable change $m \rightarrow -m$ for the second integrals on both sides leads to

$$\int_{4m_b^2}^{R^2} \frac{m \text{Im}\Pi(m)}{m_S^2 - m^2} dm^2 = \int_{4m_b^2}^{R^2} \frac{m \text{Im}\Pi^{\text{P}}(m)}{m_S^2 - m^2} dm^2. \quad (5)$$

We supply an intuitive interpretation for the above equation. The unknown spectral function $\text{Im}\Pi(m)$ on the left-hand side collects contributions from real physical particles. Take the spectral function in the case of light-quark vector currents as an example. As the momentum squared q^2 injected into the vector current increases from zero and crosses the dipion threshold, the two virtual quarks emitted from one current fragment into two real pions, whose invariant mass defines the hadronic threshold $2m_\pi$. As q^2 increases further and crosses the ρ meson threshold, the two virtual quarks emitted from one current can “annihilate” into a real ρ meson, which then “decays” into the two virtual quarks that end at the other current. One can think of the coupling between the ρ meson and the quark pair as $g_{\rho q\bar{q}}$ appearing in the Nambu-Jona-Lasinio model [18]. This real ρ meson contribution to the unknown spectral function has been parametrized as a pole term in QCD sum rules.

In the present case with the b -quark scalar currents, the hadronic threshold $2m_B$ in Eq. (4) corresponds to $2m_\pi$, and a real Higgs boson contribution corresponds to the real ρ meson contribution mentioned above, with the Yukawa coupling between the Higgs boson and the b -quark pair corresponding to $g_{\rho q\bar{q}}$. The right-hand side of Eq. (5) is evaluated in perturbation theory systematically by starting with two real b quarks. Then Eq. (5) means that the two dispersive integrals, one in terms of the unknown spectral function from the contributions of real physical particles and the other in terms of the perturbative spectral function from the contribution of real b quarks, should be equal at large m_S . Solving Eq. (5) directly, one can extract the Higgs boson mass in the same way as extracting the ρ meson mass from the correlation function of light-quark currents [9].

The idea behind our formalism is thus similar to that of QCD sum rules, but with power corrections in $(m_B - m_b)/m_S$ originating from the difference between the thresholds $2m_B$ and $2m_b$, which are necessary for establishing a physical solution [5]. Without these power corrections, i.e., if m_b is equal to m_B , there will be only the trivial solution $\text{Im}\Pi(m) = \text{Im}\Pi^{\text{P}}(m)$ and no constraint on the Higgs boson mass.

It has been proved rigorously [19] that the solution to an integral equation like Eq. (5) (the Fredholm equation of the first kind), if existing, is unique under adequate boundary conditions. We will construct a solution to Eq. (5) below. Moving the integrand on the right-hand side to the left-hand side, and regarding it as a subtraction term, we get

$$\int_{4m_b^2}^{\infty} \frac{\Delta\rho(m)}{(m_S^2 - m^2)} dm^2 = 0, \quad \Delta\rho(m) \equiv m\Delta\Pi(m), \quad \Delta\Pi(m) = \text{Im}\Pi(m) - \text{Im}\Pi^{\text{P}}(m). \quad (6)$$

The subtracted unknown function $\Delta\Pi(m)$ is fixed to $-\text{Im}\Pi^{\text{P}}(m)$ in the interval $(2m_b, 2m_B)$ of m , and approaches zero at large m , where $\text{Im}\Pi(m) \rightarrow \text{Im}\Pi^{\text{P}}(m)$. This explains why the upper bound R^2 of the integration variable m^2 can be extended to infinity. The procedure of solving Eq. (6) has been elucidated in [1], which is recaptured here for a self-contained presentation. We change m_S and m in Eq. (6) into the dimensionless variables u and v via $m_S^2 - 4m_b^2 = u\Lambda$ and $m^2 - 4m_b^2 = v\Lambda$, respectively, obtaining

$$\int_0^{\infty} dv \frac{\Delta\rho(v)}{u - v} = 0. \quad (7)$$

The purpose of introducing the arbitrary scale Λ will become transparent shortly.

Because $\Delta\rho(v)$ diminishes at large v ; namely, the major contribution to Eq. (7) comes from the region with finite v , we can expand Eq. (7) into a power series in $1/u$ for sufficiently large $|u|$ by inserting $1/(u - v) = \sum_{i=1}^{\infty} v^{i-1}/u^i$. Note that u , i.e., m_S could be a complex number, as indicated in Fig. 1. Equation (7) then dictates a vanishing coefficient for every power of $1/u$. We start with the case with N vanishing coefficients,

$$\int_0^{\infty} dv v^{i-1} \Delta\rho(v) = 0, \quad i = 1, 2, 3 \dots, N, \quad (8)$$

where N will be pushed to infinity eventually. The first N polynomials $L_0^{(\alpha)}(v), L_1^{(\alpha)}(v), \dots, L_{N-1}^{(\alpha)}(v)$ are composed of the terms $1, v, \dots, v^{N-1}$ appearing in the above expressions. Equation (8) thus implies an expansion of $\Delta\rho(v)$ in terms of the generalized Laguerre polynomials $L_j^{(\alpha)}(v)$ with degrees j not lower than N ,

$$\Delta\rho(v) = \sum_{j=N}^{N'} a_j v^\alpha e^{-v} L_j^{(\alpha)}(v), \quad N' > N, \quad (9)$$

owing to the orthogonality of the polynomials, in which a_j represent a set of unknown coefficients. The index α describes the behavior of $\Delta\rho(v)$ around $v \sim 0$. The highest degree N' can be fixed in principle by the aforementioned initial condition of $\Delta\rho(v)$ in the interval $(0, 4(m_B^2 - m_b^2)/\Lambda)$ of v . Since $-\text{Im}\Pi^P$ is a smooth function, N' needs not be infinite.

A generalized Laguerre polynomial takes the approximate form for a large j , $L_j^{(\alpha)}(v) \approx j^{\alpha/2} v^{-\alpha/2} e^{v/2} J_\alpha(2\sqrt{jv})$ [20], up to corrections of $1/\sqrt{j}$, J_α being a Bessel function of the first kind. Equation (9) becomes

$$\begin{aligned} \Delta\rho(m) &\approx \sum_{j=N}^{N'} a_j \sqrt{\frac{j(m^2 - 4m_b^2)^\alpha}{\Lambda}} e^{-(m^2 - 4m_b^2)/(2\Lambda)} \\ &\times J_\alpha\left(2\sqrt{\frac{j(m^2 - 4m_b^2)}{\Lambda}}\right), \end{aligned} \quad (10)$$

where the variable v has been written as $(m^2 - 4m_b^2)/\Lambda$ explicitly. Defining the scaling variable $\omega \equiv \sqrt{N/\Lambda}$, we have the approximation $N'/\Lambda = \omega^2 + (N' - N)/N \approx \omega^2$ for a finite $N' - N$. Equation (10) then reduces to

$$\Delta\rho(m) \approx y \left(\omega\sqrt{m^2 - 4m_b^2}\right)^\alpha J_\alpha\left(2\omega\sqrt{m^2 - 4m_b^2}\right), \quad (11)$$

as the common Bessel functions $J_\alpha(2\sqrt{j(m^2 - 4m_b^2)/\Lambda}) \approx J_\alpha(2\omega\sqrt{m^2 - 4m_b^2})$ for $j = N, N+1, \dots, N'$ are factored out, and the sum of the unknown coefficients $\sum_{j=N}^{N'} a_j$ is denoted by y . The exponential suppression factor $e^{-(m^2 - 4m_b^2)/(2\Lambda)} = e^{-\omega^2(m^2 - 4m_b^2)/(2N)}$ has been replaced by unity for a large N in the region with finite m and ω , which we are interested in.

We stress that a solution to the dispersion relation should be insensitive to the variation of the arbitrary scale Λ , i.e., of ω , which is introduced via the artificial variable changes. To realize this insensitivity, we make a Taylor expansion of $\Delta\rho(m_S)$,

$$\begin{aligned} \Delta\rho(m_S) &= \Delta\rho(m_S)|_{\omega=\bar{\omega}} + \frac{d\Delta\rho(m_S)}{d\omega} \\ &\times \left. (\omega - \bar{\omega}) + \frac{1}{2} \frac{d^2\Delta\rho(m_S)}{d\omega^2} \right|_{\omega=\bar{\omega}} \\ &\times (\omega - \bar{\omega})^2 + \dots, \end{aligned} \quad (12)$$

where the constant $\bar{\omega}$, together with the index α and the coefficient y , are fixed via the fit of the first term

$\Delta\rho(m_S)|_{\omega=\bar{\omega}}$ to $-m_S \text{Im}\Pi^P(m_S)$ in the interval $(2m_b, 2m_B)$ of m_S . The insensitivity to the scaling variable ω requires the vanishing of the first derivative in Eq. (12), $d\Delta\rho(m_S)/d\omega|_{\omega=\bar{\omega}} = 0$, from which roots of m_S are solved. Furthermore, the second derivative $d^2\Delta\rho(m_S)/d\omega^2|_{\omega=\bar{\omega}}$ should be minimal to maximize the stability window around $\bar{\omega}$, in which $\Delta\rho(m_S)$ remains almost independent of ω . It will be seen that only when m_S takes a value close to the physical Higgs boson mass, can the above requirements be satisfied. Equation (11) with this specific m_S establishes the solution to the dispersion relation in Eq. (6) with the initial condition in the interval $(2m_b, 2m_B)$. Once the solution is found, we increase the degree N for the polynomial expansion in Eq. (9) and the scale Λ arbitrarily by keeping $\omega = \sqrt{N/\Lambda}$ within the stability window. Then all the deductions based on the large- N scenario are justified.

Comparing $\Delta\rho(m_S)$ in Eq. (11) in the limit $m_S \rightarrow 2m_b$ with the input

$$-m_S \text{Im}\Pi^P(m_S) \propto (m_S^2 - 4m_b^2)^{3/2}, \quad (13)$$

we read off the index $\alpha = 3/2$. It is clear now why we considered the contour integration of $m^2\Pi(m)$; the corresponding input is proportional to a simple power of $m^2 - 4m_b^2$, so that the index α can be determined unambiguously. The boundary condition $\Delta\rho(m_S)$ at $m_S = 2m_B$ sets the coefficient

$$\begin{aligned} y &= -2m_B \text{Im}\Pi^P(2m_B) \left[\left(2\omega\sqrt{m_B^2 - m_b^2}\right)^{3/2} \right. \\ &\times \left. J_{3/2}\left(4\omega\sqrt{m_B^2 - m_b^2}\right) \right]^{-1}. \end{aligned} \quad (14)$$

The running coupling constant is given by

$$\alpha_s(\mu) = \frac{4\pi}{\beta_0 \ln(\mu^2/\Lambda_{\text{QCD}}^2)} \left[1 - \frac{\beta_1 \ln(\mu^2/\Lambda_{\text{QCD}}^2)}{\beta_0^2 \ln(\mu^2/\Lambda_{\text{QCD}}^2)} \right], \quad (15)$$

with the coefficients $\beta_0 = 11 - 2n_f/3$ and $\beta_1 = 2(51 - 19n_f/3)$. We take the QCD scale $\Lambda_{\text{QCD}} = 0.2 \text{ GeV}$ for the number of active quark flavors $n_f = 5$ [21], and choose the renormalization scale $\mu = m_S$. Adopting the b -quark mass $m_b = 4.43$ and the B -meson mass $m_B = 5.28$ [4], we derive $\bar{\omega} = 0.0254 \text{ GeV}^{-1}$ from the best fit of Eq. (11) to $-m_S \text{Im}\Pi^P(m_S)$ from Eq. (2) in the interval $(2m_b, 2m_B)$ of m_S . The fit result in terms of $-\Delta\Pi(m_S)$ is contrasted with $\text{Im}\Pi^P(m_S)$ in Fig. 2(a). Their perfect match confirms that the approximate solution in Eq. (11) works well, and that other methods for acquiring $\bar{\omega}$ should return similar values; for example, equating $\Delta\Pi(m_S)$ and $-\text{Im}\Pi^P(m_S)$ at $m_S = m_B + m_b$ yields $\bar{\omega} = 0.0256 \text{ GeV}^{-1}$, almost identical to 0.0254 GeV^{-1} from the best fit.

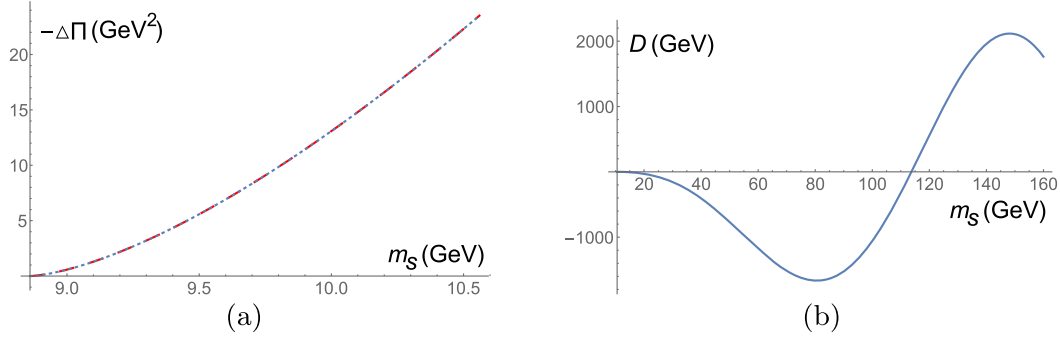


FIG. 2. (a) Comparison of $-\Delta\Pi(m_S)$ from the fit (dotted line) with the input $\text{Im}\Pi^P(m_S)$ (dashed line) in Eq. (2) in the interval $(2m_b, 2m_B)$ of m_S . (b) Dependence of the derivative $D(m_S)$ in Eq. (16) on m_S .

As postulated before, the derivative $d\Delta\rho(m_S)/d\omega$ vanishes at $\omega = \bar{\omega}$, i.e.,

$$D(m_S) \equiv \frac{d}{d\omega} \frac{J_{3/2}\left(2\omega\sqrt{m_S^2 - 4m_b^2}\right)}{J_{3/2}\left(4\omega\sqrt{m_B^2 - m_b^2}\right)} \Big|_{\omega=\bar{\omega}} = 0, \quad (16)$$

where the factors independent of ω in the solution have been dropped for simplicity. Figure 2(b) shows the dependence of the derivative $D(m_S)$ on m_S , which reveals an oscillatory behavior. The first root located at $m_S = 2m_B = 10.56$ GeV, attributed to the boundary condition of $\Delta\rho(m_S)$ at this m_S , is trivial and bears no physical significance. It has been checked that the second derivatives are larger at higher roots, as observed in [1], so a smaller root is preferred. Therefore, we identify the second root at $m_S = 114$ GeV as the physical solution of the Higgs boson mass, which deviates from the measured one $m_H = (125.25 \pm 0.17)$ GeV by only 9%. The value of $\Delta\Pi(m_S = 114$ GeV), amounting to about 10% of the perturbative input $\text{Im}\Pi^P(m_S = 114$ GeV), indicates a minor nonperturbative contribution to Higgs decays associated with the hadronic threshold $2m_B$.

We estimate the theoretical uncertainties involved in the above analysis. The variation of the b -quark mass in its well-accepted range causes a minor effect; a higher (lower) $m_b = 4.8$ (4.0) GeV yields the Higgs boson mass 116 (111) GeV. That is, an 8% change of m_b makes less than 3% impact on the outcome. We mention that the NLO QCD correction to the perturbative input is crucial for attaining a sensible Higgs boson mass. Without the $O(\alpha_s)$ piece in Eq. (2), $\bar{\omega}$ would be as small as 1.9×10^{-4} GeV $^{-1}$, and the Higgs boson mass becomes as large as 1.5×10^4 GeV. It is then necessary to examine the dependence on the renormalization scale μ : the choice $\mu = m_S/2$ ($\mu = 2m_S$) leads to the Higgs boson mass 126 (112) GeV. It hints that higher-order corrections are under control, and their inclusion into the perturbative input is likely to account for the measured m_H . This $O(10\%)$ QCD effect from the initial condition at the m_b scale is quite reasonable.

III. Z-BOSON MASS

We extract the Z-boson mass in the same formalism. A Z boson decays into a $b\bar{b}$ quark pair through the vertex $\gamma_\mu(v_b + a_b\gamma_5)$, where the vector coupling v_b and the axial-vector coupling a_b can vary independently in a mathematical point of view. It is noticed in perturbative calculations [22–25] that the axial-vector contribution is less than 10% of the vector one in the threshold region $(2m_b, 2m_B)$, and, in particular, completely negligible near the quark-level threshold $2m_b$. Hence, we work on the v_b term, namely, the two-point correlation function

$$\begin{aligned} \Pi_{\mu\nu}(q^2) &= i \int d^4x e^{iq \cdot x} \langle 0 | T [J_\mu(x) J_\nu(0)] | 0 \rangle \\ &= (q_\mu q_\nu - g_{\mu\nu} q^2) \Pi(q^2), \end{aligned} \quad (17)$$

with the momentum q being injected into the b -quark vector current $J_\mu = \bar{b}\gamma_\mu b$. Similarly, we keep only the leading term in the OPE for the correlation function $\Pi(q^2)$, i.e., the perturbative contribution [22–25], which gives the imaginary piece

$$\begin{aligned} \text{Im}\Pi^P(m_V) &\propto m_V^2 \beta(m_V) [3 - \beta^2(m_V)] \left\{ 1 + \frac{4\alpha_s(m_V)}{3} \right. \\ &\quad \left. \times \left[\frac{\pi}{2\beta(m_V)} - \frac{3 + \beta(m_V)}{4} \left(\frac{\pi}{2} - \frac{3}{4\pi} \right) \right] \right\}, \end{aligned} \quad (18)$$

with the invariant mass $m_V = \sqrt{q^2}$ and the factor $\beta(m_V) = \sqrt{1 - 4m_b^2/m_V^2}$. We have also suppressed the constant prefactor in the above expression, and taken into account the NLO QCD correction solely, where the renormalization scale μ in α_s is set to m_V .

The perturbative input in Eq. (18) consists of both even and odd functions in m_V , $\text{Im}\Pi^P(m_V) = \text{Im}\Pi_e^P(m_V) + \text{Im}\Pi_o^P(m_V)$, so the unknown function is decomposed into $\text{Im}\Pi(m_V) = \text{Im}\Pi_e(m_V) + \text{Im}\Pi_o(m_V)$ accordingly. We start with the contour integration of $\Pi(m_V)$ in this case, deriving

$$\int_{4m_b^2}^{\infty} \frac{\Delta\Pi_e(m)}{m_V^2 - m^2} dm^2 = 0, \quad \int_{4m_b^2}^{\infty} \frac{\Delta\Pi_o(m)}{m(m_V^2 - m^2)} dm^2 = 0, \quad (19)$$

for the even and odd pieces, respectively. The subtracted unknown functions are defined by $\Delta\Pi_{e,o}(m) = \text{Im}\Pi_{e,o}(m) - \text{Im}\Pi_{e,o}^{\text{P}}(m)$, which are fixed to $-\text{Im}\Pi_{e,o}^{\text{P}}(m)$ in the interval $(2m_b, 2m_B)$ of m . The solutions to Eq. (19) are constructed in a similar way. The even piece is written as

$$\Delta\Pi_e(m_V) \approx y_e \left(\omega \sqrt{m_V^2 - 4m_b^2} \right)^\alpha J_\alpha \left(2\omega \sqrt{m_V^2 - 4m_b^2} \right), \quad (20)$$

whose comparison with

$$-\text{Im}\Pi_e^{\text{P}}(m_V) \propto 8\pi\alpha_s(2m_b)m_b^2 = \text{const.}, \quad (21)$$

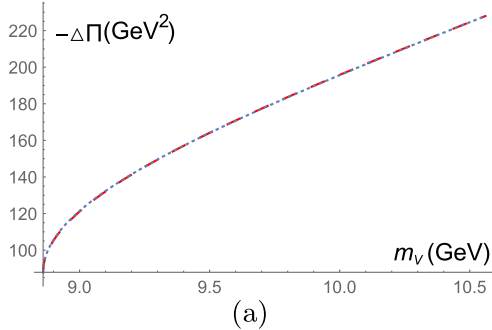
in the limit $m_V \rightarrow 2m_b$ specifies the index $\alpha = 0$. The odd piece reads

$$\frac{\Delta\Pi_o(m_V)}{m_V} \approx y_o \left(\omega \sqrt{m_V^2 - 4m_b^2} \right)^\alpha J_\alpha \left(2\omega \sqrt{m_V^2 - 4m_b^2} \right), \quad (22)$$

whose comparison with

$$-\frac{\text{Im}\Pi_o^{\text{P}}(m_V)}{m_V} \propto (m_V^2 - 4m_b^2)^{1/2}, \quad (23)$$

in the limit $m_V \rightarrow 2m_b$ places the index $\alpha = 1/2$.



The final solution takes the form

$$\begin{aligned} \Delta\Pi(m_V) &= \Delta\Pi_e(m_V) + \Delta\Pi_o(m_V) \\ &\approx y_e J_0 \left(2\omega \sqrt{m_V^2 - 4m_b^2} \right) \\ &\quad + y_o m_V \left(\omega \sqrt{m_V^2 - 4m_b^2} \right)^{1/2} \\ &\quad \times J_{1/2} \left(2\omega \sqrt{m_V^2 - 4m_b^2} \right). \end{aligned} \quad (24)$$

The second term in Eq. (24) diminishes in the limit $m_V \rightarrow 2m_b$, so the coefficient y_e is related to the initial value $-\text{Im}\Pi^{\text{P}}(m_V = 2m_b)$,

$$y_e = -\frac{\text{Im}\Pi^{\text{P}}(2m_b)}{J_0(0)} = -\text{Im}\Pi^{\text{P}}(2m_b). \quad (25)$$

The coefficient y_o is obtained from the value $-\text{Im}\Pi^{\text{P}}(m_V = 2m_B)$,

$$y_o = -\frac{\left[\text{Im}\Pi^{\text{P}}(2m_B) + y_e J_0 \left(4\omega \sqrt{m_B^2 - m_b^2} \right) \right]}{2m_B \left(2\omega \sqrt{m_B^2 - m_b^2} \right)^{1/2} J_{1/2} \left(4\omega \sqrt{m_B^2 - m_b^2} \right)}. \quad (26)$$

With the same masses $m_b = 4.43$ GeV and $m_B = 5.28$, we get $\bar{\omega} = 0.0249$ GeV⁻¹ from the best fit of Eq. (24) to $-\text{Im}\Pi^{\text{P}}(m_V)$ in the interval $(2m_b, 2m_B)$. The fit result $-\Delta\Pi(m_V)$ is contrasted with $\text{Im}\Pi^{\text{P}}(m_V)$ in Eq. (18) in Fig. 3(a), where the exact overlap of the two curves justifies the approximate solution in Eq. (24).

The stability of the solution under the variation of ω demands the vanishing of the derivative

$$D(m_V) \equiv \frac{1}{y_e} \frac{d\Delta\Pi(m_V)}{d\omega} \Big|_{\omega=\bar{\omega}} = 0, \quad (27)$$

where the additional constant $1/y_e$, rendering the differentiated function dimensionless, does not affect the locations of roots in m_V . Figure 3(b) displays the dependence of

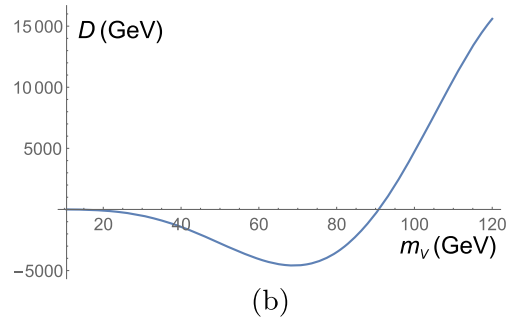


FIG. 3. (a) Comparison of $-\Delta\Pi(m_V)$ from the fit (dotted line) with the input $\text{Im}\Pi^{\text{P}}(m_V)$ (dashed line) in the interval $(2m_b, 2m_B)$ of m_V . (b) Dependence of the derivative $D(m_V)$ in Eq. (27) on m_V .

the above derivative on m_V , where the first root at $m_V = 2m_B$ has no physical significance. The second root at $m_V = 90.8$ GeV corresponds to the physical solution of the Z boson mass, which deviates from the measured one $m_Z = (91.1876 \pm 0.0021)$ GeV [4] by only 0.4%. This result is not sensitive to the variation of the b -quark mass, analogous to the Higgs boson case. It is observed that the nonperturbative contribution arising from the difference between the thresholds $2m_b$ and $2m_B$ also amounts to about 20% of the perturbative contribution to $\text{Im}\Pi(m_Z)$, i.e., the Z boson decay width. We remark that $\Upsilon(4S)$ with the quantum numbers $J^{PC} = 1^{--}$, the mass 10.58 GeV and the width 20.5 MeV can decay into the $B\bar{B}$ state. To probe such a fine resonance structure in our formalism, more precise OPE for the correlation function including QCD condensates [26] is needed.

Note that the vector coupling $v_b = -1 + 4 \sin^2 \theta_W / 3$, with the Weinberg angle θ_W and $\sin \theta_W = g' / \sqrt{g'^2 + g^2}$, is a constant, given the U(1) coupling g' and the SU(2) coupling g in the SM. The relation $\sin^2 \theta_W = 1 - m_W^2 / m_Z^2$ then implies that the W - and Z -boson masses are proportionate. It is interesting to verify this proportionality by introducing the m_V -dependent coupling

$$v_b(m_V) = -1 + \frac{4}{3} \left(1 - \frac{m_W^2}{m_V^2} \right), \quad (28)$$

for the fixed W -boson mass $m_W = 80.377$ GeV into the perturbative input in Eq. (18). The additional factor $1/m_V^2$ in Eq. (28) modifies the derivation for the odd piece of the solution, which follows the contour integration of $m_V^2 \Pi(m_V) / (m_V^2 - 4m_b^2)$ as in the Higgs boson case. The comparison of the solution with $-m_V \text{Im}\Pi_b^p(m_V) / (m_V^2 - 4m_b^2)$ in the interval $(2m_b, 2m_B)$ sets the index $\alpha = -1/2$. The straightforward steps yield $m_Z = 38$ GeV, inconsistent with the measured value. That is, the solution $m_V = 90.8$ GeV exists, only when the W - and Z -boson masses are correlated to each other, conforming to the Higgs mechanism of the electroweak symmetry breaking. It also means that the W -boson mass is known, once the Z -boson mass is extracted from our dispersion analysis and the electroweak couplings are designated.

IV. TOP-QUARK MASS

At last, we come to the determination of the top-quark mass. Because a top quark does not form a bound state, the correlation functions defined in terms of hadronic matrix elements, such as the heavy meson lifetimes considered in [1] for constraining heavy quark masses, may not be suitable. Therefore, we extend the formalism developed for neutral meson mixing in [5] to the $Q\bar{u} - \bar{Q}u$ mixing through the box diagrams with a fictitious heavy quark Q of an arbitrary mass m_Q . The Cabibbo-Kobayashi-Maskawa (CKM) factors associated with the intermediate

$b\bar{b}, b\bar{s}, s\bar{s}, \dots$ channels can vary independently in a mathematical viewpoint, so we focus on the dispersion relation for the $b\bar{b}$ channel below. All our dispersive studies on the electroweak-scale masses then depend on a single quark-level threshold $2m_b$. The box diagrams generate two effective four-fermion operators with the $(V-A)(V-A)$ and $(S-P)(S-P)$ structures. It suffices to pick up the contribution from the former, whose imaginary piece in perturbative evaluations is expressed as [27,28]

$$\begin{aligned} \text{Im}\Pi^p(m_Q) \propto C_2(m_Q) & \frac{\sqrt{m_Q^2 - 4m_b^2}}{m_Q(m_W^2 - m_b^2)^2} \\ & \times \left[2 \left(1 + \frac{m_b^4}{4m_W^4} \right) m_W^2 (m_Q^2 - 4m_b^2) \right. \\ & \left. - 6m_b^2 (m_Q^2 - 2m_b^2) \right]. \end{aligned} \quad (29)$$

We have kept only the Wilson coefficient $C_2(\mu)$, which dominates over $C_1(\mu)$ at the relevant renormalization scale $\mu = m_Q$.

The second term in the square brackets of Eq. (29) is down by a tiny ratio m_b^2/m_W^2 , so the behavior of Eq. (29) in m_Q is governed by the first term. It suggests an analysis similar to that for the Higgs boson, and the solution of the unknown subtracted function $\Delta\rho(m_Q)$ is the same as Eq. (11). The comparison of Eq. (11) with Eq. (29) in the region around $m_Q \sim 2m_b$ then specifies the index $\alpha = 3/2$. The coefficient y fixed by the boundary condition at $m_Q = 2m_B$ is identical to Eq. (14). The same b -quark and B -meson masses give $\bar{\omega} = 0.0164$ GeV $^{-1}$ from the best fit of Eq. (11) to the perturbative input $-m_Q \text{Im}\Pi^p(m_Q)$ in the interval $(2m_b, 2m_B)$ of m_Q . We present the fit result in terms of $-\Delta\Pi(m_Q)$ and the perturbative input $\text{Im}\Pi^p(m_Q)$ in Fig. 4(a) with excellent match between them. The derivative $d\Delta\rho(m_Q)/d\omega$ vanishes at $\omega = \bar{\omega}$ as in Eq. (16). Figure 4(b) exhibits the dependence of the derivative $D(m_Q)$ on m_Q , where the second root at $m_Q = 176$ GeV corresponds to the physical top-quark mass, with only 2% deviation from the measured one $m_t = (172.69 \pm 0.30)$ GeV [4].

The solved top-quark mass is more sensitive to the variation of m_b compared to the Higgs and Z -boson cases; choosing a slightly smaller $m_b = 4.42$ GeV would lower the root of m_Q to 169 GeV. Decreasing the renormalization scale in the Wilson coefficient C_2 a bit to $\mu = 0.98m_Q$ leads to $m_t = 173$ GeV in agreement with the data. It hints that the top-quark mass can be well-accommodated within the theoretical uncertainty of the calculation. To be cautious, we have repeated the above derivation for the B_d meson mixing via the $c\bar{c}$ channel. The difference is that the HQE contributions from the effective weak Hamiltonian [29,30], instead of from the box diagrams, are input. Taking the

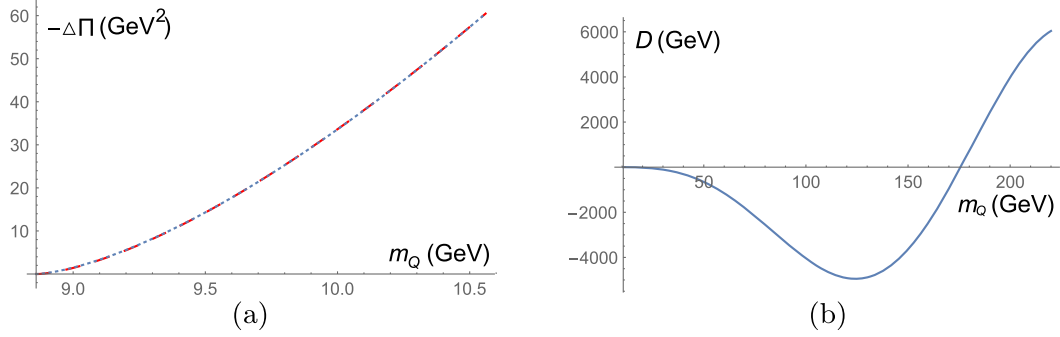


FIG. 4. (a) Comparison of $-\Delta\Pi(m_Q)$ from the fit (dotted line) with the input $\text{Im}\Pi^P(m_Q)$ (dashed line) in the interval $(2m_b, 2m_B)$ of m_V . (b) Dependence of the derivative $D(m_Q)$ in Eq. (16) on m_Q .

c -quark mass $m_c = 1.35$ GeV [1] and the D -meson mass $m_D = 1.86$ GeV [4], we deduce the b -quark mass $m_b \approx 4.7$ GeV, close to the values assumed in the present work and extracted from the study of heavy meson lifetimes [1]. This result, to be published elsewhere, solidifies the determination of the top-quark mass and the consistency of our approach.

V. CONCLUSION

We have demonstrated that dispersive analyses of physical observables involving heavy particles can disclose stringent connections between their high-energy and low-energy behaviors. The strategy is to treat the dispersion relation obeyed by a correlation function as an inverse problem with reliable perturbative inputs in the high-energy region. Two arbitrary parameters were introduced into the formalism: the lowest degree N for the Laguerre polynomial expansion and the variable Λ , which scales the heavy particle mass in dispersive integrals. The essential feature is that the solution for a physical observable depends on the ratio $\omega^2 = N/\Lambda$, which it must be insensitive to. This criterion can be met, only when the heavy particle takes a specific mass. Once the solution with the physical mass is established, both N and Λ can be extended to sufficiently high values by keeping ω in the stability window. All the large N approximations assumed in solving the dispersion relation are then justified. We emphasize that no *a priori* information of the considered heavy particle was included: the fictitious scalar mass m_S , vector mass m_V and heavy-quark mass m_Q appearing in the dispersion relations are completely arbitrary, and the b quark mass is the only input parameter. Hence, the

emergence of the Higgs-boson, Z -boson and top-quark masses as the stable solutions is not coincidence, but a necessary consequence of highly nontrivial constraints imposed by the dispersion relations.

We point out that the strong interaction plays an important role here; the initial conditions in the interval bounded by the quark- and hadron-level thresholds are indispensable for the existence of nontrivial solutions. Our previous analysis has manifested the correlation among the masses of a strange quark, charm quark, bottom quark, muon and τ lepton through the dispersion relations for heavy meson decay widths. Together with the present work, we claim that the particle masses from 0.1 GeV up to the electroweak scale may be understood by means of the internal consistency of SM dynamics, and that at least some of the SM parameters are not free, but constrained dynamically. Our observation provides a natural explanation of the flavor structures of the SM without resorting to any new symmetries, interactions or mechanism, as attempted intensively in the literature. We have taken the perturbative inputs up to NLO in QCD, and the current determinations of the particle masses deviate from the measured ones at percent level. More precise predictions will be pursued by taking into account subleading contributions in inputs, and theoretical uncertainties will be investigated more thoroughly in the future.

ACKNOWLEDGMENTS

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