Evolution kernels of twist-two operators

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The evolution kernels that govern the scale dependence of the generalized parton distributions are invariant under transformations of the $\mathrm{SL}(2,\mathrm{R})$ collinear subgroup of the conformal group. Beyond one loop the symmetry generators, due to quantum effects, differ from the canonical ones. We construct the transformation that brings the *full* symmetry generators back to their canonical form and show that the eigenvalues (anomalous dimensions) of the new, canonically invariant, evolution kernel coincide with the so-called parity respecting anomalous dimensions. We develop an efficient method that allows one to restore an invariant kernel from the corresponding anomalous dimensions. As an example, the explicit expressions for next-to-next-to-leading order invariant kernels for the twist-two flavor-nonsinglet operators in QCD and for the planar part of the universal anomalous dimension in $\mathcal{N}=4$ supersymmetric Yang-Mills are presented.

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I. INTRODUCTION

The study of deeply virtual Compton scattering (DVCS) gives one access to the generalized parton distributions (GPDs) [1–3] that encode the information on the transverse position of quarks and gluons in the proton in dependence on their longitudinal momentum. To extract the GPDs from experimental data one has to know, among other things, their scale dependence. The latter is governed by the renormalization group equations (RGEs) or, equivalently, evolution equations for the corresponding twist-two operators. Essentially the same equations govern the scale dependence of the ordinary parton distribution functions (PDFs) in the deep inelastic scattering (DIS) process. In DIS one is interested in the scale dependence of forward matrix elements of the local twist-two operators and therefore can neglect the operator mixing problem between local operators from the operator product expansion (OPE). In the nonsinglet sector, there is only one operator for a given spin/dimension. The anomalous dimensions of such operators are known currently with the three-loop accuracy [4,5], and first results at four loops are becoming available [6,7]. In contrast, the DVCS process corresponds to nonzero momentum transfer from the initial to the final state and, as a consequence, the total derivatives of the local twist-two operators have to be taken into consideration. All these operators mix under renormalization and the RGE has a matrix form. The DIS anomalous dimensions appear as the diagonal entries of the anomalous dimensions matrix which, in general, has a triangular form for the latter.

It was shown by Müller [8,9] that the off-diagonal part of the anomalous dimension matrix is completely determined by a special object, the so-called conformal anomaly. Moreover, to determine the off-diagonal part of the anomalous dimension matrix with ℓ-loop accuracy it is enough to calculate the conformal anomaly at one loop less. This technique was used to reconstruct all relevant evolution kernels/anomalous dimension matrices in QCD at two loops [10–12].

A similar approach, but based on the analysis of QCD at the critical point in noninteger dimensions, was developed in Refs. [13–15]. It was shown that the evolution kernels in $d=4$ in the $\overline{\mathrm{MS}}$-like renormalization scheme inherit the symmetries of the critical theory in $d=4-2\epsilon$ dimensions. As expected, the symmetry generators deviate from their canonical form. Corrections to the generators have a rather simple form if they are written in terms of the evolution kernel and the conformal anomaly. It was shown in Ref. [16] that by changing a renormalization scheme one can get rid of the conformal anomaly term in the generators bringing them into the so-called "minimal" form. Beyond computing the evolution kernels, the conformal approach has also been employed to calculate the next-to-next-to-leading order (NNLO) coefficient (hard) functions of vector and axial-vector contributions in DVCS [17,18], the latter

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in agreement with a direct Feynman diagram calculation [19]. Moreover, the conformal technique is also applicable to computing kinematic higher-power corrections in two-photon processes as was recently shown in Refs. [20,21].

In this paper we construct a similarity transformation that brings the full quantum generators back to the canonical form. Correspondingly, the transformed evolution kernel is invariant under the canonical $\mathrm{SL}(2,\mathrm{R})$ transformation. Moreover, we will show that the eigenvalues of this kernel are given by the so-called parity respecting anomalous dimension, $f(N)$ [22,23], which is related to the PDF anomalous dimension spectrum $\gamma(N)$ as

$$\gamma(N) = f\left(N + \bar{\beta}(a) + \frac{1}{2}\gamma(N)\right), \tag{1}$$

where $\bar{\beta}(a) = -\beta(a)/2a$ with $\beta(a)$ being the QCD beta function. The strong coupling α_s is normalized as $a = \alpha_s/(4\pi)$. We develop an effective approach to restore the canonically invariant kernel from its eigenvalues $\gamma(N)$. As an example, we present explicit expressions for three-loop invariant kernels in QCD and $\mathcal{N} = 4$ supersymmetric Yang-Mills (SYM) theory. The answers are given by linear combinations of harmonic polylogarithms [24], up to weight four in QCD and up to weight three in $\mathcal{N} = 4$ SYM. We also compare our exact result with the approximate expression for the three-loop kernels in QCD given in Ref. [16].

The paper is organized as follows: in Sec. II we describe the general structure of the evolution kernels of twist-two operators. In Sec. III we explain how to effectively recover the evolution kernel from the known anomalous dimensions and present our results for the invariant kernels in QCD and $\mathcal{N} = 4$ SYM. Section IV contains the concluding remarks. Some technical details are given in the appendixes.

II. KERNELS AND SYMMETRIES

We are interested in the scale dependence of the twist-two light-ray flavor nonsinglet operator [25]

$$\mathcal{O}(z_1, z_2) = [\bar{q}(z_1 n)\gamma_+[z_1 n, z_2 n]q(z_2 n)]_{\overline{\mathrm{MS}}}, \tag{2}$$

where n^μ is an auxiliary lightlike vector, $n^2 = 0$, $z_{1,2}$ are real numbers, $\gamma_+ = n^\mu \gamma_\mu$, $[z_1 n, z_2 n]$ stands for the Wilson line ensuring gauge invariance, and the subscript $\overline{\mathrm{MS}}$ denotes the renormalization scheme. This operator can be viewed as the generating function for local operators, $\mathcal{O}^{\mu_1\cdots\mu_N}$, that are symmetric and traceless in all Lorentz indices $\mu_1 \cdots \mu_N$.

The renormalized light-ray operator (2) satisfies the RGE

$$(\mu\partial_\mu + \beta(a)\partial_a + \mathbb{H}(a))\mathcal{O}(z_1, z_2) = 0, \tag{3}$$

where $\beta(a)$ is d-dimensional beta function

$$\beta(a) = -2a(\epsilon + \beta_0 a + \beta_1 a^2 + O(a^3)), \tag{4}$$

$\beta_0 = 11/3N_c - 2/3n_f$, etc., and $\mathbb{H}(a) = a\mathbb{H}_1 + a^2\mathbb{H}_2 + \cdots$ is an integral operator in z_1, z_2.

It follows from the invariance of the classical QCD Lagrangian under conformal transformations that the one-loop kernel $\mathbb{H}_1$ commutes with the *canonical* generators of the collinear conformal subgroup, $S_0, S_\pm$,

$$\begin{aligned} S_- &= -\partial_{z_1} - \partial_{z_2}, \\ S_0 &= z_1\partial_{z_1} + z_2\partial_{z_2} + 2, \\ S_+ &= z_1^2\partial_{z_1} + z_2^2\partial_{z_2} + 2z_1 + 2z_2. \end{aligned} \tag{5}$$

This symmetry is preserved beyond one loop albeit two of the generators, S_0, S_+ receive quantum corrections, $S_\alpha \mapsto \tilde{S}_\alpha(a) = S_\alpha + \Delta S_\alpha(a)$. The explicit form of these corrections can be found in Ref. [15].

It is quite useful to bring the generators to the following form using the similarity transformation [16]:

$$\begin{aligned} \mathbb{H}(a) &= e^{-X(a)}\mathrm{H}(a)e^{X(a)}, \\ \tilde{S}_\alpha(a) &= e^{-X(a)}\mathrm{S}_\alpha(a)e^{X(a)}, \end{aligned} \tag{6}$$

where $X(a) = aX_1 + a^2X_2 + \cdots$ is an integral operator known up to terms of $O(a^3)$ [11,16]. This transformation can be thought of as a change in a renormalization scheme.

The shift operator S_- is not modified and hence is identical to S_- in Eq. (5), and the quantum corrections to S_0 and S_+ come only through the evolution kernel

$$\mathrm{S}_0(a) = S_0 + \bar{\beta}(a) + \frac{1}{2}\mathrm{H}(a), \tag{7a}$$

$$\mathrm{S}_+(a) = S_+ + (z_1 + z_2)\left(\bar{\beta}(a) + \frac{1}{2}\mathrm{H}(a)\right), \tag{7b}$$

where $\bar{\beta}(a) = \beta_0 a + \beta_1 a^2 + \cdots$ is the beta function in four dimensions [cf. Eq. (1)]. The form of the generator $\mathrm{S}_0(a)$ is completely fixed by the scale invariance of the theory, while Eq. (7b) is the “minimal” ansatz consistent with the commutation relation $[\mathrm{S}_+, \mathrm{S}_-] = 2\mathrm{S}_0$. Since the operator $\mathrm{H}(a)$ commutes with the generators, $[\mathrm{H}(a), \mathrm{S}_\alpha(a)] = 0$, its form is completely determined by its spectrum (anomalous dimensions). However, since the generators do not have the simple form as in Eq. (5), it is yet necessary to find a way to recover the operator from its spectrum.

To this end we construct a transformation that brings the generators $\mathrm{S}_\alpha(a)$ to the canonical form S_α, Eq. (5). Let us define an operator $\mathrm{T}(\mathrm{H})$:

$$\mathrm{T}(\mathrm{H}) = \sum_{n=0}^{\infty} \frac{1}{n!}\mathrm{L}^n\left(\bar{\beta}(a) + \frac{1}{2}\mathrm{H}(a)\right)^n, \tag{8}$$

where $\mathrm{L} = \ln z_{12}$, $z_{12} \equiv z_1 - z_2$. Recall that z_1, z_2 are real variables, so for $z_{12} < 0$ it is necessary to choose a specific branch of the logarithm function. Although this choice is irrelevant for further analysis we chose the $+i0$ recipe for concreteness, i.e., $\mathrm{L} = \ln(z_{12} + i0)$. It can be shown that the operator $\mathrm{T(H)}$ intertwines the symmetry generators $\mathrm{S}_\alpha(a)$ and the canonical generators S_α. Namely,

$$\mathrm{T(H)S}_\alpha(a) = S_\alpha \mathrm{T(H)}; \tag{9}$$

see Appendix A for details. Let us also define a new kernel $\hat{\mathrm{H}}$ as

$$\mathrm{T(H)H}(a) = \hat{\mathrm{H}}(a)\mathrm{T(H)}. \tag{10}$$

It follows from Eqs. (9) and (10) that the operator $\hat{\mathrm{H}}$ commutes with the canonical generators in Eq. (5),

$$[S_\alpha, \hat{\mathrm{H}}(a)] = 0. \tag{11}$$

The problem of restoring a canonically invariant operator $\hat{\mathrm{H}}(a)$ from its spectrum is much easier than that for the operator $\mathrm{H}(a)$ and will be discussed in the next section. It can be shown that the inverse of $\mathrm{T(H)}$ takes the form

$$\mathrm{T}^{-1}(\mathrm{H}) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \mathrm{L}^n \left(\bar{\beta}(a) + \frac{1}{2}\hat{\mathrm{H}}(a)\right)^n; \tag{12}$$

see Appendix A. Further, it follows from Eq. (10) that

$$\begin{aligned}\mathrm{H}(a) &= \mathrm{T}^{-1}(\mathrm{H})\hat{\mathrm{H}}(a)\mathrm{T(H)} \\ &= \hat{\mathrm{H}}(a) + \sum_{n=1}^{\infty} \frac{1}{n!} \mathrm{T}_n(a) \left(\bar{\beta}(a) + \frac{1}{2}\mathrm{H}(a)\right)^n. \end{aligned} \tag{13}$$

The operators $\mathrm{T}_n(a)$ are defined by recursion

$$\mathrm{T}_n(a) = [\mathrm{T}_{n-1}(a), \mathrm{L}] \tag{14}$$

with the boundary condition $\mathrm{T}_0(a) = \hat{\mathrm{H}}(a)$. The nth term in the sum in Eq. (13) is of order $\mathcal{O}(a^{n+1})$ so that one can easily work out an approximation for $\mathrm{H}(a)$ with arbitrary precision, e.g.,

$$\begin{aligned}\mathrm{H}(a) = \hat{\mathrm{H}}(a) &+ \mathrm{T}_1(a)\left(1 + \frac{1}{2}\mathrm{T}_1(a)\right)\left(\bar{\beta}(a) + \frac{1}{2}\hat{\mathrm{H}}(a)\right) \\ &+ \frac{1}{2}\mathrm{T}_2(a)\left(\bar{\beta}(a) + \frac{1}{2}\hat{\mathrm{H}}(a)\right)^2 + \mathcal{O}(a^4). \end{aligned} \tag{15}$$

It can be checked that this expression coincides with that obtained in Ref. [16] [Eq. (3.9)].[1]

[1]The notations adopted here and in Ref. [16] differ slightly. To facilitate a comparison we note that the operators T_n defined here satisfy the equation $[S_+, \mathrm{T}_n] = n[\mathrm{T}_{n-1}, z_1 + z_2]$.

The evolution kernel $\hat{\mathrm{H}}(a)$ can be realized as an integral operator. It acts on a function of two real variables as follows:

$$\hat{\mathrm{H}}(a)f(z_1, z_2) = Af(z_1, z_2) + \int_+ h(\tau) f(z_{12}^\alpha, z_{21}^\beta), \tag{16}$$

where A is a constant, $z_{12}^\alpha \equiv z_1\bar{\alpha} + z_2\alpha$, $\bar{\alpha} \equiv 1 - \alpha$, and

$$\int_+ \equiv \int_0^1 d\alpha \int_0^{\bar{\alpha}} d\beta. \tag{17}$$

$\tau = \alpha\beta/\bar{\alpha}\bar{\beta}$ is called a conformal ratio. The weight function $h(\tau)$ in Eq. (16) only depends on this particular combination of the variables α, β as a consequence of invariance properties of $\hat{\mathrm{H}}$, Eq. (11).

It is easy to find that the operators T_n take the form

$$\mathrm{T}_n(a)f(z_1, z_2) = \int_+ \ln^n(1 - \alpha - \beta) h(\tau) f(z_{12}^\alpha, z_{21}^\beta) \tag{18}$$

that again agrees with the results of Ref. [16]. Note that this expression does not depend on the choice of the branch of the logarithm defining the function $\mathrm{L} = \ln z_{12}$ in Eq. (8); see Appendix A for more discussion.

III. ANOMALOUS DIMENSIONS VS KERNELS

First of all let us establish a connection between the eigenvalues of the operators H and $\hat{\mathrm{H}}$. Since both of them are integral operators of the functional form in Eqs. (16) and (18), both operators are diagonalized by functions of the form $\psi_N(z_1, z_2) = (z_1 - z_2)^{N-1}$, where N is an arbitrary complex number. One may worry that the continuation of the function ψ_N for negative z_{12} is not unique and requires special care. But it does not matter for our analysis. Indeed, $z_{12}^\alpha - z_{21}^\beta = (1 - \alpha - \beta)z_{12}$ with $\alpha + \beta < 1$, and therefore the operators do not mix the regions $z_{12} \gtrless 0$. For definiteness let us suppose that

$$\psi_N(z_1, z_2) = \theta(z_{12}) z_{12}^{N-1}. \tag{19}$$

Let $\gamma(N)$ and $\hat{\gamma}(N)$ be eigenvalues (anomalous dimensions) of the operators H and $\hat{\mathrm{H}}$ corresponding to the function ψ_N, respectively,

$$\mathrm{H}(a)\psi_N = \gamma(N)\psi_N, \tag{20}$$

$$\hat{\mathrm{H}}(a)\psi_N = \hat{\gamma}(N)\psi_N. \tag{21}$$

The anomalous dimensions $\gamma(N)$ and $\hat{\gamma}(N)$ are analytic functions of N in the right complex half-plane, $\mathrm{Re}(N) > 0$.

For integer even (odd) N, $\gamma(N)$ gives the anomalous dimensions of the local (axial) vector operators.[2]

Now let us note that the operator T(H) acts on ψ_N as follows;

$$\begin{aligned}\mathrm{T(H)}\psi_N(z_1,z_2) &= \sum_{n=0}^{\infty}\frac{\mathrm{L}^n}{n!}\left(\bar\beta(a)+\frac{1}{2}\gamma(N)\right)^n\psi_N(z_1,z_2)\\ &= z_{12}^{\bar\beta(a)+\frac{1}{2}\gamma(N)}\psi_N(z_1,z_2)\\ &= \psi_{N+\bar\beta+\frac{1}{2}\gamma(N)}(z_1,z_2). \end{aligned}\tag{22}$$

Thus, it follows from Eq. (13) that the anomalous dimensions $\gamma(N)$ and $\hat\gamma(N)$ satisfy the relation [cf. also Eq. (1)]

$$\gamma(N) = \hat\gamma\left(N+\bar\beta(a)+\frac{1}{2}\gamma(N)\right). \tag{23}$$

This relation appeared first in Refs. [22,23] as a generalization of the Gribov-Lipatov reciprocity relation [26,27]. It was shown that the asymptotic expansion of the function $\hat\gamma(N)$ for large N is invariant under the reflection $N\to -N-1$; see, e.g., Refs. [22,28–30]. This property strongly restricts harmonics sums, which can appear in the perturbative expansion of the anomalous dimension $\hat\gamma(N)$ [29]. Explicit expressions for $\hat\gamma(N)$ are known at four loops in QCD [6] and at seven loops in the $\mathcal{N}=4$ SYM; see Refs. [29,31–34].

A. Kernels from anomalous dimensions

For large N the anomalous dimension $\hat\gamma(N)$ grows as $\ln N$. This term enters with a coefficient $2\Gamma_{\mathrm{cusp}}(a)$ where $\Gamma_{\mathrm{cusp}}(a)$ is the so-called cusp anomalous dimension [35,36] whose complete form is known to the four-loop order in QCD [37,38] and in $\mathcal{N}=4$ SYM [37]. In the planar limit of $\mathcal{N}=4$ SYM, the cusp anomalous dimension is known beyond the four-loop order (e.g., as a special case of results in [33,34]), and in fact, to any loop order from Ref. [39]. Thus, we write $\hat\gamma(N)$ in the following form:

$$\hat\gamma(N) = 2\Gamma_{\mathrm{cusp}}(a)S_1(N) + A(a) + \Delta\hat\gamma(N), \tag{24}$$

where $S_1(N)=\psi(N+1)-\psi(1)$ is the harmonic sum responsible for the $\ln N$ behavior at large N and $A(a)$ is a constant term. The remaining term, $\Delta\hat\gamma(N)$, vanishes at least as $O(1/N(N+1))$ at large N. The constant $A(a)$ is exactly the same that appears in Eq. (16). The first term in Eq. (24) comes from a special SL$(2,\mathbb{R})$ invariant kernel

$$\hat{\mathcal{H}}f = \int_0^1\frac{d\alpha}{\alpha}\{2f(z_1,z_2)-\bar\alpha(f(z_{12}^\alpha,z_2)+f(z_1,z_{21}^\alpha))\}, \tag{25}$$

which in momentum space gives rise to the so-called plus distribution. The eigenvalues of this kernel are $2S_1(N)$ $[\hat{\mathcal{H}}z_{12}^{N-1}=2S_1(N)z_{12}^{N-1}]$. It corresponds to a singular contribution of the form $-\delta_+(\tau)$ to the invariant kernel $h(\tau)$; see [Ref. [16], Eq. (2.19)] for details. Thus the evolution kernel can generally be written as

$$\hat{\mathrm{H}} = \Gamma_{\mathrm{cusp}}(a)\hat{\mathcal{H}} + A(a) + \Delta\hat{\mathrm{H}}. \tag{26}$$

Here $\Delta\hat{\mathrm{H}}$ is an integral operator,

$$\Delta\hat{\mathrm{H}}f(z_1,z_2) = \int_+ h(\tau)f(z_{12}^\alpha,z_{21}^\beta), \tag{27}$$

where the weight function $h(\tau)$ is a regular function of $\tau\in(0,1)$. The eigenvalues of $\Delta\hat{\mathrm{H}}$ are equal to $\Delta\hat\gamma(N)$ and are given by the following integral:

$$\Delta\hat\gamma(N) = \int_+ h(\tau)(1-\alpha-\beta)^{N-1}. \tag{28}$$

The inverse transformation takes the form [14]

$$h(\tau) = \int_C\frac{dN}{2\pi i}(2N+1)\Delta\hat\gamma(N)P_N\left(\frac{1+\tau}{1-\tau}\right), \tag{29}$$

where P_N are the Legendre polynomials. The integration path C goes along the line parallel to the imaginary axis, $\mathrm{Re}(N)>0$, such that all poles of $\Delta\hat\gamma(N)$ lie to the left of this line. Some details of the derivation can be found in Appendix B.

One can hardly hope to evaluate the integral (29) in a closed form for an arbitrary function $\Delta\hat\gamma(N)$. However, as was mentioned before, the anomalous dimensions $\Delta\hat\gamma(N)$ in quantum field theory are rather special functions. Most of the terms in the perturbative expansion of $\Delta\hat\gamma(N)$ have the following form:

$$\eta^k(N)\Omega_{\vec m}(N), \qquad \eta^k(N)\Omega_1^p(N), \tag{30}$$

where $\eta(N)=1/(N(N+1))$ and the functions $\Omega_{\vec m}(N)=\Omega_{m_1,\ldots,m_p}(N)$ are the parity respecting harmonic sums [29], $[\Omega_{\vec m}(N)\sim\Omega_{\vec m}(-N-1)$ for $N\to\infty]$. We will assume that the sums $\Omega_{\vec m}(N)$ are "subtracted," i.e., $\Omega_{\vec m}(N)\to 0$ at $N\to\infty$. The second structure occurs only for $k>0$, since $\Omega_1(N)=S_1(N)$ grows as $\ln N$ for large N.

Since all SL$(2,\mathbb{R})$ invariant operators share the same eigenfunctions, the product of two invariant operators H_1 and H_2, $H_1H_2(=H_2H_1)$ with eigenvalues $H_1(N)$ and $H_2(N)$, respectively, has eigenvalues $H_1(N)H_2(N)$.

[2]As usual one has to consider the operators of certain parity, $\mathcal{O}_\pm(z_1,z_2)=\mathcal{O}(z_1,z_2)\mp\mathcal{O}(z_2,z_1)$; then the functions $\gamma_\pm(N)$ give the anomalous dimensions of local operators, for even and odd N, respectively.

One can use this property to reconstruct an operator with the eigenvalue (30).

First, we remark that the operator with the eigenvalues $\eta(N)$ (we denote it as $\mathcal{H}_+$) has [as follows from Eq. (28)] a very simple weight function, $h_+(\tau) = 1$. This can also be derived from Eq. (29). Since $P_N(x) = P_{-N-1}(x)$, the integral in Eq. (29) vanishes for the integration path $\mathrm{Re}(N) = -1/2$ due to the antisymmetry of the integrand. Therefore, the integral (29) can be evaluated by the residue theorem[3]

$$h_+(\tau) = \frac{2N+1}{N+1} P_N\left(\frac{1+\tau}{1-\tau}\right)\bigg|_{N=0} = 1. \qquad (31)$$

Let us consider the product $H_2 = H_+ H_1 (= H_1 H_+)$, where H_1 is an integral operator with the weight function $h_1(\tau)$. Then the weight function $h_2(\tau)$ of the operator H_2 is given by the following integral:

$$h_2(\tau) = \int_0^\tau \frac{ds}{\bar{s}^2} \ln(\tau/s) h_1(s); \qquad (32)$$

see Appendix B for details. Thus the contribution to the anomalous dimension of type (30) can be evaluated with the help of this formula if the weight function corresponding to the harmonic sums $\Omega_{\vec{m}}$ is known.

We also give an expression for another product of the operators: $H_2 = \hat{\mathcal{H}} H_1$,

$$h_2(\tau) = -\ln \tau h_1(\tau) + 2\bar{\tau} \int_0^\tau \frac{ds}{\bar{s}} \frac{h_1(\tau) - h_1(s)}{(\tau - s)}, \qquad (33)$$

which appears to be useful in the calculations as well.

B. Recurrence procedure

Let us consider the integral (29) with $\Delta\hat{\gamma} = \Omega_{\vec{m}}$,

$$h_{\vec{m}}(\tau) = \int_C \frac{dN}{2\pi i} (2N+1) \Omega_{\vec{m}}(N) P_N(z), \qquad (34)$$

where $z = (1+\tau)/(1-\tau)$. Using a recurrence relation for the Legendre functions

$$(2N+1) P_N(z) = \frac{d}{dz} (P_{N+1}(z) - P_{N-1}(z)), \qquad (35)$$

we obtain

$$h_{\vec{m}}(\tau) = -\frac{d}{dz} \int_C \frac{dN}{2\pi i} P_N(z) F_{\vec{m}}(N), \qquad (36)$$

where

$$F_{\vec{m}}(N) = (\Omega_{\vec{m}}(N+1) - \Omega_{\vec{m}}(N-1)). \qquad (37)$$

It is easy to see that the function $F_{\vec{m}}(N)$ has the negative parity under $N \to -N-1$ transformation and can be represented in the form

$$F_{m_1,\ldots,m_p}(N) = \sum_{k=2}^{p} r_k(N) \Omega_{m_k,\ldots,m_p}(N) + r(N), \qquad (38)$$

where $r_k(N)$ are rational functions of N. The harmonic sums $\Omega_{m_k,\ldots,m_p}(N)$ in Eq. (38) can be of either positive or negative parity. Therefore the coefficient $r_k(N)$ accompanying the positive parity function $\Omega_{m_k,\ldots,m_p}(N)$ has the form $r_k(N) = (2N+1) P_k(\eta)$, where P_k is some polynomial, while $r_k = P_k(\eta)$ for the harmonic sums of negative parity. The free term has the form $r(N) = (2N+1) P(\eta)$. Together, they make $F_{m_1,\ldots,m_p}(N)$ with negative parity. For example, for the harmonic sum $\Omega_{1,3}$ (see Appendix C for a definition), one gets

$$F_{1,3}(N) = (2N+1) \underline{\eta\left(\Omega_3 + \zeta_3 - \eta^2 - \frac{1}{2}\eta^3\right)}, \qquad (39)$$

while for the harmonic sum $\Omega_{2,2}$

$$F_{2,2}(N) = (2N+1) \frac{1}{2} \eta^3 (3+\eta) + \underline{\eta(2+\eta)\Omega_2}. \qquad (40)$$

Note the reappearance of the common factor $(2N+1)$ in the first case, (39). This implies that, up to the derivative d/dz, the integral (36) has the form (29). Hence, if the kernel corresponding to the underlined terms in Eq. (39) is known, the kernel corresponding to $\Omega_{1,3}$ can easily be obtained. Thus the problem of finding the invariant kernel with the eigenvalues $\Omega_{1,3}(N)$ is reduced to the problem of finding the kernel with the eigenvalues $\Omega_3(N)$ $(\Omega_{1,3} \mapsto \Omega_3)$.

However, as is seen from our second example, not all parity preserving harmonic sums share this property. Indeed, the underlined term on the right-hand side (rhs) of Eq. (40) does not have the factor $(2N+1)$. Hence, all these transformations do not help to solve the problem for $\Omega_{2,2}$.

It is easy to see that the above recurrence procedure works only if all the harmonic sums $\Omega_{m_k,\ldots,m_p}$ appearing in Eq. (38) are of positive parity. It was proven in [Ref. [29], Theorem 2] that any harmonic sum, $\Omega_{\vec{m}}$, with all indices $\vec{m}$ positive odd or negative even has positive parity (see Appendix C for explicit examples of the harmonic sums satisfying these conditions). Therefore, the rhs of Eq. (38) only contains harmonic sums of the same type. Thus the invariant kernels corresponding to the harmonic sums of positive parity can *always* be calculated recursively, using Eqs. (32), (33), (36), and (38). Crucially, only such

[3]This trick allows one to calculate the integral (29) for any function $\Delta\hat{\gamma}(N)$ with *exact* symmetry under $N \to -1-N$ reflection.

harmonic sums appear in the anomalous dimensions $\hat{\gamma}(N)$ in QCD and $\mathcal{N}=4$ SYM. All convolution integrals (32) and (33) can in turn be systematically calculated with the packages HyperInt [40] or PolyLogTools [41].

The explicit expressions for the kernels corresponding to the lowest harmonic sums are given in Appendix C for references.

C. Invariant kernels: QCD

Below we give an explicit expression for the invariant kernel of the twist-two flavor nonsinglet operator in QCD. We will not split the operator $\mathcal{O}(z_1, z_2)$ into positive (negative) parity operators. The evolution operator still takes the form (26), with $\Delta\hat{\mathrm{H}}$ given by the following integral:

$$\Delta\hat{\mathrm{H}} f(z_1, z_2) = \int_{+} (h(\tau) + \bar{h}(\tau) P_{12}) f(z_{12}^{\alpha}, z_{21}^{\beta}), \qquad (41)$$

where P_{12} is a permutation operator, $P_{12} f(z_1, z_2) = f(z_2, z_1)$.[4] For (anti)symmetric functions $f(z_1, z_2)$ the operator (41) takes a simpler form (27) with the kernel $h \pm \bar{h}$.

Our expression for the constant term $A(a)$ agrees with the constant term χ given in [Ref. [16], Eq. (5.5)], $A = \chi - 2\Gamma_{\mathrm{cusp}}$. For completeness, we provide explicit expressions for the constant $A = aA_1 + a^2 A_2 + a^3 A_3 + \cdots$,

$$\begin{aligned}
A_1 &= -6C_F,\\
A_2 &= C_F\left[n_f\left(\frac{16}{3}\zeta_2 + \frac{2}{3}\right) - N_c\left(\frac{52}{3}\zeta_2 + \frac{43}{6}\right) + \frac{1}{N_c}\left(24\zeta_3 - 12\zeta_2 + \frac{3}{2}\right)\right],\\
A_3 &= C_F\left[n_f^2\left(\frac{32}{9}\zeta_3 - \frac{160}{27}\zeta_2 + \frac{34}{9}\right) + n_f N_c\left(-\frac{256}{15}\zeta_2^2 + \frac{8}{9}\zeta_3 + \frac{2492}{27}\zeta_2 - 17\right) + \frac{n_f}{N_c}\left(\frac{232}{15}\zeta_2^2 - \frac{136}{3}\zeta_3 + \frac{20}{3}\zeta_2 - 23\right)\right.\\
&\quad + N_c^2\left(-80\zeta_5 + \frac{616}{15}\zeta_2^2 + \frac{266}{9}\zeta_3 - \frac{5545}{27}\zeta_2 + \frac{847}{18}\right) + \left(-120\zeta_5 - 16\zeta_2\zeta_3 - \frac{124}{15}\zeta_2^2 + \frac{1048}{3}\zeta_3 - \frac{356}{3}\zeta_2 + \frac{209}{4}\right)\\
&\quad \left. + \frac{1}{N_c^2}\left(120\zeta_5 + 16\zeta_2\zeta_3 - \frac{144}{5}\zeta_2^2 - 34\zeta_3 - 9\zeta_2 - \frac{29}{4}\right)\right],
\end{aligned} \qquad (42)$$

where $C_F = (N_c^2 - 1)/(2N_c)$ is the quadratic Casimir in the fundamental representation of $SU(N_c)$ and we take $T_F = 1/2$. Note that we are adopting a different color basis compared to Ref. [16].

The explicit expressions for the cusp anomalous dimensions $\Gamma_{\mathrm{cusp}}(a) = a\Gamma_{\mathrm{cusp}}^{(1)} + a^2\Gamma_{\mathrm{cusp}}^{(2)} + a^3\Gamma_{\mathrm{cusp}}^{(3)}$ up to three loops are provided in Eq. (D3). Finally we give answers for the kernels $h(\bar{h})(a) = \sum_k a^k h_k(\bar{h}_k)$. Explicit one- and two-loop expressions are known [14,16] but for completeness we give them here

$$h_1 = -4C_F, \qquad \bar{h}_1 = 0, \qquad (43)$$

and

$$\begin{aligned}
h_2 &= C_F\left\{n_f\frac{88}{9} + N_c\left(-2\mathrm{H}_1 + 8\zeta_2 - \frac{604}{9}\right)\right.\\
&\quad \left. + \frac{1}{N_c}\left(-8(\mathrm{H}_{11} + \mathrm{H}_2) + 2\left(1 - \frac{4}{\tau}\right)\mathrm{H}_1\right)\right\},\\
\bar{h}_2 &= -\frac{8C_F}{N_c}(\mathrm{H}_{11} + \tau\mathrm{H}_1),
\end{aligned} \qquad (44)$$

where $\mathrm{H}_{\vec{m}} = \mathrm{H}_{\vec{m}}(\tau)$ are the harmonic polylogarithms (HPLs) [24]. The three-loop expression[5] is more involved

$$\begin{aligned}
h_3 = C_F\Bigg\{&-\frac{64}{9}n_f^2 + n_f N_c\frac{8}{3}\left[\mathrm{H}_3 - \mathrm{H}_{110} - \mathrm{H}_{20} + \mathrm{H}_{12} + \frac{1}{\tau}\mathrm{H}_2 - \frac{1}{\tau}\mathrm{H}_{10} - \frac{19}{12}\mathrm{H}_1 + 8\zeta_3 - \frac{32}{3}\zeta_2 + \frac{5695}{72}\right]\\
&+ \frac{n_f}{N_c}\frac{16}{3}\left[3\zeta_3 - \frac{75}{16} + \mathrm{H}_3 + \mathrm{H}_{21} + \mathrm{H}_{12} + \mathrm{H}_{111} + \left(\frac{16}{3} + \frac{1}{\tau}\right)(\mathrm{H}_2 + \mathrm{H}_{11}) + \left(\frac{31}{24} + \frac{10}{3\tau}\right)\mathrm{H}_1\right]\\
&+ N_c^2 4\left[\mathrm{H}_{13} + \mathrm{H}_{112} - \mathrm{H}_{120} - \mathrm{H}_{1110} + 2\mathrm{H}_4 - 2\mathrm{H}_{30} - 2\mathrm{H}_{210} + 2\mathrm{H}_{22} + \left(\frac{8}{3} - \frac{2}{\tau}\right)(\mathrm{H}_{20} - \mathrm{H}_3 + \mathrm{H}_{110} - \mathrm{H}_{12})\right.
\end{aligned}$$

[4]To avoid possible misunderstandings we write it down explicitly, $P_{12} f(z_{12}^{\alpha}, z_{21}^{\beta}) = f(z_{21}^{\alpha}, z_{12}^{\beta})$.

[5]A file with our main results can be obtained from the preprint server http://arXiv.org by downloading the source. Furthermore, the results are available from the authors upon request.

$$
\begin{aligned}
&-\frac{5}{4}(\mathrm{H}_{10}+\mathrm{H}_{11})+\frac{2}{3\tau}(\mathrm{H}_{10}-\mathrm{H}_{2})-\frac{5}{2}\mathrm{H}_{0}+\left(\frac{115}{72}+\zeta_2+\frac{1}{\tau}\right)\mathrm{H}_1-\frac{44}{5}\zeta_2^2-\frac{22}{3}\zeta_3+\frac{436}{9}\zeta_2-\frac{4783}{27}\bigg]\\
&+16\bigg[\mathrm{H}_4-\mathrm{H}_{30}+\mathrm{H}_{13}+\mathrm{H}_{121}-\frac{3}{2}\mathrm{H}_{120}+\frac{3}{2}\mathrm{H}_{22}+\frac{3}{2}\mathrm{H}_{112}+2\mathrm{H}_{31}+2\mathrm{H}_{1111}+3\mathrm{H}_{211}-\frac{1}{2}\mathrm{H}_{1110}-\left(\frac{1}{\tau}+1\right)\mathrm{H}_{20}\\
&-\left(\frac{11}{6}-\frac{1}{\tau}\right)\mathrm{H}_3-\mathrm{H}_{110}+\left(-\frac{37}{12}+\frac{3}{2\tau}\right)\mathrm{H}_{12}-\left(\frac{7}{3}-\frac{2}{\tau}\right)\mathrm{H}_{21}+\left(-\frac{43}{12}+\frac{3}{\tau}\right)\mathrm{H}_{111}+\left(\frac{13}{8}+\frac{1}{2}\zeta_2\right)\mathrm{H}_{10}\\
&-\left(\frac{1}{2}\zeta_2+\frac{127}{9}+\frac{11}{6\tau}\right)\mathrm{H}_2-\left(\frac{899}{72}+\frac{1}{3\tau}\right)\mathrm{H}_{11}+(\zeta_2-1)\mathrm{H}_0+\left(\frac{7}{4}\zeta_2-\frac{143}{36}-\frac{1}{\tau}\left(\frac{1}{2}\zeta_2+\frac{67}{9}\right)\right)\mathrm{H}_1+\frac{5}{2}\zeta_2-\frac{47}{24}\bigg]\\
&+\frac{8}{N_c^2}\bigg[\mathrm{H}_4-\mathrm{H}_{30}-\mathrm{H}_{210}+\mathrm{H}_{112}-\mathrm{H}_{1111}-2\mathrm{H}_{120}+2\mathrm{H}_{13}+2\mathrm{H}_{31}-2\mathrm{H}_{1110}-2\mathrm{H}_{211}+3\mathrm{H}_{121}\\
&-\left(\frac{1}{2}+\frac{1}{\tau}\right)(\mathrm{H}_{20}-\mathrm{H}_3+\mathrm{H}_{110})+2\left(1+\frac{1}{\tau}\right)\mathrm{H}_{21}+\left(\frac{3}{2}-\frac{2}{\tau}\right)\mathrm{H}_{111}+\left(\frac{7}{8}+\frac{3}{2\tau}\right)\mathrm{H}_{10}-\left(\zeta_2-\frac{1}{2}+\frac{3}{2\tau}\right)\mathrm{H}_2\\
&+\left(\frac{11}{8}-\zeta_2\right)\mathrm{H}_{11}-\frac{11}{4}\mathrm{H}_0+\left(\zeta_2-\frac{107}{16}-\frac{\zeta_2}{\tau}-\frac{1}{2\tau}\right)\mathrm{H}_1+\frac{7}{2}\bigg]\bigg\}
\end{aligned}
\tag{45}
$$

and

$$
\begin{aligned}
\bar{h}_3=-8C_F\bigg\{&-\frac{2n_f}{3N_c}\bigg[\mathrm{H}_{111}+\mathrm{H}_{110}+\tau\mathrm{H}_{10}+\left(\frac{16}{3}+\tau\right)\mathrm{H}_{11}+\left(\frac{1}{2}+\frac{10}{3}\tau\right)\mathrm{H}_1+\frac{1}{2}\bigg]\\
&+\mathrm{H}_{120}+\mathrm{H}_{22}-\mathrm{H}_{1110}-\mathrm{H}_{112}-2\mathrm{H}_{121}+2\mathrm{H}_{211}-4\mathrm{H}_{1111}+\tau\mathrm{H}_{20}+\left(\frac{13}{6}-\tau\right)\mathrm{H}_{110}+\left(\frac{1}{2}-2\tau\right)\mathrm{H}_{12}\\
&+\left(\frac{5}{2}-2\tau\right)\mathrm{H}_{21}+\left(\frac{43}{6}-6\tau\right)\mathrm{H}_{111}-\left(\zeta_2-\frac{13}{6}\tau\right)\mathrm{H}_{10}-\left(3+\zeta_2+\frac{3}{2}\tau\right)\mathrm{H}_2+\left(\frac{236}{9}+\frac{2}{3}\tau\right)\mathrm{H}_{11}-\zeta_2\tau\mathrm{H}_0\\
&+\left(\frac{53}{6}+\zeta_2+3\zeta_3+\frac{134}{9}\tau+\zeta_2\tau\right)\mathrm{H}_1+\frac{11}{6}+3\left(\zeta_2+\zeta_3-\frac{1}{2}\right)\tau\\
&+\frac{1}{N_c^2}\bigg[\mathrm{H}_{1111}-\mathrm{H}_{22}-\mathrm{H}_{211}+3\mathrm{H}_{120}+3\mathrm{H}_{112}-3\mathrm{H}_{1110}+4\mathrm{H}_{121}+3\tau\mathrm{H}_{20}+3\left(\frac{1}{2}-\tau\right)\mathrm{H}_{110}-\left(\frac{7}{2}-4\tau\right)\mathrm{H}_{12}\\
&+\left(\frac{1}{2}+4\tau\right)\mathrm{H}_{21}+\left(-\frac{3}{2}+2\tau\right)\mathrm{H}_{111}-3\left(\zeta_2-\frac{1}{2}\tau\right)\mathrm{H}_{10}+\left(\zeta_2-2-\frac{3}{2}\tau\right)\mathrm{H}_2+2(\zeta_2-1)\mathrm{H}_{11}-3\zeta_2\tau\mathrm{H}_0\\
&+(5+2\zeta_2+3\zeta_3+\zeta_2\tau)\mathrm{H}_1+3\tau\left(\zeta_3-\frac{1}{2}\right)\bigg]\bigg\}.
\end{aligned}
\tag{46}
$$

The kernels are smooth functions of τ except for the end points $\tau=0$ and $\tau=1$. For $\tau\to 1$ the three-loop kernel functions behave as $\sum_{0\le k\le 4}\sum_{m>0} r_{km}\bar{\tau}^m \ln^k \bar{\tau}$. For small τ—which determines the large N asymptotic of the anomalous dimensions—the kernels (for each color structure) have the form $\sum_{k\ge 0}(a_k+b_k\ln\tau)\tau^k$. We note here that the reciprocity property of the anomalous dimension is equivalent to the statement that the small τ expansion of the kernels does not involve noninteger powers of τ, namely $h(\tau)\sim\sum_{m,k\ge 0} a_{mk}\tau^m \ln^k \tau$.

Below we compare our exact three-loop results with the approximate expressions constructed in Ref. [16]. The approximate expressions reproduce the asymptotic behaviors of the exact kernels at both $\tau\to 0,\ 1$. We therefore subtract the logarithmically divergent pieces [see Eqs. (D1) and (D2) for explicit expressions] from both the exact and the approximated expressions to highlight their (small) deviations as shown in Figs. 1 and 2. For illustrative purposes, we plot the planar contribution ($C_F N_c^2$ and C_F in h_3 and $\bar{h}_3$, respectively) and the subsubplanar contribution (C_F/N_c^2). The former is numerically dominant and generates the leading contribution in the large-N_c limit, whereas the latter shows the worst-case scenario for the previous approximation using a simple HPL function ansatz. The error of other color structures all fall between the planar and subsubplanar cases, and hence are numerically small.

D. Invariant kernels: $\mathcal{N}=4$ SYM

In this section we present the invariant kernels for the universal anomalous dimensions of the planar $\mathcal{N}=4$ SYM (see, e.g., Refs. [31,42] for expressions up to NNLO). They

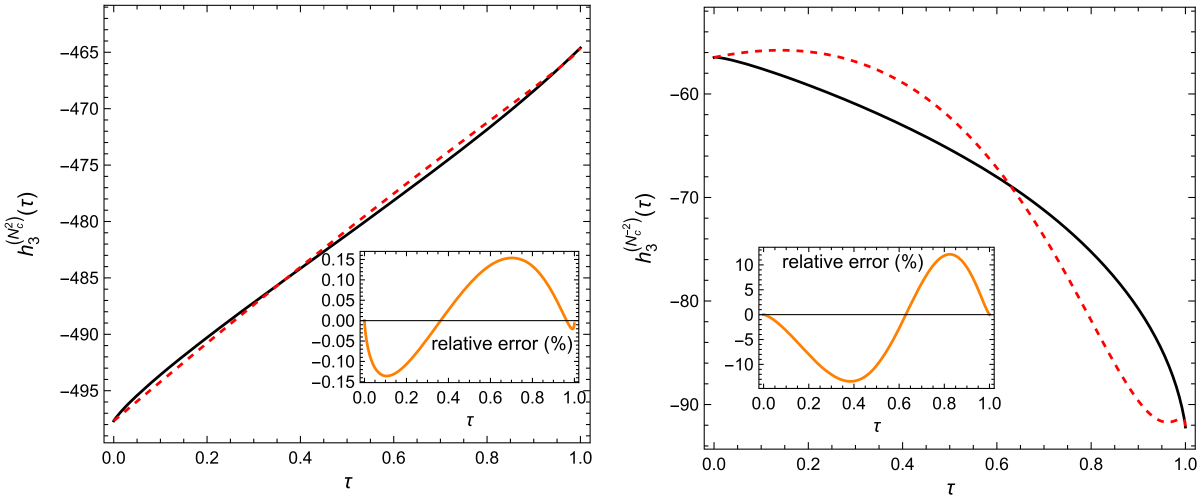

FIG. 1. Comparison of two distinct color contributions ($C_F N_c^2$ and C_F/N_c^2) in the exact (black solid line) and approximated (red dashed line) three-loop kernel h_3 (see Ref. [16] for explicit expressions of the latter). The inset curves show the relative percentage errors $[(1 - h^{(c)}_{3,\mathrm{appr}}/h^{(c)}_{3,\mathrm{exact}}) \times 100\%]$ of the approximation.

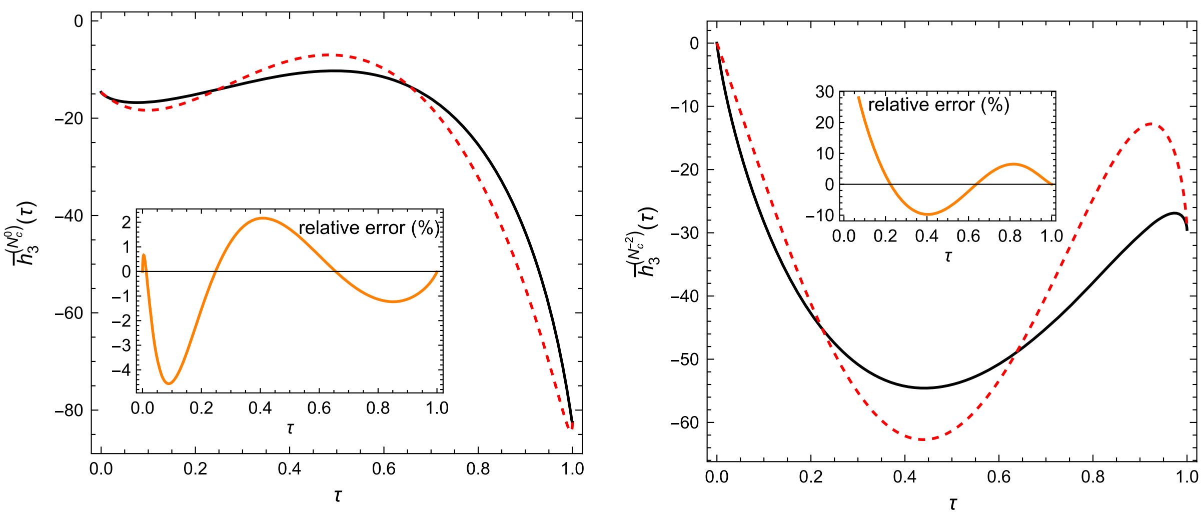

FIG. 2. Same as Fig. 1 for the color structures C_F and C_F/N_c^2 in $\bar{h}_3$.

are rather short so that we quote them here. We use the parametrization (24), where $\Gamma_{\mathrm{cusp}}(a)$ can be found in Ref. [37] and the constant term $A(a)$ is

$$A(a) = -24a^2\zeta_3 + 32a^3(\zeta_2\zeta_3 + 5\zeta_5) + O(a^4), \qquad (47)$$

where $a = \frac{N_c g^2_{\mathrm{SYM}}}{16\pi^2}$ and

$$\Delta\hat{\gamma}(N) = -a^2 16(\Omega_3 - 2\Omega_{-2,1} + 2\Omega_1\Omega_{-2}) + a^3 64(\Omega_5 + 2\Omega_{3,-2} - 8\Omega_{1,1,-2,1} + 2\Omega_{1,-4} + \Omega_1(\Omega_{-4} + \Omega_{-2}^2 + \zeta_2\Omega_{-2}) - 2\zeta_2\Omega_{-2,1}). \qquad (48)$$

For the kernels we find $h_1 = \bar{h}_1 = 0$,

$$h_2 = 8\frac{\bar{\tau}}{\tau}\mathrm{H}_1, \qquad \bar{h}_2 = -8\bar{\tau}\mathrm{H}_1, \qquad (49)$$

and

$$h_3 = -16\frac{\bar{\tau}}{\tau}(4\mathrm{H}_{111} + \mathrm{H}_{21} + \mathrm{H}_{12} + \mathrm{H}_{110}),$$
$$\bar{h}_3 = 16\bar{\tau}(4\mathrm{H}_{111} + 3(\mathrm{H}_{21} + \mathrm{H}_{12}) - \mathrm{H}_{110} + \mathrm{H}_{20} - \zeta_2\mathrm{H}_0). \qquad (50)$$

These expressions are extremely simple in comparison with the expressions in QCD of the same order. Let us notice that the two-loop kernels contain only HPLs of weight one with the three-loop kernels involving HPLs of weight three, while in QCD the corresponding kernels require HPLs of weights two and four, respectively. Note also that the kernel h is proportional to the factor $\bar{\tau}/\tau$ and the kernel $\bar{h}$ to the factor $\bar{\tau}$. It would be interesting to see if these properties persist in higher loops.

IV. SUMMARY

We have constructed a transformation that brings the evolution kernels of twist-two operators to the canonically conformal invariant form. The eigenvalues of these kernels are given by the parity respecting anomalous dimensions. We have developed a recurrence procedure that allows one to restore the weight functions of the corresponding kernels. It is applicable to a subset of the harmonic sums (with positive odd and negative even indices). It is interesting to note that exactly only such harmonic sums appear in the expressions for the reciprocity respecting anomalous dimensions.

We have calculated the three-loop invariant kernels in QCD and in $\mathcal{N}=4$ SYM (in the planar limit). In QCD it was the last missing piece to obtain the three-loop evolution kernels for the flavor-nonsinglet twist-two operators in a fully analytic form; see Ref. [16].

In the case of $\mathcal{N}=4$ SYM the lowest order expressions for the kernels are rather simple and exhibit some regularities, $h\sim\bar{\tau}/\tau$, $\bar{h}\sim\bar{\tau}$. It would be interesting to check if these properties survive at higher loops. We expect that at ℓ-loops the kernels $h^{(\ell)}(\tau)$ will be given by linear combinations (up to common prefactors) of HPLs of weight $2\ell-3$ with positive indices. Therefore going over to the invariant kernel can lead to a more compact representation of the anomalous dimensions than representing the anomalous dimension spectrum $\gamma(N)$ in terms of harmonic sums. The much smaller function basis in terms of HPLs ($\tau/\bar{\tau}H_{\vec{m}}$ and $\bar{\tau}H_{\vec{m}}$) opens the possibility of extracting the analytical expressions of the higher-order evolution kernels from minimal numerical input through the PSLQ algorithm.

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APPENDIX A

In this appendix, we describe in detail the derivations of some of the equations presented in Sec. II. Let us start with Eq. (9). For the generator $\mathrm{S}_-(a)=S_-$ the statement is trivial. Next, making use of Eq. (8) for the operator $\mathrm{T}(\mathrm{H})$ and, taking into account that $\mathrm{H}(a)$ commutes with the generators $\mathrm{S}_\alpha(a)$, one can write the left-hand side (lhs) of Eq. (9) in the form

$$\sum_{n=0}^{\infty}\frac{1}{n!}\mathrm{L}^n\mathrm{S}_\alpha(a)\mathrm{X}^n, \tag{A1}$$

where $\mathrm{X}=\bar{\beta}(a)+\frac{1}{2}\mathrm{H}(a)$. Using the representation (7) for the generators and taking into account that $[S_0,\mathrm{L}]=1$ and $[S_+,\mathrm{L}]=z_1+z_2$ (we recall that $\mathrm{L}=\ln z_{12}$) one obtains

$$\begin{aligned}\mathrm{L}^n\mathrm{S}_0&=S_0\mathrm{L}^n-n\mathrm{L}^{n-1}+\mathrm{L}^nX,\\ \mathrm{L}^n\mathrm{S}_+&=S_+\mathrm{L}^n+(z_1+z_2)(-n\mathrm{L}^{n-1}+\mathrm{L}^nX).\end{aligned} \tag{A2}$$

Substituting these expressions back into Eq. (A1) one finds that the contributions of the last two terms on the rhs of Eq. (A2) cancel each other. Hence Eq. (A1) takes the form

$$S_\alpha\sum_{n=0}^{\infty}\frac{1}{n!}\mathrm{L}^n\mathrm{X}^n=S_\alpha\mathrm{T}(\mathrm{H}) \tag{A3}$$

that finally results in Eq. (9).

Let us now show that the inverse to $\mathrm{T}(\mathrm{H})$ has the form (12). The product $\mathcal{I}=\mathrm{T}^{-1}(\mathrm{H})\mathrm{T}(\mathrm{H})$ can be written as

$$\mathcal{I}=\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\mathrm{L}^n\left(\bar{\beta}(a)+\frac{1}{2}\hat{\mathrm{H}}(a)\right)^n\mathrm{T}(\mathrm{H}). \tag{A4}$$

Moving $\mathrm{T}(\mathrm{H})$ to the left with the help of the relation (10) and then using Eq. (8) for $\mathrm{T}(\mathrm{H})$ one gets [$\mathrm{X}=\bar{\beta}(a)+\frac{1}{2}\mathrm{H}(a)$]

$$\mathcal{I}=\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\mathrm{L}^n\mathrm{T}(\mathrm{H})\mathrm{X}^n=\sum_{n,k=0}^{\infty}\frac{(-1)^n}{n!k!}\mathrm{L}^{n+k}\mathrm{X}^{n+k}=1.$$

Finally, we consider the product of operators T with a differently defined function L. Namely, let us take $\mathrm{T}_\pm(\mathrm{H})\equiv\mathrm{T}(\mathrm{L}_\pm,\mathrm{H})$, where $L_\pm=\ln(z_{12}\pm i0)$ so that $\mathrm{L}_+-\mathrm{L}_-=2\pi\theta(z_2-z_1)$. To calculate the product $\mathrm{U}=\mathrm{T}_+(H)\mathrm{T}_-(H)$ one proceeds as before: use expansion (8) for $\mathrm{T}_+(H)$, move $\mathrm{T}_-(H)$ to the left, and then expand it into a power series. It yields

$$\mathrm{U}=\sum_{n,k=0}^{\infty}\frac{(-1)^n}{n!k!}\mathrm{L}_+^n\mathrm{L}_-^k\hat{\mathrm{X}}^{n+k}, \tag{A5}$$

where $\hat{\mathrm{X}}=\bar{\beta}(a)+\frac{1}{2}\hat{\mathrm{H}}(a)$. Let $L_+=L_-+2\pi i\theta(z_2-z_1)$, and one can get for the sum in Eq. (A5)

$$\mathrm{U}=\sum_{m=0}^{\infty}\frac{(2\pi i\theta)^m}{m!}\hat{\mathrm{X}}^m(a)=1-\theta\left(1-e^{2\pi i(\bar{\beta}+\frac{1}{2}\hat{\mathrm{H}})}\right),$$

where $\theta\equiv\theta(z_2-z_1)$. Since $S_{0,+}\theta(z_{21})\sim z_{21}\delta(z_{21})=0$ one concludes that U commutes with the canonical generators S_α and hence $\mathrm{U}\hat{\mathrm{H}}=\hat{\mathrm{H}}\mathrm{U}$.

APPENDIX B

Let us check that the kernel $h(\tau)$ given by Eq. (29) has the eigenvalues $\Delta\hat{\gamma}(N)$. First, after some algebra, the integral in Eq. (28) can be brought to the following form:

$$\Delta\hat{\gamma}(N) = \int_1^\infty dt h\left(\frac{t-1}{t+1}\right) Q_N(t), \tag{B1}$$

where $Q_N(t)$ is the Legendre function of the second kind [43]. Inserting h in the form of Eq. (29) into Eq. (B1) one gets

$$\int_C \frac{dN'}{2\pi i}(2N'+1)\Delta\gamma(N') \int_1^\infty dt P_{N'}(t) Q_N(t). \tag{B2}$$

The t-integral of the product of the two Legendre functions gives [43]

$$((N-N')(N+N'+1))^{-1}. \tag{B3}$$

Then closing the integration contour in the right half-plane one evaluates the N' integral with the residue theorem at $N' = N$ yielding the desired lhs of Eq. (B1).

Finally, to verify Eq. (32) one can check that the integral (B1) with the kernel h_2, $\Delta\hat{\gamma}_2(N)$, is equal to $\Delta\hat{\gamma}_1(N)/N/(N+1)$. The simplest way to do it is to substitute the Legendre function in the form

$$Q_N(t) = -\partial_t(1-t^2)\partial_t Q_N(t)/N/(N+1) \tag{B4}$$

and perform integration by parts.

APPENDIX C

In this appendix, we collect the harmonic sums and the corresponding kernels that we have used. We split them into two parts: the first one includes the harmonic sums $\Omega_{m_1,\ldots,m_k}$ such that $\prod_i^k \mathrm{sign}(m_i) = 1$,

$$\begin{aligned}
\Omega_3 &= S_3 - \zeta_3,\\
\Omega_{3,1} &= S_{3,1} - \frac{1}{2}S_4 - \frac{3}{10}\zeta_2^2,\\
\Omega_{-2,-2} &= S_{-2,-2} - \frac{1}{2}S_4 + \frac{1}{2}\zeta_2 S_{-2} + \frac{1}{8}\zeta_2^2,\\
\Omega_{1,3,1} &= S_{1,3,1} - \frac{1}{2}S_{1,4} - \frac{1}{2}S_{4,1} + \frac{1}{4}S_5 - \frac{3}{10}\zeta_2^2 S_1 + \frac{3}{4}\zeta_5,\\
\Omega_{-2,-2,1} &= S_{-2,-2,1} - \frac{1}{2}S_{4,1} - \frac{1}{2}S_{-2,-3} + \frac{1}{4}\zeta_3 S_{-2} + \frac{5}{16}\zeta_5,\\
\Omega_5 &= S_5 - \zeta_5.
\end{aligned} \tag{C1}$$

Here $S_{\vec{m}}$ are the harmonic sums with argument N. We define the sums of negative signature, $\prod_i^k \mathrm{sign}(m_i) = -1$, with an additional sign factor:

$$\begin{aligned}
\Omega_{-2} &= (-1)^N\left[S_{-2} + \frac{\zeta_2}{2}\right],\\
\Omega_{-2,1} &= (-1)^N\left[S_{-2,1} - \frac{1}{2}S_{-3} + \frac{1}{4}\zeta_3\right],\\
\Omega_{1,-2,1} &= (-1)^N\left[S_{1,-2,1} - \frac{1}{2}S_{1,-3} - \frac{1}{2}S_{-3,1} + \frac{1}{4}S_{-4}\right.\\
&\quad\left. + \frac{1}{4}\zeta_3 S_1 - \frac{1}{80}\zeta_2^2\right],\\
\Omega_{-4,1} &= (-1)^N\left[S_{-4,1} - \frac{1}{2}S_{-5} + \frac{11}{8}\zeta_5 - \frac{1}{2}\zeta_2\zeta_3\right],\\
\Omega_{3,-2} &= (-1)^N\left[S_{3,-2} - \frac{1}{2}S_{-5} + \frac{1}{2}\zeta_2 S_3 + \frac{9}{8}\zeta_5 - \frac{3}{4}\zeta_2\zeta_3\right],\\
\Omega_{1,1,-2,1} &= (-1)^N\left[S_{1,1,-2,1} - \frac{1}{2}S_{1,1,-3} - \frac{1}{2}S_{1,-3,1} - \frac{1}{2}S_{2,-2,1}\right.\\
&\quad + \frac{1}{4}S_{2,-3} + \frac{1}{4}S_{-4,1} + \frac{1}{4}S_{1,-4} - \frac{1}{8}S_{-5}\\
&\quad\left. + \frac{1}{4}\zeta_3 S_{1,1} - \frac{1}{80}\zeta_2^2 S_1 - \frac{1}{8}\zeta_3 S_2 + \frac{1}{8}\zeta_5 - \frac{1}{16}\zeta_2\zeta_3\right],\\
\Omega_{1,-4} &= (-1)^N\left[S_{1,-4} - \frac{1}{2}S_{-5} + \frac{7}{20}\zeta_2^2 S_1 - \frac{11}{8}\zeta_5 + \frac{1}{2}\zeta_2\zeta_3\right].
\end{aligned} \tag{C2}$$

These combinations of harmonic sums are generated by the following kernels:

$$\begin{aligned}
\mathcal{H}_3 &= -\frac{1}{2}\frac{\bar{\tau}}{\tau}\mathrm{H}_1,\\
\mathcal{H}_{3,1} &= \frac{1}{4}\frac{\bar{\tau}}{\tau}(\mathrm{H}_{11} + \mathrm{H}_{10}),\\
\mathcal{H}_{-2,-2} &= \frac{1}{4}\frac{\bar{\tau}}{\tau}\mathrm{H}_{11},\\
\mathcal{H}_{1,3,1} &= -\frac{1}{8}\frac{\bar{\tau}}{\tau}(\mathrm{H}_{20} + \mathrm{H}_{110} + \mathrm{H}_{21} + \mathrm{H}_{111}),\\
\mathcal{H}_{-2,-2,1} &= \frac{1}{8}\frac{\bar{\tau}}{\tau}(\mathrm{H}_{12} - \mathrm{H}_{110}),\\
\mathcal{H}_5 &= -\frac{1}{2}\frac{\bar{\tau}}{\tau}(\mathrm{H}_{111} + \mathrm{H}_{12}),
\end{aligned} \tag{C3}$$

and

$$\begin{aligned}
\mathcal{H}_{-2} &= \frac{1}{2}\bar{\tau},\\
\mathcal{H}_{-2,1} &= -\frac{1}{4}\bar{\tau}(\mathrm{H}_1 + \mathrm{H}_0),\\
\mathcal{H}_{1,-2,1} &= \frac{1}{8}\bar{\tau}(\mathrm{H}_{10} + \mathrm{H}_{11}),\\
\mathcal{H}_{-4,1} &= -\frac{1}{4}\bar{\tau}(\mathrm{H}_{21} + \mathrm{H}_{20} + \mathrm{H}_{111} + \mathrm{H}_{110}),\\
\mathcal{H}_{3,-2} &= -\frac{1}{4}\bar{\tau}(\mathrm{H}_{21} + \mathrm{H}_{111}),
\end{aligned}$$

$$\mathcal{H}_{1,1,-2,1} = -\frac{1}{16}\bar{\tau}(\mathrm{H}_{111} + \mathrm{H}_{110}),$$
$$\mathcal{H}_{1,-4} = -\frac{1}{4}\bar{\tau}(\mathrm{H}_{12} + \mathrm{H}_{111}), \tag{C4}$$

where all HPLs have argument τ. These functions serve as a basis, and more complicated structures can be generated as products of $\Omega_{\vec{m}}$.

APPENDIX D

Here we give the small ($\tau \to 0$) and large ($\tau \to 1$) expansions of the invariant kernels $h_3, \bar{h}_3$. By $h_3^{(A)}$ ($\bar{h}_3^{(A)}$) we denote the function that appears in the expression for h_3 ($\bar{h}_3^{(A)}$) with the color factor $C_F \times A$. We will keep the logarithmically enhanced and constant terms in both limits. The former is subtracted from both the exact and the approximated three-loop kernel to obtain the two figures in Eqs. (1) and (2). At $\tau \to 0$ one gets

$$\begin{aligned}
h_3^{(n_f N_c)} &= \frac{5839}{27} - \frac{256}{9}\zeta_2 + \frac{64}{3}\zeta_3 - \frac{8}{3}\ln\tau,\\
h_3^{(n_f/N_c)} &= -\frac{17}{9} + 16\zeta_3,\\
\bar{h}_3^{(n_f/N_c)} &= \frac{8}{3},\\
h_3^{(N_c^2)} &= -\frac{18520}{27} - \frac{88}{3}\zeta_3 - \frac{176}{5}\zeta_2^2 + \frac{1744}{9}\zeta_2 - \frac{46}{3}\ln\tau,\\
h_3^{(N_c^0)} &= -\frac{1186}{9} + 32\zeta_2 + (-32 + 16\zeta_2)\ln\tau,\\
h_3^{(N_c^{-2})} &= 24 - 8\zeta_2 - 18\ln\tau,\\
\bar{h}_3^{(N_c^0)} &= -\frac{44}{3},\\
\bar{h}_3^{(N_c^{-2})} &= -48\tau\left(\zeta_2 + \zeta_3 + \frac{1}{4} - \zeta_2\ln\tau\right),
\end{aligned} \tag{D1}$$

and for $\tau \to 1$ one obtains

$$\begin{aligned}
h_3^{(n_f N_c)} &= \frac{5695}{27} - \frac{208}{9}\zeta_2 + \frac{64}{3}\zeta_3 + \left(-\frac{16}{3}\zeta_2 + \frac{38}{9}\right)\ln\bar{\tau},\\
h_3^{(n_f/N_c)} &= \frac{304}{9}\zeta_2 + 16\zeta_3 - 25 - \left(\frac{16}{3}\zeta_2 + \frac{74}{3}\right)\ln\bar{\tau} + \frac{152}{9}\ln^2\bar{\tau} - \frac{8}{9}\ln^3\bar{\tau},\\
\bar{h}_3^{(n_f/N_c)} &= \frac{16}{3}\left(\frac{1}{2} - \zeta_2 + \zeta_3\right) + \left(\frac{16}{3}\zeta_2 - \frac{184}{9}\right)\ln\bar{\tau} + \frac{152}{9}\ln^2\bar{\tau} - \frac{8}{9}\ln^3\bar{\tau},\\
h_3^{(N_c^2)} &= -\frac{72}{5}\zeta_2^2 + \frac{1741}{9}\zeta_2 - \frac{88}{3}\zeta_3 - \frac{19132}{27} + \left(\frac{4}{3}\zeta_2 - \frac{187}{18}\right)\ln\bar{\tau} + \left(-\frac{5}{2} + 4\zeta_2\right)\ln^2\bar{\tau},\\
h_3^{(N_c^0)} &= \frac{136}{5}\zeta_2^2 - \frac{2170}{9}\zeta_2 + 80\zeta_3 - \frac{94}{3} + \left(-\frac{32}{3}\zeta_2 - 24\zeta_3 + \frac{548}{3}\right)\ln\bar{\tau} + \left(16\zeta_2 - \frac{923}{9}\right)\ln^2\bar{\tau} + \frac{14}{9}\ln^3\bar{\tau} + \frac{4}{3}\ln^4\bar{\tau},\\
h_3^{(N_c^{-2})} &= -28\zeta_2^2 - 27\zeta_2 + 56\zeta_3 + 28 + \left(\frac{115}{2} - 12\zeta_2 - 40\zeta_3\right)\ln\bar{\tau} + \left(8\zeta_2 + \frac{11}{2}\right)\ln^2\bar{\tau} + \frac{2}{3}\ln^3\bar{\tau} - \frac{1}{3}\ln^4\bar{\tau},\\
\bar{h}_3^{(N_c^0)} &= -\frac{136}{5}\zeta_2^2 + \frac{88}{3}\zeta_2 - \frac{136}{3}\zeta_3 - \frac{8}{3} + \left(-\frac{16}{3}\zeta_2 + \frac{1708}{9}\right)\ln\bar{\tau} - \frac{968}{9}\ln^2\bar{\tau} + \frac{14}{9}\ln^3\bar{\tau} + \frac{4}{3}\ln^4\bar{\tau},\\
\bar{h}_3^{(N_c^{-2})} &= -\frac{216}{5}\zeta_2^2 + 40\zeta_2 + 8\zeta_3 + 12 + (40\zeta_2 - 64\zeta_3 + 40)\ln\bar{\tau} - 8(4\zeta_2 - 1)\ln^2\bar{\tau} + \frac{2}{3}\ln^3\bar{\tau} - \frac{1}{3}\ln^4\bar{\tau}.
\end{aligned} \tag{D2}$$

Here we quote the cusp anomalous dimensions up to three loops for Refs. [4,35,36],

$$\begin{aligned}
\Gamma_{\mathrm{cusp}}^{(1)} &= 4C_F,\\
\Gamma_{\mathrm{cusp}}^{(2)} &= C_F\left[N_c\left(\frac{536}{9} - 16\zeta_2\right) - \frac{40}{9}n_f\right],\\
\Gamma_{\mathrm{cusp}}^{(3)} &= C_F\left[N_c^2\left(\frac{176}{5}\zeta_2^2 + \frac{88}{3}\zeta_3 - \frac{1072}{9}\zeta_2 + \frac{490}{3}\right) + N_c n_f\left(-\frac{64}{3}\zeta_3 + \frac{160}{9}\zeta_2 - \frac{1331}{27}\right) + \frac{n_f}{N_c}\left(-16\zeta_3 + \frac{55}{3}\right) - \frac{16}{27}n_f^2\right].
\end{aligned} \tag{D3}$$

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