Neutral pion mass in a warm magnetized medium within the linear sigma model coupled to quarks framework

Aritra Das[*] and Najmul Haque[†]

School of Physical Sciences, National Institute of Science Education and Research, An OCC of Homi Bhabha National Institute, Jatni-752050, India

(Received 20 February 2023; accepted 13 July 2023; published 8 September 2023)

We study the neutral pion mass in the presence of an external arbitrary magnetic field in the framework of the linear sigma model coupled to quarks to a quark at finite temperature. In doing so, we have calculated the pion self-energy, constructed the dispersion equation via resummation, and solved the dispersion relation at zero three-momentum limit. In calculating the pion mass, we have included meson self-coupling's thermal and magnetic contribution and approximate chiral order parameter v_0. We report that the π^0 mass decreases with the magnetic field and increases with temperature.

DOI: 10.1103/PhysRevD.108.054008

I. INTRODUCTION

Recently the properties of hot and dense nuclear matter in the presence of a strong background magnetic field are drawing considerable interest. A transient magnetic field of the order of $10^{18} - 10^{19}$ G is achieved in the early stage of quark-gluon plasma (QGP) through a noncentral high-energy heavy-ion collision (HIC) [1–3]. Also, $B \sim 10^{14}$ G is predicted in the core of neutron stars, and in magnetars [4], the primordial magnetic field of 10^{22} G could have been present in the early universe due to a chiral anomaly [5,6]. This magnetic field is believed to be responsible for exotic phenomena in the QCD matter in extreme conditions, such as the chiral magnetic effect [7], magnetic catalysis, inverse magnetic catalysis [8], thermal chiral and deconfinement phase transition [9], and superconductivity of a QCD vacuum [10,11]. One of the most important hadrons from the perspective of high-energy physics is the pion, produced copiously in heavy-ion collisions.

The masses of hadrons are expected to be modified under a strong magnetic field. Considering pions as relativistic point particles in the presence of the magnetic field, we expect that $\pi^{\pm}$'s mass increases linearly with $|eB|$, whereas π^0's mass remains constant. But the predictions of the properties of the pion under the influence of a magnetic field hardly agree with its pointlike particle assumption. A recent Lattice Quantum Chromodynamics (LQCD) study [12,13] shows that a neutral pion mass decreases with the strength of the magnetic field monotonically. For low $|eB|$, the π^0 mass dies rapidly, whereas it saturates with high magnetic field values. This behavior was reproduced to a high degree of accuracy by Ayala *et al.* [14] in the strong field limit $|eB| \gg m_{\pi^0}^2$ using the linear sigma model coupled to quark (LSMq) model. Also, in Ref. [15], the authors have reported a neutral pion mass using the LSMq model at an arbitrary strength of eB. By tuning the coupling parameters λ and g of the model, they saw that mass decreases from its vacuum value, and then found a dip at an intermediate eB and again increases with eB. This qualitative nonmonotonic behavior is similar to the LQCD study of Ref. [16]. Apart from the lattice QCD studies, effective models were also invoked to study meson masses in magnetic backgrounds. For example, in Ref. [17], the authors have calculated the magnetic field-dependent pion pole mass considering pseudoscalar (PS) and pseudovector (PV) pion nucleon interaction invoking weak field approximation. They have obtained a decreasing nature for PS coupling and an increasing nature for PV coupling for the π^0 mass. Most of the work in determining the meson mass under the magnetic field was carried out in the Nambu-Jona-Laisino (NJL) model and chiral perturbation theory (ChPt). For example, a full magnetic field-independent regularization (MFIR) scheme with the random phase approximation (RPA) method was employed in Ref. [18] to calculate the meson mass. This MFIR scheme was remarkably [19] in agreement with LQCD predictions of the π^0 mass. In the ChPt framework, the charged, neutral pion mass was calculated at finite T and eB in Ref. [20].

The linear sigma model and linear sigma model coupled to quarks to quarks are recently being used to study various properties of hot and dense nuclear matter produced in heavy-ion-collision experiments. There are many interesting works in the literature regarding the transport coefficients at a nonzero temperature and the zero magnetic field

[*]aritra.das@niser.ac.in
[†]nhaque@niser.ac.in

in the framework of a quark-meson model (see [21] and references therein): The magnetic field-dependent electric charge transport in hadronic medium [22], the shear viscosity of hadronic matter at finite temperature and magnetic field [23], and the QCD phase diagram in a magnetized medium [24].

In the first part of this series [15], we calculated the neutral pion mass in a magnetic field at zero temperature. We extend the calculation to the finite temperature using the linear sigma model coupled to quarks in this part. It is extensively used to investigate from the QCD confinement/deconfinement phase transition to properties of hadrons. In the present work, we have examined the behavior of the neutral pion mass in the presence of an external magnetic field at a nonzero temperature within the framework of the LSMq model. In Sec. II, we briefly tour the LSMq model. In Sec. III, we have computed the neutral pion self-energy from the LSMq Lagrangian. In Sec. IV, the π^0 mass is obtained considering the following three scenarios: (a) bare couplings, (b) one-loop corrected meson couplings (Appendix A), and (c) one-loop corrected meson coupling, as well as the quantum corrected effective potential (details of it are elaborated in Appendix C). In Sec. V, the numerical results are discussed, and in Sec. VI, we conclude.

II. LINEAR SIGMA MODEL COUPLED TO QUARKS

The linear sigma model coupled to quarks to a quark is obtained by appending a $SU(2)$ scalar and pseudoscalar interaction of the sigma meson and pion, respectively, with light quarks (u and d flavors). The Lagrangian density for the LSMq reads

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\sigma)^2 + \frac{1}{2}(\partial_\mu\boldsymbol{\pi})^2 + \frac{a^2}{2}(\sigma^2+\boldsymbol{\pi}^2) - \frac{\lambda}{4}(\sigma^2+\boldsymbol{\pi}^2)^2 + i\bar{\psi}\gamma^\mu\partial_\mu\psi - g\bar{\psi}(\sigma + i\gamma_5\boldsymbol{\tau}\cdot\boldsymbol{\pi})\psi. \tag{1}$$

The first four terms of the last equation is the linear sigma model (LSM) part, and the other two terms are the quark part of $\mathcal{L}$. Here $\boldsymbol{\pi} = (\pi^1, \pi^2, \pi^3)$. The physical pion fields are defined as

$$\pi^\pm = \frac{1}{\sqrt{2}}(\pi^1 \pm i\pi^2), \qquad \pi^0 = \pi^3, \tag{2}$$

respectively, σ is the sigma meson, and ψ is the light quark doublet with

$$\psi = \begin{pmatrix} u \\ d \end{pmatrix}, \tag{3}$$

$\boldsymbol{\tau} = (\tau^1, \tau^2, \tau^3)$, where τ^i $(i = 1, 2, 3)$ is the ith Pauli spin matrix. Also, a^2 is the mass parameter that we take as negative in the symmetry-unbroken state. Finally, λ and g are the meson-meson coupling and meson-quark coupling, respectively. The $O(4)$ symmetry of the Lagrangian is spontaneously broken when a^2 becomes positive, and the σ field gets a nonzero vacuum expectation value (VEV). Hence, after the symmetry breaking, the σ field becomes

$$\sigma \to \sigma + v. \tag{4}$$

As a result of this shift, $\mathcal{L}$ reads

$$\mathcal{L} = \bar{\psi}(i\gamma^\mu\partial_\mu - M_f)\psi + \frac{1}{2}(\partial_\mu\sigma)^2 + \frac{1}{2}(\partial_\mu\boldsymbol{\pi})^2 - \frac{1}{2}M_\sigma^2\sigma^2 - \frac{1}{2}M_\pi^2\boldsymbol{\pi}^2 - g\bar{\psi}(\sigma + i\gamma_5\boldsymbol{\tau}\cdot\boldsymbol{\pi})\psi - V(\sigma,\pi) - V_{\text{tree}}(v), \tag{5}$$

with

$$V(\sigma,\pi) = \lambda v\sigma(\sigma^2+\boldsymbol{\pi}^2) + \frac{\lambda}{4}(\sigma^2+\boldsymbol{\pi}^2)^2, \tag{6}$$

$$V_{\text{tree}}(v) = -\frac{1}{2}a^2v^2 + \frac{1}{4}\lambda v^4. \tag{7}$$

The masses of the quarks, three pions, and sigma are given by

$$M_f = gv, \quad M_\pi^2 = \lambda v^2 - a^2, \quad M_\sigma^2 = 3\lambda v^2 - a^2. \tag{8}$$

respectively. Note that the minimum of the tree-level potential, obtained by solving $\frac{dV_{\text{tree}}(v)}{dv}\big|_{v=v_0} = 0$, is given by

$$v_0 = \sqrt{\frac{a^2}{\lambda}}. \tag{9}$$

Therefore, the masses, evaluated at v_0, are given by

$$M_f(v_0) = gv_0, \quad M_\pi^2 = 0, \quad M_\sigma^2 = 2a^2, \tag{10}$$

after symmetry breaking. To incorporate a nonvanishing pion mass into the model, an explicit symmetry-breaking term is added to the Lagrangian as

$$\mathcal{L} \to \mathcal{L}' = \mathcal{L} + \mathcal{L}_{\text{ESB}} = \mathcal{L} + \frac{1}{2}m_\pi^2 v(\sigma + v), \tag{11}$$

with $m_\pi = 0.14$ GeV. As a result, the tree-level potential V_{tree} becomes $V'_{\text{tree}} = -\frac{1}{2}(a^2+m_\pi^2)v^2 + \frac{1}{4}\lambda v^4$ and the minimum is shifted to

$$v_0 \to v_0' = \left(\frac{a^2+m_\pi^2}{\lambda}\right)^{1/2}. \tag{12}$$

The masses, evaluated at this new minimum v_0', are given by

$$M_f(v_0') = g\left(\frac{a^2+m_\pi^2}{\lambda}\right)^{1/2}, \tag{13}$$

$$M_\pi^2(v_0') = m_\pi^2,$$
$$M_\sigma^2(v_0') = 2a^2 + 3m_\pi^2. \tag{14}$$

The value of a is given by solving Eq. (14) as

$$a = \sqrt{\frac{M_\sigma^2(v_0') - 3M_\pi^2(v_0')}{2}} \simeq \sqrt{\frac{m_\sigma^2 - 3m_\pi^2}{2}}. \tag{15}$$

We consider a homogeneous time-independent background magnetic field in the z direction $\boldsymbol{\mathcal{B}} = B\hat{\boldsymbol{z}}$, which can be obtained from the electromagnetic four-potential in symmetric gauge $\mathcal{A}^\mu = \frac{B}{2}(0, -y, x, 0)$. As a result, the four derivative ∂_μ is replaced by covariant derivative D_μ for quarks and fermions. For a quark with flavor $f(= u, d)$, we write it as $D_\mu = \partial_\mu + iq_f A_\mu$. Note that $q_u = \frac{2}{3}|e|$ and $q_d = -\frac{1}{3}|e|$, with $|e|$ being the absolute charge of the electron. For charged pions, it becomes $D_\mu = \partial_\mu \pm i|e|A_\mu$, with the $\pm$ sign for $\pi^\pm$.

III. SELF-ENERGY OF THE NEUTRAL PION

If we rewrite the Feynman diagram of Eq. (5) in terms of π^+ and π^- fields, we notice that the neutral pion self-energy $\Pi_0^{(B)}(P,T)$ has contributions from $\pi^\pm$, π^0, and σ. It reads

$$\Pi_0^{(B)}(P,T) = 8\Pi_{\pi^\pm}^{(B)}(T) + 12\Pi_{\pi^0}(T) + 4\Pi_\sigma(T) + \sum_{f=u,d}\Pi_{f\bar{f}}^{(B)}(P,T). \tag{16}$$

Here, $\Pi_{\pi^\pm}^{(B)}(T)$, $\Pi_{\pi^0}(T)$, $\Pi_\sigma(T)$, and $\Pi_{f\bar{f}}^{(B)}(P,T)$ are the contributions coming from the charged pion, neutral pion, sigma meson, and quark-antiquark loop of flavor, respectively, and f to the total π^0 self-energy $\Pi_0^{(B)}(P,T)$. As mentioned in the previous section, we consider only light flavor in this article which is indicated by the flavor sum over the quark-antiquark contribution. Note that the dependence on external momentum P comes in the total self-energy solely from the quark-antiquark part. Also, we have omitted the superscript B from π^0-loop and σ-loop contributions since, being charge neutral, they are unaffected by the background magnetic field. In this section, we compute the self-energies indicated on the right-hand side (RHS) of Eq. (16). Before proceeding, we clarify some notations, conventions, and definitions that will be used repeatedly in the rest of the article.

(i) For any generic four-vectors A^μ, B^μ, we adopt the following notation and convention in which four-vectors are denoted by a capital letter (e.g., A^μ), three-vectors by small letters with boldface (e.g., $\boldsymbol{a}$), and magnitude by $|\boldsymbol{a}|$ or a. The following equations clearly express:

$$\begin{aligned}
&A^\mu = (a^0, a^1, a^2, a^3), \qquad A_\parallel^\mu = (a^0, 0, 0, a^3),\\
&A_\perp^\mu = (0, a^1, a^2, 0), \qquad (a.b)_\parallel = a^0b^0 - a^3b^3,\\
&(a.b)_\perp = a^1b^1 + a^2b^2, \qquad A.B = (a.b)_\parallel - (a.b)_\perp,\\
&\not{a}_\parallel = (\gamma.a)_\parallel = \gamma^0a^0 - \gamma^3a^3, \qquad \not{a}_\perp = (\gamma.a)_\perp = \gamma^1a^1 + \gamma^2a^2,\\
&a^0 = a_0, \qquad a_1 = -a^1, \qquad a_2 = -a^2, \qquad a_3 = -a^3,\\
&A^2 = a_0^2 - a_1^2 - a_2^2 - a_3^2, \qquad a^2 = a_1^2 + a_2^2 + a_3^2,\\
&a_\parallel^2 = a_0^2 - a_3^2, \qquad a_\perp^2 = a_1^2 + a_2^2.
\end{aligned} \tag{17}$$

(ii) For calculation involving nonzero temperature, we will work in imaginary time formalism (ITF). In ITF, the integration over the 0th component of the four-momentum running in the loop is replaced by a discrete Matsubara frequency sum. We make the following replacement:

$$\int_{-\infty}^{\infty}\frac{dk_0}{2\pi} \to iT\sum_{k_0}, \tag{18}$$

where $k_0 = i\omega_n = i2n\pi T$ for bosons and $k_0 = i\tilde{\omega}_n = \mu + i(2n+1)\pi T$ for fermions. Here T and μ denote the temperature and chemical potential of the thermal medium, respectively.

(iii) The Landau level (LL) dependent masses and the particle's energy in the presence of B-field are denoted as follow:—

$$M_{\ell,f} = \sqrt{2\ell|q_fB| + M_f^2}, \quad \Omega_{k,\ell,f} = \sqrt{k_z^2 + M_{\ell,f}^2}, \tag{19}$$

for quarks with flavor f, in Landau Level ℓ, and

$$m_{\ell,b} = \sqrt{(2\ell+1)|q_bB| + m_b^2},$$
$$E_{k,\ell,b} = \sqrt{k_z^2 + m_{\ell,b}^2}, \tag{20}$$

for mesons with species $b(=\pi^0, \pi^\pm, \sigma)$. Here we assume $m_u = m_d$ and $m_{\pi^\pm} = m_{\pi^0} = m_\pi$. Also for charged pions, $q_{\pi^\pm} = \pm e$.

(iv) In the presence of a magnetic field, the quark propagator takes the following form:

$$S_f^{(B)}(K) = \exp\left(-\frac{k_\perp^2}{|q_fB|}\right)\sum_{\ell=0}^{\infty}(-1)^\ell \times \frac{\mathcal{D}_\ell(k_\parallel, k_\perp, q_fB)}{k_\parallel^2 - 2\ell|q_fB| - M_f^2 + i\varepsilon}, \tag{21}$$

where

$$\mathcal{D}_\ell(K, q_fB) = 4\not{k}_\perp L_{\ell-1}^{(1)}\left(\frac{2k_\perp^2}{|q_fB|}\right) + (\not{k}_\parallel + M_f) \times \left[(\mathbb{1} - \mathrm{sgn}(q_fB)i\gamma^1\gamma^2)L_\ell\left(\frac{2k_\perp^2}{|q_fB|}\right) - (\mathbb{1} + \mathrm{sgn}(q_fB)i\gamma^1\gamma^2)L_{\ell-1}\left(\frac{2k_\perp^2}{|q_fB|}\right)\right]. \tag{22}$$

Here $L_\ell^{(\alpha)}(x)$ is the generalized Laguerre polynomial which is written as

$$\frac{e^{-\frac{xz}{1-z}}}{(1-z)^{1+\alpha}} = \sum_{\ell=0}^{\infty} L_\ell^{(\alpha)} z^n, \tag{23}$$

with $|z| < 1$. We note $L_\ell^{(0)}(x) = L_\ell(x)$ and $L_{-1}^{(\alpha)} = 0$. Here sgn is the sign function.

(v) In the presence of a magnetic field, the charged boson propagator becomes

$$D_b^{(B)}(K) = 2\exp\left(-\frac{k_\perp^2}{|eB|}\right)\sum_{\ell=0}^{\infty}(-1)^\ell \times \frac{L_\ell(\frac{2k_\perp^2}{|eB|})}{k_\parallel^2 - (2\ell+1)|eB| - m_b^2 + i\varepsilon}. \tag{24}$$

A. Pion to quark-antiquark loop

The quark-antiquark contribution to the neutral pion self-energy reads

$$-i\Pi_{f\bar{f}}^{(B)}(P,T) = N_c g^2 \int \frac{d^4K}{(2\pi)^4}\mathrm{Tr}[\gamma_5 iS_f^{(B)}(K)\gamma_5 iS_f^{(B)}(K-P)], \tag{25}$$

where $Q = K - P$. Here N_c denotes the number of colors, which is taken as three for QCD. Thus, from Eq. (21), the $\Pi_{f\bar{f}}^{(B)}$ in Eq. (25) becomes

$$i\Pi_{f\bar{f}}^{(B)}(P,T) = N_c g^2 \int \frac{d^4K}{(2\pi)^4}\exp\left(-\frac{k_\perp^2 + q_\perp^2}{|q_fB|}\right)\sum_{\ell,n=0}^{\infty}(-1)^{\ell+n}\frac{\mathcal{N}_{\ell,n}^{(B)}(k_\parallel, k_\perp, q_\parallel, q_\perp)}{(k_\parallel^2 - 2\ell|q_fB| - M_f^2)(q_\parallel^2 - 2n|q_fB| - M_f^2)}, \tag{26}$$

where

$$\mathcal{N}_{\ell,n}^{(B)}(k_\parallel, k_\perp, q_\parallel, q_\perp) = \mathrm{Tr}[\gamma_5\mathcal{D}_\ell(k_\parallel, k_\perp, q_fB)\gamma_5\mathcal{D}_n(q_\parallel, q_\perp, q_fB)]. \tag{27}$$

The trace in (27) is computed as

$$\mathcal{N}_{\ell,n}^{(B)}(k_\parallel, k_\perp, q_\parallel, q_\perp) = 8[M_f^2 - (k.q)_\parallel] \times \left[L_{\ell-1}\left(\frac{2k_\perp^2}{|q_fB|}\right)L_{n-1}\left(\frac{2q_\perp^2}{|q_fB|}\right) + L_\ell\left(\frac{2k_\perp^2}{|q_fB|}\right)L_n\left(\frac{2q_\perp^2}{|q_fB|}\right)\right] + 64(k^1q^1 + k^2q^2)L_{\ell-1}^1\left(\frac{2k_\perp^2}{|q_fB|}\right)L_{n-1}^1\left(\frac{2q_\perp^2}{|q_fB|}\right). \tag{28}$$

Since we are interested in modification of the π^0 mass, we take the limit $\boldsymbol{p} \to \boldsymbol{0}$ of Eq. (26):

$$\Pi_{f\bar{f}}^{(B)}(p_0, T) = N_c g^2 \int \frac{d^4K}{(2\pi)^4}\exp\left(-\frac{2k_\perp^2}{|q_fB|}\right)\sum_{\ell,n=0}^{\infty}(-1)^{\ell+n}\frac{\mathcal{N}_{\ell,n}^{(B)}(k_0, p_0, \boldsymbol{q} = \boldsymbol{k})}{(k_0^2 - \Omega_{k,\ell,f}^2)(q_0^2 - \Omega_{k,n,f}^2)}. \tag{29}$$

After performing the perpendicular momentum integral analytically, the above expression simplified to

$$\Pi^{(B)}_{f\bar{f}}(p_0,T) = -N_c 8g^2 \int_{-\infty}^{\infty} \frac{dk_z}{2\pi} \sum_{\ell,n=0}^{\infty} (-1)^{\ell+n} \frac{(\mathcal{I}^{(0)}_{\ell,n,f} + \mathcal{I}^{(0)}_{\ell-1,n-1,f})[k_0(k_0-p_0) - k_z^2 - M_f^2] - 8\mathcal{I}^{(1)}_{\ell-1,n-1,f}}{(k_0^2 - \Omega^2_{k,\ell,f})((k_0-p_0)^2 - \Omega^2_{k,n,f})}, \quad (30)$$

where $\mathcal{I}^{(\alpha)}_{l,n,f}$ (for $l, n, \alpha \in \mathbb{Z}$ and $l,\ n,\ \alpha \geq 0$) is defined in Appendix D. After performing the perpendicular integration according to Eq. (D3) and employing the following identity (under k_0 integral):

$$\frac{M_f^2 - k_0 q_0 + k_z^2}{(k_0^2 - \Omega^2_{k,\ell,f})(q_0^2 - \Omega^2_{k,\ell,f})} = \frac{p_0^2 - 4\ell|q_f B|}{2(k_0^2 - \Omega^2_{k,\ell,f})(q_0^2 - \Omega^2_{k,\ell,f})} - \frac{1}{k_0^2 - \Omega^2_{k,\ell,f}}. \quad (31)$$

It is convenient to separate the lowest Landau level (LLL) and higher Landau level (HLL) contributions as follows:

$$\Pi^{(B)}_{f\bar{f},\mathrm{LLL}} = -iN_c g^2 \frac{|q_f B|}{\pi} \int \frac{d^2k_\parallel}{(2\pi)^2} \left[\frac{p_0^2}{2} \frac{1}{(k_0^2 - \Omega^2_{k,\ell=0,f})(q_0^2 - \Omega^2_{k,\ell=0,f})} - \frac{1}{k_0^2 - \Omega^2_{k,\ell=0,f}}\right], \quad (32)$$

$$\Pi^{(B)}_{f\bar{f},\mathrm{HLL}} = -2N_c i g^2 \frac{|q_f B|}{\pi} \int \frac{d^2k_\parallel}{(2\pi)^2} \sum_{\ell=1}^{\infty} \left[\frac{p_0^2}{2} \frac{1}{(k_0^2 - \Omega^2_{k,\ell,f})(q_0^2 - \Omega^2_{k,\ell,f})} - \frac{1}{k_0^2 - \Omega^2_{k,\ell,f}}\right]. \quad (33)$$

In deriving Eqs. (32) and (33), we used the following identities:

$$\delta_{\ell-1,\ell-1} = 1 - \delta_{0,\ell}, \qquad \delta_{\ell,\ell} = 1. \quad (34)$$

Here we note that the Kronecker delta gives zero for any negative index. This kind of expression is typical in cases involving a fermion inside a loop. The degeneracy in higher Landau levels is considered by the factor $(2 - \delta_{\ell,0})$. Combining the lowest and higher Landau level terms, we write Eqs. (32) and (33) as

$$\Pi^{(B)}_{f\bar{f}}(p_0,T) = -ig^2 N_c \frac{|q_f B|}{\pi} \sum_{\ell=0}^{\infty} (2 - \delta_{\ell,0}) \int \frac{d^2k_\parallel}{(2\pi)^2} \left[\frac{p_0^2}{2} \frac{1}{(k_0^2 - \Omega^2_{k,\ell,f})(q_0^2 - \Omega^2_{k,\ell,f})} - \frac{1}{k_0^2 - \Omega^2_{k,\ell,f}}\right]. \quad (35)$$

The expression involving HLL has an overall factor of 2, which is absent in the expression of the LLL. It comes from the fact that the virtual quark-antiquark pair in HLL has spin degeneracy that is lifted in LLL. So, after replacing the k_0 integration with the frequency sum, the expression of $\Pi^{(B)}_{f\bar{f}}(p_0,T)$ becomes

$$\Pi^{(B)}_{f\bar{f}}(p_0,T) = N_c g^2 \frac{|q_f B|}{\pi} \sum_{\ell=0}^{\infty} (2 - \delta_{\ell,0}) \int_{-\infty}^{\infty} \frac{dk_z}{2\pi} T \sum_{k_0} \left[\frac{p_0^2}{2} \frac{1}{(k_0^2 - \Omega^2_{k,\ell,f})(q_0^2 - \Omega^2_{k,\ell,f})} - \frac{1}{k_0^2 - \Omega^2_{k,\ell,f}}\right]. \quad (36)$$

In the presence of the magnetic field and temperature, any loop integration contains three pieces: (i) a pure vacuum contribution, (ii) a pure magnetic field contribution, and (iii) a thermal as well as a magnetic field (i.e., thermomagnetic) contribution.[1] Now we compute each contribution separately.

1. Pure vacuum part

For the vacuum part we take $S_f(K) = \frac{\not{K} + M_f}{K^2 - M_f^2}$. The diagram in Fig. 1 gives

$$\Pi^{\mathrm{Vac}}_{f\bar{f}}(p_0) = -4N_c i g^2 \int \frac{d^4K}{(2\pi)^4} \left[\frac{p_0^2}{2} \frac{1}{k_0^2 - M_f^2} \frac{1}{(k_0 - p_0)^2 - k^2 - M_f^2} - \frac{1}{k_0^2 - M_f^2}\right]. \quad (37)$$

Employing the Feynman parametrization technique to the first term in the square brackets, we get

[1]The pure vacuum contribution contains ultraviolet divergence. In our context, the pure magnetic field and thermomagnetic part are divergence-free. As a result of taking the $B \to 0$ limit, the pure magnetic contribution vanishes, and the thermomagnetic contribution reduces to a pure thermal contribution. On the other hand, taking the $T \to 0$ limit, the thermomagnetic part vanishes.

$$\Pi^{\mathrm{Vac}}_{f\bar{f}}(p_0) = -4N_c i g^2\left[\frac{p_0^2}{2}\int_0^1 dx\int\frac{d^4K}{(2\pi)^4}\frac{1}{\{(K-xP)^2-[M_f^2-x(1-x)]\}^2}-\int\frac{d^4K}{(2\pi)^4}\frac{1}{K^2-M_f^2}\right]. \tag{38}$$

After the change of variable $K - xP \to K$, we get

$$\Pi^{\mathrm{Vac}}_{f\bar{f}}(p_0) = -4N_c i g^2\left[\frac{p_0^2}{2}\int_0^1 dx\int\frac{d^4K}{(2\pi)^4}\frac{1}{\{K^2-[M_f^2-x(1-x)]\}^2}-\int\frac{d^4K}{(2\pi)^4}\frac{1}{K^2-M_f^2}\right]. \tag{39}$$

Before regularizing the integration, we should mention a few words about the renormalizability of LSMq. The dimension of the coupling constants g and λ is 0. So according to quantum field theory, the LSMq is renormalizable [25]. The integral in Eq. (39) is ultraviolet divergent that is regularized by the method of dimensional regularization. We analytically continue the momentum integration to d dimensions. Now, it can be performed by using the following identity:

$$\int\frac{d^dK}{(2\pi)^d}\frac{1}{(K^2-\Delta)^\alpha} = i\frac{\pi^{d/2}}{(2\pi)^d}\frac{(-1)^\alpha}{\Gamma(\alpha)}\frac{\Gamma(\alpha-\frac{d}{2})}{\Delta^{\alpha-\frac{d}{2}}}. \tag{40}$$

It yields, after taking out an auxiliary scale factor Λ from g as $g \to g\Lambda^{2-\frac{d}{2}}$, to

$$\Pi^{\mathrm{Vac}}_{f\bar{f}}(p_0) = 4N_c g^2\Lambda^{4-d}\frac{1}{(4\pi)^{d/2}}\left\{\frac{p_0^2}{2}\int_0^1 dx\frac{\Gamma(2-\frac{d}{2})}{[M_f^2-x(1-x)p_0^2]^{2-\frac{d}{2}}}+\frac{\Gamma(1-\frac{d}{2})}{(M_f^2)^{1-\frac{d}{2}}}\right\}. \tag{41}$$

Let us take $\epsilon = 2 - \frac{d}{2}$ and obtain

$$\Pi^{\mathrm{Vac}}_{f\bar{f}}(p_0) = N_c\frac{g^2}{4\pi^2}\left[\frac{p_0^2}{2}\int_0^1 dx\left(\frac{1}{4\pi\Lambda^2}\right)^{-\epsilon}\frac{\Gamma(\epsilon)}{[M_f^2-x(1-x)p_0^2]^\epsilon}+\left(\frac{1}{4\pi\Lambda^2}\right)^{-\epsilon}\frac{\Gamma(\epsilon-1)}{(M_f^2)^{\epsilon-1}}\right]. \tag{42}$$

Now, we expand the above expression around $\epsilon = 0$ to get

$$\Pi^{\mathrm{Vac}}_{f\bar{f}}(p_0) = N_c\frac{g^2}{4\pi^2}\left[\left(\frac{p_0^2}{2}-M_f^2\right)\left(\frac{1}{\epsilon}-\gamma_E+\log(4\pi\Lambda^2)\right)-\left\{\frac{p_0^2}{2}\int_0^1 dx\log[M_f^2-x(1-x)p_0^2]+M_f^2-M_f^2\log(M_f^2)\right\}\right]. \tag{43}$$

In accordance with $\overline{\mathrm{MS}}$ prescription, we absorb the $\frac{1}{\epsilon}-\gamma_E+\log(4\pi)$ by introducing the counterterm. It leads to

$$\Pi^{\mathrm{Vac}}_{f\bar{f},\overline{\mathrm{MS}}}(p_0) = N_c\frac{g^2}{4\pi^2}\left[\frac{p_0^2}{2}\int_0^1 dx\log\left(\frac{\Lambda^2}{M_f^2-x(1-x)p_0^2}\right)-M_f^2\left(\log\frac{\Lambda^2}{M_f^2}+1\right)\right]. \tag{44}$$

2. Magnetic field part

After getting the pure vacuum part, we now evaluate the magnetic part. We take $\epsilon = 1 - \frac{d}{2}$ so that the integral formally diverges at $\epsilon = 0$. We get

$$\Pi^{(B)}_{f\bar{f}}(p_0) = N_c\frac{g^2}{4\pi^2}|q_fB|\sum_{\ell=0}^{\infty}(2-\delta_{\ell,0})\left[\frac{p_0^2}{2}\int_0^1 dx\frac{1}{(4\pi\Lambda^2)^{-\epsilon}}\frac{\Gamma(1+\epsilon)}{[M_f^2+2\ell|q_fB|-x(1-x)p_0^2]^{1+\epsilon}}\right.$$
$$\left.+\frac{1}{(4\pi\Lambda^2)^{-\epsilon}}\frac{\Gamma(\epsilon)}{(M_f^2+2\ell|q_fB|)^\epsilon}\right]. \tag{45}$$

The sum over the Landau levels in the RHS of the last equation can be performed as

$$\sum_{\ell=0}^{\infty}\frac{2-\delta_{\ell,0}}{[M_f^2+2\ell|q_fB|-x(1-x)p_0^2]^{1+\epsilon}} = \frac{(2|q_fB|)^{-\epsilon}}{|q_fB|}\zeta\left(1+\epsilon,\frac{M_f^2-x(1-x)p_0^2}{2|q_fB|}\right)-\frac{1}{(M_f^2-x(1-x)p_0^2)^{1+\epsilon}}, \tag{46}$$

$$\sum_{\ell=0}^{\infty}\frac{2-\delta_{\ell,0}}{(M_f^2+2\ell|q_fB|)^{\epsilon}}=2(2|q_fB|)^{-\epsilon}\zeta\left(\epsilon,\frac{M_f^2}{2|q_fB|}\right)-\frac{1}{M_f^{2\epsilon}}. \tag{47}$$

Thus, we have

$$\Pi^{(B)}_{f\bar{f}}(p_0)=N_c\frac{g^2}{4\pi^2}|q_fB|\left\{\frac{p_0^2}{2}\frac{\Gamma(1+\epsilon)}{(4\pi\Lambda^2)^{-\epsilon}}\int_0^1dx\left[\frac{(2|q_fB|)^{-\epsilon}}{|q_fB|}\zeta\left(1+\epsilon,\frac{M_f^2-x(1-x)p_0^2}{2|q_fB|}\right)\right.\right.$$
$$\left.\left.-\frac{1}{(M_f^2-x(1-x)p_0^2)^{1+\epsilon}}\right]+\frac{\Gamma(\epsilon)}{(4\pi\Lambda^2)^{-\epsilon}}\left[2(2|q_fB|)^{-\epsilon}\zeta\left(\epsilon,\frac{M_f^2}{2|q_fB|}\right)-\frac{1}{M_f^{2\epsilon}}\right]\right\}. \tag{48}$$

As usual if we expand the above expression around $\epsilon=0$, we get terms of the form $\zeta(0,x)$ and $\partial_s\zeta(s,x)|_{s=0}$. Now, using the following properties of the Hurwitz zeta function:

$$\zeta(0,x)=\frac{1}{2}-x,\qquad \zeta^{(1,0)}(0,x)\equiv\frac{d}{ds}\zeta(s,x)|_{s=0}=\log\Gamma(x)-\frac{1}{2}\log(2\pi), \tag{49}$$

and after performing some simplifications, we obtain

$$\Pi^{(B)}_{f\bar{f}}(p_0)=N_c\frac{g^2}{4\pi^2}\left\{\left(\frac{p_0^2}{2}-M_f^2\right)\left[\frac{1}{\epsilon}-\gamma_E+\log(4\pi\Lambda^2)\right]+|q_fB|\left[2\log\Gamma\left(\frac{M_f^2}{2|q_fB|}\right)+\log\left(\frac{M_f^2}{4\pi|q_fB|}\right)\right]\right.$$
$$\left.-\left(\frac{p_0^2}{2}-M_f^2\right)\log(2|q_fB|)-\frac{p_0^2}{2}\int_0^1dx\left[\psi\left(\frac{M_f^2-x(1-x)p_0^2}{2|q_fB|}\right)+\frac{|q_fB|}{M_f^2-x(1-x)p_0^2}\right]\right\}. \tag{50}$$

After performing the integral, we get

$$\int_0^1dx\frac{|q_fB|}{M_f^2-x(1-x)p_0^2}=\frac{4|q_fB|}{p_0^2}\cot^{-1}\left(\sqrt{\frac{4M_f^2}{p_0^2}-1}\right), \tag{51}$$

the (vacuum + magnetic field dependent) part of self-energy as

$$\Pi^{(B)}_{f\bar{f}}(p_0)=N_c\frac{g^2}{4\pi^2}\left(\frac{p_0^2}{2}-M_f^2\right)\left(\frac{1}{\epsilon}-\gamma_E+\log(4\pi\Lambda^2)-\log(2|q_fB|)\right)-|q_fB|\left[2\log\Gamma\left(\frac{M_f^2}{2|q_fB|}\right)+\log\left(\frac{M_f^2}{4\pi|q_fB|}\right)\right.$$
$$\left.+2\cot^{-1}\left(\sqrt{\frac{4M_f^2}{p_0^2}-1}\right)\right]-\frac{p_0^2}{2}\int_0^1dx\psi\left(\frac{M_f^2-x(1-x)p_0^2}{2|q_fB|}\right). \tag{52}$$

After subtracting the vacuum part from Eq. (52), we get the pure magnetic field-dependent contribution as

$$\Pi^{(B)}_{f\bar{f}}(p_0)-\Pi^{\mathrm{Vacuum}}_{f\bar{f}}(p_0)=N_c\frac{g^2}{4\pi^2}\left\{\frac{p_0^2}{2}\int_0^1dx\left[\log\frac{M_f^2-x(1-x)p_0^2}{2|q_fB|}-\psi\left(\frac{M_f^2-x(1-x)p_0^2}{2|q_fB|}\right)-\frac{|q_fB|}{M_f^2-x(1-x)p_0^2}\right]\right.$$
$$\left.-2|q_fB|\left[\log\Gamma\left(\frac{M_f^2}{2|q_fB|}\right)+\log\left(\frac{M_f^2}{4\pi|q_fB|}\right)\right]+M_f^2-M_f^2\log\left(\frac{M_f^2}{2|q_fB|}\right)\right\}. \tag{53}$$

3. The thermomagnetic part

To get the thermomagnetic part, we need to perform the fermionic frequency sums. It is performed in Appendix B 1. Here we quote the results:

$$T\sum_{k_0}\frac{1}{k_0^2-\Omega^2_{\ell,k,f}}\frac{1}{(k_0-p_0)^2-\Omega^2_{\ell,k,f}}=-\frac{1-\tilde{n}^+(\Omega_{\ell,k,f})-\tilde{n}^-(\Omega_{\ell,k,f})}{\Omega_{\ell,k,f}(p_0^2-4\Omega^2_{\ell,k,f})},$$
$$T\sum_{k_0}\frac{1}{k_0^2-\Omega^2_{\ell,k,f}}=-\frac{1-\tilde{n}^+(\Omega_{\ell,k,f})-\tilde{n}^-(\Omega_{\ell,k,f})}{2\Omega_{\ell,k,f}}. \tag{54}$$

Now, substituting the frequency sums in the last line in Eq. (36) and simplifying, we arrive at

$$\Pi^{(B)}_{f\bar{f}}(p_0,T) = -N_c\frac{g^2}{2\pi^2}|q_fB|\sum_{\ell=0}^{\infty}(2-\delta_{\ell,0})\int_{-\infty}^{\infty}dk_z\Omega_{\ell,k,f}\frac{1-\tilde{n}^+(\Omega_{\ell,k,f})-\tilde{n}^-(\Omega_{\ell,k,f})}{p_0^2-4\Omega^2_{\ell,k,f}}. \quad (55)$$

We have calculated the vacuum + pure B part earlier, which comes from 1 with the distribution function $\tilde{n}^{\pm}$. So dropping that term, we get the thermomagnetic part as

$$\Pi^{(B)}_{f\bar{f},\mathrm{ThM}}(p_0,T) = N_c\frac{g^2}{2\pi^2}|q_fB|\sum_{\ell=0}^{\infty}(2-\delta_{\ell,0})\int_{-\infty}^{\infty}dk_z\Omega_{\ell,k,f}\frac{\tilde{n}^+(\Omega_{\ell,k,f})+\tilde{n}^-(\Omega_{\ell,k,f})}{p_0^2-4\Omega^2_{\ell,k,f}}. \quad (56)$$

B. Pion to pion loop

1. Charged pion contribution

The tadpole diagram reads as shown in Fig. 2(a)

$$\Pi^{(B)}_{\pi^\pm} = \frac{\lambda}{4}\int\frac{d^4K}{(2\pi)^4}iD^{(B)}(K). \quad (57)$$

Substituting Eq. (24) into Eq. (57), we get

$$\Pi^{(B)}_{\pi^\pm} = i\frac{\lambda}{2}\int\frac{d^2k_\parallel}{(2\pi)^2}\sum_{\ell=0}^{\infty}\frac{\mathcal{J}_\ell}{k_\parallel^2-(2\ell+1)|eB|-m_\pi^2+i\varepsilon}, \quad (58)$$

where we have defined

$$\mathcal{J}_\ell = \int\frac{d^2k_\perp}{(2\pi)^2}(-1)^\ell\exp\left(-\frac{k_\perp^2}{|eB|}\right)L_\ell\left(\frac{2k_\perp^2}{|eB|}\right). \quad (59)$$

Here the integral can be performed analytically and shown in Appendix B. Here is the result quoted:

$$\mathcal{J}_\ell = \frac{|eB|}{4\pi}. \quad (60)$$

This leads Eq. (58) to

$$\Pi^{(B)}_{\pi^\pm} = i\frac{\lambda}{2}\frac{|eB|}{4\pi}\sum_{\ell=0}^{\infty}\int\frac{d^2k_\parallel}{(2\pi)^2}\frac{1}{k_\parallel^2-m^2_{\ell,\pi}}. \quad (61)$$

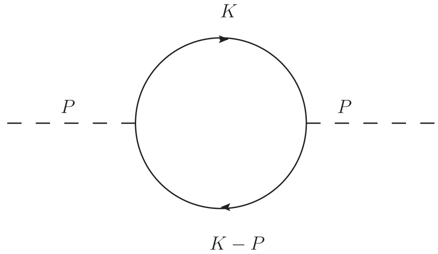

FIG. 1. Feynman diagram for one-loop quark-antiquark contribution to the π^0 self-energy.

The integral in Eq. (61) is divergent. To regulate the divergence, we go to the $d = 2-2\epsilon$ dimension. Also, from the dimensional argument, we take out a dimensional quantity via an auxiliary scale by replacing $\lambda \to \Lambda^{d-2}\lambda$. So the integral becomes

$$\Pi^{(B)}_{\pi^\pm} = i\frac{\lambda\Lambda^{2-d}}{2}\frac{|eB|}{4\pi}\sum_{\ell=0}^{\infty}\int\frac{d^dk_\parallel}{(2\pi)^d}\frac{1}{k_\parallel^2-m^2_{\ell,\pi}}. \quad (62)$$

After performing the d-dimensional integral [26], we get

$$\Pi^{(B)}_{\pi^\pm} = -\lambda\Lambda^{2-d}\frac{|eB|}{8\pi}\sum_{\ell=0}^{\infty}\frac{\pi^{d/2}}{(2\pi)^{d/2}}\frac{\Gamma(1-\frac{d}{2})}{(m^2_{\ell,\pi})^{1-\frac{d}{2}}}. \quad (63)$$

Now we write the last equation in terms of ϵ to get

$$\Pi^{(B)}_{\pi^\pm} = \frac{|eB|}{32\pi^2}\left(\frac{2|eB|}{4\pi\Lambda^2}\right)^{-\epsilon}\Gamma(\epsilon)\sum_{\ell=0}^{\infty}\frac{1}{(\ell+\frac{1}{2}+\frac{m_\pi^2}{2|eB|})^\epsilon}. \quad (64)$$

Now we expand the last equation around $\epsilon = 0$ to get

$$\begin{aligned}\Pi^{(B)}_{\pi^\pm} &= \frac{\lambda|eB|}{32\pi^2}\left\{\zeta\left(0,\frac{1}{2}+\frac{m_\pi^2}{2|eB|}\right)\right.\\ &\quad\times\left[\frac{1}{\epsilon}-\gamma_E+\log(4\pi)-\log\frac{2|eB|}{\Lambda^2}\right]\\ &\quad\left.+\zeta^{(1,0)}\left(0,\frac{1}{2}+\frac{m_\pi^2}{2|eB|}\right)\right\}\\ &= -\frac{\lambda m_\pi^2}{64\pi^2}\left[\frac{1}{\epsilon}-\gamma_E+\log 4\pi+\log\frac{\Lambda^2}{2|eB|}\right.\\ &\quad\left.+\frac{|eB|}{m_\pi^2}\log 2\pi-\frac{2|eB|}{m_\pi^2}\log\Gamma\left(\frac{1}{2}+\frac{m_\pi^2}{2|eB|}\right)\right]. \quad (65)\end{aligned}$$

We can get weak field results by using the asymptotic expansion [27]

$$\begin{aligned}\log\Gamma(t+x) &\sim \left(t+x-\frac{1}{2}\right)\log x - x + \frac{1}{2}\log(2\pi)\\ &\quad+\sum_{n=1}^{\infty}(-1)^n\frac{1}{n(n+1)}B_n(t)\frac{1}{x^n}, \quad (66)\end{aligned}$$

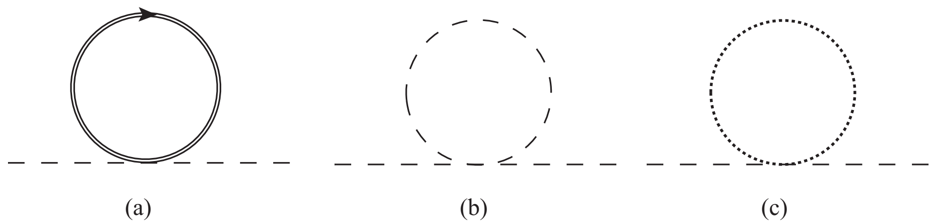

FIG. 2. Feynman diagram for one-loop (a) charged pion, (b) neutral pion, and (c) sigma meson contribution to the π^0 self-energy. The dashed line denotes π^0, the double line denotes $\pi^{\pm}$, and the dotted line denotes σ-meson. Only the charged pion is affected by the magnetic field and temperature, but the neutral pion and the sigma meson are affected by only the temperature.

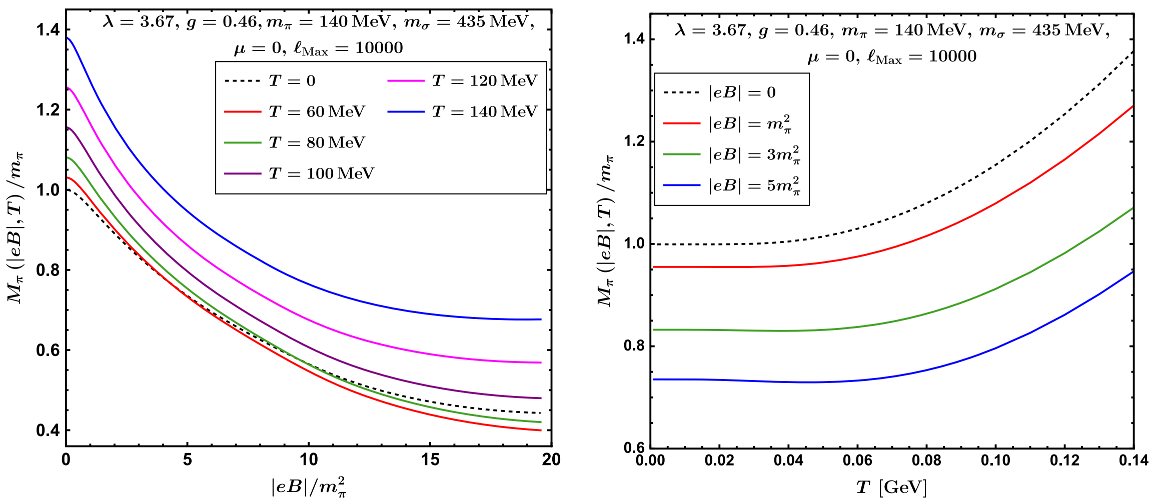

FIG. 3. The behavior of the neutral pion π^0 mass. The left panel shows the variation of the π^0 mass with background magnetic field $|eB|$ at some of the fixed values of temperature ranging from 0 to 140 MeV. The right panel shows the plot of the π^0 mass with the temperature keeping the value of the magnetic field fixed. The x axis and y axis are scaled with the square of the vacuum pion mass $m_{\pi^0}^2$ and the vacuum pion mass m_{π^0}. In this plot, we have taken the minimum of effective potential $v_B(T)$ and one-loop meson self-coupling $\lambda_{\rm eff}$ to obtain the pion mass as indicated by Eq. (76).

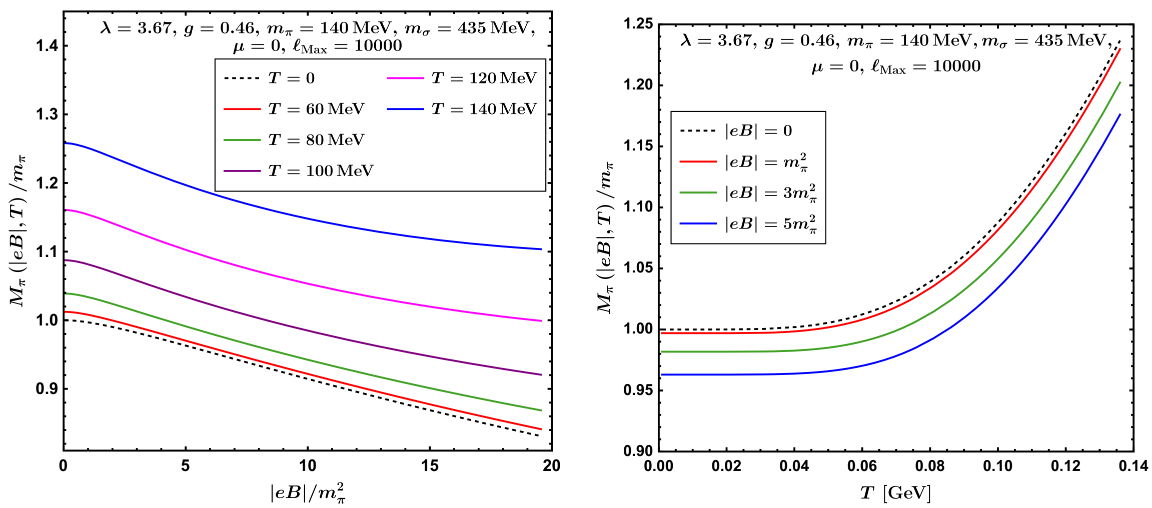

FIG. 4. The behavior of the neutral pion π^0 mass. The left panel shows the variation of the π^0 mass with background magnetic field $|eB|$ at some of the fixed values of temperature ranging from 0 to 140 MeV. The right panel shows the plot of the π^0 mass with the temperature keeping the value of the magnetic field fixed. The x axis and y axis are scaled with the square of the vacuum pion mass $m_{\pi^0}^2$ and the vacuum pion mass m_{π^0}. In this plot, we have taken the classical minimum v_0' and bare meson self-coupling λ in order to obtain the pion mass as indicated by Eq. (74).

where $B_n(t)$ are the Bernoulli polynomials defined by the generating function

$$\frac{e^{xt}}{e^t - 1} = \sum_{n=0}^{\infty} \frac{B_n(t)}{n!} x^{n-1}. \tag{67}$$

Starting from Eq. (61), we obtain

$$\Pi_{\pi^\pm}^{(B)} = -\frac{\lambda}{2}\frac{|eB|}{4\pi}\sum_{\ell=0}^{\infty}\int_{-\infty}^{\infty}\frac{dk^3}{2\pi}T\sum_{k_0}\frac{1}{k_0^2 - E_{k,\ell}^2}. \tag{68}$$

Here the frequency sum is performed in Appendix B, and the result is

$$T\sum_{k_0}\frac{1}{k_0^2 - E_{k,\ell}^2} = -\frac{1 + 2n(E_{k,\ell})}{2E_{k,\ell}}, \tag{69}$$

where

$$n(E_{k,\ell}) = \frac{1}{\exp(E_{k,\ell}/T) - 1}. \tag{70}$$

Thus after plugging Eq. (69) in Eq. (68) and dropping the nonthermal term (containing 1 in the frequency sum), we get the thermomagnetic contribution as

$$\Pi_{\pi^\pm}^{(B)} = \frac{\lambda|eB|}{16\pi^2}\sum_{\ell=0}^{\infty}\int_{-\infty}^{\infty} dk_z \frac{n(E_{k,\ell})}{E_{k,\ell}}. \tag{71}$$

2. Neutral pion and sigma loop contribution

For the neutral pion and sigma loop as shown in Figs. 2(b) and 2(c) respectively, there will be no effects from the magnetic field as they are chargeless. Again we drop the vacuum part and consider only the thermal correction;

$$\Pi_j = i\frac{\lambda}{4}\int\frac{d^4K}{(2\pi)^4}D_j(K), \tag{72}$$

where $D_j(K) = \frac{1}{K^2 - m_j^2}$ is the propagator for the j-type particle with $j = \pi^0, \sigma$. After doing the usual replacements and performing the frequency sum, we arrive at

$$\Pi_j^{\mathrm{Th}} = \frac{\lambda}{8\pi^2}\int_0^{\infty} dk k^2 \frac{n\left(\sqrt{k^2 + m_j^2}\right)}{\sqrt{k^2 + m_j^2}}. \tag{73}$$

IV. π^0 MASS

In this section, we compute the neutral pion mass. Before doing so, we must reemphasize a few important points. The vacuum π_0 mass, denoted by m_π, is one of the input parameters in the theory. We have taken it as $m_\pi = 140$ MeV as determined by experiments. Now, in the presence of the thermal medium as well as the background magnetic field, the neutral pion receives an *additional correction* over m_π, denoted by $M_\pi(eB, T)$, coming from the temperature as well as the magnetic field. Our aim in this section is to obtain $M_\pi(eB, T)$. For that, we need to solve the following equation:

$$p_0^2 - |\boldsymbol{p}|^2 - m_\pi^2 - \Pi^{(B)}(p_0, \boldsymbol{p}, T) = 0 \tag{74}$$

in the limit of $\boldsymbol{p} \to \boldsymbol{0}$ and $p_0 = M_\pi(|eB|, T)$. The self-energy is given by (16). Note that there is a factor of g^2 in the expression of the quark loop contribution as indicated in (25) and a factor of λ in the expression of the meson loop contribution. We shall solve Eq. (74) in three settings as follows.

A. Basic case

In this case, we take a numerical value of λ, g, which are two coupling parameters of the theory. Also, we consider the vacuum value of $m_\pi = 0.14$ GeV. Then we solve Eq. (74).

B. Including self-coupling

Here we consider the one-loop correction of vertex λ. In this case we take the one-loop modified vertex $\lambda_{\mathrm{eff}} = \lambda + \Delta\lambda$, where $\Delta\lambda$ is given by Eq. (A1). So the dispersion relation becomes

$$p_0^2 - (\lambda_{\mathrm{eff}}(v_0')^2 - a^2) - \Pi_{\mathrm{VM}}^{(B)}(p_0, \boldsymbol{p} \to \boldsymbol{0}, T) = 0. \tag{75}$$

Here the VM subscript in Π indicates that we replace the expression of λ_{eff} in place of λ that appears in front of the meson self-energy contribution. As mentioned earlier, v_0' is the minimum of the tree-level potential.

C. Including self-coupling and quantum corrected minimum of the effective potential

Here we solve Eq. (75) with v_0' substituted by $v_B(T)$, namely

$$p_0^2 - (\lambda_{\mathrm{eff}} v_B^2(T) - a^2) - \Pi_{\mathrm{VM}}^{(B)}(p_0, \boldsymbol{p} \to \boldsymbol{0}, T) = 0, \tag{76}$$

where $v_B(T)$ is the minimum of the effective potential. The topic of the effective potential is discussed in detail in Appendix C. Before proceeding further, we reemphasize the fact that the self-energy $\Pi_{\mathrm{VM}}^{(B)}(p_0, \boldsymbol{p} \to \boldsymbol{0}, T)$ is scale independent. In the self-energy, the vacuum and magnetic parts are scale dependent. Since we subtracted the vacuum part from the total self-energy, we get $\Pi_{\mathrm{VM}}^{(B)}(p_0, \boldsymbol{p} \to \boldsymbol{0}, T)$ scale independent. Additionally, the pure vacuum part of the effective potential carries the scale Λ as it is clear from Eqs. (C25) and (C32). However, the position of the minimum of the effective potential $v_B(T)$ and curvature *does not* change with changing the scale Λ. So, the overall mass correction over the pure-vacuum π^0 mass is scale independent as only the self-energy $\Pi_{\mathrm{VM}}^{(B)}(p_0, \boldsymbol{p} \to \boldsymbol{0}, T)$ and $v_B(T)$ enters in dispersion Eq. (76).

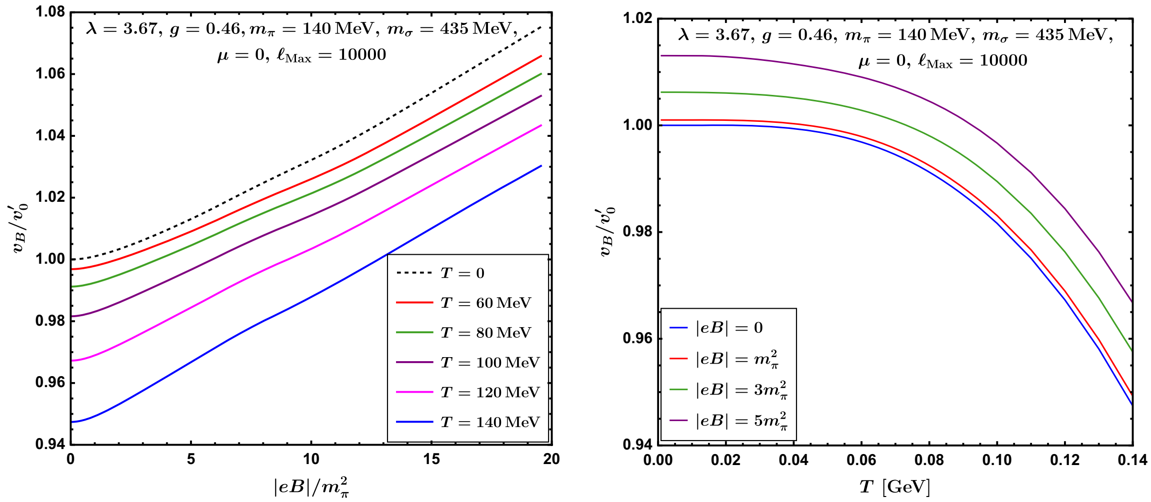

FIG. 5. The behavior of the minimum of the effective potential v_B with $|eB|$ and T shown in the left and right panels, respectively.

V. RESULTS

We have plotted the magnetic field and temperature dependence of the neutral pion mass in a warm magnetized medium as shown in Figs. 3 and 4. The Lagrangian has the following parameters: the boson self-coupling λ, the boson-fermion coupling g, the vacuum pion mass m_π, and the mass parameter a.[2] We have kept the value of m_π at 0.14 GeV and m_σ at 0.435 GeV throughout. We have kept the temperature in all of our plots up to 140 MeV, which is less than or equal to the chiral phase transition temperature (in LQCD, it is calculated to ∼156 MeV [28]). For our present purpose, we have taken $T = 0$, 60, 80, 100, 120, 140 MeV while changing the magnetic field up to $20m_\pi^2 = 0.392\ \mathrm{GeV}^2$ starting from zero as shown in the left panel of Fig. 4. Also, we have kept $T = 140$ MeV for four different values of $|eB|$ as in the right panel of Fig. 4. For the magnetic field, the value $|eB| = 20m_\pi^2$ is beyond the magnitude generated in a heavy-ion collision or inside the core of magnetars. We have taken $\lambda = 3.67$ and $g = 0.46$, which was used by Ayala *et al.* in Ref. [14] to match their result of the magnetic field dependence of the π^0 mass, calculated in the strong magnetic field approximation at zero temperature, with the LQCD data of Ref. [13]. In our calculation, we have tackled the sum over Landau levels and integration over k_z appearing in the thermomagnetic part of self-energy as well as effective potential numerically. For a very low magnetic field, one can note that the result saturates by summing over ∼5,000 LL's. In our calculation, we have taken $\ell_{\mathrm{Max}} = 10,000$; i.e., we summed over 10,000 Landau levels, which is more than enough to reach saturation.

In Fig. 3, we show the plots of the neutral pion mass with $|eB|$ (left panel) and with T (right panel) considering the effect of effective vertex and quantum corrected condensate $v_B(T)$. The mass decreases with increasing $|eB|$. The fall is rapid at low values of $|eB|$ for all temperate. But after a certain value ($15m_\pi^2$), it saturates with the field. Note that as we increase the temperature, the fall with $|eB|$ becomes more rapid in the temperature range ∼(0–100) MeV within the window of $|eB| \sim 5m_\pi^2 - 15m_\pi^2$. As a result, the plot in the right panel of Fig. 3 $T = 0$, intersects with $T = 60$ MeV and $T = 80$ MeV. Now for the variation with temperature, the mass increases with temperature, which is quite expected as the thermal contribution increases with increasing temperature. The mass remains uniform for low T but sharply increases after 60 MeV. This behavior is observed for all values of magnetic fields considered, and it can be explained from the plot of mass with the magnetic field. At low temperatures, for the magnetic field values considered in the right panel of Fig. 3, the field has a much stronger tendency to suppress the mass than the temperature to enhance it. But as the temperature increases, it gradually becomes more dominating than the magnetic field.

Considering the classical v_0' and the bare meson-meson coupling λ as well as bare quark-meson coupling g, we have shown the variation of the neutral π^0 mass with $|eB|$ and T in Fig. 4. The purpose of showing Fig. 4 was to compare it with the behavior of the neutral pion mass (shown in Fig. 3) by taking into account the effect of λ_{eff} and $v_B(T)$. The behaviour of the $v_B(T)$ with $|eB|$ and T, while keeping all other parameter fixed, is displayed in Fig. 5. As we can see clearly, the mass falls somewhat less steeply than Fig. 3 with the magnetic field. But the rise of mass with the temperature is more pronounced and steeper than that with including the effective vertex and $v_B(T)$.

VI. CONCLUSION

In conclusion, we have computed the neutral pion mass in the presence of an arbitrary background magnetic field at a nonzero temperature in the framework of the LSMq

[2]In our case a is fixed by m_π and m_σ. So we can think of m_σ as a parameter of the theory instead of a.

model. We have examined the behavior of the pion mass with the magnetic field, keeping the temperature at fixed values, and that with the temperature, keeping the field fixed. The coupling constants of the theory are the meson-meson coupling λ and quark-meson coupling g. At the same time, other parameters are vacuum pion mass m_π, vacuum sigma mass m_σ, and constituent quark mass M_f. We have incorporated the effect of meson self-coupling through $\lambda_{\rm eff}$ and the quantum correction of effective potential through $v_B(T)$ in the π^0 mass. In calculating the mass, we have shown in Fig. 6 the magnetic field and temperature dependence of the one-loop effective potential $V_{\rm eff}$ as a function of v. The general case of the arbitrary strength of the magnetic field is considered by using the general expression of the charged pion and quark propagator without invoking strong and/or weak field approximation. Also, a framework for extending the calculation in the finite density domain is incorporated by considering the constituent quark chemical potential μ. We report the decrement of the pion mass with the strength of the background magnetic field on which some LQCD and effective model studies agree. The increasing behavior of temperature agrees qualitatively with the ChPt study of Ref. [20]. To our knowledge, there is *no LQCD simulation* in the literature investigating *the pion mass with the strength of the background magnetic field at nonzero temperature*. So, our investigation of the pion mass, upon the availability of lattice data at a nonzero temperature in the near future, can shed light on the predictability of the LSMq framework. Also, the LQCD study of the neutral pion mass is hindered by the infamous sign problem while taking into account the quark chemical potential. So, in our work, we have delivered a detailed calculation of the neutral pion mass in the presence of the background magnetic field's most general settings, i.e., keeping T and μ in the framework of the relatively simple effective model.

ACKNOWLEDGMENTS

A. D. thanks Arghya Mukherjee and Md Sabir Ali for useful discussions. A. D. is supported by the school of Physical Sciences, NISER. N. H. is supported in part by the SERB-MATRICS under Grant No. MTR/2021/000939.

APPENDIX A: VERTEX CORRECTION

The Feynman diagrams that contribute to the vertex correction of π^0 is depicted in Fig. 7. The expression for $\Delta\lambda$ is given by [29]

$$\Delta\lambda = \frac{24\lambda^2}{16}[9I(P,T,m_\sigma) + I(P,T,m_\pi) + 4J^{(B)}(P,T,m_\pi)]|_{p\to\mathbf{0}}. \quad \text{(A1)}$$

As usual, $\Delta\lambda$ contains the magnetic vacuum part and the thermomagnetic part. Here we defined

$$I(P,T,m_i) = T\sum_{k_0}\int\frac{d^3k}{(2\pi)^3}D_i(K)D_i(P-K),$$
$$J^{(B)}(P,T,m_j) = T\sum_{k_0}\int\frac{d^3k}{(2\pi)^3}D_j^{(B)}(K)D_j^{(B)}(P-K), \quad \text{(A2)}$$

with $i = \pi^0, \sigma$ and $j = \pi^+, \pi^-$.

1. $\Delta\lambda_B$

There will be no magnetic vacuum part for I since its expression contains only the neutral pion and sigma loop. Only the J contribution from the magnetic field will be there due to the involvement of the charged pion propagator inside the loop. Thus for the vacuum as well as the magnetic field contributions, we write

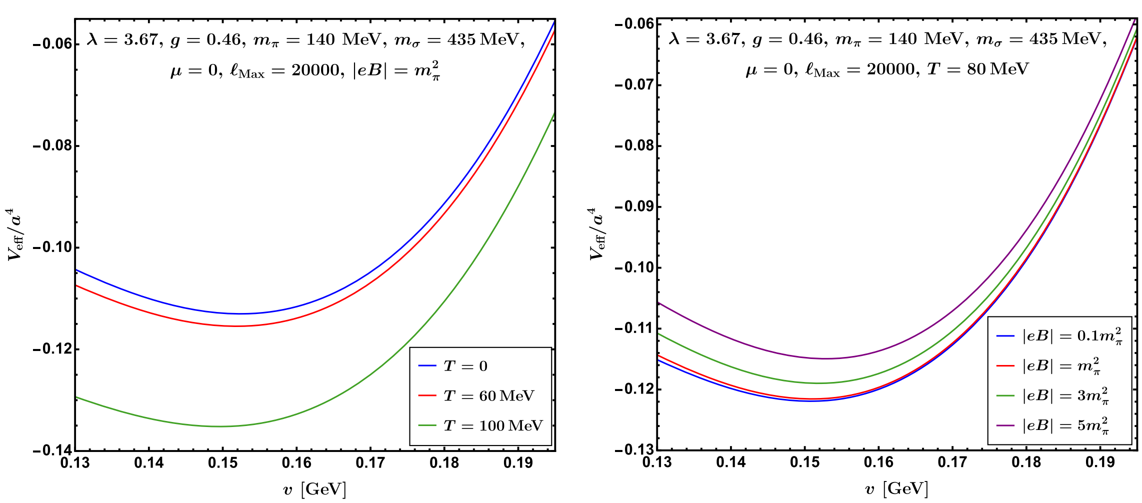

FIG. 6. Left panel: effective potential as a function of v with a fixed eB with different T. Right panel: the same with a fixed T for different eB.

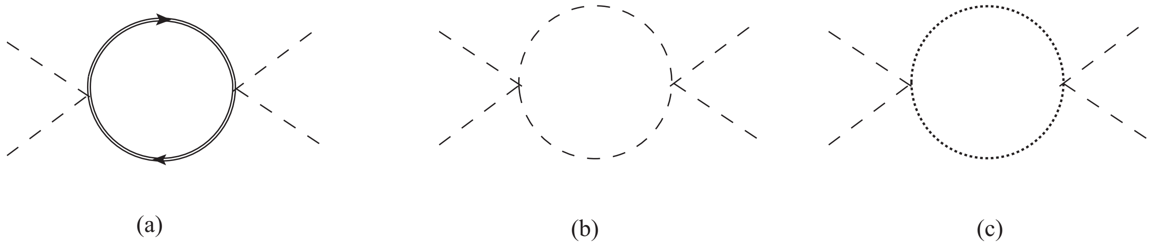

FIG. 7. Feynman diagram for a one-loop contribution to the self-coupling λ. Contributions to the self-coupling from (a) the charged pion, (b) the neutral pion, and (c) the sigma meson, respectively.

$$\begin{aligned} J^{(B)}(P,m_j) &= -i\int\frac{d^4K}{(2\pi)^4}D_j^{(B)}(K)D_j^{(B)}(P-K) \\ &= -4i\sum_{\ell,n=0}^{\infty}(-1)^{\ell+n}\int\frac{d^2k_\perp}{(2\pi)^2}\exp\left(-\frac{k_\perp^2+q_\perp^2}{|eB|}\right)L_\ell\left(\frac{2k_\perp^2}{|eB|}\right)L_n\left(\frac{2q_\perp^2}{|eB|}\right) \\ &\quad\times\int\frac{d^2k_\parallel}{(2\pi)^2}\frac{1}{k_\parallel^2-(2\ell+1)|eB|-m_j^2}\frac{1}{q_\parallel^2-(2n+1)|eB|-m_j^2}. \end{aligned} \tag{A3}$$

For $\boldsymbol{p}\to\boldsymbol{0}$, after performing the following perpendicular momentum integration and the sum over Landau level n, the above equation gets simplified to

$$\begin{aligned} J^{(B)}(p_0,\boldsymbol{p}=0,m_j) &= -i\frac{|eB|}{2\pi}\sum_{\ell=0}^{\infty}\int\frac{d^2k_\parallel}{(2\pi)^2}\frac{1}{k_\parallel^2-(2\ell+1)|eB|-m_j^2}\frac{1}{q_0^2-k_z^2-(2\ell+1)|eB|-m_j^2} \\ &= -i\frac{|eB|}{2\pi}\sum_{\ell=0}^{\infty}\int\frac{d^2k_\parallel}{(2\pi)^2}\int_0^1dx\frac{1}{\{(k_0-xp_0)^2-k_z^2-[(2\ell+1)|eB|+m_j^2-x(1-x)p_\parallel^2]\}^2} \\ &= -i\frac{|eB|}{2\pi}\sum_{\ell=0}^{\infty}\int_0^1dx\int\frac{d^2k_\parallel}{(2\pi)^2}\frac{1}{\{k_\parallel^2-[(2\ell+1)|eB|+m_j^2-x(1-x)p_0^2]\}^2}. \end{aligned} \tag{A4}$$

Now, we perform the usual dimensional regularization routine to get

$$J^{(B)}(p_0,m_j) = -i\frac{|eB|}{2\pi}\Lambda^{2\epsilon}\sum_{\ell=0}^{\infty}\int_0^1dx\int\frac{d^{2-\epsilon}k_\parallel}{(2\pi)^{2-\epsilon}}\frac{1}{\{k_\parallel^2-[(2\ell+1)|eB|+m_j^2-x(1-x)p_0^2]\}^2}. \tag{A5}$$

The momentum integration is performed as

$$\begin{aligned} J^{(B)}(p_0,m_j) &= \frac{|eB|}{2\pi}\Lambda^{2\epsilon}\sum_{\ell=0}^{\infty}\int_0^1dx\frac{1}{(4\pi)^{1-\epsilon}}\frac{\Gamma(1+\epsilon)}{[(2\ell+1)|eB|+m_j^2-x(1-x)p_0^2]^{1+\epsilon}} \\ &= \frac{|eB|}{8\pi^2}\left(\frac{1}{4\pi\Lambda^2}\right)^{-\epsilon}\Gamma(1+\epsilon)\int_0^1dx\sum_{\ell=0}^{\infty}\frac{1}{[(2\ell+1)|eB|+m_j^2-x(1-x)p_0^2]^{1+\epsilon}} \\ &= \frac{1}{16\pi^2}\left(\frac{2|eB|}{4\pi\Lambda^2}\right)^{-\epsilon}\Gamma(1+\epsilon)\int_0^1dx\zeta\left(1+\epsilon,\frac{1}{2}+\frac{m_j^2-x(1-x)p_0^2}{2|eB|}\right), \end{aligned} \tag{A6}$$

where we summed over LLs as

$$\sum_{\ell=0}^{\infty}\frac{1}{[(2\ell+1)|eB|+m_j^2-x(1-x)p_0^2]^{1+\epsilon}} = \frac{1}{2|eB|}(2|eB|)^{-\epsilon}\zeta\left(1+\epsilon,\frac{1}{2}+\frac{m_j^2-x(1-x)p_0^2}{2|eB|}\right). \tag{A7}$$

Finally, expanding $J^{(B)}$ in an equation around $\epsilon = 0$, we get

$$J^{(B)}(p_0, \boldsymbol{p} = 0, m_j) = \frac{1}{16\pi^2}\left[\frac{1}{\epsilon} - \gamma_E + \log(4\pi\Lambda^2) - \log(2|eB|) - \int_0^1 dx \psi\left(\frac{1}{2} + \frac{m_j^2 - x(1-x)p_0^2}{2|eB|}\right)\right]. \tag{A8}$$

2. $\Delta\lambda_{\mathrm{Vac}}$

Now, we compute the dimensionally regularized vacuum part by going to $d = 4 - 2\epsilon$ as

$$I^{\mathrm{Vac}}(p_0, \boldsymbol{p} = 0, m_i) = \frac{\lambda^2}{16\pi^2}\left[\frac{1}{\epsilon} - \gamma_E + \log(4\pi\Lambda^2) - \int_0^1 dx \log(m_i^2 - x(1-x)p_0^2)\right],$$
$$J^{\mathrm{Vac}}(p_0, \boldsymbol{p} = 0, m_j) = \frac{\lambda^2}{16\pi^2}\left[\frac{1}{\epsilon} - \gamma_E + \log(4\pi\Lambda^2) - \int_0^1 dx \log(m_j^2 - x(1-x)p_0^2)\right], \tag{A9}$$

and

$$J^{(B)}(p_0, \boldsymbol{p} = 0, m_j) - J^{\mathrm{Vac}}(p_0, \boldsymbol{p} = 0, m_j) = \frac{1}{16\pi^2}\int_0^1 dx\left[\log\left(\frac{m_j^2 - x(1-x)p_0^2}{2|eB|}\right) - \psi\left(\frac{1}{2} + \frac{m_j^2 - x(1-x)p_0^2}{2|eB|}\right)\right]. \tag{A10}$$

As long as $m_\pi^2 - x(1-x)p_0^2 \geq 0$,[3] the Poly-Gamma function in the above line can be expanded in the limit of $|eB| \to 0$ as

$$\psi\left(\frac{1}{2} + \frac{m_j^2 - x(1-x)p_0^2}{2|eB|}\right) = \log\left(\frac{m_j^2 - x(1-x)p_0^2}{2|eB|}\right) + \frac{1}{24}\left(\frac{2|eB|}{m_j^2 - x(1-x)p_0^2}\right)^2 + \mathcal{O}(|eB|^4). \tag{A11}$$

3. $\Delta\lambda_{\mathrm{ThM}}$

To extract the thermomagnetic contribution, we start from the expression of I and J given in Eq. (A2). For I, we perform the frequency summation, drop the term that does not contain a distribution function, and obtain the thermal part as

$$I^{\mathrm{Th}}(p_0, \boldsymbol{p} = 0, T, m_i) = \int \frac{d^3k}{(2\pi)^3}\frac{1}{E_{i,k}}\frac{1}{p_0^2 - 4E_{i,k}^2}2n(E_{i,k}), \tag{A12}$$

where $E_{i,k} = \sqrt{k^2 + m_i^2}$.

For the thermomagnetic part of vertex correction, we need to evaluate the following expression:

$$J^{(B)}(p_0, \boldsymbol{p} = 0, m_j) = \frac{|eB|}{2\pi} T\sum_{k_0}\int \frac{dk_z}{2\pi}\sum_{\ell=0}^{\infty}\frac{1}{k_\parallel^2 - (2\ell+1)|eB| - m_j^2}\frac{1}{q_\parallel^2 - (2\ell+1)|eB| - m_j^2}. \tag{A13}$$

The bosonic frequency sum is evaluated in Appendix B 2. In our case, we have the chemical potential of boson $\mu_b = 0$ and $\boldsymbol{p} = 0$, giving

$$T\sum_{k_0}\frac{1}{k_0^2 - E_{k,\ell}^2}\frac{1}{(p_0 - k_0)^2 - E_{j,k,\ell}^2} = -\frac{1 + 2n(E_{j,k,\ell})}{E_{j,k,\ell}(p_0^2 - 4E_{j,k,\ell}^2)}. \tag{A14}$$

Finally, the thermomagnetic part is written as

$$J^{\mathrm{ThM}}(p_0, \boldsymbol{p} = 0, T, m_j) = -\frac{|eB|}{2\pi}\sum_{\ell=0}^{\infty}\int_{-\infty}^{\infty}\frac{dk_z}{2\pi}\frac{1}{E_{j,\ell,k}}\frac{1}{p_0^2 - 4E_{j,\ell,k}^2}2n(E_{j,\ell,k}), \tag{A15}$$

where $E_{j,\ell,k} = \sqrt{k_z^2 + (2\ell+1)|eB| + m_j^2}$.

[3]To maintain this condition, we must choose $p_0^2 < 4m_\pi^2$ for the $J^{(B)} - J^{\mathrm{Vac}}$ to be pure real.

APPENDIX B: FREQUENCY SUMS

1. Fermionic sums

We need to calculate a summation of the form

$$I_s = T\sum_{l=-\infty}^{\infty}\tilde{\Delta}(k_0, s\Omega), \tag{B1}$$

$$I_{s_1,s_2} = T\sum_{l=-\infty}^{\infty}\tilde{\Delta}(k_0, s_1\Omega_1)\tilde{\Delta}(q_0, s_2\Omega_2), \tag{B2}$$

where

$$\tilde{\Delta}(k_0, s_1\Omega_1) \equiv \frac{s_1}{2\Omega_1}\frac{1}{k_0 - s_1\Omega_1}, \tag{B3}$$

$$\tilde{\Delta}(p_0 - k_0, s_2\Omega_2) \equiv \frac{s_1}{2\Omega_1}\frac{1}{k_0 - s_1\Omega_1}, \tag{B4}$$

with

$$k_0 = \mu + i(2l+1)\pi T, \qquad p_0 = i2\pi mT, \tag{B5}$$

where $l, m = 0, \pm1, \pm2, \ldots$. Now, we substitute

$$\frac{1}{i(2l+1)\pi T + \mu - s_1\Omega_1} = \tilde{n}^{+}(s_1\Omega_1)\int_0^{1/T} d\tau_1 \exp\left[-\tau_1\{i(2l+1)\pi T + \mu - s_1\Omega_1\}\right], \tag{B6}$$

$$\frac{1}{i2\pi mT - i(2l+1)\pi T - \mu - s_2\Omega_2} = \tilde{n}^{-}(s_2\Omega_2)\int_0^{1/T} d\tau_2 \exp\left[-\tau_2\{i2\pi mT - i(2l+1)\pi T - \mu - s_2\Omega_2\}\right] \tag{B7}$$

in Eq. (B2) and simplify the terms in the exponential to get

$$I_{s_1,s_2} = \tilde{n}_1^{+}\tilde{n}_2^{-}\int_0^{1/T} d\tau_1 d\tau_2 e^{-(i2m\pi T\tau_2 - s_1\omega_1\tau_1 - s_2\Omega_2\tau_2) - \mu(\tau_1-\tau_2)} T\sum_{l=-\infty}^{\infty}\exp\left[-(\tau_1-\tau_2)i(2l+1)\pi T\right]. \tag{B8}$$

Here $\tilde{n}_i^{\pm} \equiv \tilde{n}^{\pm}(s_i\Omega_i) = (e^{\beta(s_iE_i \mp \mu)} + 1)^{-1}$, where $\beta = T^{-1}$. Now, we use the identity

$$T\sum_{l=-\infty}^{\infty}\exp\left[-(\tau_1-\tau_2)i(2l+1)\pi T\right] = \delta(\tau_1 - \tau_2) \tag{B9}$$

to get

$$I_{s_1,s_2} = \tilde{n}_1^{+}\tilde{n}_2^{-}\int_0^{1/T} d\tau \exp\left[-\tau(i2\pi mT - s_1\Omega_1 - s_2\Omega_2)\right]. \tag{B10}$$

After performing the τ integral and simplifying by using $e^{i2m\pi} = 1$, we analytically continue back to the Minkowski p_0 by the prescription $i2\pi mT \to p_0 + i\epsilon$. Then we make use of the identity $\tilde{n}^{\pm}(x)e^{\beta(x\mp\mu)} = 1 - \tilde{n}^{\pm}(x)$ and do a little algebra to arrive at

$$I_{s_1,s_2} = -\frac{s_1s_2}{4\Omega_1\Omega_2}\frac{1 - \tilde{n}^{+}(s_1\Omega_1) - \tilde{n}^{-}(s_2\Omega_2)}{p_0 - s_1\Omega_1 - s_2\Omega_2}. \tag{B11}$$

Finally, using the identity $1 - \tilde{n}^{\pm}(-x) - \tilde{n}^{\mp}(x) = 0$, we get our desired frequency sum as

$$\sum_{s_1,s_2} I_{s_1,s_2} = -\frac{1}{4\Omega_1\Omega_2}\left[\frac{1 - \tilde{n}^{+}(\Omega_1) - \tilde{n}^{-}(\Omega_2)}{p_0 - \Omega_1 - \Omega_2} - \frac{1 - \tilde{n}^{-}(\Omega_1) - \tilde{n}^{+}(\Omega_2)}{p_0 + \Omega_1 + \Omega_2} + \frac{\tilde{n}^{+}(\Omega_1) - \tilde{n}^{+}(\Omega_2)}{p_0 - \Omega_1 + \Omega_2} - \frac{\tilde{n}^{-}(\Omega_1) - \tilde{n}^{-}(\Omega_2)}{p_0 + \Omega_1 - \Omega_2}\right]. \tag{B12}$$

Now, the following method can perform the other fermionic frequency sum as:

$$\begin{aligned}
T\sum_{k_0}\frac{1}{k_0^2 - \Omega^2} &= T\sum_{k_0}\sum_{s=\pm1}\frac{s}{2\Omega}\frac{1}{k_0 - s\Omega} = \sum_{s=\pm1}\frac{s}{2\Omega}T\sum_{l=-\infty}^{\infty}\frac{1}{i(2l+1)\pi T + \mu - s\Omega} \\
&= \sum_{s=\pm1}\frac{s}{2\Omega}\tilde{n}^{+}(s\Omega)\int_0^{1/T} d\tau e^{-\tau(\mu - s\Omega)} T\sum_{l=-\infty}^{\infty} e^{i(2l+1)\pi T\tau} = \sum_{s=\pm1}\frac{s}{2\Omega}\tilde{n}^{+}(s\Omega)\int_0^{1/T} d\tau e^{-\tau(\mu - s\Omega)}\delta(\tau) \\
&= \sum_{s=\pm1}\frac{s}{2\Omega}\tilde{n}^{+}(s\Omega).
\end{aligned} \tag{B13}$$

Thus,

$$T\sum_{k_0}\frac{1}{k_0^2-\Omega^2}=-\frac{1-\tilde{n}^+(\Omega)-\tilde{n}^-(\Omega)}{2\Omega}. \quad \text{(B14)}$$

2. Bosonic sums

The frequency sum we need to evaluate is

$$\mathcal{F}_{B,B}^{(0,0)}=T\sum_{k_0}\frac{1}{k_0^2-E_1^2}\frac{1}{(p_0-k_0)^2-E_2^2}, \quad \text{(B15)}$$

where

$$k_0=\mu+i2l\pi T. \quad \text{(B16)}$$

The summand in Eq. (B15) can be conveniently written as

$$\mathfrak{I}_{s_1,s_2}=T\sum_{l=-\infty}^{\infty}\Delta(k_0,s_1E_1)\Delta(p_0-k_0,s_2E_2), \quad \text{(B17)}$$

where

$$\Delta(k_0,s_1E_1)=\frac{s_1}{2E_1}\frac{1}{k_0-s_1E_1},$$
$$\Delta(p_0-k_0,s_2E_2)=\frac{s_2}{2E_2}\frac{1}{p_0-k_0-s_2E_2}. \quad \text{(B18)}$$

Thus, Eq. (B15) can be written as

$$\mathcal{F}_{B,B}^{(0,0)}=\sum_{\substack{s_1,s_2\\=\pm1}}\mathfrak{I}_{s_1,s_2}. \quad \text{(B19)}$$

It is easy to see that Eq. (B18) can be written in integral representation as

$$\Delta(k_0,s_1E_1)=-\frac{s_1}{2E_1}n^+(s_1E_1)\int_0^{1/T}d\tau_1e^{-\tau_1(k_0-s_1E_1)},$$
$$\Delta(p_0-k_0,s_2E_2)=-\frac{s_2}{2E_2}n^-(s_2E_2)\int_0^{1/T}d\tau_2e^{-\tau_2(p_0-k_0-s_2E_2)}. \quad \text{(B20)}$$

Thus,

$$\mathfrak{I}_{s_1,s_2}=\frac{s_1s_2n^+(s_1E_1)n^-(s_2E_2)}{4E_1E_2}\int_0^{1/T}d\tau_1d\tau_2e^{\tau_1(s_1E_1-\mu)}e^{\tau_2(s_2E_2+\mu)-\tau_2p_0}\times T\sum_{l=-\infty}^{\infty}\exp\left[-k_0(\tau_1-\tau_2)\right]. \quad \text{(B21)}$$

Using the identity

$$T\sum_{l=-\infty}^{\infty}\exp\left[-k_0(\tau_1-\tau_2)\right]=\delta(\tau_1-\tau_2), \quad \text{(B22)}$$

and integrating over the delta function, we obtain

$$\mathfrak{I}_{s1,s2}=\frac{s_1s_2n^+(s_1E_1)n^-(s_2E_2)}{4E_1E_2}\int_0^{1/T}d\tau e^{\tau(s_1E_1-\mu)}e^{\tau(s_2E_2+\mu)}e^{-\tau p_0}. \quad \text{(B23)}$$

Performing the τ integral, we get

$$\mathfrak{I}_{s1,s2}=\frac{s_1s_2}{4E_1E_2}n^+(s_1E_1)n^-(s_2E_2)\frac{e^{\beta(s_1E_1-\mu)}e^{\beta(s_2E_2+\mu)}e^{-\beta p_0}-1}{s_1E_1+s_2E_2-p_0}. \quad \text{(B24)}$$

Since $e^{-\beta p_0}=1$, we get after some algebra

$$\mathfrak{I}_{s1,s2}=-\frac{s_1s_2}{4E_1E_2}\frac{1+n^+(s_1E_1)+n^-(s_2E_2)}{p_0-s_1E_1-s_2E_2}. \quad \text{(B25)}$$

Using $n^{\pm}(E)=-[1+n^{\mp}(E)]$, we get

$$\mathcal{F}_{B,B}^{(0,0)}=-\frac{1}{4E_1E_2}\left(\frac{1+n^+(E_1)+n^-(E_2)}{p_0-E_1-E_2}-\frac{1+n^-(E_1)+n^+(E_2)}{p_0+E_1+E_2}-\frac{n^+(E_1)-n^+(E_2)}{p_0-E_1+E_2}+\frac{n^-(E_1)-n^-(E_2)}{p_0+E_1-E_2}\right). \quad \text{(B26)}$$

APPENDIX C: EFFECTIVE POTENTIAL AT NONZERO TEMPERATURE

The effective potential is a central quantity for theories with a spontaneous breakdown of continuous symmetry. In this case, the classical value of the potential is altered due to the perturbative loop correction after spontaneous symmetry breaking. As a result of this, the minimum $v_0' = \sqrt{(a^2+m_\pi^2)/\lambda}$ of the tree-level potential $V'_{\rm tree}(v)$ receives a quantum correction shifting its value to $v = v_B$. To briefly elaborate this point we note that, in the presence of interaction, the expression of VEV becomes

$$v_B = \langle\Omega|\sigma(x)|\Omega\rangle = \frac{\langle 0|\mathrm{Tr}[\sigma(x)\exp(-\beta H_{\mathrm{LSM}_q})]|0\rangle}{\langle 0|\mathrm{Tr}[\exp(-\beta H_{\mathrm{LSM}_q})]|0\rangle}, \quad \text{(C1)}$$

Here $|0\rangle$ and $|\Omega\rangle$ are the noninteracting and interacting ground states of the theory, respectively. H_{LSM_q} is the total Hamiltonian of $\mathrm{LSM_q}$, β is the inverse temperature. Equation (C1) can be expanded perturbatively in powers of λ and g as

$$\langle\Omega|\sigma(x)|\Omega\rangle = \underbrace{\langle 0|\sigma(x)|0\rangle}_{v_0'} + \text{Quantum loop correction}. \quad \text{(C2)}$$

In the lowest order in perturbation theory, the effective potential is just the classical potential v_0.

In this section, we compute the contribution of the temperature and magnetic field to the effective potential. First, the effective potential has contributions from tree level, bosonic (appearing due to the quantum fluctuations of π and σ meson), and fermionic parts (for which the quantum fluctuation of quarks is responsible). The higher-order corrections to the potential are divergent, and the incorporation of counterterm contribution $V_{\rm ct}$ is needed to remove the infinities systematically. Thus up to $\mathcal{O}(\hbar)$, it reads

$$V_{\rm eff} = V'_{\rm tree} + \sum_{b=\pi^\pm,\pi^0,\sigma} V_b^{(1)} + \sum_{f=u,d} V_f^{(1)} + V_{\rm ct} + \sum_{b=\pi^\pm,\pi^0,\sigma} V_{b,\mathrm{Ring}}^{(1)}, \quad \text{(C3)}$$

where

$$V'_{\rm tree} = -\frac{1}{2}(a^2+m_\pi^2)v^2 + \frac{1}{4}\lambda v^4, \quad \text{(C4)}$$

$$V_b^{(1,B)} = -\frac{1}{2}T\sum_{n=-\infty}^{\infty}\int\frac{d^3k}{(2\pi)^3}\log[D_{(B)}(k_0=i\omega_n,\boldsymbol{k},m_b^2)^{-1}], \quad \text{(C5)}$$

$$V_f^{(1,B)} = \frac{i}{2}\mathrm{Tr}\log[(i\not{D})^2 - M_f^2], \quad \text{(C6)}$$

$$V_{\rm ct} = \frac{1}{2}\delta m v^2 + \frac{1}{4}\delta\lambda v^4, \quad \text{(C7)}$$

and $V_{b,\mathrm{Ring}}^{(1)}$ is the ring contribution from mesons which is discussed in detail in Sec. C 3. We shall not write the v dependence, which is there in the expression of the effective potential via m_b^2 and M_f.

After a few steps of simple algebra clearly depicted in Ref [24], Eqs. (C5) and (C6) take the following form:

$$V_b^{(1,B)} = \frac{1}{2}T\sum_{n=-\infty}^{\infty}\int_0^\infty dm_b^2\int\frac{d^3k}{(2\pi)^3}\int_0^\infty ds\frac{1}{\cosh(|eB|)}\exp\left(-s\left[\omega_n^2+k_z^2+\frac{\tanh(|eB|s)}{|eB|s}k_\perp^2+m_b^2\right]\right), \quad \text{(C8)}$$

$$V_f^{(1,B)} = -\sum_{r=\pm1}T\sum_{n=-\infty}^{\infty}\int_0^\infty dm_f^2\int\frac{d^3k}{(2\pi)^3}\int_0^\infty\frac{ds}{\cosh(|q_fB|)}\exp\left(-s\left[\tilde{\omega}_n^2+k_z^2+\frac{\tanh(|q_fB|s)}{|q_fB|s}k_\perp^2+M_f^2+rq_fB\right]\right). \quad \text{(C9)}$$

Here, $\omega_n = 2\pi nT$ and $\tilde{\omega}_n = (2n+1)\pi T - i\mu$ are bosonic and fermionic Matsubara frequencies, respectively. By integrating over the proper time s in Eqs. (C8) and (C9), the expressions of $V_b^{(1)}$ and $V_f^{(1)}$ are converted to the Landau level representation,

$$V_b^{(1,B)} = \frac{T}{2}\sum_{n=-\infty}^{\infty}\int dm_b^2\int\frac{d^3k}{(2\pi)^3}\sum_{\ell=0}^{\infty}2(-1)^\ell\frac{\exp(-\frac{k_\perp^2}{|eB|})L_\ell(\frac{2k_\perp^2}{|eB|})}{\omega_n^2+k_z^2+(2\ell+1)|eB|+m_b^2}, \quad \text{(C10)}$$

$$V_f^{(1,B)} = -\sum_{r=\pm1}T\sum_{n=-\infty}^{\infty}\int dm_f^2\int\frac{d^3k}{(2\pi)^3}\sum_{\ell=0}^{\infty}2(-1)^\ell\frac{\exp(-\frac{k_\perp^2}{|eB|})L_\ell(\frac{2k_\perp^2}{|q_fB|})}{\tilde{\omega}_n^2+k_z^2+(2\ell+1+r\,\mathrm{sgn}(q_fB))|q_fB|+M_f^2}. \quad \text{(C11)}$$

After performing the sum over r, Eq. (C11) can be straightforwardly simplified by writing in terms of the spin degeneracy factor as

$$V_f^{(1,B)} = -T\sum_{n=-\infty}^{\infty}\int dm_f^2\int\frac{d^3k}{(2\pi)^3}\sum_{\ell=0}^{\infty}(2-\delta_{l,0})2(-1)^{\ell}\frac{\exp\left(-\frac{k_\perp^2}{|eB|}\right)L_\ell\left(\frac{2k_\perp^2}{|q_fB|}\right)}{\tilde{\omega}_n^2+k_z^2+2\ell|q_fB|+M_f^2}. \tag{C12}$$

1. Computation of $V_b^{(1)}$

By performing the integration over $d^2k_\perp$, we get

$$\begin{aligned} V_b^{(1,B)} &= T\sum_{n=-\infty}^{\infty}\int dm_b^2\int_{-\infty}^{\infty}\frac{dk_z}{2\pi}\frac{|eB|}{4\pi}\sum_{\ell=0}^{\infty}\frac{1}{\omega_n^2+k_z^2+(2\ell+1)|eB|+m_b^2} \\ &= \frac{|eB|}{4\pi}\int_{-\infty}^{\infty}\frac{dk_z}{2\pi}\sum_{\ell=0}^{\infty}T\sum_{n=-\infty}^{\infty}\log\left[\omega_n^2+k_z^2+(2\ell+1)|eB|+m_b^2\right]. \end{aligned} \tag{C13}$$

Now the frequency sum is performed following Ref. [30] as

$$T\sum_{n=-\infty}^{\infty}\log\left(\omega_n^2+E_{k,\ell,b}^2\right) = E_{k,\ell,b}+2T\log\left[1-\exp\left(-\frac{E_{k,\ell,b}}{T}\right)\right]. \tag{C14}$$

Substituting the above expression of the sum integration in Eq. (C13), we get

$$V_b^{(1,B)} = \frac{|eB|}{4\pi}\sum_{\ell=0}^{\infty}\int_{-\infty}^{\infty}\frac{dk_z}{2\pi}\left\{E_{\ell,k}+2T\log\left[1-\exp\left(-\frac{E_{k,\ell,b}}{T}\right)\right]\right\}. \tag{C15}$$

Now the first term containing $E_{k,\ell,b}$ is divergent, which we need to regulate. We use the following procedure to regulate the momentum integration by dimensional regularization

$$\int_{-\infty}^{\infty}\frac{dk_z}{2\pi}\to\tilde{\Lambda}^{2\epsilon}\int\frac{d^{1-2\epsilon}k}{(2\pi)^{1-2\epsilon}}. \tag{C16}$$

We use the following identity in Ref. [31]:

$$\Phi(m,d,A) = \int\frac{d^dk}{(2\pi)^d}\frac{1}{(\boldsymbol{k}^2+m^2)^A} = \frac{1}{(4\pi)^{d/2}}\frac{1}{\Gamma(A)}\Gamma\left(A-\frac{d}{2}\right)\frac{1}{(m^2)^{A-\frac{d}{2}}} \tag{C17}$$

to perform the integration

$$\begin{aligned} V_{b,\epsilon}^{(1,B)} &\equiv \frac{|eB|}{4\pi}\tilde{\Lambda}^{2\epsilon}\sum_{\ell=0}^{\infty}\int\frac{d^{1-2\epsilon}k_z}{(2\pi)^{1-2\epsilon}}E_{k,\ell,b} = \frac{|eB|}{4\pi}\tilde{\Lambda}^{2\epsilon}\sum_{\ell=0}^{\infty}\Phi\left(\sqrt{(2\ell+1)|eB|+m_b^2},1-2\epsilon,-\frac{1}{2}\right) \\ &= -\frac{|eB|}{16\pi^2}\frac{\Gamma(\epsilon-1)}{(4\pi\tilde{\Lambda}^2)^{-\epsilon}}\sum_{\ell=0}^{\infty}\frac{1}{[(2\ell+1)|eB|+m_b^2]^{\epsilon-1}}. \end{aligned} \tag{C18}$$

The sum over LL is performed by using the representation of the Hurwitz zeta function

$$\sum_{\ell=0}^{\infty}\frac{1}{(\ell+a)^{\epsilon}} = \zeta(\epsilon,a) \tag{C19}$$

as

$$V_{b,\epsilon}^{(1,B)} = -\frac{|eB|^2}{8\pi^2}\left(\frac{2|eB|}{4\pi\tilde{\Lambda}^2}\right)^{-\epsilon}\Gamma(\epsilon-1)\zeta\left(\epsilon-1,\frac{1}{2}+\frac{m_b^2}{2|eB|}\right). \tag{C20}$$

Here we used $\Gamma(-1/2) = -\sqrt{4\pi}$. This result matches exactly with [32]. Now, we expand $V_{b,\epsilon}^{(1,B)}$ around $\epsilon = 0$ and obtain

$$V_{b,\epsilon}^{(1,B)} = \frac{|eB|^2}{8\pi^2}\left\{\zeta\left(-1,\frac{1}{2}+\frac{m_b^2}{2|eB|}\right)\times\left(\frac{1}{\epsilon}-\gamma_E-\log\frac{2|eB|}{4\pi\tilde{\Lambda}^2}+1\right)+\zeta^{(1,0)}\left(-1,\frac{1}{2}+\frac{m_b^2}{2|eB|}\right)\right\}+\mathcal{O}(\epsilon). \quad (C21)$$

The finite temperature part is

$$V_{b,\mathrm{ThM}}^{(1,B)} = \frac{|eB|}{2\pi^2}T\sum_{l=0}^{\infty}\int_0^{\infty}dk_z\log\left(1-e^{-E_{k,\ell,b}/T}\right). \quad (C22)$$

For the σ and π^0 meson, we take the limit of $|eB|\to 0$ in Eq. (C8) and do the s and m_b^2 integration to get

$$V_b^{(1,B=0)} = \frac{T}{2}\sum_{n=-\infty}^{\infty}\int\frac{d^3k}{(2\pi)^3}\log(\omega_n^2+k^2+m_b^2) = \frac{1}{2}\int\frac{d^3k}{(2\pi)^3}\left\{\sqrt{k^2+m_b^2}+2T\log\left[1-\exp\left(-\frac{\sqrt{k^2+m_b^2}}{T}\right)\right]\right\}. \quad (C23)$$

For the vacuum part, we use the dimensional regularization method by modifying the measure of three-momentum integration

$$\int\frac{d^3k}{(2\pi)^3}\to\Lambda^{2\epsilon}\int\frac{d^{3-2\epsilon}k}{(2\pi)^{3-2\epsilon}}. \quad (C24)$$

As a result, we get

$$V_{b,\epsilon}^{(1,B=0)} = \frac{1}{2}\Lambda^{2\epsilon}\int\frac{d^{3-2\epsilon}k}{(2\pi)^{3-2\epsilon}}\sqrt{k^2+m_b^2} = -\frac{m_b^2}{32\pi^2}\left(\frac{m_b^2}{4\pi\Lambda^2}\right)^{-\epsilon}\Gamma(\epsilon-2) = -\frac{m_b^4}{64\pi^2}\left[\frac{1}{\epsilon}-\gamma_E+\log(4\pi)+\frac{3}{2}-\log\left(\frac{m_b^2}{\Lambda^2}\right)\right]+\mathcal{O}(\epsilon). \quad (C25)$$

The thermal part is given by

$$V_{b,\mathrm{Th}}^{(1,B=0)} = \frac{T}{2\pi^2}\int_0^{\infty}dk k^2\log\left[1-\exp\left(-\frac{\sqrt{k^2+m_b^2}}{T}\right)\right]. \quad (C26)$$

2. Computation of $V_f^{(1)}$

In this case, we start from Eq. (C12),

$$V_f^{(1)} = -2\sum_{\ell=0}^{\infty}(2-\delta_{\ell,0})T\sum_{n=-\infty}^{\infty}\int\frac{d^3k}{(2\pi)}(-1)^{\ell}\exp\left(-\frac{k_\perp^2}{|eB|}\right)L_\ell\left(\frac{2k_\perp^2}{|q_fB|}\right)\log\left[\tilde{\omega}_n^2+k_z^2+2\ell|q_fB|+M_f^2\right] = -\frac{|q_fB|}{2\pi}\sum_{l=0}^{\infty}(2-\delta_{\ell,0})\int_{-\infty}^{\infty}\frac{dk_z}{2\pi}T\sum_{n=-\infty}^{\infty}\log\left[\tilde{\omega}_n^2+k_z^2+2\ell|q_fB|+M_f^2\right]. \quad (C27)$$

The frequency sum is performed following the same method as for the bosonic part, and the result is quoted below:

$$T\sum_{n=-\infty}^{\infty}\log\left[\tilde{\omega}_n^2+\Omega_{\ell,k,f}^2\right] = \Omega_{\ell,k,f}+T\log\left[1+\exp\left(-\frac{\Omega_{\ell,k,f}-\mu}{T}\right)\right]+T\log\left[1+\exp\left(-\frac{\Omega_{\ell,k,f}+\mu}{T}\right)\right]. \quad (C28)$$

Thus, we have

$$V_f^{(1)} = -\frac{|q_fB|}{2\pi}\sum_{\ell=0}^{\infty}(2-\delta_{\ell,0})\int_{-\infty}^{\infty}\frac{dk_z}{2\pi}\left\{\Omega_{\ell,k,f}+T\log\left[1+\exp\left(-\frac{\Omega_{\ell,k,f}-\mu}{T}\right)\right]+T\log\left[1+\exp\left(-\frac{\Omega_{\ell,k,f}+\mu}{T}\right)\right]\right\}. \quad (C29)$$

The zero temperature part is written as

$$V_{f,\epsilon}^{(1,B)} = -\frac{|q_fB|}{2\pi}\tilde{\Lambda}^{2\epsilon}\sum_{\ell=0}^{\infty}(2-\delta_{\ell,0})\int\frac{d^{1-2\epsilon}k}{(2\pi)^{1-2\epsilon}}\Omega_{\ell,k,f}. \quad (C30)$$

The sum is performed, followed by the integration

$$V_{f,\epsilon}^{(1,B)} = \frac{|q_f B|^2}{2\pi^2}\Gamma(\epsilon-1)\left[\left(\frac{2|q_f B|}{4\pi\Lambda^2}\right)^{-\epsilon}\zeta\left(\epsilon-1,\frac{M_f^2}{2|q_f B|}\right) - \frac{1}{2}\frac{M_f^2}{2|q_f B|}\left(\frac{M_f^2}{4\pi\Lambda^2}\right)^{-\epsilon}\right]$$
$$= M_f^2\frac{|q_f B|}{8\pi^2}\left[\frac{1}{\epsilon}-\gamma_E-\log\left(\frac{M_f^2}{4\pi\Lambda^2}\right)+1\right] - \frac{|q_f B|^2}{2\pi^2}\left\{\zeta\left(-1,\frac{M_f^2}{2|q_f B|}\right)\left[\frac{1}{\epsilon}-\gamma_E-\log\left(\frac{2|q_f B|}{4\pi\Lambda^2}\right)+1\right]\right.$$
$$\left. + \zeta^{(1,0)}\left(-1,\frac{M_f^2}{2|q_f B|}\right)\right\} + \mathcal{O}(\epsilon). \tag{C31}$$

After applying the $\overline{\mathrm{MS}}$ scheme, by virtue of which we drop the $\frac{1}{\epsilon}-\gamma_E+\log(4\pi)$ term, we get

$$V_f^{(1,B)} = \frac{|q_f B|}{2\pi^2}3\left\{\frac{M_f^2}{4}\left[1-\log\left(\frac{M_f^2}{\Lambda^2}\right)\right] - |q_f B|\left[\zeta\left(-1,\frac{M_f^2}{2|q_f B|}\right)\left[1-\log\left(\frac{2|q_f B|}{\Lambda^2}\right)\right] + \zeta^{(1,0)}\left(-1,\frac{M_f^2}{2|q_f B|}\right)\right]\right\}. \tag{C32}$$

The thermomagnetic part is written as

$$V_{f,\mathrm{ThM}}^{(1,B)} = -T\frac{|q_f B|}{2\pi^2}\sum_{\ell=0}^{\infty}(2-\delta_{\ell,0})\int_0^\infty dk_z\left\{\log\left[1+\exp\left(-\frac{\Omega_{\ell,k,f}-\mu}{T}\right)\right]+\log\left[1+\exp\left(-\frac{\Omega_{\ell,k,f}+\mu}{T}\right)\right]\right\}. \tag{C33}$$

The counterterms are determined from the vacuum stability condition [33]. It states that the tree-level value of the position of minimum v_0' of the effective potential and the mass of the sigma meson[4] does not change after quantum correction. Mathematically,

$$\frac{1}{2v}\frac{dV_{\mathrm{vac}}}{dv}\bigg|_{v=v_0'} = 0,$$
$$\frac{d^2V_{\mathrm{vac}}}{dv^2}\bigg|_{v=v_0'} = 2a^2+3m_\pi^2. \tag{C34}$$

Applying the above conditions to the quantum corrected potential in the vacuum, we determine δa^2 and $\delta\lambda$ as

$$\delta a^2 = \frac{m_\pi^2}{2} - \frac{1}{16\pi^2\lambda}\left\{6\lambda^2(a^2+2m_\pi^2) - g^4(a^2+m_\pi^2) + 3a^2\lambda^2\left[\log\left(\frac{m_\pi^2}{a^2}\right)+\log\left(\frac{2a^2+3m_\pi^2}{a^2}\right)\right]\right\},$$
$$\delta\lambda = \frac{\lambda}{2}\frac{m_\pi^2}{a^2+m_\pi^2} - \frac{1}{16\pi^2}\left\{3\lambda^2\left[\log\left(\frac{m_\pi^2}{a^2}\right)+3\log\left(\frac{2a^2+3m_\pi^2}{a^2}\right)\right] - 8g^4\log\left(\frac{g^2}{a^2}\frac{a^2+m_\pi^2}{\lambda}\right)\right\}. \tag{C35}$$

In the magnetic part of the one-loop effective potential of pions as well as quarks, there is a scale factor Λ which we took as $a = \sqrt{\frac{m_\sigma^2-3m_\pi^2}{2}} = 255$ MeV since we choose $m_\pi = 140$ MeV and $m_\sigma = 435$ MeV.

3. Ring contributions

If we look at Eqs. (C21), (C22), (C25), and (C26), we notice that the argument of the logarithm in Eq. (C25), the argument of the Hurwitch zeta function ζ in Eq. (C21), and Eq. (C32) can become negative due to negative m_b^2 for some values of v in the range $0 < v < v_0'$. This negative argument makes the potential imaginary which is not acceptable as it can lead to the complex critical temperature (T_{chiral}) of chiral symmetry restoration [34]. Also, the meson energy $E_{\ell,k}$ and E_k can become negative for a similar reason.

Also, in the case of a small boson mass, their thermal, as well as magnetic, correction becomes of the same order as their original masses. As a result, perturbation theory breaks down, and a resummation becomes necessary. It is taken into account by incorporating the so-called *ring diagrams*. By incorporating the resummation program through the inclusion of the ring diagram, one takes the effect of plasma screening into account as well shields the effect of infrared divergence. The ring diagram contribution is added via the following term [24]:

[4]Note that the mass of the sigma meson is equal to $\frac{d^2V_{\mathrm{cl}}}{dv^2}|_{v=v_0'}$.

$$V^{(1,B)}_{b,\mathrm{Ring}} = \frac{1}{2} T \sum_{n=-\infty}^{\infty} \int \frac{d^3k}{(2\pi)^3} \log\left[1 - \Pi^{(B)}_b(k_0 = i\omega_n, \boldsymbol{k}) D_B(k_0 = i\omega_n, \boldsymbol{k}, m_b)\right]. \quad \text{(C36)}$$

Now adding Eqs. (C5) and (C36), we get

$$V^{(1,B)}_b + V^{(1,B)}_{b,\mathrm{Ring}} = \frac{1}{2} T \sum_{n=-\infty}^{\infty} \int \frac{d^3k}{(2\pi)^3} \log\left[D_B(k_0 = i\omega_n, \boldsymbol{k}, m_b)^{-1} - \Pi^{(B)}_b(k_0 = i\omega_n, \boldsymbol{k})\right]. \quad \text{(C37)}$$

Here, we rewrite the full expressions of meson self-energies as follows:

$$\Pi^{(B)}_{\pi^0}(k_0, \boldsymbol{k}) = \frac{\lambda}{4}\left[8\mathcal{I}^{(B)}(m_{\pi^\pm}) + 12\mathcal{I}(m_{\pi^0}) + 4\mathcal{I}(m_\sigma)\right] + \sum_{f=u,d} \Pi^{(B)}_{ff}(k_0, \boldsymbol{k}), \quad \text{(C38)}$$

$$\Pi^{(B)}_{\pi^\pm}(k_0, \boldsymbol{k}) = \frac{\lambda}{4}\left[16\mathcal{I}^{(B)}(m_{\pi^\pm}) + 4\mathcal{I}(m_{\pi^0}) + 4\mathcal{I}(m_\sigma)\right] + 2\Pi^{(B)}_{ud}(k_0, \boldsymbol{k}), \quad \text{(C39)}$$

$$\Pi^{(B)}_{\sigma}(k_0, \boldsymbol{k}) = \frac{\lambda}{4}\left[8\mathcal{I}^{(B)}(m_{\pi^\pm}) + 4\mathcal{I}(m_{\pi^0}) + 12\mathcal{I}(m_\sigma)\right] + 2\Pi^{(B)}_{ud}(k_0, \boldsymbol{k}), \quad \text{(C40)}$$

where the integrations $\mathcal{I}^{(B)}$ and $\mathcal{I}$ are defined as

$$\mathcal{I}^{(B)}(m_i) = T \sum_{k_0} \int \frac{d^3k}{(2\pi)^3} D^{(B)}(k_0, \boldsymbol{k}), \quad \text{(C41)}$$

$$\mathcal{I}(m_i) = T \sum_{k_0} \int \frac{d^3k}{(2\pi)^3} D(k_0, \boldsymbol{k}). \quad \text{(C42)}$$

The computation of the RHS of (C36) is analytically very challenging and numerically cumbersome. It is challenging to separate and regulate divergent contributions. Nevertheless, we can tackle it by invoking some educated approximations:

(i) First, we discard the $(k_0, \boldsymbol{k})$ dependency of $\Pi^{(B)}_b(k_0 = i\omega_n, \boldsymbol{k})$ and consider the static limit; i.e., we take $\Pi^{(B)}_b(k_0 = i\omega_n, \boldsymbol{k}) \simeq \Pi^{(B)}_b(k_0 = 0, \boldsymbol{k} \to 0)$.

(ii) Next, we observe that computing the right-hand side of Eq. (C37) is the same as computing $V^{(1,B)}_b$ defined in Eq. (C5) with m_b^2 being replaced by $m_b^2 + \Pi^{(B)}_b(k_0 = 0, \boldsymbol{k} \to 0)$ to an excellent approximation.

(iii) We have taken $\Pi^{(B)}_{\pi^0} \approx \Pi^{(B)}_{\pi^\pm} \approx \Pi^{(B)}_\sigma$ since their order of magnitude is more or less the same.

Thus, after substituting $m_b^2 \to m_b^2 + \Pi$ in Eqs. (C21), (C22), (C25), and (C26), we get the full bosonic contribution to V_{eff} as

$$V^{(1,B)}_{b,\epsilon} + V^{(1,B)}_{b,\mathrm{Ring},\epsilon} = \frac{|eB|^2}{8\pi^2}\left\{\zeta\left(-1, \frac{1}{2} + \frac{m_b^2 + \Pi}{2|eB|}\right) \times \left(\frac{1}{\epsilon} - \gamma_E - \log\frac{2|eB|}{4\pi\tilde{\Lambda}^2} + 1\right)\right. \quad \text{(C43)}$$

$$\left. + \zeta^{(1,0)}\left(-1, \frac{1}{2} + \frac{m_b^2 + \Pi}{2|eB|}\right)\right\} + \mathcal{O}(\epsilon), \quad \text{(C44)}$$

$$V^{(1,B)}_{b,\mathrm{ThM}} + V^{(1,B)}_{b,\mathrm{Ring,ThM}} = \frac{|eB|}{2\pi^2} T \sum_{\ell=0}^{\infty} \int_0^\infty dk_z \log\left[1 - \exp\left(-\frac{\sqrt{k_z^2 + (2\ell+1)|eB| + m_b^2 + \Pi}}{T}\right)\right]. \quad \text{(C45)}$$

Since $M_f = gv$ with $g, v > 0$, the quark contribution to the effective potential never becomes imaginary. Consequently, the resummation of the quark contribution is not necessary at this point.

APPENDIX D: PERPENDICULAR MOMENTUM INTEGRATIONS

In this section, we perform the general perpendicular integration

$$\mathcal{I}^{(\alpha)}_{\ell,n} \equiv \int \frac{d^2k_\perp}{(2\pi)^2} \exp\left(-\frac{2k_\perp^2}{|q_fB|}\right) k_\perp^{2\alpha} L_\ell\left(\frac{2k_\perp^2}{|q_fB|}\right) L_n\left(\frac{2k_\perp^2}{|q_fB|}\right), \quad \text{(D1)}$$

where ℓ, n, α are integers, $\ell, n, \alpha \geq 0$, and $d^2k_\perp \equiv dk^1dk^2 = d\phi dk_\perp k_\perp$. After a change of variable $\xi = 2k_\perp^2/|q_fB|$, we get

$$\mathcal{I}^{(\alpha)}_{\ell,n} = \frac{|q_fB|}{8\pi}\left(\frac{|q_fB|}{2}\right)^{\alpha}\int_0^{\infty} d\xi \xi^{\alpha} e^{-\xi} L^{(\alpha)}_{\ell}(\xi) L^{(\alpha)}_n(\xi). \quad \text{(D2)}$$

The generalized Laguerre polynomial satisfies the orthogonality relation

$$\int_0^{\infty} dx x^{\alpha} e^{-x} L^{(\alpha)}_{\ell}(x) L^{(\alpha)}_n(x) = \frac{\Gamma(n+\alpha+1)}{n!}\delta_{\ell,n},$$

Thus, we get

$$\mathcal{I}^{(\alpha)}_{\ell,n} = \frac{|q_fB|}{8\pi}\left(\frac{|q_fB|}{2}\right)^{\alpha}\frac{\Gamma(\ell+\alpha+1)}{\ell!}\delta_{\ell,n}. \quad \text{(D3)}$$

The two most important perpendicular integrals in our context are obtained by setting $\alpha = 0,\ 1$ in (D3) as

$$\int \frac{d^2k_\perp}{(2\pi)^2}\exp\left(-\frac{2k_\perp^2}{|q_fB|}\right)L_\ell\left(\frac{2k_\perp^2}{|q_fB|}\right)L_n\left(\frac{2k_\perp^2}{|q_fB|}\right) = \frac{|q_fB|}{8\pi}\delta_{\ell,n}, \quad \text{(D4)}$$

$$\int \frac{d^2k_\perp}{(2\pi)^2}k_\perp^2\exp\left(-\frac{2k_\perp^2}{|q_fB|}\right)L_\ell\left(\frac{2k_\perp^2}{|q_fB|}\right)L_n\left(\frac{2k_\perp^2}{|q_fB|}\right) = \frac{|q_fB|^2}{16\pi}(\ell+1)\delta_{\ell,n}. \quad \text{(D5)}$$

[1] V. Skokov, A. Y. Illarionov, and V. Toneev, Int. J. Mod. Phys. A **24**, 5925 (2009).
[2] W. T. Deng and X. G. Huang, Phys. Rev. C **85**, 044907 (2012).
[3] K. Tuchin, Phys. Rev. C **88**, 024911 (2013).
[4] R. C. Duncan and C. Thompson, Astrophys. J. Lett. **392**, L9 (1992).
[5] M. Joyce and M. E. Shaposhnikov, Phys. Rev. Lett. **79**, 1193 (1997).
[6] A. Brandenburg, Y. He, T. Kahniashvili, M. Rheinhardt, and J. Schober, Astrophys. J. **911**, 110 (2021).
[7] K. Fukushima, D. E. Kharzeev, and H. J. Warringa, Phys. Rev. D **78**, 074033 (2008).
[8] J. O. Andersen, W. R. Naylor, and A. Tranberg, Rev. Mod. Phys. **88**, 025001 (2016).
[9] A. J. Mizher, M. N. Chernodub, and E. S. Fraga, Phys. Rev. D **82**, 105016 (2010).
[10] M. N. Chernodub, Phys. Rev. D **82**, 085011 (2010).
[11] M. N. Chernodub, Phys. Rev. Lett. **106**, 142003 (2011).
[12] H. T. Ding, S. T. Li, S. Mukherjee, A. Tomiya, and X. D. Wang, Proc. Sci. LATTICE2019 (**2020**) 250 [arXiv:2001.05322].
[13] H. T. Ding, S. T. Li, A. Tomiya, X. D. Wang, and Y. Zhang, Phys. Rev. D **104**, 014505 (2021).
[14] A. Ayala, J. L. Hernández, L. A. Hernández, R. L. S. Farias, and R. Zamora, Phys. Rev. D **103**, 054038 (2021).
[15] A. Das and N. Haque, Phys. Rev. D **101**, 074033 (2020).
[16] Y. Hidaka and A. Yamamoto, Phys. Rev. D **87**, 094502 (2013).
[17] S. P. Adhya, M. Mandal, S. Biswas, and P. K. Roy, Phys. Rev. D **93**, 074033 (2016).
[18] S. S. Avancini, W. R. Tavares, and M. B. Pinto, Phys. Rev. D **93**, 014010 (2016).
[19] S. S. Avancini, R. L. S. Farias, M. Benghi Pinto, W. R. Tavares, and V. S. Timóteo, Phys. Lett. B **767**, 247 (2017).
[20] J. O. Andersen, J. High Energy Phys. 10 (2012) 005.
[21] P. Singha, A. Abhishek, G. Kadam, S. Ghosh, and H. Mishra, J. Phys. G **46**, 015201 (2019).
[22] R. Ghosh and M. Kurian, Phys. Rev. C **107**, 034903 (2023).
[23] R. Ghosh and N. Haque, Phys. Rev. D **105**, 11 (2022).
[24] A. Ayala, L. A. Hernández, M. Loewe, and C. Villavicencio, Eur. Phys. J. A **57**, 234 (2021).
[25] A. Sedrakian, R. A. Tripolt, and J. Wambach, Phys. Lett. B **780**, 627 (2018).
[26] A. Das, Lectures on quantum field theory, https://www.worldscientific.com/doi/10.1142/3277#t=aboutBook.
[27] Neven Elezovic, J. Math. Inequalities **9**, 1001 (2015).
[28] G. S. Bali, F. Bruckmann, G. Endrodi, Z. Fodor, S. D. Katz, and A. Schafer, Phys. Rev. D **86**, 071502 (2012).
[29] A. Ayala, R. L. S. Farias, S. Hernández-Ortiz, L. A. Hernández, D. M. Paret, and R. Zamora, Phys. Rev. D **98**, 114008 (2018).
[30] M. L. Bellac, *Thermal Field Theory* (Cambridge University Press, Cambridge, England, 1996), 10.1017/CBO9780511721700.
[31] M. Laine and A. Vuorinen, Lect. Notes Phys. **925**, 1 (2016).
[32] A. Ayala, L. A. Hernández, A. J. Mizher, J. C. Rojas, and C. Villavicencio, Phys. Rev. D **89**, 116017 (2014).
[33] M. E. Carrington, Phys. Rev. D **45**, 2933 (1992).
[34] L. Dolan and R. Jackiw, Phys. Rev. D **9**, 3320 (1974).